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On the convergence of Le Page series in Skohorod space

Youri Davydov∗ and Clément Dombry†

Abstract

We consider the problem of the convergence of the so-called Le Page series in the Skohorod space \( D^d = D([0,1],\mathbb{R}^d) \) and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an \( \alpha \)-stable distribution on \( D^d \) with given spectral measure.

Key words: stable distribution, Le Page series, Skohorod space.

AMS Subject classification. Primary: 60E07 Secondary: 60G52.

1 Introduction

We are interested in the convergence in the Skohorod space \( D^d = D([0,1],\mathbb{R}^d) \) endowed with the \( J_1 \)-topology of random series of the form

\[
X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0,1],
\]

where \( \alpha \in (0,2) \) and

- \((\Gamma_i)_{i \geq 1}\) is the increasing enumeration of the points of a Poisson point process on \([0,+\infty)\) with Lebesgue intensity;

- \((\varepsilon_i)_{i \geq 1}\) is an i.i.d. sequence of real random variables;

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- \((Y_i)_{i \geq 1}\) is an i.i.d. sequence of \(\mathbb{D}^d\)-valued random variables;
- the sequences \((\Gamma_i), (\varepsilon_i)\) and \((Y_i)\) are independent.

Note that a more constructive definition for the sequence \((\Gamma_i)_{i \geq 1}\) is given by

\[
\Gamma_i = \sum_{j=1}^{i} \gamma_j, \quad i \geq 1,
\]

where \((\gamma_i)_{i \geq 1}\) is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of \((\varepsilon_i)\) and \((Y_i)\).

Series of the form (1) are known as Le Page series. For fixed \(t \in [0, 1]\), the convergence in \(\mathbb{R}^d\) of the series (1) is ensured as soon as one of the following conditions is satisfied:

- \(0 < \alpha < 1\), \(\mathbb{E}|\varepsilon_1|^\alpha < \infty\) and \(\mathbb{E}|Y_1(t)|^\alpha < \infty\),
- \(1 \leq \alpha < 2\), \(\mathbb{E}\varepsilon_1 = 0\), \(\mathbb{E}|\varepsilon_1|^\alpha < \infty\) and \(\mathbb{E}|Y_1(t)|^\alpha < \infty\).

Here \(|.|\) denotes the usual Euclidean norm on \(\mathbb{R}\) or on \(\mathbb{R}^d\). The random variable \(X(t)\) has then an \(\alpha\)-stable distribution on \(\mathbb{R}^d\). Conversely, it is well known that any \(\alpha\)-stable distributions on \(\mathbb{R}^d\) admits a representation in terms of Le Page series (see for example Samorodnitsky and Taqqu [9] section 3.9).

There is a vast literature on symmetric \(\alpha\)-stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand [7] or Araujo and Giné [1]). In particular, any symmetric \(\alpha\)-stable distribution on a separable Banach space can be represented as an almost surely convergent Le Page series (see Corollary 5.5 in [7]). The existence of a symmetric \(\alpha\)-stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in [7] for a precise statement. In [3], Davydov, Molchanov and Zuyev consider \(\alpha\)-stable distributions in the more general framework of abstract convex cones.

The space \(\mathbb{D}^d\) equipped with the norm

\[
\|x\| = \sup\{|x_i(t)|, \quad t \in [0, 1], \quad i = 1, \ldots, d\}, \quad x = (x_1, \ldots, x_d) \in \mathbb{D}^d,
\]

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the \(J_1\)-topology. On the other hand, the space \(\mathbb{D}^d\) with the \(J_1\)-topology is Polish, i.e. there exists a metric on \(\mathbb{D}^d\) compatible with the \(J_1\)-topology that makes \(\mathbb{D}^d\) a complete and separable metric space. However, such a metric can not be compatible with the vector space structure since the addition is not continuous in the \(J_1\)-topology. These properties explains why the general theory of stable distributions on separable Banach space can not be applied to the space \(\mathbb{D}^d\).

Nevertheless, in the case when the series (1) converges, the distribution of the sum \(X\) defines an \(\alpha\)-stable distribution on \(\mathbb{D}^d\). We can determine the associated spectral measure \(\sigma\) on the unit sphere \(S^d = \{x \in \mathbb{D}^d; \|x\| = 1\}\). It is given by

\[
\sigma(A) = \frac{\mathbb{E}(|\varepsilon_1|^\alpha|Y_1||^{\alpha}1_{\{\text{sign}(\varepsilon_1)Y_1/\|Y_1\| \in A\}})}{\mathbb{E}(|\varepsilon_1|^\alpha|Y_1||^{\alpha})}, \quad A \in \mathcal{B}(S^d).
\]
This is closely related to regular variations theory (see Hult and Lindskog [5] or Davis and Mikosch [2]): for all $r > 0$ and $A \in \mathcal{B}({\mathbb{S}^d})$ such that $\sigma(\partial A) = 0$, it holds that
\[
\lim_{n \to \infty} n \mathbb{P}\left( \frac{X}{\|X\|} \in A \right) = r^{-\alpha} \sigma(A),
\]
with
\[
b_n = \inf\{r > 0; \mathbb{P}(\|X\| < r) \leq n^{-1}\}, \quad n \geq 1.
\]
The random variable $X$ is said to be regularly varying in $\mathbb{D}^d$ with index $\alpha$ and spectral measure $\sigma$.

In this framework, convergence of the Le Page series (1) in $\mathbb{D}^d$ is known in some particular cases only:

- When $0 < \alpha < 1$, $\mathbb{E}|\epsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1|^\alpha < \infty$, the series (1) converges almost surely uniformly in $[0, 1]$ (see example 4.2 in Davis and Mikoch [2]);

- When $1 \leq \alpha < 2$, the distribution of the $\epsilon_i$’s is symmetric, $\mathbb{E}|\epsilon_1|^\alpha < \infty$ and $Y_i(t) = 1_{[0, t]}(U)$ with $(U_i)_{i \geq 1}$ an i.i.d. sequence of random variables with uniform distribution on $[0, 1]$, the series (1) converges almost surely uniformly on $[0, 1]$ and the limit process $X$ is a symmetric $\alpha$-stable Lévy process (see Rosinski [8]).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in $\mathbb{D}^d$ of the random series (1). Our main result is the following:

**Theorem 1** Suppose that $1 \leq \alpha < 2$,
\[
\mathbb{E}\epsilon_1 = 0 \quad , \quad \mathbb{E}|\epsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}|Y_1|^\alpha < \infty.
\]
Suppose furthermore that there exist $\beta_1, \beta_2 > \frac{1}{2}$ and $F_1, F_2$ nondecreasing continuous functions on $[0, 1]$ such that, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,
\[
\mathbb{E}|Y_1(t_2) - Y_1(t_1)|^2 \leq |F_1(t_2) - F_1(t_1)|^{\beta_1}, \tag{2}
\]
\[
\mathbb{E}|Y_1(t_2) - Y_1(t)|^2|Y_1(t) - Y_1(t_1)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \tag{3}
\]
Then, the Le Page series (1) converges almost surely in $\mathbb{D}^d$.

The proof of this Theorem is detailed in the next section. We provide hereafter a few cases where Theorem 1 can be applied.

**Example 1** The example considered by Davis and Mikosh [2] follows easily from Theorem 1: let $U$ be a random variable with uniform distribution on $[0, 1]$ and consider $Y_1(t) = 1_{[0, t]}(U)$, $t \in [0, 1]$. Then, for $0 \leq t_1 \leq t \leq t_2 \leq 1$,
\[
\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = t_2 - t_1 \quad \text{and} \quad \mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = 0,
\]
so that conditions (2) and (3) are satisfied.
Example 2 Example 1 can be generalized in the following way: let $p \geq 1$, $(U_i)_{1 \leq i \leq p}$ independent random variables on $[0, 1]$ and $(R_i)_{1 \leq i \leq p}$ random variables on $\mathbb{R}^d$. Consider

$$Y_1(t) = \sum_{i=1}^{p} R_i \mathbf{1}_{[0,t]}(U_i).$$

Assume that for each $i \in \{1, \ldots, p\}$, the cumulative distribution function $F_i$ of $U_i$ is continuous on $[0, 1]$. Assume furthermore that there is some $M > 0$ such that for all $i \in \{1, \ldots, p\}$

$$\mathbb{E}[R_i^4 | \mathcal{F}_U] \leq M \quad \text{almost surely},$$

where $\mathcal{F}_U = \sigma(U_1, \ldots, U_p)$. This is for example the case when the $R_i$’s are uniformly bounded by $M^{1/4}$ or when the $R_i$’s have finite fourth moment and are independent of the $U_i$’s. Simple computations entails that under condition (4), it holds for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 \leq M^{1/2}p^2|F(t_2) - F(t_1)|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 \leq Mp^4|F(t_2) - F(t_1)|^4.$$ 

with $F(t) = \sum_{i=1}^{p} F_i(t)$. So conditions (2) and (3) are satisfied and Theorem (1) can be applied in this case.

Example 3 A further natural example is the case when $Y_1(t)$ is a Poisson process with intensity $\lambda > 0$ on $[0, 1]$. Then, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = \lambda|t_2 - t_1| + \lambda^2|t_2 - t_1|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = (\lambda|t_2 - t| + \lambda^2|t_2 - t|^2)(\lambda|t - t_1| + \lambda^2|t - t_1|^2)$$

and we easily see that conditions (2) and (3) are satisfied.

2 Proof

For the sake of clarity, we divide the proof of Theorem 1 into five steps.

Step 1. According to Lemma 1.5.1 in [9], it holds almost surely that for $k$ large enough

$$|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}| \leq 2\alpha^{-1}k^{-1/\alpha} \sqrt{\ln \ln k / k}. \quad (5)$$

This implies the a.s. convergence of the series

$$\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}| \|\varepsilon_i\|Y_i\| < \infty. \quad (6)$$
The series (6) has indeed nonnegative terms, and (5) implies that the following conditional expectation is finite,

\[
E \left[ \sum_{i=1}^{\infty} \left| \Gamma_i^{-1/\alpha} - i^{-1/\alpha} \right| |\epsilon_i| \|Y_i\| \right | \mathcal{F}_\Gamma = E|\epsilon_1| E\|Y_1\| \sum_{i=1}^{\infty} \left| \Gamma_i^{-1/\alpha} - i^{-1/\alpha} \right| 
\]

where \( \mathcal{F}_\Gamma = \sigma(\Gamma_i, i \geq 1) \).

This proves that (6) holds true and it is enough to prove the a.s. convergence in \( \mathbb{D}^d \) of the series

\[
Z(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \epsilon_i Y_i(t), \quad t \in [0, 1], \quad (7)
\]

**Step 2.** Next, consider

\[
\tilde{Z}(t) = \sum_{i=1}^{\infty} i^{-1/\alpha} \tilde{\epsilon}_i Y_i(t), \quad t \in [0, 1]. \quad (8)
\]

with

\[
\tilde{\epsilon}_i = \epsilon_i \mathbf{1}_{|\epsilon_i| \leq i^\alpha}, \quad i \geq 1.
\]

We prove that the series (7) and (8) differ only by a finite number of terms. We have indeed

\[
\sum_{i=1}^{\infty} \mathbb{P}(\tilde{\epsilon}_i \neq \epsilon_i) = \sum_{i=1}^{\infty} \mathbb{P}(|\epsilon_i|^\alpha > i) \leq \mathbb{E}|\epsilon_1|^\alpha < \infty
\]

and the Borel-Cantelli Lemma implies that almost surely \( \tilde{\epsilon}_i = \epsilon_i \) for \( i \) large enough. So, both series (7) and (8) have the same nature and it is enough to prove the convergence in \( \mathbb{D}^d \) of the series (8).

**Step 3.** As a preliminary for step 4, we prove several estimates involving the moments of the random variables \( (\tilde{\epsilon}_i)_{i \geq 1} \). First, for all \( m > \alpha \),

\[
C(\alpha, m) := \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(|\tilde{\epsilon}_i|^m) < \infty. \quad (9)
\]

We have indeed

\[
C(\alpha, m) = \sum_{i=1}^{\infty} i^{-m/\alpha} \mathbb{E}(\min(|\epsilon_i|^m, |\epsilon_i|^\alpha))
\]

\[
= \mathbb{E}(\min(|\epsilon_1|^m, \sum_{i=1}^{\infty} i^{-m/\alpha} 1_{|\epsilon_i|^\alpha}))
\]

\[
\leq C \mathbb{E}(|\epsilon_1|^m |\epsilon_1|^{m-\alpha}) = C \mathbb{E}(|\epsilon_1|^\alpha) < \infty
\]
where the constant $C = \sup_{x>0} x^{m/\alpha - 1} \sum_{i \geq x} i^{-m/\alpha}$ is finite since for $m > \alpha$

$$\lim_{x \to \infty} x^{m/\alpha - 1} \sum_{i \geq x} i^{-m/\alpha} = \frac{\alpha}{m - \alpha}. $$

Similarly, we also have

$$C(\alpha, 1) := \sum_{i=1}^{\infty} i^{-1/\alpha}\|\mathbb{E}(\tilde{\varepsilon}_i)\| < \infty. \quad (10)$$

Indeed, the assumption $\mathbb{E}\varepsilon_i = 0$ implies $\mathbb{E}(\tilde{\varepsilon}_i) = \mathbb{E}(\varepsilon_i1_{\{\varepsilon_i|>1\}})$. Hence,

$$\sum_{i=1}^{\infty} i^{-1/\alpha}\|\mathbb{E}(\tilde{\varepsilon}_i)\| \leq \sum_{i=1}^{\infty} i^{-1/\alpha}\mathbb{E}(\varepsilon_i1_{\{\varepsilon_i|>i^{1/\alpha}\}}) = \mathbb{E}\left(\left\{\mathbb{E}(\varepsilon_i1_{\{\varepsilon_i|>i^{1/\alpha}\}})\right\} \right) \leq C'\mathbb{E}|\varepsilon_1|^\alpha < \infty$$

where the constant $C' = \sup_{x>0} x^{1/\alpha - 1} \sum_{i=1}^{x} i^{-1/\alpha}$ is finite.

Step 4. For $n \geq 1$, consider the partial sum

$$\tilde{Z}_n(t) = \sum_{i=1}^{n} i^{-1/\alpha}\tilde{\varepsilon}_i Y_i(t), \quad t \in [0, 1]. \quad (11)$$

We prove that the sequence of processes $(\tilde{Z}_n)_{n\geq1}$ is tight in $\mathbb{D}^d$. Following Theorem 3 in Gikhman and Skohorod [4] chapter 6 section 3, it is enough to show that there exists $\beta > 1/2$ and a non decreasing continuous function $F$ on $[0, 1]$ such that

$$\mathbb{E}|\tilde{Z}_n(t_2) - \tilde{Z}_n(t_1)|^2 \leq |F(t_2) - F(t_1)|^{2\beta}, \quad (12)$$

for all $0 \leq t_1 \leq t \leq t_2 \leq 1$. Remark that in Gikhman and Skohorod [4], the result is stated only for $F(t) \equiv t$. However, the case of a general continuous non decreasing function $F$ follows easily from a simple change of variable.

We use the notations $Y(t) = (Y^p(t))_{1 \leq p \leq d}$, $[1, n] = \{1, \cdots, n\}$ and
\( i = (i_1, i_2, i_3, i_4) \in [1, n]^4 \). We have

\[
\mathbb{E}[\tilde{Z}_n(t_2) - \tilde{Z}_n(t_1)]^2 \mathbb{E}[\tilde{Z}_n(t) - \tilde{Z}_n(t_1)]^2
= \sum_{i=1}^n i^{-1/\alpha} \mathbb{E}[\tilde{Y}_i(t) - \tilde{Y}_i(t_1)]^2 \left| \sum_{j=1}^n j^{-1/\alpha} \tilde{Y}_j(t_2) - \tilde{Y}_j(t) \right|^2
= \sum_{1 \leq p, q \leq d \leq [1, n]^4} (i_1^2 i_2 i_3 i_4)^{-1/\alpha} \mathbb{E}[\tilde{Y}_{i_1}(t) - \tilde{Y}_{i_1}(t_1)] \mathbb{E}[\tilde{Y}_{i_2}(t_2) - \tilde{Y}_{i_2}(t)] (Y_{i_2}(t) - Y_{i_2}(t_1))(Y_{i_3}(t_2) - Y_{i_3}(t))(Y_{i_4}(t_2) - Y_{i_4}(t))
\leq d^2 \sum_{i \in [1, n]^4} (i_1 i_2 i_3 i_4)^{-1/\alpha} \mathbb{E}[\tilde{Y}_{i_1}(t) - \tilde{Y}_{i_1}(t_1)] \mathbb{E}[\tilde{Y}_{i_2}(t_2) - \tilde{Y}_{i_2}(t)] D_1(t, t_1, t_2)
\]

where

\[
D_1(t, t_1, t_2) = \mathbb{E}[Y_{i_1}(t) - Y_{i_1}(t_1)] |Y_{i_2}(t) - Y_{i_2}(t_1)| |Y_{i_3}(t_2) - Y_{i_3}(t)| |Y_{i_4}(t_2) - Y_{i_4}(t)|.
\]

Consider \( \sim_1 \) the equivalence relation on \( \{1, \cdots, 4\} \) defined by

\[
j \sim_1 j' \quad \text{if and only if} \quad i_j = i_{j'}.
\]

Let \( \mathcal{P} \) be the set of all partitions of \( \{1, \cdots, 4\} \) and \( \tau(i) \) be the partition of \( \{1, 2, 3, 4\} \) given by the equivalence classes of \( \sim_1 \). We introduce these definitions because, since the \( Y_i \)'s are i.i.d., the term \( D_1(t, t_1, t_2) \) depends on \( i \) only through the associated partition \( \tau(i) \). For example, if \( \tau(i) = \{1, 2, 3, 4\} \), i.e. if \( i_1 = i_2 = i_3 = i_4 \), then

\[
D_1(t, t_1, t_2) = \mathbb{E}[Y_1(t) - Y_1(t_1)]^2 |Y_1(t_2) - Y_1(t)|^2.
\]

Or if \( \tau(i) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \), i.e. if the indices \( i_1, \cdots, i_4 \) are pairwise distinct, then

\[
D_1(t, t_1, t_2) = (\mathbb{E}[Y_1(t) - Y_1(t_1)] |\mathbb{E}[Y_1(t_2) - Y_1(t)]|^2.
\]

For \( \tau \in \mathcal{P} \), we denote by \( D_\tau(t, t_1, t_2) \) the common value of the terms \( D_1(t, t_1, t_2) \) corresponding to indices \( i \) such that \( \tau(i) = \tau \). Define also

\[
S_{n, \tau} = \sum_{i \in \{1, \cdots, n\}^4; \tau(i) = \tau} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}[\tilde{Y}_{i_1}(t) - \tilde{Y}_{i_1}(t_1)] |\mathbb{E}[\tilde{Y}_{i_2}(t_2) - \tilde{Y}_{i_2}(t)]|.
\]

With these notations, equation (15) can be rewritten as

\[
\mathbb{E}[\tilde{Z}_n(t_2) - \tilde{Z}_n(t)]^2 |\tilde{Z}_n(t) - \tilde{Z}_n(t_1)|^2 \leq d^2 \sum_{\tau \in \mathcal{P}} S_{n, \tau} D_\tau(t, t_1, t_2).
\]

Under conditions (2) and (3), we will prove that for each \( \tau \in \mathcal{P} \), there exist \( \beta_\tau > 1/2 \), a non decreasing continuous function \( F_\tau \) on \([0, 1]\) and a constant \( S_\tau > 0 \) such that

\[
D_\tau(t, t_1, t_2) \leq |F_\tau(t_1) - F_\tau(t_2)|^{2\beta_\tau}, \quad 0 \leq t_1 \leq t \leq t_2, \quad (17)
\]
and
\[ S_{n, \tau} \leq S_{\tau}, \quad n \geq 1. \tag{18} \]

Equations (16), (17) and (18) together imply inequality (12) for some suitable choices of \( \beta > 1/2 \) and \( F \).

It remains to prove inequalities (17) and (18). If \( \tau = \{1, 2, 3, 4\} \),
\[ D_{\tau}(t, t_1, t_2) \leq \mathbb{E}|Y_1(t) - Y_1(t_1)|^2|Y_1(t_2) - Y_1(t)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2/3} \]
and
\[ S_n^\tau = \sum_{i=1}^n i^{-4/\alpha} \mathbb{E}\tilde{\varepsilon}_i^4 \leq C(\alpha, 4). \]

If \( \tau = \{1\} \cup \{2\} \cup \{3\} \cup \{4\} \), Cauchy-Schwartz inequality entails
\[ D_{\tau}(t, t_1, t_2) \leq (\mathbb{E}|Y_1(t) - Y_1(t_1)|^2|Y_1(t_2) - Y_1(t)|^2 \leq |F_1(t_2) - F_1(t_1)|^{2/3} \]
and
\[ S_n^\tau \leq \sum_{1 \leq i \neq j \leq n} (i^3 j^3)^{-1/\alpha} \mathbb{E}\tilde{\varepsilon}_i^3 \mathbb{E}\tilde{\varepsilon}_j \leq C(\alpha, 3) C(\alpha, 3). \]

or for \( \tau = \{1, 2\} \cup \{3\} \cup \{4\} \),
\[ D_{\tau}(t, t_1, t_2) = \mathbb{E}|Y_1(t) - Y_1(t_1)|^2(\mathbb{E}|Y_1(t_2) - Y_1(t)|^2 \leq |F_1(t) - F_1(t_1)|^{2/3} |F_1(t_2) - F_1(t)|^{2/3} \]
and
\[ S_n^\tau \leq \sum_{1 \leq i \neq j \neq k \leq n} (i^2 j^2 k)^{-1/\alpha} \mathbb{E}\tilde{\varepsilon}_i^2 \mathbb{E}\tilde{\varepsilon}_j \mathbb{E}\tilde{\varepsilon}_k \leq C(\alpha, 2) C(\alpha, 1)^2. \]

Similar computations can be checked in all remaining cases. The cardinality of \( \mathcal{P} \) is equal to 13.

**Step 5.** We prove Theorem 1. For each fixed \( t \in [0, 1] \), Kolmogorov’s three-series Theorem implies that \( \tilde{Z}_n(t) \) converge almost surely as \( n \to \infty \). So the finite-dimensional distributions of \( (\tilde{Z}_n)_{n \geq 1} \) converge. The tightness in \( \mathbb{D}^d \) of the sequence has already been proved in step 4, so \( (\tilde{Z}_n)_{n \geq 1} \) weakly convergence in \( \mathbb{D}^d \) as \( n \to \infty \). We then apply Theorem 1 in Kallenberg [6] and deduce that \( \tilde{Z}_n \) converges almost surely in \( \mathbb{D}^d \). In view of step 1 and step 2, this yields the almost sure convergence of the series (1). \( \square \)
References


