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ACCEPTED MANUSCRIPT

ON THE CONNECTIONS BETWEEN WEAKLY STABLE AND PSEUDO-ISOTROPIC DISTRIBUTIONS

B.H. JASIULIS ¹ AND J.K. MISIEWICZ ²

ABSTRACT. A random vector \mathbf{X} is weakly stable iff for all $a, b \in \mathbb{R}$ there exists a random variable Θ such that $a\mathbf{X}+b\mathbf{X}' \stackrel{d}{=} \mathbf{X}\Theta$. This is equivalent (see [7]) with the condition that for all random variables Q_1, Q_2 there exists a random variable Θ such that

$$\mathbf{X}Q_1 + \mathbf{X}'Q_2 \stackrel{d}{=} \mathbf{X}\Theta, \qquad (*)$$

where $\mathbf{X}, \mathbf{X}', Q_1, Q_2, \Theta$ are independent. Some of the weakly stable distributions turn out to be the extreme points for the class of pseudoisotropic distributions, where the distribution is pseudo-isotropic if all its one-dimensional projections are the same up to a scale parameter. We show here that the scaling function for pseudo-isotropic distribution can define a generalized distribution iff it is an α -norm for some $\alpha > 0$.

Key words: weakly stable distribution, symmetric stable distribution, ℓ_{α} -symmetric distribution, scale mixture

Mathematics Subject Classification: 60A10, 60B05, 60E05, 60E07, 60E10.

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B.H. JASIULIS $^{\rm 1}$ AND J.K. MISIEWICZ $^{\rm 2}$

1. INTRODUCTION

The idea of generalized convolutions was studied by K. Urbanik in a series of papers, see [10, 11, 12, 13, 14]. This area turned out to be interesting and it has been developed extensively till now.

In seventies of the last century there appeared a series of papers written mainly by Kucharczak and Urbanik (see e.g. [3, 10]), where the authors introduced the idea of weakly stable distributions on $[0, \infty)$. More precisely, a probability measure μ on $[0, \infty)$ was called weakly stable if

$$\forall a, b > 0 \exists \lambda \text{ on } [0, \infty) \quad T_a \mu * T_b \mu = \mu \circ \lambda,$$

where $(T_a\mu)(A) = \mu(A/a)$, $(\mu \circ \lambda)(A) = \int \mu(A/s)\lambda(ds)$ for every Borel set A in $[0, \infty)$. Currently measures μ for which this condition holds are rather called \mathbb{R}_+ -weakly stable. They are extensively studied, see e.g. [4, 16, 17, 18]. Recently Misiewicz, Oleszkiewicz and Urbanik (see [7]) gave a definition and showed some basic properties of weakly stable distributions on \mathbb{R}^n , or on a separable Banach spaces \mathbb{E} . The definition of weakly stable distributions is almost the same as the one given by Kucharczak and Urbanik, except for the fact that constants a, b can be any real numbers. The authors obtained in [7] the full characterization of weakly stable distributions with non-trivial discrete part. In [8] J.K. Misiewicz studied basic properties of the weak generalized convolution based on weakly stable distribution.

In this paper we present some new results on weak generalized convolutions and their connections with pseudo-isotropic distributions, i.e. multidimensional symmetric distributions having all one-dimensional margins the same

WEAK STABILITY AND PSEUDO-ISOTROPY

up to a scale parameter. We show that pseudo-isotropy defines a weak generalized convolution iff the corresponding scale function is the ℓ_p norm on \mathbb{R}^n . This result links the geometry of L_p -spaces with functional equations, theory of pseudo-isotropic distributions and generalized convolutions.

Throughout this paper we denote by $\mathcal{L}(\mathbf{X})$ the distribution of the random vector \mathbf{X} . If random vectors \mathbf{X} and \mathbf{Y} have the same distribution we will write $\mathbf{X} \stackrel{d}{=} \mathbf{Y}$. By $\mathcal{P}(I\!\!E)$ we denote the set of all probability measures on a separable Banach space (or on a set) $I\!\!E$. We will use the simplified notation $\mathcal{P}(I\!\!R) = \mathcal{P}, \ \mathcal{P}([0, +\infty)) = \mathcal{P}^+$. For every $a \in I\!\!R$ and every probability measure μ we define the rescaling operator $T_a: \mathcal{P}(I\!\!E) \to \mathcal{P}(I\!\!E)$ as follows

$$T_a \mu(A) = \begin{cases} \mu(A/a) & \text{for} \quad a \neq 0; \\ \delta_0(A) & \text{for} \quad a = 0, \end{cases}$$

for every Borel set $A \in \mathbb{E}$. Equivalently $T_a \mu$ is the distribution of the vector $a\mathbf{X}$ if μ is the distribution of the vector \mathbf{X} . The scale mixture $\mu \circ \lambda$ of a measure $\mu \in \mathcal{P}(\mathbb{E})$ with respect to the measure $\lambda \in \mathcal{P}$ is defined by:

$$\mu \circ \lambda(A) \stackrel{def}{=} \int_{I\!\!R} T_s \mu(A) \,\lambda(ds).$$

It is easy to see that $\mu \circ \lambda$ is the distribution of the random vector $\mathbf{X}\Theta$ if $\mu = \mathcal{L}(\mathbf{X}), \ \lambda = \mathcal{L}(\Theta), \ \mathbf{X}$ and Θ are independent. In the language of characteristic functions we obtain

$$\widehat{\mu \circ \lambda}(\mathbf{t}) = \int_{I\!\!R} \widehat{\mu}(\mathbf{t}s) \lambda(ds).$$

Notice that for a symmetric random vector \mathbf{X} independent of random variable Θ we have $\mathbf{X}\Theta \stackrel{d}{=} \mathbf{X}|\Theta|$. From this property we obtain that if μ is a

B.H. JASIULIS ¹ AND J.K. MISIEWICZ ²

symmetric probability distribution then for $|\lambda|=\mathcal{L}(|\Theta|)$ we have

$$\mu \circ \lambda = \mu \circ |\lambda|.$$

Definition 1. A probability measure $\mu \in \mathcal{P}(\mathbb{E})$ is weakly stable (or \mathbb{R}_+ weakly stable) if for every choice of $\lambda_1, \lambda_2 \in \mathcal{P}$ ($\lambda_1, \lambda_2 \in \mathcal{P}_+$) there exists $\lambda \in \mathcal{P}$ ($\lambda \in \mathcal{P}_+$) such that

$$(\lambda_1 \circ \mu) * (\lambda_2 \circ \mu) = \lambda \circ \mu.$$

If μ is not symmetric then the measure λ is uniquely determined. This fact was proven in [7] for a weakly stable measure μ , and in [15] for a \mathbb{R}_+ -weakly stable measure μ . If the measure μ is symmetric then only the symmetrization of λ is uniquely determined (see [7], Remark 1). In this case we can always replace the measure λ by its symmetrization $(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}) \circ \lambda$. For convenience in this paper we assume that for symmetric μ the measure λ is concentrated on $[0, \infty)$ taking if necessary $|\lambda|$ instead of λ .

2. Generalized weak convolution

Definition 2. Let $\mu \in \mathcal{P}(\mathbb{E})$ be a nontrivial weakly stable measure, and let λ_1, λ_2 be probability measures on \mathbb{R} . If

$$(\lambda_1 \circ \mu) * (\lambda_2 \circ \mu) = \lambda \circ \mu$$

then the weak generalized convolution of the measures λ_1, λ_2 with respect to the measure μ (notation $\lambda_1 \oplus_{\mu} \lambda_2$) is defined as follows

$$\lambda_1 \oplus_{\mu} \lambda_2 = \begin{cases} \lambda & \text{if } \mu \text{ is not symmetric;} \\ |\lambda| & \text{if } \mu \text{ is symmetric.} \end{cases}$$

In the case of symmetric weakly stable distribution μ we take $\lambda_1 \oplus_{\mu} \lambda_2 = |\lambda|$ in order to get uniqueness of this measure. If Θ_1, Θ_2 are random variables with distributions λ_1, λ_2 respectively then the random variable with distribution $\lambda_1 \oplus_{\mu} \lambda_2$ will be denoted as $\Theta_1 \oplus_{\mu} \Theta_2$. Thus we have

$$\Theta_1 \mathbf{X}' + \Theta_2 \mathbf{X}'' \stackrel{d}{=} (\Theta_1 \oplus_{\mu} \Theta_2) \mathbf{X},$$

where $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ have distribution μ , $\Theta_1, \Theta_2, \mathbf{X}', \mathbf{X}''$ and $\Theta_1 \oplus_{\mu} \Theta_2, \mathbf{X}$ are independent. Now it is easy to see that the following lemma holds.

Lemma 1. If the weakly stable measure $\mu \in \mathcal{P}(\mathbb{E})$ is not trivial then \oplus_{μ} is commutative and associative. Moreover for all $\lambda_1, \lambda_2, \lambda \in \mathcal{P}$ $\lambda_1 \oplus_{\mu} \lambda_2$ is uniquely determined and the following conditions hold:

(i) $\lambda \oplus_{\mu} \delta_0 = \lambda \ (\lambda \oplus_{\mu} \delta_0 = |\lambda| \text{ if } \mu \text{ is symmetric});$

(ii)
$$T_a(\lambda_1 \oplus_{\mu} \lambda_2) = (T_a \lambda_1) \oplus_{\mu} (T_a \lambda_2);$$

(iii)
$$(p\lambda_1 + (1-p)\lambda_2) \oplus_{\mu} \lambda = p(\lambda_1 \oplus_{\mu} \lambda) + (1-p)(\lambda_2 \oplus_{\mu} \lambda) \quad \forall p \in [0,1];$$

(iv) if $\lambda_n \to \lambda$ then $\lambda_n \oplus_{\mu} \nu \to \lambda \oplus_{\mu} \nu$ for all $\nu \in \mathcal{P}$.

In 1964 K. Urbanik introduced the definition of a generalized convolution as an commutative and associative binary operator \diamond on the space \mathcal{P}^2_+ of all probability measures on $[0, \infty)$ taking values in \mathcal{P}_+ such that the properties (i)÷(iv) of the previous lemma hold with \oplus_{μ} replaced by \diamond and

(v) there exists a sequence of positive numbers (c_n) such that $T_{c_n}\delta_1^{\diamond n}$ converges weakly to a measure $\nu \neq \delta_0$ (here $\lambda^{\diamond n} = \lambda \diamond \cdots \diamond \lambda$ denotes the generalized convolution of n identical measures λ).

Notice that for a weakly stable measure μ we have that $(T_{c_n}\delta_1^{\diamond n}) \circ \mu = T_{c_n}(\mu^{\diamond n})$ thus if the condition (v) holds for the weak generalized convolution,

B.H. JASIULIS ¹ AND J.K. MISIEWICZ ²

then μ is in the domain of attraction of some stable measure, which we do not want to assume.

Urbanik calls the generalized convolution \diamond regular if there exists a nontrivial homomorphism $h: \mathcal{P}^+ \to \mathbb{R}$ such that $h(\lambda_1 \diamond \lambda_2) = h(\lambda_1)h(\lambda_2)$ and $h(p\lambda_1 + (1-p)\lambda_2) = ph(\lambda_1) + (1-p)h(\lambda_2)$. It is easy to see that the weak generalized convolution \oplus_{μ} is regular with $h(\delta_x) = \hat{\mu}(x)$.

3. PSEUDO-ISOTROPIC DISTRIBUTIONS

We give here only the definition and an elementary description of pseudoisotropic distributions in the scope required for this paper. More about pseudo-isotropic distributions one can find in [5], [9].

Definition 3. The random vector $\mathbb{X} = (X_1, \ldots, X_n)$ or its distribution μ on \mathbb{R}^n is pseudo-isotropic if there exists a function $c \colon \mathbb{R}^n \to [0, \infty)$ and a function f on \mathbb{R} , f(-t) = f(t) such that for each $\xi \in \mathbb{R}^n$, and every $t \in \mathbb{R}$

$$\mathbf{E}\exp\left\{i < \xi t, \mathbf{X} > \right\} = f\left(c(\xi)|t|\right).$$

It was shown (see e.g. [9]) that c is a continuous quasi-norm on \mathbb{R}^n , i.e. it has all the properties of a norm except that the triangular inequality holds with some constant. Moreover, by putting $\xi = (1, 0, ..., 0)$ we see that

$$\mathbf{E} \exp \{itX_1\} = f(c(1, 0, \dots, 0))|t|)$$

thus f(|t|) is, up to a scale parameter, equal the characteristic function of the variable X_1 . We can always assume that c(1, 0, ..., 0) = 1. Then

$$\forall \xi \in \mathbb{R}^n \quad \langle \xi, \mathbf{X} \rangle = \sum_{k=1}^n \xi_k X_k \stackrel{d}{=} c(\xi) X_1.$$

WEAK STABILITY AND PSEUDO-ISOTROPY

This property reminds the one in the definition of strictly stable variables, except that here we do not have independent and identically distributed components of the random vector \mathbf{X} . Let

$$\mathcal{M}(c,n) = \{ \mu \in \mathcal{P}(\mathbb{R}^n) \colon \widehat{\mu}(\xi) = f(c(\xi)), f \in \Phi \},\$$

where Φ is the set of all real characteristic functions on \mathbb{R} . Thus $\mathcal{M}(c, n)$ is the set of all pseudo-isotropic distributions on \mathbb{R}^n with the fixed quasinorm c. It is evident that $\mathcal{M}(c, n)$ is closed under rescaling, convex linear combinations and weak limits since the set of characteristic functions of the form $f(c(\xi))$ with the fixed function c has these properties, thus $\mathcal{M}(c, n)$ is convex and weakly closed. In some cases this set contain only one element δ_0 , for example $\mathcal{M}(\|\cdot\|_{\alpha}, n) = \{\delta_0\}$ if $\alpha \in (2, \infty]$ and $n \geq 3$ (see [9]). In view of this fact we say that the quasi-norm c on \mathbb{R}^n is admissible if the set $\mathcal{M}(c, n)$ contains more than one element.

Proposition 1. Let $c: \mathbb{R}^n \to [0, \infty)$ be an admissible quasi-norm. If there exists $\mu \in \mathcal{M}(c,n)$ such that the set $\mathcal{E} = \{T_a\mu: a \ge 0\}$ is the set of all extreme points of $\mathcal{M}(c,n)$, then μ is weakly stable. Moreover, if there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}^n} \|\mathbf{x}\|^{\varepsilon} \mu(dx) < \infty$$

then for the binary operation \oplus_{μ} the condition (v) from the Urbanik definition of generalized convolution holds.

Proof. Let $\hat{\mu}(\xi) = \varphi(c(\xi))$. Notice that the measure $T_a \mu * T_b \mu$ has the characteristic function $\varphi(|a|c(\xi))\varphi(|b|c(\xi)) = \psi_{a,b}(c(\xi))$ which is also some function dependent on $c(\xi)$ only. Thus the corresponding measure ν , $\hat{\nu}(\xi) =$

B.H. JASIULIS ¹ AND J.K. MISIEWICZ ²

 $\psi_{a,b}(c(\xi))$ belongs to $\mathcal{M}(c,n)$. Since \mathcal{E} is the set of all extreme points of $\mathcal{M}(c,n)$ then there exists a measure λ on $[0,\infty)$ such that

$$\psi_{a,b}(c(\xi)) = \int_0^\infty \varphi(c(\xi)t)\lambda(dt)$$

This means that $\nu = \mu \circ \lambda$ and consequently $T_a \mu * T_b \mu = \mu \circ \lambda$. Assume now that $\int \|\mathbf{x}\|^{\varepsilon} \mu(d\mathbf{x}) < \infty$ and let $\alpha = \sup\{p \leq 2: \int \|\mathbf{x}\|^p \mu(d\mathbf{x}) < \infty\}$. Since μ is symmetric we see that μ belongs to the domain of attraction (see e.g. [?]) of some non-trivial symmetric α -stable measure γ_{α} and

$$T_{n^{-1/\alpha}}\mu^{*n} \to \gamma_{\alpha}. \tag{(**)}$$

This means, in particular, that $\gamma_{\alpha} \in \mathcal{M}(c, n)$ and there exists $\lambda \in \mathcal{P}$ such that $\gamma_{\alpha} = \mu \circ \lambda$. The measure λ is not equal δ_0 since otherwise we would have $\gamma_{\alpha} = \delta_0$ which is excluded. Now the equation (**) together with the trivial equality $\mu = \mu \circ \delta_1$ implies finally that $T_{n^{-1/\alpha}} \delta_1^{\oplus \mu^n} \to \lambda$.

4. Pseudo-isotropic generalized convolutions

Let (X, Y) be a pseudo-isotropic random vector with the scaling function (a quasi-norm) $c \colon \mathbb{R}^2 \to [0, \infty)$, i.e. for every $a, b \in \mathbb{R}$ we have that $aX + bY \stackrel{d}{=} c(a, b)X$. We define a binary operation \oplus_c on the set \mathcal{P}_+ by the following

$$\lambda_1 \oplus_c \lambda_2 = \mathcal{L}\left(c(\Theta_1, \Theta_2)\right),\,$$

where $\mathcal{L}(\Theta_1) = \lambda_1$, $\mathcal{L}(\Theta_2) = \lambda_2$. It is easy to see that if (Θ_1, Θ_2) and (X, Y)are independent then $\Theta_1 X + \Theta_2 Y \stackrel{d}{=} c(\Theta_1, \Theta_2) X$. The question arises under which conditions the operation \oplus_c defines an generalized convolution?

Theorem 1. Let $c: \mathbb{R}^2 \to [0, \infty)$ be the quasi-norm defined by a pseudoisotropic random vector (X, Y). Then the binary operation \oplus_c defines an generalized convolution on \mathcal{P}_+ if and only if there exists $\alpha \in (0, \infty]$ such that $c(t, s) = \|(t, s)\|_{\alpha}$.

Proof. Notice first that for constant random variables $\Theta_1 \equiv t, \Theta_2 \equiv s$ we have $c(\Theta_1, \Theta_2) \equiv c(t, s)$. This means in particular that $\delta_t \oplus_c \delta_s = \delta_{c(t,s)}$. Now we shall check under what conditions the operation \oplus_c is such that the conditions (i) \div (v) hold for \oplus_{μ} and \diamond replaced by \oplus_c . The condition (i) states that δ_0 is the unit element for \oplus_c , i.e. $\delta_0 \oplus_c \lambda = \lambda$ for each $\lambda \in \mathcal{P}_+$. Applying this condition for the measure $\lambda = \delta_s$ we would have that $\delta_s = c(\delta_0, \delta_s) = \delta_{c(0,s)}$, which means that for every s > 0 we shall have c(0,s) = s. Since the operation \oplus_c is supposed to be commutative then we have also c(t,s) = c(s,t) for all $s, t \ge 0$. Associativity of the operation \oplus_c (applied only to measures concentrated at one point) means that for the function c the following condition holds

$$\forall \ r,s,t \geq 0 \qquad c(c(r,s),t) = c(r,c(s,t)).$$

Now we came to the assumptions of the spectacular result of Bohnenblust (see Th. 4.1 in [1]), which states that if c is any function defined over the quadrant $s, t \ge 0$ having the following properties:

- 1) c(rs, rt) = rc(s, t) for all $r, s, t \ge 0$;
- 2) $c(s,t) \le c(s_1,t_1)$ for all $0 \le s \le s_1, 0 \le t \le t_1$;
- 3) c(s,t) = c(t,s) for all $s, t \ge 0$;

- 4) c(0,1) = 1;
- 5) c(c(r,s),t) = c(r,c(s,t)) for all $r, s, t \ge 0$

then there exists $\alpha \in (0, \infty]$ such that

$$c(s,t) = \begin{cases} (s^{\alpha} + t^{\alpha})^{1/\alpha} & \text{for} \quad \alpha < \infty;\\\\ \max\{s,t\} & \text{for} \quad \alpha = \infty. \end{cases}$$

The assumption 2) in the Bohnenblust theorem was supposed to be the weakest possible substitute of the continuity of the function c. In our case the function c is continuous as the quasi-norm on \mathbb{R}^2 , thus we can directly apply this result. Now, with this particular function c the conditions $(ii) \div (iv)$ hold trivially. To see the opposite implication notice first that for $\alpha \in (0, 2]$ it is enough to take X, Y independent, identically distributed symmetric α -stable random variables. Then (X, Y) is pseudo-isotropic with the quasi-norm $c(s,t) = ||(s,t)||_{\alpha}$. For $\alpha = \infty$ we shall take two independent symmetric standard Cauchy random variables Z_1, Z_2 , and define $(X, Y) = (Z_1 + Z_2, Z_1 - Z_2)$. Then we have

$$aX + bY = (a+b)Z_1 + (a-b)Z_2 \stackrel{d}{=} (|a+b| + |a-b|)Z_1 \stackrel{d}{=} \max\{|a|, |b|\}X.$$

For $\alpha \in (2, \infty)$ we use the fact that every two dimensional Banach space embeds isometrically into some L_1 -space (see e.g. [6]). Thus for each such α the function $\exp\{-\|(s,t)\|_{\alpha}\}$ is a characteristic function of some symmetric random vector, say (X, Y). Obviously the vector (X, Y) is pseudo-isotropic with the quasi-norm $\|(s,t)\|_{\alpha}$.

Remark. Notice that the considered generalized convolution $\bigoplus_{\|\cdot\|_p}$ is regular with the homomorphism e^{-t^p} . If p > 2 then the generalized convolution

WEAK STABILITY AND PSEUDO-ISOTROPY

 $\bigoplus_{\|\cdot\|_p}$ is not weak since in this case $e^{-|t|^p}$ cannot be a characteristic function for any probability measure.

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ACCEPTED MANUSCRIPT

B.H. JASIULIS 1 AND J.K. MISIEWICZ 2

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