

AFFINE OPEN SUBSETS IN \mathbb{A}^3 WITHOUT THE CANCELLATION PROPERTY

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ABSTRACT. We give families of examples of principal open subsets of the affine space \mathbb{A}^3 which do not have the cancellation property. We show as a by-product that the cylinders over Koras-Russell threefolds of the first kind have a trivial Makar-Limanov invariant.

INTRODUCTION

The generalized Cancellation Problem asks if two algebraic varieties X and Y with isomorphic cylinders $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are isomorphic themselves. Although the answer turns out to be affirmative for a large class of varieties including the case when one of the varieties is the affine plane \mathbb{A}^2 [10, 14], counter-examples exist for affine varieties in any dimension ≥ 2 , and the particular case when one of the two varieties is an affine space \mathbb{A}^n , $n \geq 3$, still remains a widely open problem.

The first counter-example for complex affine varieties has been constructed by Danielewski [1] in 1989: he exploited the fact that the non isomorphic affine surfaces $S_1 = \{xz = y^2 - 1\}$ and $S_2 = \{x^2z = y^2 - 1\}$ in $\mathbb{A}_{\mathbb{C}}^3$ can be equipped with free actions of the additive group \mathbb{G}_a admitting geometric quotients in the form of non trivial \mathbb{G}_a -bundles $\rho_i : S_i \rightarrow \tilde{\mathbb{A}}^1$, $i = 1, 2$ over the affine line with a double origin. It then follows that the fiber product $S_1 \times_{\tilde{\mathbb{A}}^1} S_2$ inherits simultaneously the structure of a \mathbb{G}_a -bundle over S_1 and S_2 via the first and the second projection respectively, but since S_1 and S_2 are both affine, the latter are both trivial, providing isomorphisms $S_1 \times \mathbb{A}^1 \simeq S_1 \times_{\tilde{\mathbb{A}}^1} S_2 \simeq S_2 \times \mathbb{A}^1$. Since then, Danielewski's fiber product trick has been the source of many new counter-examples in any dimension [7, 3, 8, 5], some of these being very close to affine spaces either from an algebraic or a topological point of view.

However, a counter-example over the field of real numbers was constructed earlier by Hochster [9] using the algebraic counterpart of the classical fact from differential geometry that the tangent bundle of the real sphere S^2 is non trivial but 1-stably trivial. His argument actually applies more generally to the situation when a finitely generated domain R over a field k admits a non trivial projective module M of rank $n - 1 \geq 1$ such that $M \oplus R \simeq R^{\oplus n} = R^{\oplus n-1} \oplus R$. Indeed, these hypotheses immediately imply that the varieties $X = \text{Spec}_R(\text{Sym}(M))$ and $Y = \text{Spec}(R[x_1, \dots, x_n])$ are not isomorphic as schemes over $Z = \text{Spec}(R)$ while their cylinders are. Of course, there is no reason in general that X and Y are not isomorphic as k -varieties, but this holds for instance when Z does not admit any dominant morphism from an affine space \mathbb{A}_k^n since then any isomorphism between X and Y necessarily descends to an automorphism of Z ([10, 2]). Recently, Jelonek [11] gave revival to Hochster's idea by constructing families of examples of non uniruled affine open subsets of affine spaces of any dimension ≥ 8 with 1-stably trivial but non trivial vector bundles, which fail the cancellation property.

While affine open subsets of affine spaces of dimension ≤ 2 always have the cancellation property (see e.g. *loc.cit.*), we derive in this note from a variant of Danielewski's fiber product trick that cancellation already fails for suitably chosen principal open subsets of \mathbb{A}^3 .

As an application of our construction we also obtain that all cylinders over Koras-Russell threefolds $X_{d,k,l} = \{x^d z = y^l + x - t^k = 0\} \subset \mathbb{A}^4$, $d \geq 2$ and $2 \leq l < k$ relatively prime [12], have a trivial Makar-Limanov invariant [13].

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1. PRINCIPAL OPEN SUBSETS IN \mathbb{A}^3 WITHOUT THE CANCELLATION PROPERTY

For every $d \geq 1$ and $l \geq 2$, we denote by $B_{d,l}$ the surface in $\mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$ defined by the equation $f_{d,l} = y^l + x - x^d z = 0$ and by $U_{d,l} = \mathbb{A}^3 \setminus B_{d,l} \simeq \text{Spec}(\mathbb{C}[x, y, z]_{f_{d,l}})$ its open complement. By construction, $U_{d,l}$ comes equipped with a flat isotrivial fibration $f_{d,l}|_{U_{d,l}}: U_{d,l} \rightarrow \mathbb{A}_*^1 = \text{Spec}(\mathbb{C}[t^{\pm 1}])$ with closed fibers isomorphic to the surface $S_{d,l} \subset \mathbb{A}^3 = \text{Spec}(\mathbb{C}[X, Y, Z])$ defined by the equation $X^d Z = Y^l + X - 1$. A surface $S_{d,l}$ having no non constant invertible function, an isomorphism $\varphi: U_{d,l} \xrightarrow{\sim} U_{d',l'}$ necessarily maps closed fibers of $f_{d,l}$ isomorphically onto that of $f_{d',l'}$. But since $S_{d,l}$ is isomorphic to $S_{d',l'}$ if and only if $(d', l') = (d, l)$ (see e.g. [6, Theorem 3.2 and Proposition 3.6]), it follows that the threefolds $U_{d,l}$, $d \geq 1$, $l \geq 2$, are pairwise non isomorphic. In contrast, we have the following result:

Theorem 1. *For every fixed $l \geq 2$, the fourfolds $U_{d,l} \times \mathbb{A}^1$, $d \geq 1$, are all isomorphic.*

Proof. We exploit the fact that every $U_{d,l}$ admits a free \mathbb{G}_a -action defined by the locally nilpotent derivation $x^d \partial_y + ly^{l-1} \partial_z$ of its coordinate ring $\mathbb{C}[x, y, z]_{f_{d,l}}$. A free \mathbb{G}_a -action being locally trivial in the étale topology, it follows that a geometric quotient $\nu_{d,l}: U_{d,l} \rightarrow \mathfrak{S}_{d,l} = U_{d,l}/\mathbb{G}_a$ exists in the form of an étale locally trivial \mathbb{G}_a -bundle over a certain algebraic space $\mathfrak{S}_{d,l}$. Then it is enough to show that for every fixed $l \geq 2$, the algebraic spaces $\mathfrak{S}_{d,l}$ are all isomorphic, say to a fixed algebraic space \mathfrak{S}_l . Indeed, if so, then for every $d, d' \geq 1$, the fiber product $U_{d,l} \times_{\mathfrak{S}_l} U_{d',l}$ will be simultaneously a \mathbb{G}_a -bundle over $U_{d,l}$ and $U_{d',l}$ via the first and second projection respectively whence will be simultaneously isomorphic to the trivial \mathbb{G}_a -bundles $U_{d,l} \times \mathbb{A}^1$ and $U_{d',l} \times \mathbb{A}^1$ as $U_{d,l}$ and $U_{d',l}$ are both affine.

The algebraic spaces $\mathfrak{S}_{d,l}$ can be described explicitly as follows: one checks that the isotrivial fibration $f_{d,l}: U_{d,l} \rightarrow \mathbb{A}_*^1$ becomes trivial on the Galois étale cover $\xi_l: \mathbb{A}_*^1 = \text{Spec}(\mathbb{C}[u^{\pm 1}]) \rightarrow \mathbb{A}_*^1$, $u \mapsto t = u^l$, with isomorphism $\Phi_{d,l}: S_{d,l} \times_{\mathbb{A}_*^1} \mathbb{A}_*^1 \xrightarrow{\sim} U_{d,l} \times_{\mathbb{A}_*^1} \mathbb{A}_*^1$ given by $(X, Y, Z, u) \mapsto (u^l X, uY, u^{(1-d)l} Z, u)$. The group μ_l of l -th roots of unity acts freely on $S_{d,l} \times_{\mathbb{A}_*^1}$ by $\varepsilon \cdot (X, Y, Z, u) = (X, \varepsilon^{-1} Y, Z, \varepsilon u)$ and the μ_l -invariant morphism $\pi_{d,l} = \text{pr}_1 \circ \Phi_{d,l}: S_{d,l} \times_{\mathbb{A}_*^1} \rightarrow U_{d,l}$ descends to an isomorphism $(S_{d,l} \times_{\mathbb{A}_*^1})/\mu_l \simeq U_{d,l}$. The \mathbb{G}_a -action on $U_{d,l}$ lifts via the proper étale morphism $\pi_{d,l}$ to the free \mathbb{G}_a -action on $S_{d,l} \times_{\mathbb{A}_*^1}$ commuting with that of μ_l defined by the locally nilpotent derivation $u^{ld-1}(X^d \partial_Y + lY^{l-1} \partial_Z)$ of its coordinate ring $\mathbb{C}[X, Y, Z]/(X^d Z - Y^l - X + 1)[u^{\pm 1}]$. The principal divisor $\{X = 0\}$ of $S_{d,l} \times_{\mathbb{A}_*^1}$ is \mathbb{G}_a -invariant and it decomposes into the disjoint union of irreducible divisors $D_\eta = \{X = Y - \eta = 0\}_{\eta \in \mu_l} \simeq \text{Spec}(\mathbb{C}[Z][u^{\pm 1}])$ on which μ_l acts by $D_\eta \ni (Z, u) \mapsto (Z, \varepsilon u) \in D_{\varepsilon \eta}$. Now a similar argument as in [7, Lemma 1.2] implies that for every $\eta \in \mu_l$, the \mathbb{G}_a -invariant morphism $\text{pr}_X \times \text{id}: S_{d,l} \times_{\mathbb{A}_*^1} \rightarrow \mathbb{A}^1 \times \mathbb{A}_*^1$ restricts on $(S_{d,l} \times_{\mathbb{A}_*^1}) \setminus \bigcup_{\varepsilon \in \mu_l \setminus \{1\}} D_\varepsilon$ to a trivial \mathbb{G}_a -bundle over $\mathbb{A}^1 \times \mathbb{A}_*^1$. Letting $C(l)$ be the scheme over $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[X])$ obtained by gluing l copies C_η , $\eta \in \mu_l$, of $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[X])$ outside their respective origins, it follows that $\text{pr}_X \times \text{id}$ factors through a μ_l -equivariant \mathbb{G}_a -bundle $\rho_{d,l} \times \text{id}: S_{d,l} \times_{\mathbb{A}_*^1} \rightarrow C(l) \times \mathbb{A}_*^1$, where μ_l acts freely on $C(l) \times \mathbb{A}_*^1$ by $C_\eta \times \mathbb{A}_*^1 \ni (X, u) \mapsto (X, \varepsilon u) \in C_{\varepsilon \eta} \times \mathbb{A}_*^1$.

A quotient $(C(l) \times \mathbb{A}_*^1)/\mu_l$ exist in the category of algebraic spaces in the form of a principal μ_l -bundle $\sigma_l: C(l) \times \mathbb{A}_*^1 \rightarrow \mathfrak{S}_l$, and the above description implies that $\rho_{d,l} \times \text{id}$ descends to an étale locally trivial \mathbb{G}_a -bundle $\tilde{\nu}_{d,l}: U_{d,l} \rightarrow \mathfrak{S}_l$ for which the diagram

$$\begin{array}{ccc} S_{d,l} \times_{\mathbb{A}_*^1} & \xrightarrow{\pi_{d,l}} & U_{d,l} \simeq (S_{d,l} \times_{\mathbb{A}_*^1})/\mu_l \\ \rho_{d,l} \times \text{id} \downarrow & & \downarrow \tilde{\nu}_{d,l} \\ C(l) \times \mathbb{A}_*^1 & \xrightarrow{\sigma_l} & \mathfrak{S}_l, \end{array}$$

is cartesian. By virtue of the universal property of categorical quotients one has necessarily $\mathfrak{S}_{d,l} \simeq \mathfrak{S}_l$ for every $d \geq 1$. In particular, the isomorphy type of $\mathfrak{S}_{d,l}$ depends only on l , which completes the proof. \square

Remark 2. The algebraic spaces $\mathfrak{S}_l = (C(l) \times \mathbb{A}_*^1)/\mu_l$, $l \geq 2$, considered in the proof above cannot be schemes: indeed, otherwise the image in \mathfrak{S}_l of the point $(0, 1) \in C_1 \times \mathbb{A}_*^1 \subset C(l) \times \mathbb{A}_*^1$ would have a Zariski open affine neighborhood V . But then the inverse image of V by the finite étale cover $\sigma_l: C(l) \times \mathbb{A}_*^1 \rightarrow \mathfrak{S}_l$ would be a μ_l -invariant affine open neighborhood of $(0, 1)$ in $C(l) \times \mathbb{A}_*^1$, which

is absurd since $(0, 1)$ does not even have a separated μ_l -invariant open neighborhood in $C(l) \times \mathbb{A}_*^1$. This implies in turn that the free \mathbb{G}_a -action on $U_{d,l}$ defined by the locally nilpotent derivation $x^d \partial_y + ly^{l-1} \partial_z$ is not locally trivial in the Zariski topology. In contrast, the latter property holds for its lift to $S_{d,l} \times \mathbb{A}_*^1$ via the étale Galois cover $\pi_{d,l} : S_{d,l} \times \mathbb{A}_*^1 \rightarrow U_{d,l}$.

Remark 3. In Danielewski's construction for the surfaces $S_i = \{x^i z = y^2 - 1\} \subset \mathbb{A}^3$, $i = 1, 2$, the geometric quotients $S_i/\mathbb{G}_a \simeq \tilde{\mathbb{A}}^1$, $i = 1, 2$, were obtained from the categorical quotients $S_i//\mathbb{G}_a = \text{Spec}(\mathbb{C}[x])$ taken in the category of affine schemes by replacing the origin by two copies of itself, one for each orbit in the zero fiber of the quotient morphism $q = \text{pr}_x : S_i \rightarrow \mathbb{A}^1$. For the threefolds $U_{d,l}$, the difference between the quotients $U_{d,l}//\mathbb{G}_a = \text{Spec}(\mathbb{C}[x, y, z]_{f_{d,l}}^{\mathbb{G}_a})$ taken in the category of (affine) schemes and the geometric quotients $\mathfrak{S}_l = U_{d,l}/\mathbb{G}_a$ is very similar : indeed, we may identify $U_{d,l}$ with the closed subvariety of $\mathbb{A}^3 \times \mathbb{A}_*^1 = \text{Spec}(\mathbb{C}[x, y, z][t^{\pm 1}])$ defined by the equation $x^d z = y^l + x - t$ in such a way that $f_{d,l} : U_{d,l} \rightarrow \mathbb{A}_*^1$ coincides with the projection $\text{pr}_t|_{U_{d,l}}$. Then, the kernel of the locally nilpotent derivation $x^d \partial_y + ly^{l-1} \partial_z$ of the coordinate ring of $U_{d,l}$ coincides with the subalgebra $\mathbb{C}[x, t^{\pm 1}]$ and so, the \mathbb{G}_a -invariant morphism $q = \text{pr}_{x,t} : U_{d,l} \rightarrow \mathbb{A}^1 \times L = \text{Spec}(\mathbb{C}[x][t^{\pm 1}])$ is a categorical quotient in the category of affine schemes. One checks easily that q restricts to a trivial \mathbb{G}_a -bundle over the principal open subset $\{x \neq 0\}$ of $\mathbb{A}^1 \times \mathbb{A}_*^1$ whereas the inverse image of the punctured line $\{x = 0\} \simeq L$ is isomorphic to $\tilde{L} \times \mathbb{A}^1 = \text{Spec}(\mathbb{C}[y, t^{\pm 1}]/(y^l - t)[z])$ where \mathbb{G}_a acts by translations on the second factor. So we may interpret the geometric quotient $\mathfrak{S}_l = U_{d,l}/\mathbb{G}_a$ as being obtained from $U_{d,l}//\mathbb{G}_a = \mathbb{A}^1 \times L$ by replacing the punctured line $\{x = 0\} \simeq L$ not by l disjoint copies of itself but, instead, by the total space \tilde{L} of the nontrivial étale Galois cover $\text{pr}_t : \tilde{L} \rightarrow L$.

The Koras-Russell threefolds $X_{d,k,l}$ are smooth complex affine varieties defined by equations of the form $x^d z = y^l + x - t^k$, where $d \geq 2$ and $2 \leq l < k$ are relatively prime.¹ While all diffeomorphic to the euclidean space \mathbb{R}^6 , none of these threefold is algebraically isomorphic to the affine \mathbb{A}^3 . Indeed, it was established by Kaliman and Makar-Limanov [13, 12] that they have fewer algebraic \mathbb{G}_a -actions than the affine space \mathbb{A}^3 : the subring $\text{ML}(X_{d,k,l})$ of their coordinate ring consisting of regular functions invariant under all algebraic \mathbb{G}_a -actions on $X_{d,k,l}$ is equal to the polynomial ring $\mathbb{C}[x]$, while $\text{ML}(\mathbb{A}^3)$ is *trivial*, consisting of constants only. However, it was observed by the author in [4] that the Makar-Limanov invariant ML fails to distinguish the cylinder over the so-called Russell cubic $X_{2,2,3}$ from the affine space \mathbb{A}^4 . This phenomenon holds more generally for cylinders over all Koras-Russell threefolds $X_{d,k,l}$:

Corollary 4. *All the cylinders $X_{d,k,l} \times \mathbb{A}^1$ have a trivial Makar-Limanov invariant.*

Proof. We consider $X_{d,k,l} \times \mathbb{A}^1$ as the subvariety of $\text{Spec}(\mathbb{C}[x, y, z, t][v])$ defined by the equation $f_{d,l} - t^k = 0$. Since $\text{ML}(X_{d,k,l} \times \mathbb{A}^1) \subset \text{ML}(X_{d,k,l}) = \mathbb{C}[x]$, it is enough to construct a locally nilpotent derivation of $\mathbb{C}[x, y, z][v]/(f_{d,l} - t^k)$ which does not have x in its kernel. One checks easily that $\text{ML}(U_{1,l}) = \mathbb{C}[f_{1,l}^{\pm 1}]$ is the intersection of the kernels of the locally nilpotent derivations $x \partial_y + ly^{l-1} \partial_z$ and $ly^{l-1} \partial_x + (z - 1) \partial_y$ of $\mathbb{C}[x, y, z]_{f_{1,l}}$. Theorem 1 above implies in particular that $\text{ML}(U_{d,l} \times \mathbb{A}^1) \simeq \text{ML}(U_{1,l} \times \mathbb{A}^1) = \mathbb{C}[f_{1,l}^{\pm 1}]$ and so, there exists a locally nilpotent derivation $\delta_{d,l}$ of $\Gamma(U_{d,l} \times \mathbb{A}^1, \mathcal{O}_{U_{d,l} \times \mathbb{A}^1}) = \mathbb{C}[x, y, z]_{f_{d,l}}[v]$ which does not have x in its kernel. Up to multiplying it by a suitable power of $f_{d,l} \in \text{Ker}(\delta_{d,l})$, we may further assume that $\delta_{d,l}$ is the extension to $\mathbb{C}[x, y, z]_{f_{d,l}}[v]$ of a locally nilpotent derivation of $\mathbb{C}[x, y, z][v]$ which has $f_{d,l}$ but not x in its kernel. This implies in particular that $B_{d,l} \times \mathbb{A}^1 = \text{Spec}(\mathbb{C}[x, y, z]/(f_{d,l})[v])$ is invariant under the corresponding \mathbb{G}_a -action on $\mathbb{A}^4 = \text{Spec}(\mathbb{C}[x, y, z][v])$. The projection $p = \text{pr}_{x,y,z,v} : X_{d,k,l} \times \mathbb{A}^1 \rightarrow \mathbb{A}^4$ being a finite Galois cover with branch locus $B_{d,l} \times \mathbb{A}^1$, it follows that the \mathbb{G}_a -action on \mathbb{A}^4 lifts to a one on $X_{d,k,l} \times \mathbb{A}^1$ for which $p : X_{d,k,l} \times \mathbb{A}^1 \rightarrow \mathbb{A}^4$ is \mathbb{G}_a -equivariant. By construction, the corresponding locally nilpotent derivation of $\mathbb{C}[x, y, z][v]/(f_{d,l} - t^k)$ does not have x in its kernel. \square

In the proof above, we used the following classical fact that we include here because of a lack of an appropriate reference.

¹These are called Koras-Russell threefolds of the first kind in [15].

Lemma 5. *Let X be a variety defined over a field of characteristic zero and equipped with a non trivial \mathbb{G}_a -action, let Z be a normal variety and let $p : Z \rightarrow X$ be a finite surjective morphism. Suppose that there exists a \mathbb{G}_a -invariant affine open subvariety U of X over which p restricts to an étale morphism. Then there exists a unique \mathbb{G}_a -action on Z for which $p : Z \rightarrow X$ is a \mathbb{G}_a -equivariant morphism.*

Proof. The induced \mathbb{G}_a -action on the invariant affine open subvariety U of X is determined by a locally nilpotent derivation ∂ of $\Gamma(U, \mathcal{O}_U)$. Since $p : p^{-1}(U) \rightarrow U$ is étale and proper, $p^{-1}(U)$ is an affine open subvariety of Z and ∂ lifts in a unique way to a derivation of $\Gamma(p^{-1}(U), \mathcal{O}_{p^{-1}(U)})$ which is again locally nilpotent by virtue of [17]. By construction, the latter defines a \mathbb{G}_a -action on $p^{-1}(U)$ for which the restriction of p to $p^{-1}(U)$ is equivariant. Now the assertion follows from [16, Lemma 6.1] which guarantees that the \mathbb{G}_a -action on $p^{-1}(U)$ can be uniquely extended to a one on Z with the desired property. \square

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