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Abstract. This paper is devoted to describe the structure of singular codimension one foliations with numerically trivial canonical bundle on projective manifolds. To achieve this goal we study the reduction modulo p of foliations, describe the structure of first integrals of (semi-)stable foliations with (negative) zero canonical bundle, establish a criterium for uniruledness of projective manifolds, and investigate the deformation of free morphisms along foliations. This paper also contains a classification of the irreducible components of the space of foliations with $K\mathcal{F} \leq 0$ on Fano 3-folds with rank one Picard group, and new information about the structure of codimension one foliations on $\mathbb{P}^n$ of degree smaller than or equal to $2n-3$.

Contents

1. Introduction and Statement of Results

Let $\mathcal{F}$ be a singular holomorphic foliation on a compact complex manifold X, and $K\mathcal{F}$ be its canonical line bundle. In strict analogy with the case of complex manifolds, the canonical line bundle of $\mathcal{F}$ is the line bundle on X which, away from the singular set of $\mathcal{F}$, coincides with bundle of differential forms of maximal degree along the leaves of $\mathcal{F}$.

As in the case of manifolds, one expects that $K\mathcal{F}$ governs much of the geometry of $\mathcal{F}$. When X is a projective surface, this vague expectation has already been turned into precise results. There is now a birational classification of foliations on projective surfaces, very much in the spirit of Enriques-Kodaira classification of projective surfaces, in terms of numerical properties of $K\mathcal{F}$, see [49, 11].

Key words and phrases. Foliation, Transverse Structure, Birational Geometry.

In this paper we investigate the structure of codimension one foliations on projective manifolds with $K\mathcal{F}$ numerically equivalent to zero. We were dragged into this problem by a desire to better understand/generalize two previous results on the subject. The first, by Cerveau and Lins Neto [15], concerns the description of irreducible components of the space of foliations on $\mathbb{P}^3$ with $K\mathcal{F} = 0$. While the second, by the third author of this paper [63], classifies smooth codimension one foliations with numerically trivial canonical bundle on compact Kähler manifolds.

Further motivation comes from the study of holomorphic Poisson manifolds as studied in [56], and from [55, Corollary 4.6] which says that foliations with numerically trivial canonical bundle naturally appears when studying the obstructions for a projective variety to have Ω^1_X generically ample.

1.1. **Previous results on foliations with $c_1(K\mathcal{F}) = 0$.** Cerveau and Lins Neto proved that the space of foliations on $\mathbb{P}^3$ with $K\mathcal{F} = 0$ has six irreducible components. Their result not only counts the number of irreducible components, but also give a rather precise description of them which we now proceed to recall. Four of the irreducible components parametrize foliations defined by logarithmic 1-forms with poles on a reduced divisor of degree four, the different irreducible components correspond to the different partitions of 4 with at least two summands. One of the components parametrizes pull-back under linear projections of foliations on $\mathbb{P}^2$ with canonical bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. The remaining component is rigid in the sense that its generic element correspond to a unique foliation up to automorphisms of $\mathbb{P}^3$. If we fix a point p in $\mathbb{P}^1$ and identify $\mathrm{Aff}(\mathbb{C})$ with the isotropy group of this point under the natural action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1$ then this foliation corresponds to the induced action of $\mathrm{Aff}(\mathbb{C})$ on $\mathbb{P}^3 = \mathrm{Sym}^3 \mathbb{P}^1$. From this description one promptly sees that foliations on $\mathbb{P}^3$ with trivial canonical bundle are either defined by closed rational 1-forms, or come from $\mathbb{P}^2$ by means of a linear pull-back. Notice that in the latter case the leaves are covered by rational curves, indeed lines.

In 1997, one year after the publication of Cerveau-Lins Neto paper, appeared a paper [56] by Polishchuk which, among other things, contains a classification of Poisson structures on $\mathbb{P}^3$ under restrictive hypothesis on their singular set. But (non-zero) Poisson structures on 3-folds are nothing more than foliations with trivial canonical bundle, thus Polishchuk's result is a particular case of Cerveau-Lins Neto classification.

The third author of this paper proved in [63] that a smooth codimension one foliation $\mathcal{F}$ with numerically trivial canonical bundle on compact Kähler manifold X fits into at least one of the following descriptions.

(1) After a finite étale covering, X is the product of Calabi-Yau variety Y and a complex torus T and $\mathcal{F}$ is the pull-back under the natural projection to T of a linear codimension one foliation on T.
(2) The manifold X is fibration by rational curves over a compact Kähler variety Y with $c_1(Y) = 0$, and $\mathcal{F}$ is a foliation everywhere transverse to the fibers of the fibration.
(3) The foliation $\mathcal{F}$ is an isotrivial fibration by hypersurfaces with zero first Chern class.

The particular case of smooth Poisson structures on projective 3-folds was treated before by Druel in [26]. When the third author of this paper proved the classification above he was not aware of Druel's work.

1.2. **Rough structure of foliations with $c_1(K\mathcal{F}) = 0$.** While some of the foliations described above are defined by closed meromorphic 1-forms, some are not. Nevertheless, all of them are either defined by closed meromorphic 1-forms with coefficients in a torsion line bundle, or through a generic point of the ambient space there exists a rational curve contained in a leaf. Foliations having the latter property will be called uniruled foliations.

In face of these examples one is naturally lead to enquire if this pattern persists for arbitrary codimension one foliations with numerically trivial canonical bundle. One of our main results gives a positive answer to this question.

Theorem 1. *Let $\mathcal{F}$ be a codimension one foliation with numerically trivial canonical bundle on a projective manifold X. Then at least one of following assertions hold true.*

(a) *The foliation $\mathcal{F}$ is defined by a closed rational 1-form with coefficients in a torsion line bundle and without divisorial components in its zero set.*
(b) *All the leaves of $\mathcal{F}$ are algebraic.*
(c) *The foliation $\mathcal{F}$ is uniruled.*

Moreover, if $\mathcal{F}$ is not uniruled then $K\mathcal{F}$ is a torsion line bundle.

Our proof of Theorem 1 combines a variety of techniques: reduction modulo p of foliations, basic Hodge theory, deformations of morphisms along foliations, and the theory of transversely homogeneous structures for foliations. In the course of our investigations we stumble upon results with a somewhat broader scope, which we will now proceed to present.

1.3. **Semi-stable foliations and reduction modulo p.** In a joint work with Cerveau and Lins Neto, we have proved that codimension one foliations in positive characteristic are, as a rule, defined by closed rational 1-forms, see [17, Section 6]. Given a complex foliation $\mathcal{F}$ on a projective manifold X, then both $\mathcal{F}$ and X are defined over a finitely generated $\mathbb{Z}$-algebra R and we can reduce modulo a maximal prime $\mathfrak{p} \subset R$ to obtain a foliation on a variety over a field of characteristic $p > 0$. Applying the above mentioned result we obtain that this reduction is indeed defined a closed rational 1-form. In general, one does not expect to be able to lift this information back to characteristic zero, since foliations on complex projective surfaces defined by closed 1-forms are quite rare. Nevertheless, under the additional assumption that $T\mathcal{F}$ is semi-stable and $K\mathcal{F}$ is numerically trivial we prove the following Theorem.

Theorem 2. *Let (X, H) be a n-dimensional polarized projective complex manifold, and $\mathcal{F}$ be a semi-stable foliation of codimension one on X. If $K\mathcal{F} \cdot H^{n-1} = 0$ then at least one of the following assertions hold true*

(a) *for almost every maximal prime $\mathfrak{p} \subset R$, the reduction modulo $\mathfrak{p}$ of $\mathcal{F}$ is p-closed;*
(b) *$\mathcal{F}$ is induced by a closed rational 1-form with coefficients in a flat line bundle and without divisorial components in its zero set.*

In case (a) of Theorem 2, we expect that the foliation $\mathcal{F}$ admits a rational first integral in characteristic zero. We are not alone in this hope. Ekehdal, Shepherd–Barron, and Taylor conjectured that this is the case for any foliation of any codimension [28]. Indeed their conjecture is a non-linear version of a previous conjecture by Grothendieck–Katz about the reduction modulo p of rational flat connections. To

the best of our knowledge, and despite of the recent advances [9], both conjectures are still wide-open up-to-date.

In a number of cases, we can deal with the p-closedness given by item (a) using some (basic) index theory for singularities of holomorphic foliations. For example, when there exists an ample divisor such that $KX^2 \cdot H^{n-2} > 0$ we can prove that leaves of foliations as in (a) are covered by rationally connected varieties, see Section 3.4.

1.4. **Number of reducible fibers of first integrals.** Let $\mathcal{F}$ be a codimension one foliation on a projective manifold X defined by the levels of a rational map $F : X \dashrightarrow C$ from X to some algebraic curve C. If we further assume that F has irreducible generic fiber (what can always be done after replacing F by its Stein factorization) and, following [66], define its base number as

$$r(\mathcal{F}) = r(F) = \sum_{x \in C} \left(\#\{ \text{ irreducible components of } F^{-1}(x) \} - 1 \right) .$$

then we obtain a rather strong bound on $r(\mathcal{F})$ under the additional assumption that $T\mathcal{F}$ is stable/semi-stable and has zero/positive first Chern class.

Theorem 3. *Let $\mathcal{F}$ be such a codimension one foliation on a polarized projective manifold (X, H) of dimension at least three. If $T\mathcal{F}$ is H-semi-stable and $K\mathcal{F} \cdot H^{n-1} < 0$, or $T\mathcal{F}$ is H-stable and $K\mathcal{F} \cdot H^{n-1} = 0$, then*

$$r(\mathcal{F}) \leq \operatorname{rank} NS(X) - 1 ,$$

where $NS(X)$ is the Neron-Severi group of X. In particular, if $X = \mathbb{P}^n$, $n \geq 3$, then $r(\mathcal{F}) = 0$.

Combining this result with a classical Theorem by Halphen about pencils on projective spaces (which we generalize to simply connected projective manifolds in Theorem 4.3) we are able to control the first integrals of (semi)-stable foliations on Fano manifolds with rank one Picard group having (negative) zero canonical bundle. This immediately gives the classification of foliations with $K\mathcal{F} < 0$ on these 3-folds, see Proposition 4.7, and will be essential in the classification of foliations with $K\mathcal{F} = 0$ in the same class of manifolds.

1.5. **Spaces of foliations on Fano 3-folds.** Unfortunately Theorem 2 does not give a very clear picture of the structure of foliations with numerically trivial $K\mathcal{F}$, unless we assume that Ekehdal–Sheperd–Barron–Taylor conjecture holds true. Nevertheless it is sufficient to obtain a pretty precise description of these foliations on Fano 3-folds with rank one Picard group.

Theorem 4. *Let X be a Fano 3-fold with $\operatorname{Pic}(X) = \mathbb{Z}$, and let $\mathcal{F}$ be a codimension one foliation on X with trivial canonical bundle. If $\mathcal{F}$ is unstable then $X = \mathbb{P}^3$ and $\mathcal{F}$ is the linear pull-back of a degree two foliation on $\mathbb{P}^2$. If $\mathcal{F}$ is semi-stable then at least one of the following assertions holds true:*

1. *$T\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\mathcal{F}$ is induced by an algebraic action;*
2. *$\mathcal{F}$ is tangent to an algebraic action by $\mathbb{C}$ or $\mathbb{C}^*$ with non-isolated fixed points;*
3. *$\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

Manifold	**Irreducible component**	dim
Projective space $\mathbb{P}^3$	$\mathrm{Rat}(1,3)$	21
	$\mathrm{Rat}(2,2)$	16
	$\mathrm{Log}(1,1,1,1)$	14
	$\mathrm{Log}(1,1,2)$	17
	LPB(2)	17
	Aff	13
Hyperquadric Q^3	$\mathrm{Rat}(1,2)$	17
	$\mathrm{Log}(1,1,1)$	14
	Aff	8
Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	$\mathrm{Rat}(1,1) \simeq \mathbb{P}^2$	2
Hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,4)$	4
Cubic in $\mathbb{P}^4$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,5)$	6
Intersection of quadrics in $\mathbb{P}^5$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,6)$	8
X_5	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,7)$	10
	Aff $\simeq \mathbb{P}^1$	1
Mukai-Umemura 3-fold	Aff $\simeq \mathbb{P}^1$	1

TABLE 1. Irreducible components of the space of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

From this result, combined with Theorem 3, we are able to obtain a new proof of Cerveau-Lins Neto classification of foliations with $K\mathcal{F} = 0$ on $\mathbb{P}^3$, and generalize it to the other Fano 3-folds with rank one Picard group. We summarize the results in the Table 1. We are also able to recover Cerveau-Lins Neto classification of foliations of degree 2 on $\mathbb{P}^n$, $n \geq 4$; and to classify Poisson Fano 3-folds with rank one Picard group.

1.6. **A criterium for uniruledness.** Perhaps the first examples of foliations with $K\mathcal{F} = 0$ that come to mind are those with trivial tangent bundle. Foliations with trivial tangent bundle are exactly those induced by (analytic) actions of complex Lie groups which are locally free outside an analytic subset of codimension at least two. If the action is not locally free then it is well-known that the manifold must be uniruled. We are able to generalize this well-known fact, confirming a recent conjecture of Peternell [55, Conjecture 4.23].

Theorem 5. *Let X be a projective manifold and L be a pseudo-effective line bundle on X. If there exists $v \in H^0(X, \bigwedge^p TX \otimes L^*)$ vanishing at some point then X is uniruled. In particular, if there exists a foliation $\mathcal{F}$ on X with $c_1(T\mathcal{F})$ pseudo-effective and $\mathrm{sing}(\mathcal{F}) \neq \emptyset$ then X is uniruled.*

Theorem 5 brings the task of classifying codimension one foliations with $c_1(K\mathcal{F}) = 0$ to the realm of uniruled manifolds, as smooth foliations satisfying these assumptions have already been classified by the third author. It is a consequence of Boucksom-Demailly-Paun-Peternell characterization of uniruledness [10] combined with the following result.

Theorem 6. *Let X be a projective manifold with KX pseudo-effective and L be a pseudo-effective line bundle on X. If $v \in H^0(X, \bigwedge^p TX \otimes L^*)$ is a non-zero section then the zero set of v is empty. Moreover, if $\mathcal{D}$ is a codimension q distribution on X with $c_1(T\mathcal{D}) = 0$ then $\mathcal{D}$ is a smooth foliation (i.e. $T\mathcal{D}$ is involutive) with torsion canonical bundle, and there exists another smooth foliation $\mathcal{G}$ of dimension q on X such that $TX = T\mathcal{D} \oplus T\mathcal{G}$.*

Theorem 6 also holds true in the Kähler realm, except for the claim that $K\mathcal{D}$ is torsion as we use Simpson's Theorem [61], which is only available in the algebraic category, to prove it.

Using similar ideas we are able to prove that codimension one foliations with $c_1(K\mathcal{F}) = 0$ having the so called division property are automatically smooth, see Theorem 6.6. In particular, if $\mathrm{sing}(\mathcal{F}) \neq \emptyset$ then it has an irreducible component of codimension two with non-vanishing Baum-Bott index. This corollary will be used in the proof of Theorem 1, and gives some evidence (rather weak we might say) toward Beauville's generalization of Bondal's conjecture on the degeneracy locus of holomorphic Poisson structures, see Remark 6.8.

1.7. **Foliations on uniruled manifolds.** On a uniruled variety we know there exist morphisms $f : \mathbb{P}^1 \to X$ such that f^*TX is generated by global sections – the so called free morphisms. At a neighborhood of any free morphism f the irreducible component $M = M_f$ of $\mathrm{Mor}(\mathbb{P}^1, x)$ containing f is smooth and has dimension $h^0(\mathbb{P}^1, f^*TX)$. A foliation $\mathcal{F}$ on X naturally defines a foliation $\mathcal{F}_{tang}$ on M_f. Intuitively, its leaves correspond to maximal families of morphisms which map points on $\mathbb{P}^1$ to leaves of $\mathcal{F}$. The dimension of $\mathcal{F}_{tang}$ is equal to $h^0(\mathbb{P}^1, f^*T\mathcal{F})$ where f is a generic element of the irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ containing it. When $c_1(K\mathcal{F}) = 0$, we promptly see that $h^0(\mathbb{P}^1, f^*T\mathcal{F}) \geq n - 1$, $n = \dim(X)$, and it is natural to expect that the study of $\mathcal{F}_{tang}$ should shed light into the structure of $\mathcal{F}$. Indeed, this is true even if we do not assume $K\mathcal{F} = 0$. All we have to ask is the non-triviality of $\mathcal{F}_{tang}$, i.e. $\dim \mathcal{F}_{tang} > 0$, to be able to infer properties of the original foliation $\mathcal{F}$.

Theorem 7. *Let $\mathcal{F}$ be a codimension one foliation on a n-dimensional uniruled projective manifold X. If $f : \mathbb{P}^1 \to X$ is a generic free morphism, $\delta_0 = h^0(\mathbb{P}^1, f^*T\mathcal{F})$, and $\delta_{-1} = h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ then at least one of the following assertions hold true.*

(a) *The foliation $\mathcal{F}$ is transversely projective.*
(b) *The foliation $\mathcal{F}$ is the pull-back by a rational map of a foliation $\mathcal{G}$ on a projective manifold of dimension $\leq n - \delta_0 + \delta_{-1}$, and if $\delta_{-1} > 0$ then $\mathcal{F}$ is uniruled.*

Moreover, if X is rationally connected and f is an embedding with ample normal bundle then we can replace transversely projective by transversely affine in item (a).

To prove this result we use techniques germane to our previous joint works with Cerveau and Lins Neto [16, 17] combined with Bogomolov-McQuillan's graphic neighborhood [7]. In the case of rationally connected manifolds we add to the mixture a study of the variation of projective structures, see §8.5, together with Hartshorne's results on extension of meromorphic functions [35]. Using a Lefschetz-type Theorem due to Kollar [44], we derive from Theorem 7 the following consequence.

Corollary 8. *Let $\mathcal{F}$ be a codimension one foliation on a rationally connected manifold X. If $K\mathcal{F} = 0$ then $\mathcal{F}$ is uniruled or defined by a closed rational 1-form.*

While it is still not our final word about the structure of foliations with $c_1(K\mathcal{F}) = 0$, the result above goes a long way in that direction, proving Theorem 1 when the ambient manifold is rationally connected.

Theorem 7 also admits as a Corollary a refinement of a recent result by Cerveau and Lins Neto [18] on the structure of foliations on $\mathbb{P}^3$ with $K\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1)$ (foliations of degree three). They proved that a foliation of degree 3 on $\mathbb{P}^3$ is either a pull-back of a foliation on $\mathbb{P}^2$ by a rational map, or is transversely affine.

Corollary 9. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$ of degree d. If $3 \leq d \leq 2n-3$ then $\mathcal{F}$ is a pull-back by a rational map of a foliation on a projective manifold of dimension at most $\frac{d}{2}+1$ or $\mathcal{F}$ is defined by a closed rational 1-form.*

Our generalization is not only more precise, as we say defined by a closed rational 1-form while they say transversely affine, but also more general since it applies to projective spaces of every dimension greater than or equal to three.

1.8. **Ingredients of the proof of Theorem 1.** The results obtained from reduction modulo p (Theorem 2) together with the ones obtained from the study of deformations of free rational curves (Theorem 7) do not seem to imply directly Theorem 1. To deal with the cases not covered by them, we use Theorem 2 to restrain the type of singularities, and explore the existence of a projective structure given by Theorem 7 to infer the existence of an invariant divisor with good combinatorial properties which allow us to conclude. To conclude we explore methods similar to the ones used in the proof of our criterium for uniruledness (Theorem 5).

1.9. **Plan of the paper.** In Section 2 we have collected basic results about foliations that will be used in the sequel. Section 3 is devoted to the reduction modulo p of foliations with $K\mathcal{F} = 0$. It starts by recalling results from [17] about the existence of invariant hypersurfaces and then proceeds to the proof of Theorem 2. This sections finishes with the study of singularities of p-closed foliations and its implications to the structure of foliations with $K\mathcal{F} = 0$. Section 4 studies the relationship between the existence of invariant hypersurfaces and the semi-stability of the tangent sheaf. Besides the proof of Theorem 3, it contains a generalization of a classical result of Halphen, and the classification of the irreducible components of the space of foliations with $K\mathcal{F} < 0$ on Fano 3-folds with rank one Picard group. Section 5 deals with the classification of foliations with $K\mathcal{F} = 0$ in Fano 3-folds with rank one Picard group. It starts with the proof of a rough classification (Theorem 4), and proceeds to a case-by-case analysis in order to detail the classification according to the index of Fano 3-folds. Section 6 establishes our uniruledness criterium (Theorem 5). Section 7 reviews some of the theory of transversely projective foliations preparing the ground for Sections 8 and 9. Section 8 starts by recalling the basic theory of deformation of free morphisms from $\mathbb{P}^1$ to projective manifolds, then it uses this theory to obtain naturally defined foliations on the space of such morphisms. The study of these foliations uncover some of the structure of the original foliation, and allow us to obtain Theorem 7. In Section 9 we put together information provided by Theorems 2, 5, and 7 and theirs proofs in order to establish our Theorem 1. In Section 10 we present a conjecture refining Theorem 1, together with some evidence toward it.

2. Preliminaries

2.1. **Foliations as subsheaves of the tangent and cotangent bundles.** A foliation $\mathcal{F}$ on a complex manifold is determined by a coherent subsheaf $T\mathcal{F}$ of the tangent sheaf TX of X which

(1) is closed under the Lie bracket (involutive), and
(2) the inclusion $T\mathcal{F} \to TX$ has torsion free cokernel.

The locus of points where $TX/T\mathcal{F}$ is not locally free is called the singular locus of $\mathcal{F}$, denoted here by $\mathrm{sing}(\mathcal{F})$.

Condition (1) allow us to apply Frobenius Theorem to ensure that for every point x in the complement of $\mathrm{sing}(\mathcal{F})$, the germ of $T\mathcal{F}$ at x can be identified with relative tangent bundle of a germ of smooth fibration $f : (X, x) \to (\mathbb{C}^q, 0)$. The integer $q = q(\mathcal{F})$ is the codimension of $\mathcal{F}$. Condition (2) is of different nature and is imposed to avoid the existence of *removable* singularities. In particular it implies that the codimension of $\mathrm{sing}(\mathcal{F})$ is at least two.

The dual of $T\mathcal{F}$ is the cotangent sheaf of $\mathcal{F}$ and will be denoted by $T^*\mathcal{F}$. The determinant of $T^*\mathcal{F}$, i.e. $(\wedge^p T^*\mathcal{F})^{**}$ where $\dim(X) = n = p+q$, in its turn is what we will call the canonical bundle of $\mathcal{F}$ and will be denoted by $K\mathcal{F}$.

There is a dual point of view where $\mathcal{F}$ is determined by a subsheaf $N^*\mathcal{F}$ of the cotangent sheaf $\Omega^1_X = T^*X$ of X. The involutivity asked for in condition (1) above is replace by integrability: if d stands for the exterior derivative then $dN^*\mathcal{F} \subset N^*\mathcal{F} \wedge \Omega^1_X$ at the level of local sections. Condition (2) is unchanged: $\Omega^1_X/N^*\mathcal{F}$ is torsion free.

The normal bundle of $\mathcal{F}$ is defined as the dual of $N^*\mathcal{F}$. Over the smooth locus $X - \mathrm{sing}(\mathcal{F})$ we have the following exact sequence

$$0 \to T\mathcal{F} \to TX \to N\mathcal{F} \to 0\,,$$

but this is no longer true over the singular locus. Anyway, as the singular set has codimension at least two we obtain the adjunction formula

$$KX = K\mathcal{F} \otimes \det N^*\mathcal{F}$$

valid in the Picard group of X.

The definitions above adapt verbatim to define algebraic foliations on smooth algebraic varieties defined over an arbitrary field. But be aware that the geometric interpretation given by Frobenius Theorem will no longer hold, especially over fields of positive characteristic.

2.2. **Foliations as q-forms and spaces of foliations.** If $\mathcal{F}$ is a codimension q foliation on a complex variety X then the q-th wedge product of the inclusion

$$N^*\mathcal{F} \longrightarrow \Omega^1_X$$

determines a differential q-form ω with coefficients in the line bundle $\det N\mathcal{F} = (\wedge^q N\mathcal{F})^{**}$ having the following properties:

- **Local decomposability**: the germ of ω at the generic point of X decomposes as the product of q germs of holomorphic 1-forms

$$\omega = \omega_1 \wedge \cdots \wedge \omega_q.$$

- **Integrability**: the decomposition of ω at the generic point of X satisfies Frobenius integrability condition

$$d\omega_i \wedge \omega = 0 \quad \text{for every } i = 1, \ldots, q\,.$$

The tangent bundle of $\mathcal{F}$ can be recovered as the kernel of the morphism

$$TX \to \Omega_X^{q-1} \otimes \det N\mathcal{F}$$

defined by contraction with ω.

Reciprocally, if $\omega \in H^0(X, \Omega^q \otimes \mathcal{L})$ is a twisted q-form with coefficients in a line bundle $\mathcal{L}$ which is locally decomposable and integrable then the kernel of ω has generic rank $\dim X - q$, and it is the tangent bundle of a holomorphic foliation $\mathcal{F}$. Moreover, if the zero set of ω has codimension at least two then $\mathcal{L} = \det N\mathcal{F}$.

Example 2.1 (Foliations on $\mathbb{P}^n$ and homogeneous forms)**.** Let $\mathcal{F}$ be a codimension q-foliation on $\mathbb{P}^n$ given by $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n} \otimes \mathcal{L})$. If $i : \mathbb{P}^q \to \mathbb{P}^n$ is a generic linear immersion then $i^*\omega \in H^0(\mathbb{P}^q, \Omega^q_{\mathbb{P}^q} \otimes \mathcal{L})$ is a section of a line bundle, and its zero divisor reflects the tangencies between $\mathcal{F}$ and $i(\mathbb{P}^q)$. The degree of $\mathcal{F}$ is, by definition, the degree of such tangency divisor. It is commonly denoted by $\deg(\mathcal{F})$. Since $\Omega^q_{\mathbb{P}^q} \otimes \mathcal{L} = \mathcal{O}_{\mathbb{P}^q}(\deg(\mathcal{L}) - q - 1)$, it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{F}) + q + 1)$.

The Euler sequence implies that a section ω of $\Omega^q_{\mathbb{P}^n}(\deg(\mathcal{F})+q+1)$ can be thought as a polynomial q-form with homogeneous coefficients of degree $\deg(\mathcal{F}) + 1$, which we will still denote by ω, satisfying (*) $i_R\omega = 0$ where $R = x_0\frac{\partial}{\partial x_0} + \cdots + x_n\frac{\partial}{\partial x_n}$ is the radial vector field. Thus the study of foliations of degree d on $\mathbb{P}^n$ reduces to the study of locally decomposable, integrable homogeneous q-forms of degree $d+1$ on $\mathbb{C}^{n+1}$ satisfying the relation (*).

For a fixed variety X, and a fixed line bundle N we will consider the space of foliations of codimension q having $\det N\mathcal{F} = N$ on X as the locally closed subvariety $\mathrm{Fol}^q(X, N) \subset \mathbb{P}H^0(X, \Omega^q_X \otimes N)$ corresponding to locally decomposable, integrable q-forms having zero set of codimension at least two. The study of irreducible components of these spaces has been initiated by Jouanolou in [39], where the irreducible components of $\mathrm{Fol}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ (codimension one foliations of degree zero) and $\mathrm{Fol}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3))$ (codimension one foliations of degree one) are described. The irreducible components of $\mathrm{Fol}^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(4))$, $n \geq 3$, have been classified by Cerveau and Lins Neto in [15]. In §5.6.1 we generalize Jouanolou's classification of degree one foliations on $\mathbb{P}^n$ to arbitrary codimensions, and we give an alternative proof of Cerveau-Lins Neto classification in §5.3 (degree two foliations on $\mathbb{P}^3$) and §5.6 (degree two foliations on $\mathbb{P}^n, n > 3$).

2.3. **Harder-Narasimhan filtration.** Let $\mathcal{E}$ be a torsion free coherent sheaf on a n-dimensional smooth projective variety X polarized by the ample line bundle H. The slope of $\mathcal{E}$ (more precisely the H-slope of $\mathcal{E}$) is defined as the quotient

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\mathrm{rank}(\mathcal{E})}\,.$$

If the slope of every subsheaf $\mathcal{E}'$ of $\mathcal{E}$ satisfies $\mu(\mathcal{E}') < \mu(\mathcal{E})$ (respectively $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$) then $\mathcal{E}$ is called stable (respectively semi-stable). A sheaf which is semi-stable but not stable is said to be strictly semi-stable.

If $\mathcal{E}$ is not semistable then there exists a unique filtration of $\mathcal{E}$ by torsion free subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable, and $\mu(\mathcal{G}_1) > \mu(\mathcal{G}_2) > \ldots > \mu(\mathcal{G}_r)$. This filtration is called the Harder–Narasimhan filtration of $\mathcal{E}$. Usually one writes $\mu_{max}(\mathcal{E}) = \mu(\mathcal{G}_1)$ and $\mu_{min}(\mathcal{E}) = \mu(\mathcal{G}_r)$. The sheaf $\mathcal{E}_1$ is called the maximal destabilizing subsheaf of $\mathcal{E}$.

When $\mathcal{E}$ is the tangent sheaf of a foliation $\mathcal{F}$, the proof of [42, Chapter 9, Lemma 9.1.3.1] implies the following result.

Proposition 2.2. *Let $\mathcal{F}$ be a foliation on a polarized smooth projective variety (X, H) satisfying $\mu(T\mathcal{F}) \geq 0$. If $T\mathcal{F}$ is not semi-stable then the maximal destabilizing subsheaf of $T\mathcal{F}$ is involutive. Thus there exists a semi-stable foliation $\mathcal{G}$ tangent to $\mathcal{F}$ and satisfying $\mu(T\mathcal{G}) > \mu(T\mathcal{F})$.*

Proof. Let $\mathcal{E}$ be the maximal destabilizing subsheaf of $T\mathcal{F}$. Since $T\mathcal{F}$ is involutive, the Lie bracket of local sections of $\mathcal{E}$ lies in $T\mathcal{F}$. Thus the Lie bracket defines a morphism of $\mathcal{O}_X$-modules

$$[\cdot,\cdot] : \bigwedge^2 \mathcal{E} \longrightarrow \frac{T\mathcal{F}}{\mathcal{E}}\,.$$

On the one hand $\mu(\bigwedge^2 \mathcal{E}) = 2\mu(\mathcal{E})$ and, since $\mathcal{E}$ is semi-stable, $\bigwedge^2 \mathcal{E}$ is semi-stable. On the other hand $\mu_{max}(T\mathcal{F}/\mathcal{E}) < \mu_{\max}(T\mathcal{F})$. Therefore $\mu(T\mathcal{F}) \geq 0$ implies $\mu(\mathcal{E}) > 0$ and, consequently

$$\mu_{min}(\bigwedge^2 \mathcal{E}) = \mu(\bigwedge^2 \mathcal{E}) > \mu(\mathcal{E}) = \mu_{max}(T\mathcal{F}) > \mu_{max}(T\mathcal{F}/\mathcal{E})\,.$$

But $\mu_{min}(\mathcal{A}) > \mu_{\max}(\mathcal{B})$ implies $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B}) = 0$ for any pair of torsion free sheaves. We conclude that $\mathcal{E}$ is involutive and must be equal to the tangent sheaf of a foliation $\mathcal{G}$. □

Example 2.3. If $\mathcal{F}$ is a foliation of $\mathbb{P}^n$ then the slope of $T\mathcal{F}$ is

$$\mu(T\mathcal{F}) = \frac{\dim(\mathcal{F}) - \deg(\mathcal{F})}{\dim(\mathcal{F})}\,.$$

Therefore $T\mathcal{F}$ is semi-stable if and only if for every distribution $\mathcal{D}$ tangent to $\mathcal{F}$ we have $\frac{\deg(\mathcal{D})}{\dim(\mathcal{D})} \geq \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})}$. Of course, $T\mathcal{F}$ is stable if and only if the strict inequality holds for every distribution $\mathcal{D}$.

If $\mathcal{F}$ is unstable and $\deg(\mathcal{F}) \leq \dim(\mathcal{F})$ then there exists a foliation $\mathcal{G}$ contained in $\mathcal{F}$ satisfying

$$\frac{\deg(\mathcal{G})}{\dim(\mathcal{G})} < \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})}\,.$$

2.4. **Miyaoka-Bogomolov-McQuillan Theorem.** The result stated below is a particular case of a more general result by Bogomolov and McQuillan proved in [7] . It generalizes a Theorem of Miyaoka, see [51, Theorem 8.5], [42, Chapter 9], or [40].

Theorem 2.4. *Let $\mathcal{F}$ be foliation on a projective manifold X. If there exists a curve $C \subset X$ disjoint from the singular set of $\mathcal{F}$ for which $T\mathcal{F}_{|C}$ is ample then the leaves of $\mathcal{F}$ are algebraic and the closure of a generic leaf of $\mathcal{F}$ is a rationally connected variety.*

We recall that a variety Y is rationally connected if through any two points $x, y \in Y$ there exists a rational curve C in Y containing x and y. Foliations satisfying the conclusions of Miyaoka-Bogomolov-McQuillan Theorem will be called rationally connected foliations. Notice that this does not mean that every leaf is rationally connected but that the generic leaf is rationally connected. For example, if we consider the codimension one foliation on $\mathbb{P}^3$ determined by a pencil of cubics generated by $3H$, an hyperplane with multiplicity three, and a cone V over a smooth planar cubic transverse to H then the generic leaf is a smooth cubic surface, and therefore rationally connected, but V is not rationally connected but only rationally chain–connected. Thus this foliation is rationally connected but it has one leaf which is not.

We will use Theorem 2.4 in the following form, closer to Miyaoka's original statement.

Corollary 2.5. *Let $\mathcal{F}$ be a semi-stable foliation on a n-dimensional polarized projective variety (X, H). If $K\mathcal{F} \cdot H^{n-1} < 0$ then $\mathcal{F}$ is a rationally connected foliation.*

Proof. If $m \gg 0$ and C is a very generic curve defined as a complete intersection of elements of $|mH|$ then $T\mathcal{F}_{|C}$ is a semi-stable vector bundle of positive degree. Therefore every quotient bundle of $T\mathcal{F}_{|C}$ has positive degree and we can apply [34, Theorem 2.4] to see that $T\mathcal{F}_{|C}$ is ample. We apply Theorem 2.4 to conclude. □

In the particular case of $\mathbb{P}^n$ this corollary reads as below.

Corollary 2.6. *Let $\mathcal{F}$ be a semi-stable foliation on $\mathbb{P}^n$. If $\deg(\mathcal{F}) < \dim(\mathcal{F})$ then $\mathcal{F}$ is a rationally connected foliation.*

2.5. **Tangent subvarieties and pull-backs.** Let $\mathcal{F}$ be a singular foliation on a projective manifold X of dimension n. We will say that $\mathcal{F}$ is the pull-back of a foliation $\mathcal{G}$ defined on a lower dimensional variety Y, say of dimension $k < n$, if there exists a dominant rational map $\pi : X \dashrightarrow Y$ such that $\mathcal{F} = \pi^*\mathcal{G}$. In this case, the leaves of $\mathcal{F}$ are covered by algebraic subvarieties of dimension $n - k$, the fibers of π.

Actually, the converse holds true. Suppose that through a generic point of X there exists an algebraic subvariety tangent to $\mathcal{F}$. Since tangency to $\mathcal{F}$ imposes a closed condition on the Hilbert scheme, it follows that the leaves of $\mathcal{F}$ are covered by q-dimensional algebraic subvarieties, $q = n - k$. More precisely, there exists an irreducible algebraic variety Y and an irreducible subvariety $Z \subset X \times Y$ such that the natural projections

$$\begin{array}{ccc} Z & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \\ X & & \end{array}$$

are both dominants; the generic fiber of π_2 has dimension q; and the generic fiber of π_2 projects to X as a subvariety tangent to $\mathcal{F}$. By resolution of singularities and Stein factorization Theorem, we can moreover assume that Z and Y are smooth and π_2 has generic irreducible fibers. Keep in mind that Z is no more in $X \times Y$ after doing so, but it is still birational to its image by $\pi_1 \times \pi_2 : Z \to X \times Y$.

Throughout the paper we will make use of the following result.

Lemma 2.7. *Let $\mathcal{F}$ be a foliation on a projective manifold X of dimension n. Assume that $\mathcal{F}$ is covered by a family of $(n-k)$-dimensional algebraic subvarieties as above. Then $\mathcal{F}$ is the pull-back of a foliation defined on a variety Y having dimension $\leq k$.*

When $\pi_1 : Z \to X$ is birational, which means that through a generic point passes exactly one subvariety Z_y of the family, then $\pi_2 \circ \pi_1^{-1} : X \dashrightarrow Y$ is the pull-back map. On the other hand, when π_1 is not finite, or even finite but not of degree 1, then the pull-back dimension jumps to lower dimension $< k$ as we shall see in the proof.

Proof. We want to prove that there is an intermediate family of algebraic subvarieties (containing the $Z_y = \pi_2^{-1}(y)$ and contained into the leaves of $\mathcal{F}$) that foliate (through a generic point passes exactly one subvariety); in other words, there is a rational map $\pi : Y \to Y'$ such that π_1 maps the image $\pi_1 \times (\pi \circ \pi_2)(Z)$ birationally to X.

Since a generic Z_y is irreducible, any two points $x, x' \in X \setminus \mathrm{sing}(\mathcal{F})$ belonging to Z_y are contained into the same leaf for $\mathcal{F}$. The idea is to consider the equivalence relation on $X \setminus \mathrm{sing}(\mathcal{F})$ generated by the Z_y's. For this, we consider, for any positive integers n, the set:

$$\Sigma_n = \left\{ \underline{z} = (z_1, \ldots, z_{2n}) \in Z^{2n} \; ; \; \begin{array}{c} \pi_1(z_k) \in X \setminus \mathrm{sing}(\mathcal{F}) \\ \pi_2(z_{2k-1}) = \pi_2(z_{2k}) \\ \pi_1(z_{2k}) = \pi_1(z_{2k+1}) \end{array} \right\}$$

which are irreducible quasi-projective subvarieties of Z^{2n}. Now define

$$\begin{array}{ccc} \pi : \Sigma & \to & X \\ \underline{z} & \mapsto & \pi_1(z_1) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi' : \Sigma & \to & X \\ \underline{z} & \mapsto & \pi_1(z_{2n}) \end{array}$$

and set

$$R_n := \pi \times \pi'(\overline{\Sigma_n})$$

which are thus irreducible quasi-projective subvarieties $R_n \subset X \times X$. By construction, a pair $(x, x') \in (X - \mathrm{sing}(\mathcal{F}))^2$ belongs to R_n if and only if x and x' can be connected in their common leaf by a chain of n elements Z_y of the family (intersecting outside of $\mathrm{sing}(\mathcal{F})$). It is then easy to check that the relation

$$x \sim_{R_n} x' \quad \Leftrightarrow \quad (x, x') \in R_n$$

is reflexive (R_n contains the diagonal) and symmetric (R_n is invariant under $(x, x') \mapsto (x', x)$); moreover, we have inclusions $R_n \subset R_{n+1}$ and $R = \cup_n R_n$ is certainly the graph of the equivalence relation generated by the Z_i's on $X \setminus \mathrm{sing}(\mathcal{F})$. In fact, if some $R_n = R_{n+1}$, then $R_n = R$ already defines the equivalence relation; this actually occurs and the reason is that $\dim(R_{n+1}) > \dim(R_n)$ whenever $R_{n+1} \neq R_n$, by irreducibility: the dimension is bounded by $\dim(X \times X)$.

Finally, the equivalence relation, whose graph is algebraic, defines a singular foliation by algebraic subvarieties on X; following [30], there exists a rational map $\pi'' : X \to Y'$ whose fibers coincide with the (connected) cosets on $X \setminus \mathrm{sing}(\mathcal{F})$. By construction, $\mathcal{F}$ is the pull-back of a foliation $\mathcal{G}$ on Y'. □

Lemma 2.8. *Given a codimension one foliation $\mathcal{F}$ on a projective manifold X of dimension n, let $\pi : (X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ be a pull-back having minimal dimension $k = \dim(Y)$. By Stein Factorization Theorem, one can assume that π has connected fibers (after lifting π to a finite cover of Y). Then, π is the minimal pull-back in the*

following sense: if $\pi' : (X, \mathcal{F}) \dashrightarrow (Y', \mathcal{G}')$ is another pull-back map with connected fibers, then it factors through a rational map $\phi : (Y', \mathcal{G}') \dashrightarrow (Y, \mathcal{G})$.

In other words, the fibers of π form the unique family of subvarieties of X tangent to $\mathcal{F}$ having maximal dimension.

Proof. If the fibers of $\pi' : X \dashrightarrow Y'$ are contained in those of $\pi : X \dashrightarrow Y$, then we get a factorization like in the statement. If not, then the fibers of π' project on Y as dimension > 0 submanifolds contained into the leaves of $\mathcal{G}$. We can apply the previous lemma and this contradict the minimality of π. □

We immediately deduce that the minimal pull-back dimension is invariant under dominant rational maps: if $\pi : (X, \mathcal{F}) \dashrightarrow (Y, \mathcal{G})$ is such a map, then $\mathcal{F}$ and $\mathcal{G}$ have the same minimal pull-back dimension.

3. Reduction modulo p

In this section we start our study of foliations with numerically trivial canonical bundle on projective manifolds. We are bound to restrict ourselves to the algebraic category as our results do depend on the reduction modulo p of foliations defined on complex projective manifolds.

3.1. **A few words about reduction modulo p.** Let $\mathcal{F}$ be a foliation defined on a complex projective manifold X. The variety X and the subsheaf $T\mathcal{F} \subset TX$, can be both viewed as objects defined over a ring R of characteristic zero finitely generated over $\mathbb{Z}$. If $\mathfrak{p} \subset R$ is a maximal ideal then $R/\mathfrak{p}$ is a finite field k of characteristic $p > 0$. The reduction modulo $\mathfrak{p}$ of $\mathcal{F}$ is the *foliation* $\mathcal{F}_\mathfrak{p}$ determined by the subsheaf $T\mathcal{F}_\mathfrak{p} = T\mathcal{F} \otimes_R k$ of the tangent sheaf of the projective variety $X_\mathfrak{p} = X \otimes_R k$. In layman terms, we are just reducing modulo $\mathfrak{p}$ the equations (which have coefficients in R) defining X and $\mathcal{F}$. For more on the reduction modulo p see, [28] and [52, Chapter 1, §2.5].

Here we will use reduction modulo $\mathfrak{p}$ to in order to find invariant hypersurfaces and integrating factors for semi-stable complex foliations with numerically trivial canonical bundle. We will implicitly make use of the following result.

Proposition 3.1. *Let $\mathcal{F}$ be a foliation on a polarized projective manifold (X, H). If there are integers M, m, and an infinite set of primes $\mathcal{P}$ such that $\mathcal{F}_\mathfrak{p}$ has an invariant subvariety of dimension m and degree at most M for every $\mathfrak{p} \in \mathcal{P}$ then $\mathcal{F}$ has an invariant hypersurface of dimension m degree at most M.*

Proof. For a fixed Hilbert polynomial χ, the subschemes of X invariant by $\mathcal{F}$ with Hilbert polynomial χ form a closed subscheme $Hilb_\chi(X, \mathcal{F})$ of $Hilb_\chi(X)$, see [19]. Moreover, its formation commutes with base change. Thus $Hilb_\chi(X, \mathcal{F})$ is non-empty if and only if $Hilb_\chi(X_\mathfrak{p}, \mathcal{F}_\mathfrak{p})$ is non-empty for infinitely many primes $\mathfrak{p}$, see for instance [52, Lecture I, Proposition 2.6]. To conclude it suffices to remind that irreducible reduced subvarieties of $X_\mathfrak{p}$ of bounded degree have bounded Hilbert polynomial, independently of $\mathfrak{p}$. □

If v is vector field on a manifold of positive characteristic then its p-th power is also a vector field since it satisfies Leibniz's rule:

$$v^p(f \cdot g) = \sum_{i=0}^{p} \binom{p}{i} v^i(f) v^{p-i}(g) = f v^p(g) + v^p(f) g \mod p\,.$$

A foliation $\mathcal{F}$ on a manifold X defined over a field of characteristic $p > 0$ is said to be p-closed if and only if for every local section v of $T\mathcal{F}$ its p-th power v^p is also a local section of $T\mathcal{F}$. The p-closed foliations of codimension q are precisely those that can be defined by q rational functions $f_1, \ldots, f_q$ in the sense that $df_1 \wedge \ldots \wedge df_q$ is a non-zero rational section of $\det N^*\mathcal{F} \subset \Omega^q_X$. This illustrates what is perhaps the most astonishing contrast between foliations in positive/zero characteristic: the easiness/toughness to decide whether or not $\mathcal{F}$ has first integrals.

If $\mathcal{F}$ is a foliation on a projective manifold defined over a finitely generated $\mathbb{Z}$-algebra $R \subset \mathbb{C}$ then the behavior of $X_{\mathfrak{p}}$ and $\mathcal{F}_{\mathfrak{p}}$ may vary wildly when $\mathfrak{p}$ varies among the maximal primes of R. Thus in order to have some hope to read properties of $\mathcal{F}$ on its reductions modulo $\mathfrak{p}$ one has to discard the *bad primes*. When a foliation $\mathcal{F}$ on a complex projective manifold has $\mathfrak{p}$-closed reduction modulo $\mathfrak{p}$ for every maximal prime ideal $\mathfrak{p}$ lying in an open subset $U \subset \mathrm{Spec}(R)$ then we will simply say that $\mathcal{F}$ is p-closed.

As already mentioned in the Introduction, Ekehdal, Shepherd-Barron, and Taylor [28] conjectured that p-closed foliations are foliations by algebraic leaves. This generalizes a previous conjecture by Grothendieck and Katz about the reduction modulo p of flat connections. Despite the recent advances, notably [9], both conjectures are still wide open.

3.2. **Integrating factors in positive characteristic.** In this section we collect some preliminary results, simple variations of the ones in [17, Section 5], which will be essential in what follows.

Lemma 3.2. *Let X be a smooth affine variety of dimension n defined over an algebraically closed field of arbitrary characteristic. If ω is an integrable q-form which is non zero at a closed point $x \in X$ then there exists $n-q$ regular vector fields $v_1, \ldots, v_{n-q}$ at an affine neighborhood of x such that*

1. *$v_1 \wedge \cdots \wedge v_{n-q}(x) \neq 0$;*
2. *$[v_i, v_j] = 0$ for every $i, j \in \{1, \ldots, n-q\}$;*
3. *$i_{v_i}\omega = 0$ for every $i \in \{1, \ldots, n-q\}$.*

Proof. As ω is integrable and $\omega(x) \neq 0$, we can write

$$\omega = \omega_n \wedge \omega_{n-1} \wedge \cdots \wedge \omega_{n-q+1}$$

where ω_i are rational 1-forms regular at a neighborhood of x. Let $f_1, \ldots, f_{n-q} \in k(X)$ be rational functions, regular at a neighborhood of x, such that

$$(\omega \wedge df_1 \wedge \cdots \wedge df_{n-q})(x) \neq 0\,.$$

If we set $\omega_i = df_i$, for $i = 1, \ldots, n-q$ then $\{\omega_i\}_{i=1}^n$ form a basis of the $k(M)$-vector space of rational 1-forms over X, and $\{\omega_i(x)\}_{i=1}^n$ for a basis of the k-vector space T^*_pX.

Let $\{v_i\}_{i=1}^n$ be a basis of the space of rational vector fields on X dual to $\{\omega_i\}_{i=1}^n$, i.e., $\omega_i(v_j) = \delta_{ij}$. It is clear that $i_{v_i}\omega = 0$ for every $i = 1 \ldots n-q$. We claim that $[X_i, X_j] = 0$ for every $i, j = 1 \ldots n-q$. It is sufficient to show that

$$\omega_k([X_i, X_j]) = 0 \text{ for every } k = 1 \ldots n \tag{1}$$

For $k > n-q$ the integrability of ω implies that (1) holds. For $k \leq n-q$ we have that

$$\begin{aligned} \omega_k([X_i, X_j]) &= X_i(\omega_k(X_j)) - X_j(\omega_k(X_i)) + d\omega_k(X_i, X_j) = \\ &= X_i(\delta_{kj}) - X_j(\delta_{ki}) + d^2 f_k(X_i, X_j) = 0\,. \end{aligned}$$

The lemma follows. □

The underlying idea in the next result is that p-th powers of vector fields tangent to a integrable q-form give rise to infinitesimal automorphisms, and the abundance of these allow us to find integrating factors.

Proposition 3.3. *Let X be a smooth variety defined over a field k of characteristic $p > 0$ and ω be a rational q-form on X. If ω is integrable and there exists rational vector fields $\xi_1, \ldots, \xi_q$ such that*

1. *$i_{\xi_i}\omega = 0$ for every $i = 1, \ldots, q$; and*
2. *$F = \omega(\xi_1^p, \ldots, \xi_q^p) \neq 0$*

then the q-form $F^{-1} \cdot \omega$ is closed.

Proof. Let n be the dimension of X and $v_1, \ldots, v_{n-q}$ be the rational vector fields given by Lemma 3.2. Thus $\xi_i = \sum_{j=1}^{n-q} a_{ij} v_j$ for suitable rational functions a_{ij}. By a formula of Jacobson [37, page 187] we can write $\xi_i^p = \sum_{j=1}^{n-q} a_{ij}^p v_j^p + P(a_{i,1}v_1, \ldots, a_{i,n-q}v_{n-q})$ with P being a Lie polynomial. Since $[v_i, v_j] = 0$ it follows that

$$\xi_i^p = \sum_{j=1}^{n-q} a_{ij}^p v_j^p\,, \quad \mod\ < v_1, \ldots, v_{n-q} > \,.$$

As we are interested in contracting ξ_i^p with ω we will replace ξ_i^p by $\zeta_i = \sum_{j=1}^{n-q} a_{ij}^p v_j^p$. Notice that $[\zeta_i, v_j] = 0$ for $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, n-q\}$; and $[\zeta_i, \zeta_j] = 0$ for $i, j \in \{1, \ldots, q\}$.

Set $\Xi_i = (-1)^i \zeta_1 \wedge \cdots \wedge \widehat{\zeta_i} \wedge \cdots \wedge \zeta_q$ and $\alpha_i = F^{-1} \cdot i_{\Xi_i}\omega$. The 1-forms α_i are integrable, i.e. $\alpha_i \wedge d\alpha_i = 0$ and, up to a non-zero multiplicative constant, the following identity holds true

$$F^{-1}\omega = \alpha_1 \wedge \cdots \wedge \alpha_q\,.$$

To prove the proposition it suffices to show that the 1-forms α_i are closed.

Fix an arbitrary $i \in \{1, \ldots, q\}$. The integrability of α_i together with $\alpha_i(\xi_i^p) = \alpha_i(\zeta_i) = 1$ implies

$$0 = i_{\zeta_i}(\alpha_i \wedge d\alpha_i) = d\alpha_i - \alpha_i \wedge i_{\zeta_i} d\alpha_i\,.$$

Hence to prove α_i is closed, it suffices to verify that the 1-form $i_{\zeta_i} d\alpha_i$ is zero. As the vector fields $v_1, \ldots, v_{n-q}, \zeta_1, \ldots, \zeta_q$ commute, then for every vector field w in the previous list we have

$$i_{\zeta_i} d\alpha_i(w) = \alpha_i([\zeta_i, w]) - \zeta_i(\alpha_i(w)) + w(\alpha_i(\zeta_i)) = 0\,.$$

This ensures $d\alpha_i = 0$ and settles the proposition. □

Corollary 3.4. *Hypothesis as in Proposition 3.3. If $\tilde{\xi}_1, \ldots, \tilde{\xi}_q$ is another collection of rational vector fields satisfying (1) and (2) then the rational functions $F = \omega(\xi_1^p, \ldots, \xi_q^p)$ and $\tilde{F} = \omega(\tilde{\xi}_1^p, \ldots, \tilde{\xi}_q^p)$ differ by the multiplication of a p-th power of a rational function, i.e., $F = H^p \tilde{F}$, for some rational function H.*

3.3. **Lifting integrating factors.** We will now focus on foliations of codimension one.

Theorem 3.5. *Let (X, H) be a polarized projective complex manifold, and $\mathcal{F}$ a semi-stable foliation of codimension one on X. If $K\mathcal{F} \cdot H^{n-1} = 0$ then at least one of the following assertions holds true*

1. *the foliation $\mathcal{F}$ is p-closed;*
2. *$\mathcal{F}$ is induced by a closed rational 1-form with coefficients in a flat line bundle and without divisorial components in its zero set.*

Proof. Suppose the set of primes $\mathcal{P}$ for which $\mathcal{F}_{\mathfrak{p}}$ – the reduction mod $\mathfrak{p}$ of $\mathcal{F}$ – is not p-closed is infinite, and fix $\mathfrak{p} \in \mathcal{P}$.

To raise germs of vector fields v in $T\mathcal{F}_{\mathfrak{p}}$ to theirs p-th powers provides a non-zero global section $S_{\mathfrak{p}}$ of

$$\mathrm{Hom}_{\mathcal{O}_X}(F^*T\mathcal{F}_{\mathfrak{p}}, N\mathcal{F}_{\mathfrak{p}}) = (F^*T\mathcal{F}_{\mathfrak{p}})^* \otimes N\mathcal{F}_{\mathfrak{p}}$$

where F is the absolute Frobenius.

Let us explicitly describe $S_{\mathfrak{p}}$ at sufficiently small Zariski open subsets U_i disjoint from the singular set of $\mathcal{F}_{\mathfrak{p}}$. Let $v_{1,i}, ..., v_{n-1,i}$ be $n-1$ vector fields tangent to the foliation and such that $v_{1,i} \wedge ... \wedge v_{n-1,i}$ does not vanish on U_i. We can also assume that $\mathcal{F}_{\mathfrak{p}}$ is defined on the same domain by a 1-form ω_i without divisorial components in its singular set. Take another open set U_j with the same properties. On overlapping charts, we have

$$\begin{pmatrix} v_{1,i} \\ \vdots \\ v_{n-1,i} \end{pmatrix} = M_{ij} \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{n-1,j} \end{pmatrix}$$

where the matrix cocycle $\{M_{ij}\}$ represents the cotangent bundle $T^*\mathcal{F}_{\mathfrak{p}}$ of the foliation outside $\mathrm{sing}(\mathcal{F}_{\mathfrak{p}})$.

As a consequence, using Jacobson's formula [37], we obtain

$$\begin{pmatrix} v_{1,i}^p \\ \vdots \\ v_{n-1,i}^p \end{pmatrix} = N_{ij} \begin{pmatrix} v_{1,j}^p \\ \vdots \\ v_{n-1,j}^p \end{pmatrix} \ mod\ T\mathcal{F}_{\mathfrak{p}}$$

where the matrix N_{ij} is obtained from M_{ij} by replacing each entry by its p^{th} power. If we set $s_{k,i} = \omega_i(v_{k,i}^p)$ then we gain the following equality

$$\begin{pmatrix} s_{1,i} \\ \vdots \\ s_{n-1,i} \end{pmatrix} = g_{ij} N_{ij} \begin{pmatrix} s_{1,j} \\ \vdots \\ s_{n-1,j} \end{pmatrix}$$

where g_{ij} is the cocycle representing the normal bundle of $\mathcal{F}_{\mathfrak{p}}$. The collection of vector $\{(s_{1,i}s_{2,i} \dots s_{n-1,i})^T\}$ represents $S_{\mathfrak{p}}$ on $X_{\mathfrak{p}} - \mathrm{sing}(\mathcal{F}_{\mathfrak{p}})$.

Let $D_{\mathfrak{p}}$ be the zero divisor of the section $S_{\mathfrak{p}}$. Over U_i, $D_{\mathfrak{p}}$ is given by the common zeros of $s_{1,i}, \dots, s_{n-1,i}$. Since $\mathcal{F}_{\mathfrak{p}}$ is not p-closed, there is at least one among these functions which do not vanish identically. Choose one and denote it by s_i. Corollary 3.4 implies that over $U_i \cap U_j$, $s_i = g_{ij} h_{ij}^p s_j$ for some rational function h_{ij} in $U_i \cap U_j$. Therefore

$$\frac{ds_i}{s_i} - \frac{ds_j}{s_j} = \frac{dg_{ij}}{g_{ij}}\,.$$

If $C_{\mathfrak{p}}$ is an arbitrary irreducible curve on $X_{\mathfrak{p}}$ then

$$D_{\mathfrak{p}} \cdot C_{\mathfrak{p}} = N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}} \mod p \tag{2}$$

as one can verify by writing $g_{ij} = g_i/g_j$ as a quotient of sufficiently general rational functions and applying the Residue formula to the logarithmic 1-form $\eta_{\mathfrak{p}}$ obtained by patching together the 1-forms $\frac{ds_i}{s_i} - \frac{dg_i}{g_i}$. Notice that if we set $\omega = \omega_i/g_i$ then

$$d\omega = \omega \wedge \eta_{\mathfrak{p}} \quad \text{and} \quad d\eta_{\mathfrak{p}} = 0\,.$$

In order to obtain further restrictions on $D_{\mathfrak{p}}$, we will use the following result by Shepherd-Barron, [60, Corollary 2^p] and [42].

Lemma 3.6. *(char p) Suppose that $\mathcal{E}$ is a semi-stable vector bundle of rank r over a curve C of genus g. Consider $F^*\mathcal{E} = \tilde{\mathcal{E}}$, the pull-back of $\mathcal{F}$ under the absolute Frobenius, then there exists $M = M(r,g) > 0$ independent of p such that*

$$\mu_{max}(\tilde{\mathcal{E}}) - \mu_{min}(\tilde{\mathcal{E}}) \leq M$$

Now, return to the original $\mathcal{F}$ on the complex manifold X. Consider a general complete intersection curve C cut out by elements of $|mH|$ ($m \gg 0$) for which the $T\mathcal{F}_{|C}$ is semi-stable. Notice that this semi-stability is preserved under specialization *mod* $\mathfrak{p}$ for almost all $\mathfrak{p}$.

Restricting $S_{\mathfrak{p}}$ to $C_{\mathfrak{p}}$ and cleaning up its zero divisor we get a section of

$$\mathrm{Hom}_{\mathcal{O}_{C_{\mathfrak{p}}}}(F^*\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}, N\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}} \otimes \mathcal{O}_{C_{\mathfrak{p}}}(-D_{\mathfrak{p}})).$$

Since $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A},\mathcal{B}) = 0$ whenever $\mu_{min}\mathcal{A} > \mu_{max}\mathcal{B}$, we deduce that

$$\mu_{min}(F^*\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}) \leq \mu_{max}(N\mathcal{F}_{\mathfrak{p}|C_{\mathfrak{p}}}) - D_{\mathfrak{p}} \cdot C_{\mathfrak{p}}\,.$$

Lemma 3.6 and the fact that $\mu_{max}(F^*\mathcal{F}_{|C_{\mathfrak{p}}}) \geq 0$ implies

$$D_{\mathfrak{p}} \cdot C_{\mathfrak{p}} \leq M + N\mathcal{F}_{\mathfrak{p}} \cdot C_{\mathfrak{p}}$$

with M uniform in $\mathfrak{p}$. Combined with (2) this last inequality implies $D_{\mathfrak{p}}$ is numerically equivalent to $N\mathcal{F}_{\mathfrak{p}}$. In particular, the degree of $D_{\mathfrak{p}}$ is uniformly bounded, and the same holds true for the degree of the polar locus of $\eta_{\mathfrak{p}}$. Thus we can lift the integrating factor $\eta_{\mathfrak{p}}$ to characteristic zero and obtain a logarithmic 1-form η such that $\eta + \frac{dg_i}{g_i}$ has residues in $\mathbb{Z}_{>0}$. Therefore, in characteristic zero, the 1-form

$$\frac{\omega}{\exp\int \eta}$$

is a rational 1-form with isolated zeros and coefficients in a flat line bundle. □

3.4. **Singularities of p-closed foliations.** McQuillan observed in [49, Proposition II.1.3] that isolated singularities of p-closed foliations of dimension one with non-nilpotent linear part are fairly special. We state below his result.

Lemma 3.7. *Let $\mathcal{F}$ be a p-closed foliation by curves on a projective variety X. If $x \in \mathrm{sing}(\mathcal{F})$ is an isolated singularity with non-nilpotent linear part then there exist formal coordinates at x where $\mathcal{F}$ is generated by the linear vector field*

$$v = \sum_{i=1}^{n} \lambda_i x_i$$

where $\lambda_1, \ldots, \lambda_n$ are non-zero integers.

Theorem 3.8. *Let (X, H) be a polarized projective complex manifold and $\mathcal{F}$ be a codimension one semi-stable foliation on X with numerically trivial canonical bundle. Suppose $c_1(TX)^2 \cdot H^{n-2} > 0$. If $\mathcal{F}$ is p-closed then*

1. *$\mathcal{F}$ is a rationally connected foliation, i.e., the generic leaf of $\mathcal{F}$ is a rationally connected algebraic variety; or*
2. *$\mathcal{F}$ is strictly semi-stable and there is a rationally connected foliation $\mathcal{H}$ tangent to $\mathcal{F}$ and with $K\mathcal{H} \cdot H^{n-1} = 0$.*

Proof. As $c_1(T\mathcal{F}) = 0$, we have that $c_1(TX) = c_1(N\mathcal{F})$. Thus $c_1(TX)^2 \cdot H^{n-2} = c_1(N\mathcal{F})^2 \cdot H^{n-2} > 0$ and Baum-Bott index Theorem implies the existence of a codimension two component S of the singular set of $\mathcal{F}$ which has positive Baum-Bott index.

Take a generic surface $\Sigma \subset X$ intersecting S transversally. As p-closedness is preserved by birational transformations and restrictions to subvarieties, it follows from Lemma 3.7 that the singularities of $\mathcal{F}_{|\Sigma}$ on $S \cap \Sigma$ either have zero linear part or are linearizable with quotient of eigenvalues rational positive.

Therefore, if we first resolve the singularities of S and then blow-up its strict transforms sufficiently many times we obtain a sequence of blow-ups $\pi : Y \to X$ such that the canonical bundle of $\mathcal{G} = \pi^*\mathcal{F}$ is of the form

$$K\mathcal{G} = \pi^* K\mathcal{F} - E - D\,,$$

where E is an effective divisor supported on an irreducible hypersurface such that $\pi(E) = S$ and D is a divisor (not necessarily effective) such that $\pi(|D|) \subset \mathrm{sing}(S)$. In particular $\pi(|D|)$ has codimension at least three.

Let A be an ample line bundle on Y. We claim that for $\varepsilon > 0$ sufficiently small, the divisor $H_\varepsilon = \pi^* H + \varepsilon A$ satisfies $(K\mathcal{G}) \cdot H_\varepsilon^{n-1} < 0$. Indeed,

$$(K\mathcal{G}) \cdot H_\varepsilon^{n-1} = K\mathcal{F} \cdot H^{n-1} + \varepsilon(n-1)(\pi^* H)^{n-2} \cdot A \cdot (-D - E) \quad mod\ \varepsilon^2.$$

Since $\pi(|D|)$ has codimension at least three, the intersection of $(\pi^* H)^{n-2} \cdot A$ with D is zero. The claim follows.

If $\mathcal{G}$ is H_ε-stable for some choice of A and $\varepsilon > 0$ then Theorem 2.4 implies that the leaves of $\mathcal{G}$ are rationally connected varieties. If not then $\mathcal{H}$, the maximal destabilizing foliation of $\mathcal{G}$, will satisfy

$$\mu(T\mathcal{H}) < \mu(T\mathcal{G})$$

where the slope μ is computed as a function of A and ε. Making ε arbitrarily small we deduce that $\mu(\pi_* \mathcal{H}) \leq \mu(T\mathcal{F})$ (now μ is computed using H), and we deduce that $\mathcal{F}$ is strictly semi-stable. The Theorem follows. □

From the proof of Theorem 3.8 we can promptly deduce the following result.

Corollary 3.9. *Let $\mathcal{F}$ be a p-closed foliation with $c_1(K\mathcal{F}) = 0$ on a projective manifold X. If $\mathcal{F}$ is not uniruled and S is an irreducible component of* $\mathrm{sing}(\mathcal{F})$ *of codimension two then at a generic point of S the foliation $\mathcal{F}$ is locally defined by a holomorphic 1-form of type*

$$pxdy + qydx$$

with p, q relatively prime positive integers.

Later in Section 6 we will show that under the same hypothesis of the corollary above we can ensure the existence of at least one irreducible component of $\mathrm{sing}(\mathcal{F})$ having non-zero Baum-Bott index, i.e., on this component $p \neq q$.

4. First integrals of (semi)-stable foliations

Miyaoka-Bogomolov-McQuillan Theorem (Theorem 2.4) tells us that semi-stable foliations with negative canonical bundle have algebraic leaves and that the generic one is rationally connected. The goal of this section is to complement this result for codimension one foliations by giving more information about the first integral. We also deal with stable foliations with numerically trivial canonical bundle having rational first integrals, and the results here presented will play an important role in proof of the classification of codimension one foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

4.1. **Invariant hypersurfaces and subfoliations.** Let $\mathcal{F}$ be a foliation of codimension q on a compact Kähler manifold X. Let $\mathrm{Div}(\mathcal{F}) \subset \mathrm{Div}(X)$ be the subgroup of the group of divisors of X generated by irreducible hypersurfaces invariant by $\mathcal{F}$. The arguments used in [31] to prove Jouanolou's theorem lead us to the following result.

Lemma 4.1. *Suppose the dimension of $\mathcal{F}$ is greater than or equal to two. If $D \in \mathrm{Div}(\mathcal{F})$ satisfies $c_1(D) = m \cdot c_1(N\mathcal{F})$ for a suitable $m \in \mathbb{Z}$ then at least one of the following assertions holds true:*

(a) *the integer m is non-zero and $\mathcal{F}$ is, after a ramified abelian covering of degree m and a bimeromorphic morphism, defined by a meromorphic closed q-form with coefficients in a flat line bundle; or*
(b) *the integer m is zero and $\mathcal{F}$ is tangent to a codimension one logarithmic foliation with poles at the support of D and integral residues; or*
(c) *there exists a foliation $\mathcal{G}$ of codimension $q+1$ tangent to $\mathcal{F}$ with normal sheaf satisfying*

$$\det N\mathcal{G} = \det N\mathcal{F} \otimes \mathcal{O}_X(-\Delta)$$

for some effective divisor $\Delta \geq 0$.

Proof. Let $N = \det N\mathcal{F}$ and $\omega \in H^0(X, \Omega^q_X \otimes N)$ be a twisted q-form defining $\mathcal{F}$. Write D as $\sum \lambda_\alpha H_\alpha$ with $\lambda_\alpha \in \mathbb{Z}$.

Our hypothesis ensure the existence of an open covering of $\mathcal{U} = \{U_i\}$ where

$$H_\alpha \cap U_i = \{h^{(i)}_\alpha = 0\} \quad \text{and} \quad \sum \lambda_\alpha \left(\frac{dh^{(i)}_\alpha}{h^{(i)}_\alpha} - \frac{dh^{(j)}_\alpha}{h^{(j)}_\alpha} \right) = m \frac{dg_{ij}}{g_{ij}}$$

where $\{g_{ij}\} \in H^1(\mathcal{U}, \mathcal{O}^*_X)$ is a cocycle defining N, i.e. ω is defined by a collection of q-forms $\{\omega_i \in \Omega^q_X(U_i)\}$ which satisfies $\omega_i = g_{ij}\omega_j$.

On U_i, set $\eta_i = \sum \lambda_\alpha \frac{dh^{(i)}_\alpha}{h^{(i)}_\alpha}$ and define

$$\theta_i = \eta_i \wedge \omega_i + m \cdot d\omega_i \, .$$

As the hypersurfaces H_α are invariant by $\mathcal{F}$, θ_i is a holomorphic $(q+1)$-form. It is also clear that θ_i is locally decomposable and integrable. Moreover, on $U_i \cap U_j$ we have the identity

$$\theta_i = \left(\eta_j - m \frac{dg_{ij}}{g_{ij}} \right) \wedge g_{ij}\omega_j + m \cdot d(g_{ij}\omega_j) = g_{ij}\theta_j \, .$$

Hence the collection $\{\theta_i\}$ defines a holomorphic section θ of $\Omega^{q+1}_X \otimes N$. If this section is non-zero then it defines a foliation $\mathcal{G}$ with $\det N\mathcal{G} = \det N\mathcal{F} \otimes \mathcal{O}_X(-(\theta)_0)$. We are in case (c).

Suppose now that θ is identically zero. If $m = 0$ then $\eta_i = \eta_j$ on $U_i \cap U_j$ and we can patch then together to obtain a logarithmic 1-form η with poles at the support of D. Clearly we are in case (b).

If $m \neq 0$ then, on U_i the (multi-valued) meromorphic q-form

$$\Theta_i = \exp\left(\int \frac{1}{m}\eta_i\right)\omega_i = \left(\prod h_{\alpha_i}^{\lambda_\alpha/m}\right)\omega_i$$

is closed. Moreover, if $U_i \cap U_j \neq \emptyset$ then $\Theta_i = \mu_{ij}\Theta_j$ for suitable $\mu_{ij} \in \mathbb{C}^*$. It is a simple matter to see that we are in case (a). □

4.2. **Number of reducible fibers of first integrals.** Let $\mathcal{F}$ be a codimension one foliation on a polarized projective manifold (X, H) having a rational first integral. Stein's factorization ensures the existence of a rational first integral $F : X \dashrightarrow C$ with irreducible generic fiber. We are interested in bounding the number of non-irreducible fibers of f. More precisely we want to bound the number

$$r(\mathcal{F}) = r(F) = \sum_{x\in C} \left(\#\{ \text{ irreducible components of } F^{-1}(x)\} - 1\right) .$$

This problem, for rational functions $F : X \dashrightarrow \mathbb{P}^1$ has been investigated by A. Vistoli and others. In [66] he obtains a bound in function of the rank of the Neron-Severi group of X and what he calls the base number of F. In particular, when X is $\mathbb{P}^n$, he proves that $r(F) \leq \deg(F)^2 - 1$ where $\deg(F)$ is the degree of a generic fiber of F. Our result below gives much stronger bounds for the first integrals obtained through Theorem 2.4 when $\dim X \geq 3$.

Theorem 4.2. *Suppose the dimension X is at least three. If $\mathcal{F}$ is semi-stable and $c_1(T\mathcal{F}) \cdot H^{n-1} > 0$, or $\mathcal{F}$ is stable and $c_1(T\mathcal{F}) \cdot H^{n-1} = 0$ then*

$$r(\mathcal{F}) \leq \operatorname{rank} NS(X) - 1 ,$$

where $NS(X)$ is the Neron-Severi group of X. In particular, if $X = \mathbb{P}^n$, $n \geq 3$, then $r(\mathcal{F}) = 0$.

Proof. Let $x_1, \ldots, x_k$ be the points of C for which $F^{-1}(x)$ is non-irreducible, and let $n_1, \ldots, n_k$ be the number of irreducible components of $F^{-1}(x_i)$. Choose $n_i - 1$ irreducible components in each of the non-irreducible fibers and denote them by $F_1, \ldots, F_{r(\mathcal{F})}$. If $r(\mathcal{F}) \geq \operatorname{rank} NS(X)$ then an irreducible fiber F_0 is numerically equivalent to a $\mathbb{Q}$-divisor supported on $F_1 \cup \cdots \cup F_{r(\mathcal{F})}$. Therefore we can construct a logarithmic 1-form η with polar divisor supported on $F_0, F_1, \ldots, F_{r(\mathcal{F})}$. According to Lemma 4.1 either there exists a codimension two foliation $\mathcal{G}$ contained in $\mathcal{F}$ with $\det N\mathcal{G} = N\mathcal{F}(-\Delta)$, for some $\Delta \geq 0$; or $\mathcal{F}$ is defined by η. We will now analyze these two possibilities.

If there exists $\mathcal{G}$ as above and $c_1(T\mathcal{F}) \cdot H^{n-1} < 0$ then

$$0 < c_1(T\mathcal{F}) \cdot H^{n-1} = (c_1(T\mathcal{G}) - \Delta) \cdot H^{n-1} \leq c_1(T\mathcal{G}) \cdot H^{n-1} ,$$

which implies $\mu(T\mathcal{F}) < \mu(T\mathcal{G})$. Similarly, when $c_1(T\mathcal{F}) \cdot H^{n-1} = 0$ we deduce $\mu(T\mathcal{F}) \geq \mu(T\mathcal{G})$. In both cases we have a contradiction.

Suppose now that $\mathcal{F}$ is defined by η. As the generic fiber of $\mathcal{F}$ is irreducible, there exists a logarithmic 1-form η' on C such that $\eta = F^*\eta'$. But this implies that the

polar set of η is set-theoretically equal to union of fibers of F. This contradicts the choice of the irreducible components of the fibers of F, and concludes the proof. □

4.3. **Multiple fibers of rational maps to $\mathbb{P}^1$.** A classical result of Halphen [33, Chapitre 1] says that a rational map $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ with irreducible generic fiber has at most two multiple fibers. In this section we follow closely the exposition of Lins Neto [46] to establish the following generalization.

Theorem 4.3. *Let X be a simply-connected compact Kähler manifold and $F : X \dashrightarrow \mathbb{P}^1$ be meromorphic map. If the generic fiber of F is irreducible then F has at most two multiple fibers.*

We will say that a line bundle $\mathcal{L}$ is primitive if its Chern class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ generates a maximal rank 1 submodule of $H^2(X, \mathbb{Z})$. To adapt Lins Neto's proof of Halphen's Theorem to other manifolds we will need the following lemma.

Lemma 4.4. *Let X be a simply-connected compact complex manifold. If $\mathcal{L} \in \mathrm{Pic}(X)$ is a primitive line bundle on X then the total space of $\mathcal{L}$ minus its zero section is simply-connected.*

Proof. Let E be the total space of $\mathcal{L}$ minus its zero section. As E is a $\mathbb{C}^*$-bundle, we can use Gysin sequence

$$H^1(X, \mathbb{Z}) \to H^1(E, \mathbb{Z}) \to H^0(X, \mathbb{Z}) \stackrel{\wedge c_1(\mathcal{L})}{\longrightarrow} H^2(X, \mathbb{Z})$$

to deduce that the fundamental group of E is torsion. If E is not simply-connected then its universal covering is a $\mathbb{C}^*$-bundle over X, and the associated line bundle divides $\mathcal{L}$. This contradicts the primitiveness of $\mathcal{L}$. □

Proof of Theorem 4.3. Let $\mathcal{L}$ be a primitive line bundle such that $\mathcal{L}^{\otimes k} = F^*\mathcal{O}_{\mathbb{P}^1}(1)$, for some positive integer k. If E is the total space of the $\mathbb{C}^*$-bundle defined by $\mathcal{L}^*$ then sections of $\mathcal{L}$ and its positive powers naturally define holomorphic functions on E. Moreover, if $f \in H^0(X, \mathcal{L}^{\otimes k})$ then the element of $H^0(E, \mathcal{O}_E)$ determined by f, which we still denote by f, is homogeneous of degree k with respect to $\mathbb{C}^*$-action on E given by fiberwise multiplication.

Now let $F : X \dashrightarrow \mathbb{P}^1$ be a rational map with three multiple fibers, of multiplicity p, q, r. Assume that they are over the points $[0 : 1], [1 : 0], [1 : -1]$. Thus we can write $F = f^p/g^q$ with

$$f^p + g^q + h^r = 0, \tag{3}$$

and $f^p, g^q, h^r \in H^0(X, \mathcal{L}^{\otimes k})$. If we interpret f, g, h now as functions on E then taking the differential of the relation (3) we get

$$pf^{p-1}df + qg^{q-1}dg + rh^{r-1}dr = 0\,.$$

Taking the wedge product with df, dg and dh we deduce the following equalities between holomorphic 2-forms

$$\frac{df \wedge dg}{h^{r-1}} = \frac{dg \wedge dh}{f^{p-1}} = \frac{df \wedge dh}{g^{q-1}}$$

where we have deliberately omitted irrelevant constants. Let now R be a vector field tangent to the $\mathbb{C}^*$-action on E. The contraction of this 2-form with R induce a non-zero section of $\Omega^1_X \otimes \mathcal{L}^a$ for a suitable integer a. As they are holomorphic we have

that $a \geq 0$. Moreover, since X is Kähler and simply-connected, $H^0(X, \Omega^1_X) = 0$ and, consequently, $a > 0$. Therefore

$$\frac{k}{p} + \frac{k}{q} - \frac{(r-1)k}{r} = \frac{k}{q} + \frac{k}{r} - \frac{(p-1)k}{p} = \frac{k}{p} + \frac{k}{r} - \frac{(q-1)k}{q} = a > 0$$

which implies

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + a > 1\,.$$

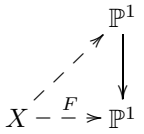
Therefore the triple (p, q, r), after reordering, must be one of the following: $(2, 2, m), (2, 3, 3), (2, 3, 4)$, or $(2, 3, 5)$. If $S_{p,q,r} = \{(x, y, z) \in \mathbb{C}^3 \setminus \{0\} | x^p + y^q + z^r = 0\}$ then $S_{p,q,r}$ is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by one of the following subgroups $G \subset SL(2, \mathbb{C})$ (see for instance [46, Example 1.11])

- a dihedral group of order $2m$ when $(p, q, r) = (2, 2, m)$;
- the tetrahedral group when $(p, q, r) = (2, 3, 3)$;
- the octahedral group when $(p, q, r) = (2, 3, 4)$; and
- the icosahedral group when $(p, q, r) = (2, 3, 5)$.

Since E is simply-connected and the base locus of F has codimension two, we can lift the map

$$\begin{aligned} \varphi : E \setminus \{f = g = 0\} &\longrightarrow S_{p,q,r} \\ x &\longmapsto (f, g, h)\,. \end{aligned}$$

to $\mathbb{C}^2 \setminus \{0\}$, the universal covering of $S_{p,q,r}$. Since the lift of F to E factors through φ, we deduce that $F : X \dashrightarrow \mathbb{P}^1$ fits into the diagram



where the vertical arrow has positive degree. Therefore the generic fiber of F is not irreducible. With this contradiction we conclude the proof. □

4.4. **Codimension one stable foliations with first integrals.** Having Theorem 4.3 at hand we are able to give precisions about the structure of the first integrals of semi-stable foliations of codimension one having negative canonical bundle on projective manifolds with rank one Picard group.

Proposition 4.5. *Let X be a projective manifold with $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathcal{F}$ be a codimension one foliation on X. Suppose*

(a) *$\mathcal{F}$ is semi-stable and $K\mathcal{F} < 0$, or*
(b) *$\mathcal{F}$ is stable, has a rational first integral, and $K\mathcal{F} = 0$.*

Then $\mathcal{F}$ admits a rational first integral of the form $(f^p : g^q) : X \dashrightarrow \mathbb{P}^1$ where f, g are sections of line bundles $\mathcal{L}_1, \mathcal{L}_2$ which satisfy

$$\mathcal{L}_1^{\otimes p} = \mathcal{L}_2^{\otimes q} \quad \text{and} \quad N\mathcal{F} = \mathcal{L}_1 \otimes \mathcal{L}_2.$$

In particular $\mathcal{F}$ is defined by a logarithmic 1-form without divisorial components in its zero set.

Proof. Let $F : X \dashrightarrow \mathbb{P}^1$ be a rational first integral for $\mathcal{F}$ with generic irreducible generic fiber. Notice that the target has to be $\mathbb{P}^1$ since $\mathrm{Pic}(X) = \mathbb{Z}$. Theorem 4.2 implies that every fiber of F is irreducible, and Theorem 4.3 tells us that there are at most two non-reduced fibers. Assume that they are over $0, \infty \in \mathbb{P}^1$ and write $F^{-1}(0) = pH_0$, $F^{-1}(\infty) = qH_\infty$ where H_0 and H_∞ are reduced and irreducible hypersurfaces. If we take the logarithmic 1-form on $\mathbb{P}^1$ given in homogeneous coordinates by $dx/x - dy/y$ and we pull-back it by F then the resulting logarithmic 1-form, which defines $\mathcal{F}$, has polar divisor equal to $H_0 + H_\infty$ and empty zero divisor. Therefore $N\mathcal{F} = \mathcal{O}_X(H_0 + H_\infty)$ and the F can be written as $(f^p : g^q)$ with $f \in H^0(X, \mathcal{O}_X(H_0))$, $g \in H^0(X, \mathcal{O}_X(H_\infty))$. The proposition follows. □

Corollary 4.6. *Let $\mathcal{F}$ be a semi-stable codimension one foliation on $\mathbb{P}^n$, $n \geq 3$. If $\deg(\mathcal{F}) < n - 1$ then $\mathcal{F}$ admits a rational first integral of form $(F^p : G^q)$ where F and G are homogeneous polynomials and p, q are relatively prime positive integers such that $p\deg(F) = q\deg(G)$ and $\deg(F) + \deg(G) - 2 = \deg(\mathcal{F})$.*

4.5. **Very negative foliations on Fano manifolds.** A projective manifold X is Fano if its anticanonical bundle $-KX$ is ample. Let H be an ample generator of the Picard group of a Fano manifold with $\rho(X) = 1$ ($\rho(X)$ is the rank of the Picard group of X). The **index** of X, denoted by $i(X)$, is defined through the relation $-KX = i(X)H$. The index of a Fano manifold of dimension n is bounded by $n + 1$ and the extremal cases are $\mathbb{P}^n$ ($i(X) = n + 1$) and hyperquadrics $Q^n \subset \mathbb{P}^{n+1}$ ($i(X) = n$), see [41].

A codimension one foliation of degree one on $\mathbb{P}^n$ has canonical bundle $K\mathcal{F}$ equal to $\mathcal{O}_{\mathbb{P}^n}(2-n)$, see Example 2.1. Our next result can be thought as a generalization of Jouanolou's classification of codimension one foliations of degree one on $\mathbb{P}^n$ to arbitrary Fano manifolds with $\rho(X) = 1$.

Proposition 4.7. *Let X be a Fano manifold of dimension $n \geq 3$ and Picard number $\rho(X) = 1$. Let H be an ample generator of $\mathrm{Pic}(X)$. If $\mathcal{F}$ is a codimension one foliation on X with $K\mathcal{F} = (2-n)H$ then $\mathcal{F}$ is a foliation of degree one on $\mathbb{P}^n$, or $\mathcal{F}$ is the restriction of a pencil of hyperplanes on $\mathbb{P}^{n+1}$ to a hyperquadric Q^n.*

Proof. Assume first that $\mathcal{F}$ is semi-stable. Theorem 2.4, or rather Corollary 2.5, implies $\mathcal{F}$ has a rational first integral. Proposition 4.5 implies $N\mathcal{F} \geq 2H$. Since $KX = K\mathcal{F} - N\mathcal{F}$, it follows that $KX \leq -nH$. Therefore $KX = -(n+1)H$, $N\mathcal{F} = 3H$ and $X = \mathbb{P}^n$, or $KX = -nH$, $N\mathcal{F} = 2H$ and $X = Q^n$. Proposition 4.5 implies $\mathcal{F}$ is a pencil of quadrics with a non-reduced member in the first case, and a pencil of hyperplane sections of Q^n in the second case.

Suppose now that $\mathcal{F}$ is unstable and let $\mathcal{G}$ be its maximal destabilizing foliation. Therefore

$$-K\mathcal{G} = c_1(T\mathcal{G}) > \frac{-K\mathcal{F}}{\dim(\mathcal{F})} \cdot \dim(\mathcal{G}) \geq (\dim(\mathcal{G}) - 1)H\,.$$

and, consequently, $-K\mathcal{G} \geq \dim(\mathcal{G})H$ and we can produce a non-zero section of $\wedge^{\dim(\mathcal{G})}TX \otimes \mathcal{O}_X(-\dim(\mathcal{G})H)$. It follows from [2, Theorem 1.2] that $X = \mathbb{P}^n$ and $\mathcal{G}$ is a foliation of degree zero on $\mathbb{P}^n$. Therefore $\mathcal{F}$ is the linear pull-back of a foliation of degree one on $\mathbb{P}^{n-\dim(\mathcal{G})}$. □

5. Foliations on Fano threefolds with rank one Picard group

The goal of this section is to classify codimension one foliations with $K\mathcal{F} = 0$ on Fano 3-folds with $\rho(X) = 1$. We will also describe the irreducible components of the corresponding moduli spaces.

5.1. **Rough Classification.** Before dealing with specific examples, we will prove the following rough classification.

Theorem 5.1. *[Theorem 4 of the Introduction] Let X be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}$, and let $\mathcal{F}$ be a codimension one foliation on X with trivial canonical bundle. If $\mathcal{F}$ is unstable then $X = \mathbb{P}^3$ and $\mathcal{F}$ is the linear pull-back of a degree two foliation on $\mathbb{P}^2$. If $\mathcal{F}$ is semi-stable then at least one of the following assertions holds true:*

1. *$T\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ and $\mathcal{F}$ is induced by an algebraic action;*
2. *$\mathcal{F}$ is tangent to an algebraic action of $\mathbb{C}$ or $\mathbb{C}^*$ with non-isolated fixed points;*
3. *$\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

5.1.1. *Division Lemma.* To prove Theorem 5.1 we will use the following division lemma.

Lemma 5.2. *Let X be a projective 3-fold, $\mathcal{G}$ be a one-dimensional foliation on X with isolated singularities, and $\mathcal{F}$ a codimension one foliation containing $\mathcal{G}$. If $H^1(X, KX \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F}) = 0$ then $T\mathcal{F} \cong T\mathcal{G} \oplus T\mathcal{H}$ for a suitable one-dimensional foliation $\mathcal{H}$.*

Proof. Let $v \in H^0(X, TX \otimes K\mathcal{G})$ be a twisted vector field defining $\mathcal{G}$. By hypothesis v has isolated singularities. Therefore (see for instance [27, Exercise 17.20]) contraction of differential forms with v defines a resolution of the singular scheme $\mathrm{sing}(\mathcal{G})$ of $\mathcal{G}$:

$$0 \to \Omega^3_X \to \Omega^2_X \otimes K\mathcal{G} \xrightarrow{\Phi} \Omega^1_X \otimes K\mathcal{G}^{\otimes 2} \to K\mathcal{G}^{\otimes 3} \to \mathcal{O}_{\mathrm{sing}(\mathcal{G})} \to 0\,.$$

After tensoring by $N\mathcal{F} \otimes K_{\mathcal{G}}^{\otimes -2}$, we obtain from the exact sequence above the following exact sequences

$$0 \to Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to \Omega^1_X \otimes N\mathcal{F} \to K\mathcal{G} \otimes N\mathcal{F}\,,$$

and

$$0 \to \Omega^3_X \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to \Omega^2 \otimes N\mathcal{F} \otimes K\mathcal{G}^{-1} \to Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} \to 0\,.$$

If $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ defines $\mathcal{F}$ then, since $\mathcal{F}$ contains $\mathcal{G}$, ω belongs to the kernel of

$$H^0(X, \Omega^1_X \otimes N\mathcal{F}) \to H^0(X, K\mathcal{G} \otimes N\mathcal{F})\,.$$

The first sequence tells us that we can lift ω to $H^0(X, Im\Phi \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F})$. The second exact sequence, together with our cohomological hypothesis, ensures the existence of $\theta \in H^0(X, \Omega^2_X \otimes N\mathcal{F} \otimes K\mathcal{G}^{-1})$ such that $\omega = i_v\theta$. Clearly θ defines the sought foliation $\mathcal{H}$. □

5.1.2. *Automorphisms of a foliation.* Let $\mathcal{F}$ be a codimension one foliation on a projective manifold X. The automorphism group of $\mathcal{F}$, $\mathrm{Aut}(\mathcal{F})$, is the subgroup of $\mathrm{Aut}(X)$ formed by automorphisms of X which send $\mathcal{F}$ to itself. It is a closed subgroup of $\mathrm{Aut}(X)$, and therefore the connected component of the identity is a finite dimensional connected Lie group. We will denote by $\mathfrak{aut}(\mathcal{F})$ its Lie algebra, which can be identified with a subalgebra of $\mathfrak{aut}(X) = H^0(X, TX)$. If $\mathcal{F}$ is defined by $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ then we define the $\mathfrak{fix}(\mathcal{F})$ as the subalgebra of $\mathfrak{aut}(\mathcal{F})$ annihilating ω, i.e.

$$\mathfrak{fix}(\mathcal{F}) = \{v \in \mathfrak{aut}(\mathcal{F}) \,|\, i_v\omega = 0\}\,.$$

Notice that $\mathfrak{fix}(\mathcal{F})$ is nothing more than $H^0(X, T\mathcal{F})$. We also point out that $\mathfrak{fix}(\mathcal{F})$ is an ideal of $\mathfrak{aut}(\mathcal{F})$, and that subgroup $\mathrm{Fix}(\mathcal{F}) \subset \mathrm{Aut}(\mathcal{F})$ generated by $\mathfrak{fix}(\mathcal{F})$ is not necessarily closed.

Lemma 5.3. *The following assertions hold true:*

1. *If $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ then $\mathcal{F}$ is tangent to an algebraic action.*
2. *If $\mathfrak{fix}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$ then $\mathcal{F}$ is generated by a closed rational 1-form without divisorial components in its zero set.*

Proof. The connected component of the identity of $\mathrm{Aut}(\mathcal{F})$ is closed. If $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ then $\mathrm{Fix}(\mathcal{F})$ is also closed and therefore correspond to an algebraic subgroup of $\mathrm{Aut}(X)$. Item (1) follows. To prove Item (2), let v be a vector field in $\mathfrak{aut}(\mathcal{F}) - \mathfrak{fix}(\mathcal{F})$. If $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ is a twisted 1-form defining $\mathcal{F}$ then [54, Corollary 2] implies $(i_v\omega)^{-1}\omega$ is a closed meromorphic 1-form. Since the singular set of ω has codimension at least two, the same holds for $(i_v\omega)^{-1}\omega$. □

Proof of Theorem 5.1. If $T\mathcal{F}$ is unstable then Proposition 2.2 implies the existence of a foliation by curves $\mathcal{G}$ tangent to $\mathcal{F}$ and with $T\mathcal{G} > 0$. Consequently $\mathcal{G}$ is defined by a vector field vanishing along an ample divisor. According to Wahl's Theorem [67], X is isomorphic to $\mathbb{P}^3$ and $T\mathcal{G} = \mathcal{O}_{\mathbb{P}^3}(1)$. Thus $\mathcal{G}$ is a foliation of degree zero and, consequently, its leaves are the lines through a point $p \in \mathbb{P}^3$. It follows that $\mathcal{F}$ is a pullback under the linear projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ determined by $\mathcal{G}$.

Suppose now that $T\mathcal{F}$ is stable. If $\mathcal{F}$ is not p-closed then Theorem 3.5 implies that $\mathcal{F}$ is defined by a closed rational 1-form without divisorial components in its zero set, since the only flat line bundle over X is the trivial one. If $\mathcal{F}$ is p-closed then Theorem 3.8 implies $\mathcal{F}$ has a rational first integral. Corollary 4.6 implies that also in this case $\mathcal{F}$ is defined by a logarithmic 1-form without codimension one components in its zero set. Thus if $T\mathcal{F}$ is stable, $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.

Finally, we will deal with the case where $T\mathcal{F}$ is strictly semistable. Now we have a foliation by curves $\mathcal{G}$ tangent to $\mathcal{F}$ with $T\mathcal{G} = \mathcal{O}_X$. In other words, $\mathcal{G}$ is induced by a vector field $v \in H^0(X, TX)$ with zeros of codimension at least two. Notice that $\mathbb{C}v \subset \mathfrak{fix}(\mathcal{F})$.

Suppose $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$. If $\mathfrak{fix}(\mathcal{F}) = \mathbb{C}v$ then we claim $\mathcal{G}$ is defined by an algebraic action of $\mathbb{C}$ or $\mathbb{C}^*$ with non-isolated fixed points. Indeed Lemma 5.3 implies $\mathcal{F}$ is tangent to action of a one-dimension Lie group. Moreover, if the action has only isolated fixed points then we can apply Lemma 5.2 to deduce that the tangent bundle of $\mathcal{F}$ is $\mathcal{O}_X \oplus \mathcal{O}_X$. Notice that the hypothesis of Lemma 5.2 are satisfied since $KX \otimes K\mathcal{G}^{\otimes -2} \otimes N\mathcal{F} = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$ for varieties with discrete Picard group.

If we still assume $\mathfrak{fix}(\mathcal{F}) = \mathfrak{aut}(\mathcal{F})$ but now with $\dim \mathfrak{fix}(\mathcal{F}) > 1$ then, as v has no divisorial components in its zero set, any two elements in it will generate $T\mathcal{F}$. Thus $T\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X$ in this case and $\mathcal{F}$ is defined by an algebraic action since $\mathrm{Aut}(\mathcal{F})$ is closed.

Finally, if $\mathfrak{fix}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$ then Lemma 5.3 implies $\mathcal{F}$ is given by a closed meromorphic 1-form with zero set of codimension at least two. □

5.2. **Closed 1-forms without divisorial components in theirs zero sets.** Below we collect a couple of useful results concerning foliations defined by closed rational 1-forms. The first is a straightforward adaptation of [15, Lemma 8].

Lemma 5.4. *Let X be a projective manifold with $H^0(X, \Omega^1_X) = 0$. Let $\mathcal{F}$ be codimension one foliation on X defined by a closed rational 1-form ω with zero set of codimension at least two and polar divisor $(\omega)_\infty = \sum_{i=1}^k r_i D_i$. Then there exists a holomorphic family of foliations $\mathcal{F}_t$, $t \in (\mathbb{C}, 0)$, such that*

1. *$\mathcal{F}_0 = \mathcal{F}$;*
2. *$N\mathcal{F}_t = N\mathcal{F} = \mathcal{O}_X(\sum_{i=1}^k r_i D_i)$ for every $t \in \mathbb{C}$; and*
3. *$\mathcal{F}_t$ is defined by a logarithmic 1-form for every $t \neq 0$.*

The next proposition will allow us to guarantee that the irreducible components of the space of foliations with $K\mathcal{F} = 0$ on Fano 3-folds of index two are indeed smooth.

Proposition 5.5. *Let X be a manifold with $\mathrm{Pic}(X) = \mathbb{Z}H$ of dimension at least three. If $h^0(X, \mathcal{O}_X(H)) > 1$ then the set of foliations defined by a closed 1-form without codimension one zeros and with polar divisor linearly equivalent to $2H$ is a smooth and irreducible subvariety of $\mathbb{P}H^0(X, \Omega^1_X \otimes \mathcal{O}_X(2H))$ isomorphic to the Grassmannian $\mathrm{Gr}(2, h^0(X, \mathcal{O}_X(H)))$.*

Proof. Consider the linear map

$$\begin{aligned} \Psi : \bigwedge^2 H^0(X, \mathcal{O}_X(H)) &\to H^0(X, \Omega^1_X(2H)) \\ F \wedge G &\mapsto FdG - GdF\,. \end{aligned}$$

We claim that the kernel of Ψ intersects the set of decomposable elements of $\bigwedge^2 H^0(X, \mathcal{O}_X(H))$ only at zero. If $\Psi(F \wedge G) = 0$ then the rational map $\varphi = (F : G) : X \dashrightarrow \mathbb{P}^1$ must be constant since $FdG - GdF = \varphi^*(x_0 dx_1 - x_1 dx_0)$. Therefore $F = \lambda G$ for some $\lambda \in \mathbb{C}$. Thus Ψ induces a linear projection from $\bigwedge^2 H^0(X, \mathcal{O}_X(H))$ to $\mathbb{P}(X, \Omega^1_X(2H))$ with center disjoint from the Grassmannian $\mathrm{Gr} = \mathrm{Gr}(2, h^0(X, \mathcal{O}_X(H)))$.

Notice that $\mathcal{O}_X(H)$ is a primitive line bundle. Thus for every non-zero $F \wedge G \in \bigwedge^2 H^0(X, \Omega^1_X)$, every fiber of the rational map $\varphi = (F : G) : X \dashrightarrow \mathbb{P}^1$ is reduced and irreducible. If $\Psi(F_1 \wedge G_1) = \Psi(F_2 \wedge G_2)$ then the set of fibers of $\varphi_1 = (F_1 : G_1)$ and $\varphi_2 = (F_2 : G_2)$ must be equal, and consequently the rational maps must differ by a right composition with an automorphism of $\mathbb{P}^1$. Thus $F_1 \wedge G_1 = \lambda F_2 \wedge G_2$ for some $\lambda \in \mathbb{C}$. In other words, the linear projection induced by Ψ has kernel disjoint from the secant variety of Gr

$$Sec(\mathrm{Gr}) = \bigcup_{x,y \in \mathrm{Gr}^2, x \neq y} \ell_{x,y}\,, \qquad \text{where } \ell_{x,y} \text{ is the line joining } x \text{ and } y\,.$$

Notice that elements in the projective tangent space of $\mathrm{Gr} \subset \mathbb{P}\bigwedge^2 H^0(X, \mathcal{O}_X(H))$ at $[F \wedge G]$ can be written as $[F \wedge G + F \wedge G' + F' \wedge G] = [F \wedge (G + G') + F' \wedge G]$ and therefore are in the secant variety of Gr. Thus $Sec(\mathrm{Gr}) = \overline{Sec(\mathrm{Gr})}$ is a closed subvariety of $\mathbb{P}\bigwedge^2 H^0(X, \mathcal{O}_X(H))$ disjoint from the center of the linear projection determined by Ψ. It follows that $[\Psi] : \mathrm{Gr} \to \mathbb{P}H^0(X, \Omega^1_X(2H))$ is an embedding. □

5.3. **Foliations on $\mathbb{P}^3$.** We will now proceed to present our proof of the classification of foliations on $\mathbb{P}^3$ with $K\mathcal{F} = 0$ originally due to Cerveau and Lins Neto [15].

Theorem 5.6. *The irreducible components of the space of foliations on $\mathbb{P}^3$ with $K\mathcal{F} = 0$ are:*

1. $\mathrm{Rat}(2,2)$ *: the generic element is a pencil of quadrics with irreducible elements;*
2. $\mathrm{Rat}(1,3)$ *: the generic element is a pencil of cubics having a triple hyperplane as a member;*
3. $\mathrm{Log}(1,1,1,1)$*: the generic element is defined by a logarithmic* 1*-form with poles on four hyperplanes;*
4. $\mathrm{Log}(2,1,1)$*: the generic element is defined by a logarithmic* 1*-form with poles on two hyperplanes and a quadric;*
5. $Exc(2)$*: the generic element is isomorphic to the foliation defined by the natural action of* $\mathrm{Aff}(\mathbb{C})$ *on* $\mathbb{P}^3 = \mathrm{Sym}^3 \mathbb{P}^1$*;*
6. $\mathrm{LPB}(2)$*: every element is the pull-back of a degree* 2 *foliation on* $\mathbb{P}^2$ *under a linear projection.*

5.3.1. *Foliations tangent to algebraic actions.* We start by analyzing the foliations tangent to algebraic $\mathbb{C}^*$-actions.

Proposition 5.7. *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^3$ with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}^*$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational* 1*-form without divisorial components in its zero set.*

Proof. Let $\varphi : \mathbb{C}^* \times \mathbb{P}^3 \to \mathbb{P}^3$ by a $\mathbb{C}^*$-action. For a suitable choice of coordinates this action is determined by a homogeneous vector field

$$\xi = \sum_{i=0}^{3} \lambda_i x_i \frac{\partial}{\partial x_i}$$

with the coefficients $\lambda_i \in \mathbb{Z}$, which we can assume ordered $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$. Adding $-\lambda_0 R$ to ξ, we can further assume that $\lambda_0 = 0$. With this assumption the foliation $\mathcal{L}_v$ determined by v is defined on the affine neighborhood $\{x_0 = 1\}$ by the vector field

$$v = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z}\,.$$

We have the following possibilities for the triple $(\lambda_1, \lambda_2, \lambda_3)$:

(a) $0 < \lambda_1 < \lambda_2 < \lambda_3$;
(b) $0 < \lambda_1 = \lambda_2 < \lambda_3$;
(c) $0 < \lambda_1 < \lambda_2 = \lambda_3$;
(d) $0 = \lambda_1 < \lambda_2 = \lambda_3$.

In case (a) the action has isolated fixed points contrary to our assumptions. In the remaining cases we do have non-isolated fixed points. As the three remaining cases can be treated by similar arguments, we will analyze in detail only case (b).

We start by constructing a rational map with generic fibers equal to orbits of φ. In the affine neighborhood $x_0 = 1$, we can take the natural quotient map $\Phi : \mathbb{C}^3 \dashrightarrow \mathbb{P}(\lambda_1, \lambda_1, \lambda_3)$ where $\mathbb{P}(\lambda_1, \lambda_1, \lambda_3)$ stands for the corresponding weighted projective space. Recall from [25, 36] that, as a projective variety, $\mathbb{P}(\lambda_1, \lambda_1, \lambda_3)$ is isomorphic to $\mathbb{P}(1, 1, \lambda_3)$ and can hence be identified with a cone over the rational normal curve of degree λ_3. The blow-up of the vertex of this cone yields Hirzebruch's surface $F_{\lambda_3} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_3))$. Explicitly, we can present Φ as the rational map

$$\begin{aligned} \Phi : \mathbb{P}^3 &\dashrightarrow \mathbb{P}(\lambda_1, \lambda_1, \lambda_3) \subset \mathbb{P}^{\lambda_3+1} \\ (x : y : z : w) &\mapsto (x^{\lambda_3} : x^{\lambda_3-1}y : x^{\lambda_3-2}y^2 : \cdots : y^{\lambda_3} : z^{\lambda_1}w^{\lambda_3-\lambda_1}) \end{aligned}$$

Notice that

(1) the indeterminacy locus of Φ is composed by two points $(0 : 0 : 0 : 1)$ and $(0 : 0 : 1 : 0)$;
(2) the pre-image of the vertex is the line $\{x = y = 0\}$; and
(3) the divisorial components of its critical locus are $\{z = 0\}$ (only when $\lambda_1 > 1$) and $\{w = 0\}$ (only when $\lambda_3 - \lambda_1 > 1$). Both components are mapped to the curve $C \subset F_{\lambda_3}$ which has self-intersection $\lambda_3 > 1$.

It will be more convenient to blow-up the vertex and work with the induced rational map from $\mathbb{P}^3$ to F_{λ_3} which we will still denote by Φ. We will denote the unique section of F_{λ_3} with self-intersection $-\lambda_3$ (which is the pre-image of the vertex under the blow-up) by σ_0.

Let $\mathcal{G}$ be a foliation on F_{λ_3} such that $\mathcal{F} = \Phi^*\mathcal{G}$. Its normal bundle can be written as $N\mathcal{G} = aF + b\sigma_0$, where F is a general fiber of the natural fibration $\pi : F_{\lambda_3} \to \mathbb{P}^1$ and a, b are integers. If $\mathcal{G}$ coincides with the foliation defined by π then $N\mathcal{G} = 2F$ and $\Phi^*\mathcal{G}$ is the pencil of hyperplane containing the line $\{x = y = 0\}$. Otherwise, the general fiber is not $\mathcal{G}$-invariant and we can write, see for instance [11, Chapter 2, Proposition 2],

$$b = N\mathcal{G} \cdot F = \chi(F) + \operatorname{tang}(\mathcal{G}, F) = 2 + \operatorname{tang}(\mathcal{G}, F) \geq 2$$

with equality holding if and only if $\mathcal{G}$ is a Riccati foliation with respect to the natural fibration.

If σ_0 is not $\mathcal{G}$-invariant then

$$N\mathcal{G} \cdot \sigma_0 = a - b\lambda_3 = 2 + \operatorname{tang}(\mathcal{G}, \sigma_0) \geq 2 \implies a \geq 2 + b\lambda_3 \,. \tag{4}$$

If instead σ_0 is $\mathcal{G}$-invariant then

$$N\mathcal{G} \cdot \sigma_0 = a - b\lambda_3 = \sigma_0^2 + Z(\mathcal{G}, \sigma_0) \implies a = \lambda_3(b-1) + Z(\mathcal{G}, \sigma_0) \,. \tag{5}$$

Since all the divisorial components of the critical locus of Φ are mapped to C, the pull-back of $\mathcal{G}$ under Φ has normal bundle given by

$$N\Phi^*\mathcal{G} = \begin{cases} \mathcal{O}_{\mathbb{P}^3}(a) & \text{when} \quad C \text{ is not } \mathcal{G}\text{-invariant} \\ \mathcal{O}_{\mathbb{P}^3}(a - \lambda_3 + 2) & \text{when} \quad C \text{ is } \mathcal{G}\text{-invariant} \,. \end{cases} \tag{6}$$

If we impose that $\Phi^*\mathcal{G}$ is a foliation of degree two, $N\Phi^*\mathcal{G} = \mathcal{O}_{\mathbb{P}^3}(4)$, then we deduce from (4), (5), and (6) that σ_0 must be $\mathcal{G}$-invariant, and

(b.1) if C is not $\mathcal{G}$-invariant then $a = 4$ and $b = Z(\mathcal{G}, \sigma_0) = \lambda_3 = 2$; and
(b.2) if C is $\mathcal{G}$-invariant then $a = \lambda_3 + 2$ and $b = Z(\mathcal{G}, \sigma_0) = 2$.

Notice that $\mathcal{G}$ is a Riccati foliation ($b = 2$), and that there are one or two fibers of π invariant under $\mathcal{G}$ ($b = Z(\mathcal{G}, \sigma_0)$).

In case (b.1) we have that the pull-back of $\mathcal{G}$ is in the irreducible component $\mathrm{Log}(2,1,1)$. Indeed, a generic element in $\mathbb{P}H^0(F_2, \Omega^1_{F_2}(4F + 2\sigma_0)$ is defined by a logarithmic one-form with poles in σ_0, two distinct fibers of π and another section of π distinct from σ_0. The pull-back of σ_0 is not a divisor, the pull-back of the fibers are two distinct hyperplanes, and the pull-back of the other section is a quadric. Putting all together we deduce that the pull-back foliation lies in $\mathrm{Log}(2,1,1)$. Similarly, in case (b.2), we have that the pull-back of $\mathcal{G}$ is in the irreducible component $\mathrm{Log}(1,1,1,1)$. In either cases we deduce that $\mathcal{F}$ is defined by a closed rational 1-form (not necessarily logarithmic) without divisorial components in theirs zero sets.

It is perhaps worth noticing that in the first case we have that almost every element of $\mathrm{Log}(2,1,1)$ is projectively equivalent to the pull-back of some $\mathcal{G}$, while in case (b.2) this is no longer true: there is a resonance between the residues of the hyperplanes $z = 0$ and $w = 0$. □

We will now carry a similar analysis for foliations tangent to algebraic $\mathbb{C}$-actions.

Proposition 5.8. *Let $\mathcal{F}$ be a codimension one semi-stable foliation on $\mathbb{P}^3$ with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

Proof. We start by pointing out that algebraic $\mathbb{C}$-actions on $\mathbb{P}^3$ are of the form

$$\begin{aligned} \varphi : \mathbb{C} \times \mathbb{P}^3 &\longrightarrow \mathbb{P}^3 \\ (t, x) &\longmapsto exp(t\xi)(x) \end{aligned}$$

where the vector field ξ is nilpotent. In suitable homogeneous coordinates it takes one of the following forms:

(a) $\xi = x_1 \frac{\partial}{\partial x_0}$; or
(b) $\xi = x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$; or
(c) $\xi = x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1}$; or
(d) $\xi = x_1 \frac{\partial}{\partial x_0} + x_3 \frac{\partial}{\partial x_2}$.

If $\mathcal{F}$ is a codimension one foliation on $\mathbb{P}^3$ tangent to an action as in (a) then $\mathcal{F}$ is tangent to the radial foliation by lines determined by ξ. Thus $\mathcal{F}$ is the linear pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ and, in particular, $\mathcal{F}$ is unstable contrary to our assumptions. In case (b) all the fixed points of the action are isolated also contrary to our assumptions. The remaining cases are subtler, though rather similar to foliations invariant by $\mathbb{C}^*$-actions and will be briefly analyzed below.

Case (c): $\xi = x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1}$. The orbits of φ are tangents to the fibers of the rational map

$$\begin{aligned} \Phi : \mathbb{P}^3 &\longrightarrow \mathbb{P}(1,1,2) \subset \mathbb{P}^3 \\ (x_0 : x_1 : x_2 : x_3) &\longmapsto (x_2^2 : x_2 x_3 : x_3^2 : x_1^2 - 2x_0 x_2)\,. \end{aligned}$$

Notice that $\mathbb{P}(1,1,2)$ is nothing more than a cone over a conic, and its minimal resolution is the Hirzebruch surface $F_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. The indeterminacy locus of Φ is the point $x_1 = x_2 = x_3 = 0$, the pre-image of the vertex is the line $x_2 = x_3 = 0$, and Φ has no divisorial components in its critical locus.

We will now argue as in the proof of Proposition 5.7. If $\mathcal{G}$ is a foliation on F_2 with normal bundle equal to $N\mathcal{G} = aF + b\sigma_0$ then the normal bundle of $\mathcal{F} = \Phi^*\mathcal{G}$ is $\mathcal{O}_{\mathbb{P}^3}(a)$. Consequently we must have $a = 4$, σ_0 must be $\mathcal{G}$-invariant, and $b = 2$. It follows that $\mathcal{F}$ is in $\mathrm{Log}(2, 1, 1)$.

Case (d): $\xi = x_1\frac{\partial}{\partial x_0} + x_3\frac{\partial}{\partial x_2}$. We are in a situation very similar to case (c). Now, the orbits of φ are tangent to the fibers of

$$\Phi : \mathbb{P}^3 \longrightarrow \mathbb{P}(1, 1, 2) \subset \mathbb{P}^3$$
$$(x_0 : x_1 : x_2 : x_3) \longmapsto (x_1^2 : x_1x_3 : x_3^2 : x_1x_2 - x_0x_3)\,.$$

The indeterminacy locus of Φ and the pre-image of the vertex are both set-theoretically equal to the line $x_1 = x_3 = 0$, and Φ has no divisorial components in its critical locus as in the previous case. Thus, as before, if $\mathcal{F}$ is a degree 2 foliation tangent to the fibers of Φ then it must be in $\mathrm{Log}(2, 1, 1)$. □

Proof of Theorem 5.6. Let $\mathcal{F}$ be a foliation of degree 2 on $\mathbb{P}^3$. Theorem 5.1 tell us that $\mathcal{F}$ is (a) given by a closed meromorphic 1-form with codimension two zero set, or (b) $\mathcal{F}$ is the linear pull-back of a foliation on $\mathbb{P}^2$, or (c) $\mathcal{F}$ is induced by an algebraic action of a two dimensional Lie group and $\dim \mathfrak{iso}(\mathcal{F}) = \dim \mathfrak{aut}(\mathcal{F}) = 2$. If we are in case (a) then Lemma 5.4 implies that $\mathcal{F}$ can be deformed to a foliation defined by a logarithmic 1-form. Thus $\mathcal{F}$ belongs to (at least) one of the following irreducible components: $\mathrm{Rat}(2, 2), \mathrm{Rat}(1, 3), \mathrm{Log}(1, 1, 1, 1), \mathrm{Log}(2, 1, 1)$. If we are in case (b) then $\mathcal{F}$ belongs to $\mathrm{LPB}(2)$.

Assume that we are in case (c). If the Lie algebra $\mathfrak{f} = \mathfrak{fix}(\mathcal{F}) \subset \mathfrak{aut}(\mathbb{P}^3) = \mathfrak{sl}(4, \mathbb{C})$ is abelian then writing down Jordan normal forms for the generators of $\mathfrak{f}$ as in [24, page 54] we see that $\mathfrak{f}$ is contained in an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{sl}(4, \mathbb{C})$ having dimension at least three. Notice that $\mathfrak{a}$ is contained in $\mathfrak{aut}(\mathcal{F})$. Therefore when $\mathfrak{f}$ is abelian we have $\mathfrak{iso}(\mathcal{F}) \neq \mathfrak{aut}(\mathcal{F})$. Lemma 5.3 implies that $\mathcal{F}$ is induced by a closed meromorphic 1-form without divisorial components in its zero set.

Suppose now that $\mathfrak{f}$ is the affine Lie algebra generated by $x, y \in \mathfrak{sl}(4, \mathbb{C})$ satisfying $[x, y] = y$ with y nilpotent. There are four possible Jordan normal forms for y: one with rank 3, two with rank 2, and one with rank one. The case of rank one corresponds to a vector field in $H^0(\mathbb{P}^3, T\mathcal{F}) \subset H^0(\mathbb{P}^3, T\mathbb{P}^3)$ vanishing on a hyperplane. This case is excluded because $T\mathcal{F}$ is assumed to be semi-stable. The other cases have been analyzed in detail in [24, pages 58–65]. In particular it can be verified that there is only one case up to conjugacy where $\mathfrak{aut}(\mathcal{F}) = \mathfrak{fix}(\mathcal{F})$:

$$x = \frac{1}{3}\left(-3x_0\frac{\partial}{\partial x_0} - x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 3x_3\frac{\partial}{\partial x_3}\right) \text{ and}$$
$$y = x_1\frac{\partial}{\partial x_0} + x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}.$$

It is a simple matter to verify that the foliation corresponds to the natural action of $\mathrm{Aff}(\mathbb{C})$ in $\mathbb{PC}_3[x] = \mathbb{P}^3$. In all the remaining cases, $\mathfrak{fix}(\mathcal{F}) \subsetneqq \mathfrak{aut}(\mathcal{F})$ and therefore, according to Lemma 5.3, $\mathcal{F}$ is induced by a closed meromorphic 1-form without divisorial components in its zero set. In particular, these foliations fit not only in case (c) but also in case (a), which has already been analyzed. □

5.4. **Foliations on Q^3.** We will now classify the foliations with $K\mathcal{F} = 0$ on the 3-dimensional quadric. We start by presenting an example.

Example 5.9. Identify $\mathbb{P}^4$ with the set of 4 unordered points in $\mathbb{P}^1$. This identification gives a natural action of $PSL(2,\mathbb{C}) \simeq \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^4$. Let $p_0 \in \mathbb{P}^4$ be the point defined by the set $\{1,-1,i,-i\} \subset \mathbb{P}^1$. The closure of the $PSL(2,\mathbb{C})$-orbit of p_0 is a smooth quadric $Q^3 \subset \mathbb{P}^4$, see [53]. This quadric can be decomposed as the union of three orbits of $PSL(2,\mathbb{C})$: a closed orbit of dimension one isomorphic to a rational normal curve of degree 4 corresponding to points on $\mathbb{P}^1$ counted with multiplicity 4; an orbit of dimension two corresponding to two distinct points on $\mathbb{P}^1$, one with multiplicity three and the other with multiplicity one; and the open orbit of dimension three corresponding to 4 distinct points isomorphic to $\{1,-1,i,-i\}$ The affine subgroup $\mathrm{Aff}(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{P}^1)$ acts on Q^3 and defines on it a codimension one foliation $\mathcal{F}$ with trivial tangent bundle. Notice that the singular set of $\mathcal{F}$ has three irreducible components: a rational normal curve of degree 4 corresponding to four equal points; a twisted cubic corresponding to points of the form $3p+\infty$; and a line corresponding to points of the form $p+3\infty$.

Besides the example above there are only two further families of foliations with $K\mathcal{F} = 0$ on Q^3.

Theorem 5.10. *The irreducible components of space of codimension one foliations with $K\mathcal{F} = 0$ on the hyperquadric Q^3 are:*

1. $\mathrm{Rat}(2,1)$*: the generic element is a pencil of hypersurfaces of degree* 4 *and having a double hyperplane section as element;*
2. $\mathrm{Log}(1,1,1)$*: the generic element is defined by a logarithmic* 1*-form with poles on three hyperplane sections;*
3. Aff*: the generic element is conjugated to the foliation presented in Example 5.9.*

The strategy of proof is the same as before. Theorem 5.10 follows from the next three propositions combined with Lemma 5.4.

Proposition 5.11. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}^*$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational* 1*-form without divisorial components in its zero set.*

Proof. We can assume that $Q^3 \subset \mathbb{P}^4$ is given by the equation $\{x_0^2+x_1x_2+x_3x_4 = 0\}$ and that $\mathbb{C}^* \subset \mathrm{Aut}(Q^3)$ is a subgroup of the form

$$\varphi_\lambda(x_0 : x_1 : x_2 : x_3 : x_4) = (x_0 : \lambda^a x_1 : \lambda^{-a} x_2 : \mu^b x_3 : \mu^{-b} x_4),,$$

with $a, b \in \mathbb{N}$ relatively prime, since $\mathrm{Aut}(Q^3) = \mathbb{P}O(5,\mathbb{C})$ has rank two. If both a and b are distinct non-zero natural numbers then the fixed points of the action are isolated. Thus we have to analyze only two cases: $(a,b) = (0,1)$ and $(a,b) = (1,1)$.

Let us start with the case $(a,b) = (0,1)$. Consider the rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}(1,1,1,2) \subset \mathbb{P}^6$$

$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2 : x_3x_4)\,,$$

which identifies $\mathbb{P}(1,1,1,2)$ with a cone over the Veronese surface in $\mathbb{P}^5$. Notice that the quadric Q^3 is mapped to a hyperplane section of $\mathbb{P}(1,1,1,2)$ not passing through the vertex $(0:0:0:0:0:0:1)$, which is of course isomorphic to $\mathbb{P}^2$. We will denote by Φ_0 the induced rational map $\Phi_0 : Q^3 \dashrightarrow \mathbb{P}^2$. The generic fiber of Φ_0 is an orbit of φ, and therefore the foliation $\mathcal{F}$ must be the pull-back of a foliation $\mathcal{H}$ on $\mathbb{P}^2$. Notice also that $\Phi_0^*\mathcal{O}_{\mathbb{P}^2}(1)$ is equal to $\mathcal{O}_{Q^3}(1)$. A simple

computation shows that the critical set of Φ_0 has codimension greater than two. Thus $\mathcal{O}_{Q^3}(3) = N\mathcal{F} = \Phi_0^* N\mathcal{H}$. It follows that $N\mathcal{H} = \mathcal{O}_{\mathbb{P}^2}(3)$, i.e., $\mathcal{H}$ has degree one. Since every foliation of degree one on $\mathbb{P}^2$ is induced by a closed meromorphic 1-form with isolated singularities the proposition follows in this case.

Suppose now that $(a,b) = 1$, and consider the rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$$
$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0^2 : x_1x_2 : x_1x_4 : x_2x_3 : x_3x_4)\,.$$

Its image is contained in a cone over a smooth quadric surface in $\mathbb{P}^3$. The quadric Q^3 is mapped into a smooth hyperplane section of this cone which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. If we denote by $\Phi_0 : Q^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ the induced rational map then $\Phi_0^* \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,d) = \mathcal{O}_{Q^3}(c+d)$. It can be checked that the only divisorial component of the critical set of Φ_0 is the intersection of the hyperplane $\{x_0 = 0\}$ with Q^3. The image of this critical set is a $(1,1)$ curve C in $\mathbb{P}^1 \times \mathbb{P}^1$. If $\mathcal{G}$ is a foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{NG} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,d)$ then

$$N\Phi_0^*\mathcal{G} = \begin{cases} \mathcal{O}_{Q^3}(c+d) & \text{if} \quad C \text{ is not } \mathcal{G}\text{-invariant} \\ \mathcal{O}_{Q^3}(c+d-1) & \text{if} \quad C \text{ is } \mathcal{G}\text{-invariant}\,. \end{cases}$$

Therefore if $\mathcal{F} = \Phi_0^*\mathcal{G}$ and $N\mathcal{F} = \mathcal{O}_{Q^3}(3)$ then $c = d = 2$ and C is $\mathcal{G}$-invariant. A foliation $\mathcal{G}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with $N\mathcal{G} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,2)$ is given by a closed rational 1-form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$ where $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the natural projections and the 1-forms ω_i have polar set of degree two. Since the $(1,1)$-curve C is $\mathcal{G}$-invariant, we must have $\omega_1 = -\omega_2 = dx_0/x_0 - dx_1/x_1$ in a suitable choice of coordinates where $C = \{x_0y_1 - y_0x_1 = 0\}$. Therefore

$$\omega = \frac{dx_0}{x_0} - \frac{dx_1}{x_1} - \frac{dy_0}{y_0} + \frac{dy_1}{y_1}\,.$$

Notice that ω is proportional to

$$\alpha = \left(\frac{d(x_0y_1 - y_0x_1)}{x_0y_1 - y_0x_1} - \frac{dx_0}{x_0} - \frac{dy_1}{y_1}\right).$$

and the pull-back of α under Φ_0 is closed 1-form without divisorial components in its zero set. □

Proposition 5.12. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with $K\mathcal{F} = 0$. If $\mathcal{F}$ is tangent to an algebraic $\mathbb{C}$-action with non-isolated fixed points then $\mathcal{F}$ is given by a closed rational 1-form without divisorial components in its zero set.*

Proof. Let $\varphi : \mathbb{C} \times Q^3 \to Q^3$ be an algebraic $\mathbb{C}$-action. As such, it must be of the form $\varphi(t) = \exp(t \cdot n)$ when $n \in \mathfrak{aut}(Q^3) = \mathfrak{so}(5,\mathbb{C})$. In $\mathfrak{so}(5,\mathbb{C})$ there are exactly three $\mathrm{Aut}(Q^3) = \mathbb{P}O(5,\mathbb{C})$-conjugacy classes of non-zero nilpotent elements. The Jordan normal forms of the corresponding matrices in $End(\mathbb{C}^5)$ have: (1) only one Jordan block of order 5; (2) one Jordan block of order 3 and two trivial (order one) Jordan blocks ; or (3) two Jordan blocks of order 2 and one trivial Jordan block.

Case (1) is excluded by hypothesis. To deal with case (2) we can assume that $n = x_1\frac{\partial}{\partial x_0} + x_2\frac{\partial}{\partial x_1}$ and that the quadric Q^3 is $\{x_1^2 - 2x_0x_2 + x_3^2 + x_4^2 = 0\}$. The generic fiber of the rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$$
$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_1^2 - 2x_0x_2 : x_2^2 : x_2x_3 : x_2x_4 : x_3^2 : x_3x_4 : x_4^2)$$

coincides with an orbit of φ, and sends $\mathbb{P}^4$ to a cone over the second Veronese embedding of $\mathbb{P}^2$. The image of the quadric Q^3 avoids the vertex of this cone and is isomorphic to $\mathbb{P}^2$. Moreover, the critical set of $\Phi_0 : Q \dashrightarrow \mathbb{P}^2$ (the restriction of Φ to Q) has no divisorial components. Therefore every foliation $\mathcal{F}$ on Q^3 tangent to φ is of the form $\Phi_0^*\mathcal{G}$ for some foliation on $\mathbb{P}^2$ and its normal bundle satisfies $N\mathcal{F} = \Phi_0^* N\mathcal{G}$. Since $\Phi_0^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_Q(1)$, it follows that $\mathcal{F}$ is the pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ of degree one and, as such, is given by a closed 1-form without zeros of codimension one.

Case (3) is very similar to case (2). Now the vector field n is of the form $x_1\frac{\partial}{\partial x_0} + x_3\frac{\partial}{\partial x_2}$, the quadratic is $Q = \{x_0x_3 - x_1x_2 + x_4^2 = 0\}$ and the quotient map is

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$$
$$(x_0 : x_1 : x_2 : x_3 : x_4) \mapsto (x_0x_3 - x_1x_2 : x_1^2 : x_1x_3 : x_1x_4 : x_3^2 : x_3x_4 : x_4^2)\,.$$

The restriction of Φ to Q has critical set of codimension at least two, and therefore the conclusion is the same: $\mathcal{F}$ is the pull-back under $\Phi_{|Q}$ of a foliation on $\mathbb{P}^2$ of degree one. □

Proposition 5.13. *Let $\mathcal{F}$ be a codimension one foliation on Q^3 with trivial canonical bundle. Suppose that $\mathcal{F}$ is induced by an algebraic action of a two dimensional Lie subgroup of* $\mathrm{Aut}(Q^3)$. *Then $\mathcal{F}$ is defined by a closed* 1*-form without zeros of codimension one, or $\mathcal{F}$ is conjugated to the foliation presented in Example 5.9.*

Proof. Let $G \subset \mathrm{Aut}(Q^3)$ be the subgroup defining $\mathcal{F}$, and $\mathfrak{g} \subset \mathfrak{so}(5,\mathbb{C})$ the corresponding Lie subalgebra. If G is abelian then it must be of the form $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{C} \times \mathbb{C}^*$, or $\mathbb{C} \times \mathbb{C}$. In the first case every element in $\mathfrak{g}$, the Lie algebra of G, is a semi-simple element of $\mathfrak{so}(5,\mathbb{C})$. Since the rank of $\mathfrak{so}(5,\mathbb{C})$ is two, $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{so}(5,\mathbb{C})$. Therefore, we can find $\mathbb{C}^* \subset G$ inducing an algebraic action with non-isolated fixed points tangent to $\mathcal{F}$. We can apply Proposition 5.11 to conclude that $\mathcal{F}$ is induced by a closed 1-form without codimension one zeros. In the two remaining cases, $\mathfrak{g}$ contains a nilpotent element n which defines an algebraic subalgebra $\mathbb{C} \subset G$. If the corresponding action has non-isolated fixed points then Proposition 5.12 implies $\mathcal{F}$ is defined by a closed rational 1-form without divisorial components in its zero set.

If the corresponding action has only isolated fixed points then we can assume that Q is defined by the quadratic form $q = x_2^2 - 2x_1x_3 + 2x_0x_4$ and that n, seen as an element of $\mathfrak{so}(q,\mathbb{C})$, have only one Jordan block of order 5. The centralizer $C(n)$ of n in $\mathfrak{so}(q,\mathbb{C})$ is thus formed by nilpotent matrices of the form

$$\begin{pmatrix} 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, since $\mathfrak{g} \subset C(n)$, $\mathfrak{g}$ contains another nilpotent element which defines a $\mathbb{C}$-action with non-isolated fixed points. Proposition 5.12 implies $\mathcal{F}$ is defined by a closed 1-form without codimension one zeros.

Suppose now that G is not abelian. Its Lie algebra $\mathfrak{g}$ is isomorphic to the affine Lie algebra $\mathbb{C}x \oplus \mathbb{C}y$ with the relation $[x,y] = y$. This relation implies that y is a

nilpotent element of $\mathfrak{so}(5,\mathbb{C}) \subset \mathfrak{sl}(5,\mathbb{C})$. As before, using Proposition 5.12, we can reduce to the case where y is in Jordan normal form and has only one Jordan block of order 5. The elements $x \in \mathfrak{so}(q,\mathbb{C})$ satisfying $[x,y] = y$ are of the form

$$\begin{pmatrix} 2 & \alpha & 0 & \beta & 0 \\ 0 & 1 & \alpha & 0 & \beta \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

After one last conjugation by an element of $SO(q,\mathbb{C})$ we can suppose that $\beta = 0$ is in the form above. We have just proved that there is only one foliation defined by an algebraic action of an algebraic subgroup $G \subset \mathrm{Aut}(Q^3)$ which is not tangent to an algebraic action without isolated fixed points. Therefore it must be equal to the foliation described in Example 5.9. □

5.5. **Foliations on Fano 3-folds with index ≤ 2.** It remains to deal with foliations with $K\mathcal{F} = 0$ on Fano 3-folds of index one and two. Unlikely in the cases where the index is four (projective space) or three (quadric), these 3-folds have moduli. As will be seen below the space of foliations with $K\mathcal{F} = 0$ on them behaves rather uniformly with respect to the moduli, with only two exceptions. The exceptions are the quasi-homogeneous $PSL(2,\mathbb{C})$–manifolds of index one and two.

5.5.1. *Index two.* Let X be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}H$ and index $i(X) = 2$ which means, by definition, $-KX = 2H$. In this case the classification is very precise and says that X is isomorphic to a 3-fold fitting in one of the following classes:

(1) $H^3 = 1$. Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$;
(2) $H^3 = 2$. Hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$;
(3) $H^3 = 3$. Cubic in $\mathbb{P}^4$;
(4) $H^3 = 4$. Intersection of two quadrics in $\mathbb{P}^5$;
(5) $H^3 = 5$. Intersection of the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a $\mathbb{P}^6$.

Although not evident from the description above, the 3-folds falling in class (5) are all isomorphic to a 3-fold $X_5 \subset \mathbb{P}^6$. In [53] X_5 is described as an equivariant compactification of $\mathrm{Aut}(\mathbb{P}^1)/\Gamma$ where Γ is the octahedral group. Explicitly, if we consider the point $p_0 \in \mathrm{Sym}^6 \mathbb{P}^1$ defined by the polynomial $xy(x^4 - y^4)$ then X_5 is the closure of the $\mathrm{Aut}(\mathbb{P}^1)$-orbit of p_0 under the natural action.

Theorem 5.14. *Let X be a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}H$ and index $i(X) = 2$. If $X \neq X_5$ then the space of codimension one foliations on X with trivial canonical bundle is smooth and irreducible. If $X = X_5$ then the space of codimension one foliations on X with trivial canonical bundle is smooth and has two irreducible components.*

As we will see from its proof the result is much more precise as it describes quite precisely the irreducible components. We summarize the description in the Table below.

Manifold	**Irreducible component**	dim
Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	$\mathrm{Rat}(1,1) \simeq \mathbb{P}^2$	2
Hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,4)$	4
Cubic in $\mathbb{P}^4$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,5)$	6
Intersection of quadrics in $\mathbb{P}^5$	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,6)$	8
X_5	$\mathrm{Rat}(1,1) \simeq \mathrm{Gr}(2,7)$	10
	$\mathrm{Aff} \simeq \mathbb{P}^1$	1

Lemma 5.15. *The dimension of $H^0(X_5, TX_5)$ is 3, and every $v \in H^0(X_5, TX_5)$ has isolated singularities.*

Proof. Let Σ be the variety of lines contained in X_5. According to [29], Σ is isomorphic to $\mathbb{P}^2$. The induced action of $\mathrm{Aut}(\mathbb{P}^1)$ on it has one closed orbit isomorphic to a conic $C \subset \mathbb{P}^2$, and one open orbit isomorphic to $\mathbb{P}^2 \setminus C$. It can be identified with the natural action of $\mathrm{Aut}(\mathbb{P}^1)$ in $\mathrm{Sym}^2 \mathbb{P}^1 \simeq \mathbb{P}^2$. If an automorphism of X_5 induces the identity on Σ then it must be identity since through every point of every line contained in X_5 passes at least another line, loc. cit. Corollary 1.2. This suffices to show that $h^0(X_5, TX_5) = 3$.

Let now $v \in H^0(X_5, TX_5)$ be a non-zero vector field, and $H = \exp(\mathbb{C}v) \subset \mathrm{Aut}(X_5)$ be the one-parameter subgroup generated by it. The description of the induced action of $\mathrm{Aut}(X)$ on Σ implies that the induced action of H on Σ has isolated fixed points. Therefore, if the zero set of v has positive dimension then it must be contained in a finite union of lines. If we take ℓ as one of these lines then the action of H on Σ would fix all the lines intersecting ℓ. This contradicts the description of the induced action of $\mathrm{Aut}(X)$ on Σ. □

Lemma 5.16. *Let $\mathbb{P} = \mathbb{P}(q_0, q_1, q_2, q_3, q_4)$ be a well-formed weighted projective space of dimension four with $q_0 \leq q_1 \leq q_2 \leq q_3 \leq q_4$, and $X \subset \mathbb{P}$ be a smooth hypersurface. If $\deg(X) \geq q_2 + q_3 + q_4$ then $h^0(X, TX) = 0$.*

Proof. Set $d = \deg(X)$ and $Q = \sum_{i=0}^4 q_i$. By [25, Theorem 3.3.4], $\Omega^3_X = \mathcal{O}_X(d-Q)$. Consequently $TX = \Omega^2_X \otimes \mathcal{O}_X(Q-d)$. From the long exact sequence associated to

$$0 \to \Omega^1_X \otimes N^*_X \otimes \mathcal{O}_X(Q-d) \to \Omega^2_{\mathbb{P}_X}(Q-d) \to \Omega^2_X(Q-d) \to 0$$

we see that $h^0(X, TX) = 0$ when $h^0(X, \Omega^2_{\mathbb{P}_X}(Q-d)) = h^1(X, \Omega^1_X(Q-2d)) = 0$.

To compute $h^1(X, \Omega^1_X(Q-2d))$, consider the conormal sequence of $X \subset \mathbb{P}$ tensored by $\mathcal{O}_X(Q-2d)$

$$0 \to N^*_X(Q-2d) \to \Omega^1_{\mathbb{P}_X}(Q-2d) \to \Omega^1_X(Q-2d) \to 0\,.$$

On the one hand, as the intermediary cohomology of $\mathcal{O}_X(n)$ vanishes for every $n \in \mathbb{Z}$ [25, Theorem 3.2.4 (iii)], $H^2(X, N^*_X(Q-2d)) = H^2(X, \mathcal{O}_X(Q-3d)) = 0$. On the other hand $H^1(X, \Omega^1_{\mathbb{P}_X}(Q-2d))$ can be computed with the exact sequence

$$0 \to \Omega^1_{\mathbb{P}}(Q-3d) \to \Omega^1_{\mathbb{P}}(Q-2d) \to \Omega^1_{\mathbb{P}_{|X}}(Q-2d) \to 0\,.$$

Now [25, Theorem 2.3.2] tell us that $H^2(\mathbb{P}, \Omega^1_{\mathbb{P}}(n)) = 0$ for every $n \in \mathbb{Z}$, and $H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}(n)) = 0$ if and only if $n \neq 0$. But $d \geq q_2 + q_3 + q_4$, as we have assumed, implies $2d > Q$. Thus $H^1(X, \Omega^1_{\mathbb{P}_X}(Q-2d)) = 0$ as wanted.

It remains to show that $H^0(X, \Omega^2_{\mathbb{P}_{|X}}(Q-d)) = 0$. To do it, consider the exact sequence

$$0 \to \Omega^2_{\mathbb{P}}(Q-2d) \to \Omega^2_{\mathbb{P}}(Q-d) \to \Omega^2_{\mathbb{P}_{|X}}(Q-d) \to 0\,.$$

The vanishing of $H^1(\mathbb{P}, \Omega^2_{\mathbb{P}}(Q-2d))$ is insured by [25, Theorem 2.3.4]. Finally, [25, Corollary 2.3.4] implies $H^0(\mathbb{P}, \Omega^2_{\mathbb{P}}(Q-d)) \neq 0$ if and only if

$$d < Q - q_0 - q_1.$$

The Lemma follows. □

We deduce from the classification of Fano 3-folds of index two the following corollary.

Corollary 5.17. *If X is a Fano 3-fold with $\rho(X) = 1$ and $i(X) = 2$ then $h^0(X, TX) \neq 0$ if and only if X is isomorphic to X_5.*

Proof of Theorem 5.14. Let X be a Fano 3-fold of index two with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, where H is an ample divisor, and $\mathcal{F}$ a codimension one foliation on X with $K\mathcal{F} = 0$. If $H^3 \leq 4$ then Corollary 5.17 implies X has no vector fields. Therefore by Theorem 5.1 any foliation on X with $K\mathcal{F} = 0$ is given by a closed 1-form without codimension one zeros and with polar divisor linearly equivalent to $2H$. The result follows in this case from Proposition 5.5. Notice that the dimension of $H^0(X, \mathcal{O}_X(H))$ is equal to $H^3 + 2$, [43, Chapter V, Exercise 1.12.6]. Therefore the irreducible component $\mathrm{Rat}(1,1)$ on X is isomorphic to the Grassmannian of lines on $\mathbb{P}^{H^3+1}$.

Suppose now that $H^3 = 5$, i.e., $X = X_5$. Lemma 5.15 implies that every algebraic action of $\mathbb{C}$ or $\mathbb{C}^*$ has isolated fixed points. Theorem 5.1 tells us that a foliation on X_5 with trivial canonical bundle is either induced by an algebraic action of two dimensional Lie group or is given by a closed 1-form without codimension one zeros and with polar divisor linearly equivalent to $2H$. The Lie algebra of regular vector fields on X_5 is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ (Lemma 5.15) and the two dimension subalgebras are all $PSL(2,\mathbb{C})$-conjugated, and isomorphic to the affine Lie algebra $\mathfrak{aff}(\mathbb{C})$. Let $\mathcal{F}$ be a foliation of X_5 determined by any of the affine Lie algebras contained into $\mathfrak{sl}(2,\mathbb{C})$. The induced action of $\mathrm{Aff}(\mathbb{C}) \subset \mathrm{Aut}(X)$ on $\mathbb{P}H^0(X, \Omega^1_X(H))$ has only one fixed point, therefore $\mathrm{Aff}(\mathbb{C})$ is tangent to only one hyperplane section of $X_5 \subset \mathbb{P}^6$. It follows that $\mathcal{F}$ is not defined by a closed 1-form without codimension one zeros since in this case the action would have to preserve a pencil of hyperplane sections. As there is a smooth $\mathbb{P}^1$ of affine Lie subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ we conclude that the space of foliations on X_5 with $K\mathcal{F} = 0$ has two disjoint irreducible components: one corresponding to foliations defined by closed 1-forms and the other defined by affine subalgebras of $\mathfrak{aut}(X_5)$. Notice that they are both smooth, with the second one corresponding to a closed orbit of $\mathrm{Aut}(X_5)$ in $\mathbb{P}H^0(X_5, \Omega^1_{X_5}(2H))$. □

5.5.2. *Index one.* Much of the work for the classification of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with $\mathrm{Pic}(X) = \mathbb{Z}$ and of index one has already been done by Jahnke and Radloff in [38]. In [38, Proposition 1.1] it is proved that $h^0(X, \Omega^1_X(1)) \neq 0$ implies that the genus of X, which by definition is $g(X) = h^0(X, -KX) + 2 = \frac{1}{2}KX^3 + 1$, is 10 or 12. This considerably reduces the amount of work to prove the final bit in the classification of foliations with $K\mathcal{F} = 0$ on Fano 3-folds with rank one Picard group.

Theorem 5.18. *If $\mathcal{F}$ is a codimension one foliation with trivial canonical bundle on a Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z}$ and $i(X) = 1$ then X is Mukai-Umemura 3-fold and $\mathcal{F}$ is induced by an algebraic action of the affine group.*

Proof. In [57] the Fano 3-folds of index one and $g \geq 7$ carrying vector fields are classified. There are two rigid examples (Mukai-Umemura 3-fold with $\mathrm{Aut}^0(X) = \mathbb{P}SL(2,\mathbb{C})$ and a 3-fold with $\mathrm{Aut}^0(X) = (\mathbb{C},+)$) and a one parameter family of examples with $\mathrm{Aut}^0(X) = (\mathbb{C}^*,\cdot)$. All the cases can be obtained from X_5, the Fano 3-fold of index two and degree 5, by means of a birational transformation defined by a linear system on X_5 of the form $|3H-2Y|$ where Y is the closure of a $(\mathbb{C},+)$ or $(\mathbb{C}^*,\cdot)$-orbit in X_5. Thus Lemma 5.15 implies that the vector fields in X have, exactly as the vector fields in X_5, isolated fixed points.

Theorem 5.1 implies that any codimension one foliation on X with $K\mathcal{F} = 0$ must be induced by an algebraic group. It follows that X is Mukai-Umemura 3-fold and that $\mathcal{F}$ is induced by an action of the affine group. □

Remark 5.19. In the main result of [38] there is an imprecision. They claim that a general section of $H^0(X,\Omega^1_X(1))$ for a general deformation of Mukai-Umemura 3-fold is integrable. This cannot happen since $h^0(X,\Omega^1_X(1)) = 3$ for any sufficiently small deformation of Mukai-Umemura 3-fold ([38, Proposition 2.6]) and therefore the closedness of Frobenius integrability condition would imply that every element of $H^0(X,\Omega^1_X(1)) \simeq (\mathfrak{sl}_2)^*$ is integrable. Apparently, their mistake is at the proof of their Proposition 2.16. More specifically, at the determination of the integer a from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-a+1) \oplus \tau \to 0$, where τ is a torsion sheaf.

5.6. **Classification of codimension one foliations of degree 2 on $\mathbb{P}^n$, $n > 3$.** Our methods also allow us to recover the classification of codimension one foliations of degree 2 on $\mathbb{P}^n$, $n > 3$, due to Cerveau and Lins Neto [15].

Theorem 5.20. *The irreducible components of the space of codimension one foliations on $\mathbb{P}^n$, $n > 3$, with $K\mathcal{F} = 0$ are* $\mathrm{Rat}(2,2)$, $\mathrm{Rat}(1,3)$, $\mathrm{Log}(1,1,1,1)$, $\mathrm{Log}(2,1,1)$, $\mathrm{LPB}(2)$, *and $Exc(2)$. The generic element of $Exc(2)$ is a linear pull-back from the foliation on $\mathbb{P}\mathbb{C}_3[x]$ induced by the natural action of the affine group.*

Its proof will use the classification of degree one foliations of arbitrary codimension on $\mathbb{P}^n$ which we now proceed to present.

5.6.1. *The space of foliations on $\mathbb{P}^n$ of degree zero and one (arbitrary codimension).* Foliations of degree zero have been classified by Cerveau and Deserti [24, Théorème 3.8]: a codimension q foliation of degree zero on $\mathbb{P}^n$ is defined by a linear projection from $\mathbb{P}^n$ to $\mathbb{P}^q$. The classification of foliations of degree one can be easily deduce from Medeiros classification of locally decomposable integrable homogeneous q-forms of degree one stated below.

Theorem 5.21. [23, Theorem A] *If ω is a locally decomposable integrable homogeneous q-form of degree 1 on $\mathbb{C}^{n+1}$ then*

(a) *there exist $q-1$ linearly independent linear forms $L_1,\ldots,L_{q-1}$ and a quadratic form Q such that $\omega = dL_1 \wedge \cdots \wedge dL_{q-1} \wedge dQ$, or*
(b) *there exist a linear projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^{q+1}$, and a locally decomposable integrable homogeneous q-form $\widetilde{\omega}$ of degree 1 on $\mathbb{C}^{q+1}$ such that $\omega = \pi^*\widetilde{\omega}$.*

Theorem 5.22. *If $\mathcal{F}$ be a foliation of degree 1 and codimension q on $\mathbb{P}^n$ then we are in one of following cases:*

(1) *$\mathcal{F}$ is defined a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$ with irreducible generic fiber determined by q linearly independent linear forms and one quadratic form; or*
(2) *$\mathcal{F}$ is the linear pull back of a foliation of induced by a global holomorphic vector field on $\mathbb{P}^{q+1}$.*

Proof. Let ω be a locally decomposable, integrable homogeneous q-form on $\mathbb{C}^{n+1}$ defining $\mathcal{F}$. Since $\mathcal{F}$ has degree 1, the degree of the coefficients of ω is 2. It is immediate from the definitions that the differential of a locally decomposable integrable q-form is also locally decomposable and integrable. Therefore we can apply Theorem 5.21 to $d\omega$. To recover information about ω we will use that $i_R\omega = 0$ implies $i_R d\omega = (q+2)\cdot\omega$.

If $d\omega$ is case (a) of Theorem 5.21, i.e.

$$d\omega = dL_1 \wedge dL_2 \wedge \cdots dL_q \wedge dQ$$

then $d\omega$ is the pull-back of $dx_0 \wedge \cdots \wedge dx_q$ under the map

$$\mathbb{C}^{n+1} \ni (x_0, \ldots, x_n) \mapsto (L_1, \ldots, L_q, Q) \in \mathbb{C}^{q+1},$$

and $(q+2)\omega = i_R d\omega$ is the pull-back of $i_{R(1^q,2)}dy_0 \wedge \cdots \wedge dy_q$ where $R(1^q, 2) = y_0\frac{\partial}{\partial y_0} + \cdots + y_q\frac{\partial}{\partial y_q} + 2y_{q+1}\frac{\partial}{\partial y_{q+1}}$. We are clearly in case (a) of the statement with rational map from $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$ described in homogeneous coordinates as above. It still remains to check that the generic fiber is irreducible. As ω has zero set of codimension at least two, the same holds true for $d\omega$ and consequently the map considered does not ramify in codimension one. Since $\mathbb{P}^n$ is simply-connected, the irreducibility of the generic fiber follows.

If $d\omega$ is case (b) of Theorem 5.21 then, in suitable coordinates, $d\omega$ depends only on $q+2$ variables, say $x_0, \ldots, x_{q+1}$. Being a $(q+1)$-form with coefficients of degree 1, there exists a linear vector field X such that $d\omega = i_X dx_0 \wedge \cdots \wedge dx_{q+1}$. The result follows. □

Corollary 5.23. *The space of foliations of degree 1 and codimension q on $\mathbb{P}^n$ has two irreducible components.*

Proof of Theorem 5.20. Notice that when $n > 3$, a foliation of degree two has negative canonical bundle. Thus, if $\mathcal{F}$ is semi-stable Proposition 4.5 implies that $\mathcal{F}$ is either a pencil of quadrics or a pencil of cubics with a hyperplane of multiplicity three as a member.

Suppose now that $\mathcal{F}$ is unstable and let $\mathcal{G}$ be its maximal destabilizing foliation. Recall from Example 2.3 that

$$\frac{\deg(\mathcal{G})}{\dim(\mathcal{G})} < \frac{\deg(\mathcal{F})}{\dim(\mathcal{F})}.$$

Therefore $\deg(\mathcal{G}) < 2$. If $\mathcal{G}$ has degree zero then $\mathcal{F}$ is a linear pull-back of a foliation of degree two on a lower-dimensional projective space and we can proceed inductively. Suppose now that the degree of $\mathcal{G}$ is one. The classification of foliations of degree one, Theorem 5.22, implies that the semi-stable foliations of degree one are either defined by a rational map to $\mathbb{P}(1^q, 2)$ or have dimension one. The maximal destabilizing foliation $\mathcal{G}$, which is semi-stable by definition, does not fit into the former case as we would have $1 < \deg(\mathcal{F})/\dim(\mathcal{F})$. Thus $\mathcal{G}$ must be defined by a rational map to $\mathbb{P}(1^q, 2)$. It is not hard to verify that in this case the foliation $\mathcal{F}$ must be in the component $\mathrm{Log}(1, 1, 2)$. □

Manifold	Irreducible component	dim
Projective space $\mathbb{P}^3$	$\mathrm{Rat}(1,1) \times S_2$	13
	$\mathrm{Log}(1,1,1) \times S_1$	14
	$\mathrm{Rat}(1,2) \times S_1$	14
Hyperquadric Q^3	$\mathrm{Rat}(1,1) \times S_1$	10

TABLE 2. Poisson structures on Fano 3-folds with rank one Picard group degenerating on hypersurfaces. In the description of the irreducible components the product stands for a Segre like embedding with S_i being $|\mathcal{O}_X(i)|$.

5.7. **Holomorphic Poisson structures.** A (non-trivial) holomorphic Poisson structure on projective manifold X is an element of $[\Pi] \in \mathbb{P}H^0(X, \bigwedge^2 TX)$ such that $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket. In dimension three, a Poisson structure is equivalent to a pair $(\mathcal{F}, D)$ where $\mathcal{F}$ is a codimension one foliation with $K\mathcal{F} = \mathcal{O}_X(-D)$ and a divisor $D \geq 0$. Our classifications of irreducible components of the space of foliations with $K\mathcal{F}$ very negative (Proposition 4.7) and with $K\mathcal{F} = 0$ on Fano 3-folds of rank one implies at once a description of the irreducible components of the space of Poisson structures

$$\mathrm{Poisson}(X) = \left\{ \Pi \in \mathbb{P}H^0(X, \bigwedge^2 TX) \;\middle|\; [\Pi, \Pi] = 0 \right\}$$

on these manifolds.

Theorem 5.24. *If X is a Fano 3-fold with rank one Picard group then* $\mathrm{Poisson}(X)$ *has 9 irreducible components when $X = \mathbb{P}^3$; 4 irreducible components when $X = Q^3$; 2 irreducible components when $X = X_5$; 1 irreducible component when X Mukai-Umemura 3-fold; 1 irreducible component when X has index two and is distinct from X_5; and is empty when X has index one and is not Mukai-Umemura 3-fold.*

6. Criterium for Uniruledness

We now turn back to the problem of describing the structure of an arbitrary codimension one foliation with numerically trivial canonical bundle. The main goal of this section is to obtain information about the ambient manifold when the singular set of $\mathcal{F}$ is not empty.

6.1. **Pseudo-effectiveness of the canonical bundle implies smoothness.** A particular case of the result below ($p = \dim X - 1$) already appeared in [65]. The arguments here are a simple generalization of the arguments therein.

Theorem 6.1 (First part of Theorem 6 of the Introduction)**.** *Let X be a compact Kähler manifold with KX pseudo-effective, L be a flat line bundle on X, p a positive integer and $v \in H^0(X, \bigwedge^p TX \otimes L)$ a non-zero section; then the zero set of v is empty.*

Proof. To not overburden the notation we will suppose that L is trivial. There are no extra difficulties to prove the general case. Let $\{U_i\}$ be an open covering of X, $\Omega_i \in KX(U_i)$ be holomorphic n-forms trivializing KX. If we write $\Omega_i = g_{ij}^{-1}\Omega_j$

then $\{g_{ij}\}$ is a cocycle defining KX. Let ω_i be the contraction of v with the n-form Ω_i. Notice that the collection $\{\omega_i\}$ defines a holomorphic section of $\Omega^q_X \otimes KX^*$ with $q = n - p$.

The pseudo-effectiveness of KX implies the existence of singular hermitian metric on it with non-negative curvature. Concretely, there exists plurisubharmonic functions φ_i on U_i such that

$$|g_{ij}|^2 = \exp(\varphi_i - \varphi_j)\,.$$

Thus the (q,q)-form

$$\eta = \sqrt{-1}\exp(\varphi_i)\omega_i \wedge \overline{\omega_i}$$

is a global well-defined real (q,q)-form with coefficients in L^∞_{loc}. Demailly, in [22], proved that the identity of currents $d\omega_i = -\partial\varphi_i \wedge \omega_i$ holds true, see also the proof of [12, Proposition 2.1] for the case of distributions. Consequently, $d\eta = 0$ as a current, and η defines a class in $H^{q,q}(X,\mathbb{R})$. By Serre duality there exists $[\beta] \in H^{n-q,n-q}(X,\mathbb{R})$ such that $[\eta]\wedge[\beta] \neq 0$.

Decompose η as the product $\big(-\sqrt{-1}\exp(\varphi_i)\overline{\omega_i}\big)\wedge\big(\omega_i\big)$. Since the identity

$$\exp(\varphi_i)\overline{\omega_i} = |g_{ij}|^2\exp(\varphi_j)\,\overline{g_{ij}}^{-1}\overline{\omega_j} = g_{ij}\exp(\varphi_j)\overline{\omega_j}.$$

holds true, the first factor is a $\overline{\partial}$-closed $(0,q)$-form with values on KX. Similarly the second factor is a holomorphic q-form with values on KX^*. Notice that

$$0 \neq [\sqrt{-1}\exp(\varphi_i)\overline{\omega_i}]\wedge[\omega_i\wedge\beta]$$

and that we can interpret the first factor as a class in $H^{0,q}_{\overline{\partial}}(KX)$, and the second factor as a class in $H^{n,n-q}_{\overline{\partial}}(KX^*)$. Serre duality provides a non-zero zero class in $H^{0,n-q}_{\overline{\partial}}(KX\otimes KX^*) = H^{0,n-q}_{\overline{\partial}}(X)$.

Let γ be a harmonic representative of $[\omega_i\wedge\beta]$ in $H^{0,n-q}_{\overline{\partial}}(X)$. Hodge symmetry implies $\overline{\gamma}$ is a holomorphic $(n-q)$-form. Therefore

$$[\sqrt{-1}\exp(\varphi_i)\overline{\omega_i}\wedge\gamma] \neq 0$$

and consequently $\{\overline{\gamma}\wedge\omega_i\}$ is a non-zero section of $KX\otimes KX^* = \mathcal{O}_X$. It follows that ω_i has no zeros, and the same holds for v. □

Theorem 6.2 (Second part of Theorem 6 of the Introduction)**.** *Let $\mathcal{D}$ be a distribution of codimension q on a compact Kähler manifold X. If $c_1(T\mathcal{D}) = 0$ and KX is pseudoeffective then $\mathcal{D}$ is a smooth foliation. Moreover, there exists a smooth foliation $\mathcal{G}$ on X of dimension q such that $TX = T\mathcal{D}\oplus T\mathcal{G}$. Finally, if X is projective, then the canonical bundle of $\mathcal{D}$ is torsion.*

Proof. The integrability follows from [22]. The previous theorem implies that $\mathrm{sing}(\mathcal{D}) = \emptyset$ and that there exists a holomorphic $(n-q)$-form $\overline{\gamma}$ which restricts to a volume form on the leaves of the foliation defined by $\mathcal{D}$.

In order to prove the result we just need to modify $\overline{\gamma}$ to obtain that its kernel is the expected complementary subbundle defining $\mathcal{G}$. This can be done as follows. There is a natural monomorphism of sheaves

$$\psi : \bigwedge^{n-q-1} T\mathcal{D} \to \Omega^1_X,$$

defined by the contraction of $\overline{\gamma}$ with $n-q-1$ vectors fields tangent to $\mathcal{D}$. Notice that the projection morphism of Ω^1_X onto $T^*\mathcal{D}$ is actually an isomorphism in restriction to $Im\ \psi$. Its inverse provides a splitting of the exact sequence

$$0 \to N^*\mathcal{D} \to \Omega^1_X \to T^*\mathcal{D} \to 0\,.$$

Since $\det T^*\mathcal{D}$ is numerically trivial, $Im\ \psi$ is an integrable subbundle of Ω^1_X. This subbundle defines the conormal bundle of the sought foliation $\mathcal{G}$.

Let $L = K\mathcal{D}^*$ and $\gamma \in H^0(X, \Omega^p_X \otimes L)$, $p = n - q$, be a twisted p-form defining $\mathcal{G}$. After passing to a finite étale covering we can assume that the integral Chern class of L is zero, i.e., $L \in \mathrm{Pic}^0(X)$.

Since L is flat, Hodge symmetry implies that $H^0(X, \Omega^p_X \otimes L) \cong H^p(X, L^*)$. Let $m = h^p(X, L^*)$ and consider the Green-Lazarsfeld set

$$S = \{E \in \mathrm{Pic}^0(X) \,|\, h^p(X, E) \geq m\}\,.$$

According to [61], if X is projective then S is a finite union of translates of subtori by torsion points. To conclude the proof of the Theorem it suffices to show that L^* is an isolated point of S. Let $\Sigma \subset \mathrm{Pic}^0(M)$ be an irreducible component of S passing through L. If $\mathcal{P}$ is the restriction of the Poincaré bundle to $\Sigma \times X$ and $\pi : \Sigma \times X \to \Sigma$ is the natural projection then, by semi-continuity, $R^p\pi_*\mathcal{P}$ is locally free at a neighborhood of L. Therefore we can extend the element $H^p(X, L^*)$ determined by γ to a holomorphic family of non-zero elements with coefficients in line bundles $E \in \Sigma$ close to L^*. Hodge symmetry gives us a family of holomorphic p-forms with coefficients in the duals of these line bundles. Taking the wedge product of these p-forms with a q-form defining $\mathcal{D}$ we obtain, by transversality of $\mathcal{D}$ and $\mathcal{G}$, non-zero sections of $H^0(X, KX \otimes N\mathcal{D} \otimes E^*)$ for E varying on a small neighborhood of L^* at Σ. Since $KX \otimes N\mathcal{D} \otimes E^* \in \mathrm{Pic}^0(X)$, this implies that $E \in \Sigma$ if and only if $E = KX \otimes N\mathcal{D} = L^*$. Thus Σ reduces to a point. □

Remark 6.3. On the same vein, one can prove that the flat line bundle L in theorem 6.1 is actually a torsion one (under the exta assumption that X is projective). The Theorem above provides evidence toward the following conjecture of Sommese ([62]): if $\mathcal{F}$ is a smooth foliation of dimension p with trivial canonical bundle on a compact Kähler manifold X then there exists a holomorphic p-form on X which is non-trivial when restricted to the leaves of $\mathcal{F}$.

Theorem 6.2 reduces the classification of codimension one foliations with numerically trivial canonical bundle on Kähler manifolds with pseudo-effective canonical bundle to the work done by the third author in [63] and recalled in the Introduction. The case of smooth foliations of higher codimension with numerically trivial canonical bundle on compact Kähler manifolds have also been treated by the third author in [64], but the results are not as complete as in the codimension one case. As shown by Peternell in his recent preprint ([55]) already mentioned in the introduction, such foliations arise naturally, at least on non uniruled projective manifolds, when the cotangent bundle fails to be generically ample. The same author makes also some conjectures about the structure of those manifolds, generalizing former results by Lieberman, and which are confirmed in very special cases in [64]. We redirect the interested reader to the above mentioned papers.

6.2. **Criterium for uniruledness.** The structure of projective manifolds carrying non-zero holomorphic vector fields are fairly well understood, see for instance [1,

Theorem 0.1]. In particular, projective manifolds having a non-zero vector field with non-empty zero set are uniruled. A similar result was proved by Lieberman in the compact Kähler case (loc. cit. Section 1). More recently, Campana and Peternell proved the following result [13, Corollary 1.12]: if the m-th tensor power $TX^{\otimes m}$ of the tangent sheaf of a projective manifold X admits a subsheaf $\mathcal{E}$ of rank r such that $\det \mathcal{E}$ is pseudo-effective and the induced section $\wedge^r TX^{\otimes m} \otimes \det \mathcal{E}^*$ vanishes along a divisor then X is uniruled. Theorem 6.1 allow us to deduce a strictly related result which confirms [55, Conjecture 4.23].

Theorem 6.4 (Theorem 5 of the Introduction)**.** *Let X be a projective manifold and L be a pseudo-effective line bundle on X. If there exists $v \in H^0(X, \bigwedge^p TX \otimes L^*)$ vanishing at some point then X is uniruled. In particular, if there exists a foliation $\mathcal{F}$ on X with $c_1(T\mathcal{F})$ pseudo-effective and* $\mathrm{sing}(\mathcal{F}) \neq \emptyset$ *then X is uniruled.*

Proof. If X is not uniruled then KX is pseudo-effective [10, Corollary 0.3]. Miyaoka's Theorem (Corollary 2.5) together with Mehta-Ramanathan Theorem [50] imply that the Harder-Narasimhan filtration of the restriction of TX to curves obtained as complete intersections of sufficiently ample divisors has no subsheaf of positive degree. Therefore the same holds true for $\bigwedge^p TX$, and consequently L cannot intersect ample divisors positively. This property together with its pseudo-effectiveness imply $c_1(L) = 0$. Theorem 6.1 implies $\mathrm{sing}(v) = \emptyset$. This contradiction concludes the proof. □

6.3. **Division property implies smoothness.**

Proposition 6.5. *Let $\mathcal{F}$ be a codimension one foliation with numerically trivial canonical bundle on a Kähler manifold X. If $h^1(X, N^*\mathcal{F}) \neq 0$ then $\mathcal{F}$ is smooth or there exists a foliation $\mathcal{G}$ by rational curves tangent to $\mathcal{F}$.*

Proof. Let $\theta \in H^1(X, N^*\mathcal{F})$ be a non-zero element. Serre duality produces a non-zero element in $H^{n-1}(X, KX \otimes N\mathcal{F}) \cong H^{n-1}(X, K\mathcal{F})$. Since $K\mathcal{F}$ is numerically trivial by hypothesis, we can apply Hodge symmetry to obtain a non-zero element $v_\theta \in H^0(X, \Omega_X^{n-1} \otimes -K\mathcal{F}) \cong H^0(X, TX \otimes N^*\mathcal{F})$. Contract v_θ with a twisted 1-form $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$ defining $\mathcal{F}$ to obtain a section of $\mathcal{O}_X$. It is either nowhere zero or identically zero. In the first case $\mathcal{F}$ must be smooth and in the second we have a foliation by curves $\mathcal{G}$ tangent to $\mathcal{F}$ with canonical bundle $K\mathcal{G} = N^*\mathcal{F} - \Delta$, where Δ is the divisor of zeros of v_θ. If $K\mathcal{G}$ is pseudo-effective then the same is true for $N^*\mathcal{F} = KX$ (numerically) and we can apply Theorem 6.1 to conclude. If $K\mathcal{G}$ is not pseudo-effective then $\mathcal{G}$ must be a foliation by rational curves according to Brunella's Theorem (if X is projective then we can apply Miyaoka's Theorem). □

Theorem 6.6. *Let $\mathcal{F}$ be a codimension one foliation with numerically trivial canonical bundle on a Kähler manifold X defined by $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$. If for every point $x \in X$ there exists a germ of holomorphic 1-form η at x such that $d\omega = \eta \wedge \omega$ holds in a neighborhood of x then $\mathcal{F}$ is smooth.*

Proof. Since $K\mathcal{F}$ is numerically zero we obtain that KX is numerically equivalent to $N^*\mathcal{F}$.

Let $\{U_i\}$ be an open covering of X and $\omega_i \in \Omega^1_X(U_i)$ local representatives of ω. At the intersections $\omega_i = g_{ij}\omega_j$, where $\{g_{ij}\}$ is a cocycle defining $N\mathcal{F}$ in $H^1(X, \mathcal{O}_X^*)$. Notice that the logarithmic derivative $\frac{dg_{ij}}{g_{ij}}$ represents (up to a constant factor) the

Chern class of $N^*\mathcal{F}$ in $H^1(X, \Omega^1_X)$. If $c_1(N^*\mathcal{F}) = 0$ then KX is numerically equivalent to zero; in particular, it is pseudo-effective, and we can thus apply Theorem 6.1 to deduce the smoothness of $\mathcal{F}$. From now on, we will assume that $N^*\mathcal{F}$ has non-zero Chern class.

We will now use the division property to obtain another representative in $H^1(X, \Omega^1_X)$ for $c_1(N^*\mathcal{F})$. By hypothesis there exists $\eta_i \in \Omega^1_X(U_i)$ satisfying $d\omega_i = \eta_i \wedge \omega_i$. A simple computation shows that the 1-forms

$$\theta_{ij} = \frac{dg_{ij}}{g_{ij}} - (\eta_i - \eta_j)$$

vanish along the leaves of $\mathcal{F}$. Therefore the collection $\{\theta_{ij}\}$ defines an element $\theta \in H^1(X, N^*\mathcal{F})$ with image in $H^1(X, \Omega^1_X)$ equal to the first Chern class of $N^*\mathcal{F}$, in particular $h^1(X, N^*\mathcal{F}) \neq 0$. We can apply Proposition 6.5 to ensure that $\mathcal{F}$ is smooth, or there exists a foliation by rational curves $\mathcal{G}$ tangent to $\mathcal{F}$. To conclude the proof it remains to exclude the latter possibility.

Let $i : \mathbb{P}^1 \to X$ be a generically injective morphism to a generic leaf of $\mathcal{G}$. Since it is tangent to $\mathcal{F}$, then the 1-forms $i^*\theta_{ij}$ vanish identically. Therefore $i^*N^*\mathcal{F}$ is the trivial line bundle. Consequently $i^*KX = i^*K\mathcal{F} \otimes i^*N^*\mathcal{F}$ is also trivial. On the other hand, since $i(\mathbb{P}^1)$ moves in a family of rational curves which cover X,

$$i^*TX = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$$

with $a_1 \geq 2$ and $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Thus $i^*KX = \mathcal{O}_{\mathbb{P}^1}(-\sum a_i)$ contradicting its triviality. Therefore the foliation by rational curves $\mathcal{G}$ cannot be contained in $\mathcal{F}$. □

Corollary 6.7. *Let $\mathcal{F}$ be a foliation of codimension one with $c_1(K\mathcal{F}) = 0$ on a projective manifold X. If $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ then it has an irreducible component of codimension two.*

Proof. If $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ and $\operatorname{codim} \operatorname{sing}(\mathcal{F}) \geq 3$ then $\mathcal{F}$ satisfies the division property of Theorem 6.6 thanks to de Rham-Saito Lemma [58] and we get a contradiction. □

Remark 6.8. Among the examples of codimension one foliations with $c_1(K\mathcal{F}) = 0$, one finds the foliations defined by Poisson structures of corank one. Corollary 6.7 implies that these either have constant rank or that the rank drops in a codimension two subvariety. Therefore it generalizes and gives a conceptual proof of [5, Proposition 4 item 3] as asked by Beauville. It is also in accordance with Bondal's conjecture [5, Conjecture 4].

Combining Corollaries 3.9 and 6.7 we obtain the following result which will be useful later.

Corollary 6.9. *Let $\mathcal{F}$ be a p-closed codimension one foliation with $c_1(K\mathcal{F}) = 0$ on a projective manifold X. If $\mathcal{F}$ is not uniruled and $\operatorname{sing}(\mathcal{F}) \neq \emptyset$ then there exists an irreducible component S of $\operatorname{sing}(\mathcal{F})$ having codimension two and at a generic point of S the foliation $\mathcal{F}$ is locally defined by a holomorphic 1-form of type*

$$pxdy + qydx$$

with p, q relatively prime **distinct** *positive integers.*

7. Projective structures and transversely projective foliations

7.1. **Projective structures on $\mathbb{P}^1$.** On a Riemann surface, a projective structure is defined by an atlas taking values in $\mathbb{P}^1$ with transition charts in $\mathrm{Aut}(\mathbb{P}^1)$. On $\mathbb{P}^1$ there is only one projective structure. Therefore it is natural to allow meromorphic singularities. There are at least two equivalent ways to define a meromorphic projective structure on $\mathbb{P}^1$. One can first define it by a rational quadratic form η. The poles are the singular points of the structure. Given a *linear* coordinate x, one can write $\eta = \phi(x)dx^2$ and the charts φ of the structure are the solutions of the equation

$$\{\varphi, x\} = \phi$$

where $\{\varphi, x\}$ stands for the Schwarzian derivative of φ with respect to x:

$$\{\varphi, x\} = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2.$$

This is the classical approach which can be traced back to Schwarz and Poincaré.

One can also define the structure by the data of a $\mathbb{P}^1$-bundle $P \to \mathbb{P}^1$, a Riccati foliation $\mathcal{H}$ on P (i.e. a singular foliation on the ruled surface P which is transversal to a generic fibre of the ruling) and a section $\sigma : \mathbb{P}^1 \to P$ which is not $\mathcal{H}$-invariant. At the neighborhood of a fibre transversal to $\mathcal{H}$, one can find coordinates (x, z) such that x defines the ruling, and $z \in \mathbb{P}^1$ defines the foliation. Then $\varphi := z \circ \sigma$ is a chart of the structure. If $(P', \mathcal{H}', \sigma')$ is derived in the natural way by a birational modification $P \dashrightarrow P'$ of the bundle, it then defines the same projective structure. From this point view, a singular projective structure is the data of the triple $(P, \mathcal{H}, \sigma)$ up to birational modification. There is however a representative which is well-defined up to bundle isomorphisms. It is characterized by the following conditions :

(1) the (effective) polar divisor has minimal degree,
(2) the section σ does not intersect the singular locus of $\mathcal{H}$.

This is the minimal model discussed in [48]. The first condition characterizes the relative minimal model for $(P, \mathcal{H})$ discussed in [11], chapter 5; there are countably many except in some particular cases and the second condition fixes this freedom.

The equivalence between the two point of view described above is as follows. Given the quadratic form η defining the projective structure, one can associate the foliation $\mathcal{H}$ defined on the trivial bundle $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates (x, z) by the Riccati equation $\frac{dz}{dx} + z^2 + \frac{\phi(x)}{2} = 0$ and the section $z = \infty$.

Conversely, given a triple $(P, \mathcal{H}, \sigma)$, one can assume, up to birational transformation, that P is the trivial bundle with coordinate z and moreover σ is defined by $z = \infty$. The foliation $\mathcal{H}$ is thus defined by a 1-form

$$\Omega = dz + (f(x)z^2 + g(x)z + h(x))dx\,.$$

Using a birational map of the form $(x, z) \mapsto (x, a(x)z + b(x))$, we are able to suppose that $f(x) = 1$ and $g(x) = 0$. Precisely, if

$$\begin{aligned} \vartheta : \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \\ (x, z) &\longmapsto \left(x, \frac{2z + \frac{f'}{f} - g}{2f}\right) \end{aligned} \tag{7}$$

then

$$\vartheta^*\Omega = \frac{1}{f}\left(dz + \left(z^2 + \frac{\phi(x)}{2}\right)dx\right)$$

where

$$\phi = \{f, x\} + 2fh - g' + g\frac{f'}{f} - \frac{1}{2}g^2. \tag{8}$$

One easily check from relative minimal models of singular fibres described in [11] p.52-56 that the order of poles of η at a singular fibre of $\mathcal{H}$ is $2(\kappa+1)$ where κ is the Katz index: in the relative minimal model, the order of the pole is $\kappa+1$ in the unramified case (non nilpotent fibre) and $\kappa+\frac{3}{2}$ in the ramified (nilpotent) case; the other poles of η are double, at each tangency point of σ with $\mathcal{H}$.

For a more thorough account of the material presented here see [47].

7.2. **Transversely projective, affine, and Euclidean foliations.** We now turn to transversely projective foliations following [59, 16, 17, 48]. A *transversely projective* structure for a foliation $\mathcal{F}$ on a projective manifold X is the data $(P, \mathcal{H}, \sigma)$ of a $\mathbb{P}^1$-bundle $P \to X$, a Riccati foliation $\mathcal{H}$ on P, and a meromorphic section $\sigma : X \dashrightarrow P$ such that $\sigma^*\mathcal{H} = \mathcal{F}$.

Another triple $(P', \mathcal{H}', \sigma')$ defines the same transversely projective structure if it is derived from the initial one by a birational bundle transformation $P \dashrightarrow P'$. Up to such birational bundle transformation, one can always assume that P is the trivial bundle $X \times \mathbb{P}^1$ with vertical coordinate z and σ is the section $\{z = 0\}$ at infinity. The foliation $\mathcal{H}$ is defined by a 1-form

$$dz + \omega_0 + \omega_1 z + \omega_2 z^2 \tag{9}$$

where $\omega_0, \omega_1, \omega_2$ are rational 1-forms on X. The integrability of $\mathcal{H}$ is equivalent to the equations

$$\begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \ \omega_1 \wedge \omega_2 \end{cases} \tag{10}$$

Here, $\mathcal{F}$ is defined by ω_0. A foliation $\mathcal{F}$ on $\mathbb{P}^n$ is transversely projective if, and only if, there exist rational 1-forms $\omega_0, \omega_1, \omega_2$ on X satisfying (10) where ω_0 defines the foliation $\mathcal{F}$.

If there exists a transversely projective structure $(P, \mathcal{H}, \sigma)$ for $\mathcal{F}$ in which $\omega_2 = 0$, or equivalently, there exists a section $\tilde{\sigma} : X \dashrightarrow P$ invariant by $\mathcal{G}$, then we say that $\mathcal{F}$ is a transversely affine foliation.

If there exists a transversely projective structure $(P, \mathcal{H}, \sigma)$ for $\mathcal{F}$ in which $\omega_2 = \omega_1 = 0$ then $\mathcal{F}$ is a transversely Euclidean foliation. In particular, a foliation $\mathcal{F}$ is transversely Euclidean if and only if $\mathcal{F}$ is given by a closed rational 1-form.

Let C be a smooth curve and $f : C \to X$ a morphism which is generically transverse to a foliation $\mathcal{F}$. If $(P, \mathcal{H}, \sigma)$ is a transverse projective structure and the image of f is not contained in the indeterminacy locus of $\sigma : X \dashrightarrow P$ then the naturally defined triple $(f^*P, f^*\mathcal{H}, f^*\sigma)$ gives a projective structure on C. In Section 8 we are going to explore this fact in order to define a foliation $\mathcal{P}$ on $\mathrm{Mor}(\mathbb{P}^1, X)$ with leaves corresponding to morphisms inducing the same projective structure on $\mathbb{P}^1$. There we will need a couple of basic facts about transversely projective, affine, and Euclidean foliations that we state below as lemmas.

Lemma 7.1. *Let $F : Y \dashrightarrow X$ be a dominant rational map between projective manifolds, $\mathcal{F}$ be a codimension one foliation on X, and $\mathcal{G} = F^*\mathcal{F}$ be the foliation on Y induced by $\mathcal{F}$. Then the following assertions hold true.*

1. *The foliation $\mathcal{G}$ is transversely projective if and only if $\mathcal{F}$ is transversely projective.*
2. *The foliation $\mathcal{G}$ is transversely affine if and only if $\mathcal{F}$ is transversely affine.*

We note that $\mathcal{G}$ might be transversely Euclidean while $\mathcal{F}$ being only transversely affine.

Proof. If $\mathcal{F}$ is transversely projective (resp. affine or Euclidean) then $\mathcal{G} = F^*\mathcal{F}$ is transversely projective (resp. affine or Euclidean) since such a structure $(P, \mathcal{H}, \sigma)$ for $\mathcal{F}$ pulls back to a similar structure $(F^*P, F^*\mathcal{H}, F^*\sigma)$ for $\mathcal{G}$.

Suppose now that $\mathcal{G}$ is transversely projective (resp. affine). Restrict $\mathcal{G}$ and its projective structure to a sufficiently general submanifold having the same dimension as X. This reduces the problem to case where F is a generically finite rational map and we can apply [14, Lemme 2.1, Lemme 3.1] to conclude. □

8. Deformation of free morphisms along foliations

8.1. **Deformation of free morphisms.** Let X be a projective manifold of dimension n. The morphisms from $\mathbb{P}^1$ to X are parametrized by a locally Noetherian scheme $\mathrm{Mor}(\mathbb{P}^1, X)$, [43, Theorem I.1.10]. The Zariski tangent space of $\mathrm{Mor}(\mathbb{P}^1, X)$ at a given morphism $f : \mathbb{P}^1 \to X$ is canonically identified with $H^0(X, f^*TX)$ [20, Proposition 2.4] [43, Theorem I.2.16]. To understand this, suppose $\mathrm{Mor}(\mathbb{P}^1, X)$ is smooth at a point $[f]$, and let $\gamma : (\mathbb{C}, 0) \to \mathrm{Mor}(\mathbb{P}^1, X)$ be a germ of holomorphic curve in $\mathrm{Mor}(\mathbb{P}^1, X)$ such that $\gamma(0) = [f]$. If we fix $x \in \mathbb{P}^1$ and compute $\gamma'(0)(x)$ we obtain a vector at $T_{f(x)}X \simeq (f^*TX)_x$. Thus $\gamma'(0) \in H^0(\mathbb{P}^1, f^*TX)$.

For an arbitrary morphism f, the local structure of $\mathrm{Mor}(\mathbb{P}^1, X)$ at a neighborhood of $[f]$ can be rather nasty, but if $h^1(X, f^*TX) = 0$ then $\mathrm{Mor}(\mathbb{P}^1, X)$ is smooth and has dimension $h^0(\mathbb{P}^1, f^*TX)$ at a neighborhood of $[f]$, see [43, Theorem I.2.16] or [20, Theorem 2.6].

If $[f] \in \mathrm{Mor}(\mathbb{P}^1, X)$ then Birkhoff-Grothendieck's Theorem implies that f^*TX splits as a sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. The morphism f is called free when $a_n \geq 0$. Notice that $h^1(\mathbb{P}^1, f^*TX) = 0$ when f is a free morphism. Therefore $\mathrm{Mor}(\mathbb{P}^1, X)$ is smooth of dimension $h^0(\mathbb{P}^1, f^*TX) = n + \sum_{i=1}^n a_i$ at a neighborhood of $[f]$.

The scheme $\mathrm{Mor}(\mathbb{P}^1, X)$ comes together with an evaluation map

$$\begin{aligned} F : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, X) &\longrightarrow X \\ (x, [f]) &\longmapsto f(x)\,. \end{aligned}$$

Let f be a free morphism and $M = M_f$ be the irreducible component of $\mathrm{Mor}(\mathbb{P}^1, X)$ containing $[f]$. The restriction of F to $\mathbb{P}^1 \times \{[f]\}$ has maximal rank at any point of a neighborhood of $\mathbb{P}^1 \times \{[f]\}$ in $\mathbb{P}^1 \times M$ [43, Corollary II.3.5.4]. Indeed, it has maximal rank at a neighborhood of any point of the $\mathrm{Aut}(\mathbb{P}^1)$-orbit of $[f]$ under the natural action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathrm{Mor}(\mathbb{P}^1, X)$ defined by right composition.

Example 8.1. When $X = \mathbb{P}^n$ and f is a linear embedding, then $f^*TX = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, f is free and M_f has dimension $2n + 1$.

8.2. **Tangential foliation.** Let $\mathcal{D}$ be a distribution on X. We will say that a germ of deformation $f_t : \mathbb{P}^1 \to X$, $t \in (\mathbb{C}, 0)$ of a free morphism $f = f_0 : \mathbb{P}^1 \to X$ is tangent to $\mathcal{D}$ if the curves $f_t(x) : (\mathbb{C}, 0) \to X$ are tangent to the distribution $\mathcal{D}$ for every x in $\mathbb{P}^1$. These deformations correspond to germs of curves on M tangent to a distribution $\mathcal{D}_{tang} = \mathcal{D}_{tang}(f)$ on M which we will call the tangential distribution of $\mathcal{D}$.

The construction of $\mathcal{D}_{tang}$ is rather simple. From the identification $T_{[f]}M \simeq H^0(\mathbb{P}^1, f^*TX)$ it follows that $TM \simeq \pi_* F^*TX$. Thus the inclusion $T\mathcal{D} \hookrightarrow TX$ gives rise to a morphism $\pi_* F^*T\mathcal{D} \to TM$. If $\mathcal{I}$ denotes its image then we define $\mathcal{D}_{tang}$ as the distribution on M determined by the saturation of $\mathcal{I}$ inside TM, i.e., $T\mathcal{D}_{tang}$ is the smallest subsheaf of TM containing $\mathcal{I}$ and with torsion free cokernel.

Proposition 8.2. *For a generic morphism $g : \mathbb{P}^1 \to X$, $[g] \in M$, any germ of deformation $g_t : \mathbb{P}^1 \to X$, $t \in (\mathbb{C}, 0)$, of $g = g_0$ tangent to $\mathcal{D}$ gives rise to a germ of curve $[g_t] : (\mathbb{C}, 0) \to M$ tangent to $\mathcal{D}_{tang}$.*

Proof. By semi-continuity, there exists a non-empty open subset $U \subset M$ where the sheaf $\pi_* F^*T\mathcal{D}$ is locally free. If a germ of deformation $g_t : \mathbb{P}^1 \to X$, $t \in (\mathbb{C}, 0)$, of a morphism $g = g_0 \in U$ is tangent to $\mathcal{D}$ then the corresponding germ of curve $g_t : (\mathbb{C}, 0) \to M$ is clearly tangent to $\mathcal{D}_{tang}$. □

Proposition 8.3. *If the distribution $\mathcal{D}$ is closed under Lie brackets then the same holds true for $\mathcal{D}_{tang}$.*

Proof. It suffices to verify at a neighborhood of a generic morphism $[g] \in M$. We can assume for instance that the image of g is disjoint from the singular set of $\mathcal{D}$, thus $\mathcal{D}$ foliates a neighborhood of $g(\mathbb{P}^1)$. If ξ is a germ of section of $T\mathcal{D}_{tang}$ at $[g]$ then the orbits of the corresponding vector field give rise to deformations of morphisms $\phi_t : \mathbb{P}^1 \times (\mathbb{C}, 0)$, $[\phi_0] \in (M, [g])$, which map $x \times (\mathbb{C}, 0)$ to the leaves of $\mathcal{D}$. The involutivity of $\mathcal{D}$ promptly implies the involutivity of $\mathcal{D}_{tang}$. □

8.3. **Interpretation.** Throughout this section we will suppose $f : \mathbb{P}^1 \to X$ is free and generically transverse to $\mathcal{F}$, i.e., the generic morphism in $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ is not tangent to $\mathcal{F}$. In what follows $\mathcal{G} = F^*\mathcal{F}$ denotes the pull-back of $\mathcal{F}$ under the evaluation morphism $F : \mathbb{P}^1 \times M \to X$ and $\mathcal{F}_{tang}$ is the tangential foliation induced on M as defined above.

According to [8], there is a foliation $\overline{\mathcal{F}}_{tang} \supset \mathcal{F}_{tang}$ with algebraic leaves such that for a generic $x \in X$, the leaf of $\overline{\mathcal{F}}_{tang}$ passing through x is algebraic and coincides with the Zariski closure of the corresponding leaf of $\mathcal{F}_{tang}$. We shall discuss the codimension of $\mathcal{F}_{tang}$ in $\overline{\mathcal{F}}_{tang}$, and see how it controls the transverse geometry of $\mathcal{G}$.

Theorem 8.4. *The foliation $\mathcal{F}_{tang}$ has codimension at most three in $\overline{\mathcal{F}}_{tang}$. Moreover,*

1. *If $\mathcal{F}_{tang} = \overline{\mathcal{F}}_{tang}$ then $\mathcal{G}$ is a pull-back from a manifold of dimension $\dim M + 1 - \dim \mathcal{F}_{tang}$.*
2. *If $\mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 1$ or 2 then $\mathcal{G}$ is the pull-back of a transversely affine Riccati foliation on $\mathbb{P}^1 \times M$ by a rational map of the form $(z, x) \mapsto (\alpha(x, z), x)$. In particular, $\mathcal{G}$ is transversely affine.*
3. *If $\mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 3$ then $\mathcal{G}$ is the pull-back of a Ricatti foliation on $\mathbb{P}^1 \times M$ by a rational map of the form $(z, x) \mapsto (\alpha(x, z), x)$. In particular, $\mathcal{G}$ is transversely projective.*

If $f^*N\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(N+2)$ then $\mathcal{G}$ is defined by a 1-form

$$\Omega = \left(\sum_{i=0}^{N} a_i z^i\right) dz + \sum_{i=0}^{N+2} z^i \omega_i \quad =: \quad p(z)dz + \omega(z) \tag{11}$$

where $p \in \mathbb{C}(M)[z]$ is a polynomial in z of degree at most N whose coefficients a_i are rational functions on M and $\omega_0, \omega_1, \ldots, \omega_{N+2}$ are rational 1-forms on M with $\omega_{N+2} \neq 0$. Notice that the zeros of p correspond to the tangencies between $\mathcal{G}$ and the fibration $\mathbb{P}^1 \times M \to M$ or, equivalently, to the tangencies between the corresponding morphisms and the foliation $\mathcal{F}$. Due to the natural action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^1 \times M$ and the above geometrical interpretation, both leading coefficients a_N and a_0 are non zero. After dividing Ω by a_N, we can assume $p(z)$ monic (and $a_0 \neq 0$). This will be assumed from now on.

When $N = 0$, then $\mathcal{G}$ is transversal to the fibration $\mathbb{P}^1 \times M \to M$, thus a Riccati foliation. When $X = \mathbb{P}^n$ and M is the variety of linear morphisms (the lines of $\mathbb{P}^n$ equipped with a projective coordinate), then N is precisely the degree of $\mathcal{F}$.

The foliation $\mathcal{F}_{tang}$ is the foliation on M defined by the 1-forms $\omega_0, \omega_1, \ldots, \omega_{N+2}$, as the restriction of $\mathcal{G}$ to $\mathbb{P}^1 \times L$, where L is a leaf of $\mathcal{F}_{tang}$, is by construction determined by the 1-form dz.

The Frobenius integrability condition $\Omega \wedge d\Omega = 0$ for $\mathcal{G}$ can be reduced to

$$d\Omega = \Omega \wedge L_v\Omega \quad \text{with} \quad v = \frac{1}{p(z)}\frac{\partial}{\partial z}$$

where $L_v\Omega := d(i_v\Omega) + i_v(d\Omega)$ is the Lie derivative of Ω with respect to the vector field v. Indeed, $i_v\Omega = \Omega(v) = 1$ and we get

$$0 = i_v(\Omega \wedge d\Omega) = (i_v\Omega) \cdot d\Omega - \Omega \wedge (i_v\Omega) = d\Omega - \Omega \wedge L_v\Omega;$$

the converse is obvious. If we denote by d_M the exterior differential of M and by $'$ the derivative with respect to z, then integrability condition can be succinctly written as

$$p(z)d_M\omega(z) = \omega(z) \wedge \omega'(z) + d_M p(z) \wedge \omega(z). \tag{12}$$

Let $K \subset \mathbb{C}(M)$ be the field of first integrals for $\mathcal{F}_{tang}$.

Lemma 8.5. *The coefficients of the polynomial $p(z) = z^N + \sum_{i=0}^{N-1} a_i z^i$ belong to the field K.*

Proof. It suffices to prove that $L_v a_i = 0$ for $i = 0, \ldots, N-1$ for any rational vector field v on M tangent to $\mathcal{F}_{tang}$. Such a vector field, characterized by $\omega_i(v) = 0$ for $i = 0, \ldots, N+2$, can be lifted as an horizontal vector field $\tilde{v}$ on $\mathbb{P}^1 \times M$ tangent to $\mathcal{G}$. Of course we have $\Omega(\tilde{v}) = 0$. The integrability condition implies

$$i_{\tilde{v}}(\Omega \wedge d\Omega) = L_{\tilde{v}}\Omega \wedge \Omega = 0.$$

Thus $L_{\tilde{v}}\Omega$ is zero, or it defines the foliation $\mathcal{G}$. But

$$L_{\tilde{v}}\Omega = L_v p(z)dz + L_v\omega(z) = \left(\sum_{i=0}^{N} (L_v a_i) z^i\right) dz + \sum_{i=0}^{N+2} z^i L_v\omega_i$$

cannot be a non zero multiple of Ω since $L_v p$ has degree $< N$ (recall that p is monic). Therefore, we are in the former case and $L_v a_i = 0$ for $i = 0, \ldots, N-1$ as wanted. □

Lemma 8.6. *For $k = 0, \ldots, N+2$, we have*

$$d\omega_k = \sum_{\substack{0\le i<j\le k+1\\ i+j\le k+1}} \lambda_{ij}\omega_i \wedge \omega_j + \sum_{\substack{0\le i<j\le k\\ i+j\le k}} \mu_{ij} da_i \wedge \omega_j$$

for suitable $\lambda_{ij}, \mu_{ij} \in K$. Moreover, if $i+j = k+1$ then λ_{ij} is non-zero.

Proof. Equation (12) implies

$$(13)\qquad \begin{cases} a_0 d\omega_0 & = & \omega_0\wedge\omega_1 + da_0\wedge\omega_0\,,\\ a_0 d\omega_1 + a_1 d\omega_0 & = & 2\omega_0\wedge\omega_2 + da_0\wedge\omega_1 + da_1\wedge\omega_0\,,\\ a_0 d\omega_2 + a_1 d\omega_1 + a_2 d\omega_0 & = & 3\omega_0\wedge\omega_3 + \omega_1\wedge\omega_2 \\ & & \quad + da_0\wedge\omega_2 + da_1\wedge\omega_1 + da_2\wedge\omega_0\,,\\ \vdots & = & \vdots\\ \sum_{i=0}^k a_i d\omega_{k-i} & = & \sum_{i=0}^{k+1}(k-i+1)\omega_i\wedge\omega_{k-i+1}\\ & & \quad + \sum_{i=0}^k da_i\wedge\omega_{k-i} \end{cases}$$

The lemma follows inductively. □

Lemma 8.7. *Let n be the codimension of $\mathcal{F}_{tang}$ in M. Then $\Theta := \omega_0\wedge\ldots\wedge\omega_{n-1}$ is a non trivial closed n-form defining $\mathcal{F}_{tang}$. Moreover, for $k = n, \ldots, N+2$ we can write $\omega_k = \sum_{i=0}^{n-1} f_i\omega_i$ with $f_i \in K$.*

Proof. Let n be the largest integer such that $\Theta := \omega_0\wedge\ldots\wedge\omega_{n-1} \neq 0$. We have to prove that $\Theta\wedge\omega_i = 0$ for any i. From the natural action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^1\times M$, we have a 1-parameter transformation group Φ_t inducing $z\mapsto z+t$ on the $\mathbb{P}^1$-coordinate and say $\phi_t : M\to M$ on the basis. This action preserves $\mathcal{G}$ as well as Ω since the dz coefficient of $\Phi_t^*\Omega$ is still monic. Therefore, for any $t\in\mathbb{C}$, the identity below holds true:

$$\Phi_t^*\Omega = \left(\sum_{i=0}^N a_i\circ\phi_t(z+t)^i\right)dz + \sum_{i=0}^{N+2}(z+t)^i\phi_t^*\omega_i = \Omega\ .$$

From Taylor formula we have

$$\sum_{i=0}^{N+2}(z+t)^i\phi_t^*\omega_i = \phi_t^*\omega(z+t) = \phi_t^*\left(\sum_{i=0}^{N+2} z^i\omega^{(i)}(t)\right)$$

(where $\omega^{(i)}$ denotes the i^{th} derivative of $\omega(z)$ with respect to z) and we deduce that $\omega(t)\wedge\ldots\wedge\omega^{(n)}(t) = (\phi_t^{-1})^*(\omega_0\wedge\ldots\wedge\omega_n) = 0$ for all t. Expanding this equality in t-powers shows inductively that $\omega_0\wedge\ldots\wedge\omega_{n-1}\wedge\omega_k = 0$ (i.e. $\omega_k = \sum_{i=0}^{n-1} f_i\omega_i$) for all $k = n, \ldots, N+2$.

Now, we claim that $d\omega_i\wedge\hat\Theta_i = 0$ for any $i = 0,\ldots,n-1$ where

$$\hat\Theta_i := \omega_0\wedge\ldots\wedge\widehat{\omega_i}\wedge\ldots\wedge\omega_{n-1} = 0$$

(here, $\widehat{\omega_i}$ means that this term is omitted in the wedge product). In order to see this, go back to Lemma 8.6 and its proof, and notice that

$$a_0 d\omega_k = (k+1)\omega_0\wedge\omega_{k+1} + da_0\wedge\omega_k + \cdots$$

where $(\cdots)$ is a sum of wedge products of 1-forms involving at least one ω_l, with $l<k$ (here we assume $n>1$). Therefore

$$d\omega_i\wedge\hat\Theta_i = da_0\wedge\Theta$$

which is zero by Lemma 8.5. On the other hand, if $n = 1$, then $\omega_0 \wedge \omega_1 = 0$ and $a_0 d\omega_0 = da_0 \wedge \omega_0 = 0$ (same reason). We promptly deduce that $d\Theta = \sum_{i=0}^{n-1} d\omega_i \wedge \hat{\Theta}_i = 0$.

Now, let ω_k be any other coefficient of Ω; one can write $\omega_k = \sum_{i=0}^{n-1} f_i \omega_i$ for some (unique) $f_i \in \mathbb{C}(M)$. Then

$$d\omega_k \wedge \omega_0 \wedge \ldots \wedge \widehat{\omega_k} \wedge \ldots \wedge \omega_{n-1} = \pm df_k \wedge \Theta$$

must be zero by the very same argument, and f_k actually belongs to K. □

The field K may now be defined as follows:

$$K = \{f \in \mathbb{C}(M) \ ; \ df \wedge \Theta = 0\}.$$

It is an integrally closed field and, according to Siegel's Theorem, there exists a dominant rational map $\Phi : M \dashrightarrow N$ onto a projective variety N such that $K = \Phi^*\mathbb{C}(N)$. The dimension of N coincides with the transcendence degree $p := [K : \mathbb{C}]$. The generic fibers of Φ are the (algebraic closures of) the leaves of $\overline{\mathcal{F}}_{tang}$. One can choose $f_1, \ldots, f_p \in K$ such that the closed p-form $df_1 \wedge \cdots \wedge df_p \neq 0$ defines $\overline{\mathcal{F}}_{tang}$, or equivalently K is the integral closure of $\mathbb{C}(f_1, \ldots, f_p)$. Let now $q = \mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}]$, $n = p + q$. By very similar arguments, one can prove the alternate K-relative version of the previous lemma.

Lemma 8.8. *The foliation $\mathcal{F}_{tang}$ is defined by the non trivial closed n-form*

$$\Theta := \omega_0 \wedge \ldots \wedge \omega_{q-1} \wedge df_1 \wedge \cdots \wedge df_p, \quad q + p = n$$

while the algebraic closure $\overline{\mathcal{F}}_{tang}$ is defined by $df_1 \wedge \cdots \wedge df_p$. Moreover, every other ω_k in (11) can be written as $\omega_k = \sum_{i=0}^{q-1} b_i \omega_i + \sum_{i=q}^{n-1} b_i df_{i-q+1}$ with $b_i \in K$.

Lemma 8.9. *The codimension $q = \mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}]$ is at most 3.*

Proof. Following Lemma 8.8, we can write

$$\omega_q = \sum_{i=0}^{q-1} b_i \omega_i + \sum_{i=q}^{n-1} b_i df_{i-q+1}.$$

According to equation (13), we have

$$a_0 d\omega_{q-1} + \cdots + a_{q-1} d\omega_0 \ = \ q\omega_0 \wedge \omega_q + (q-2)\omega_1 \wedge \omega_{q-1} + (q-4)\omega_2 \wedge \omega_{q-2} + \ldots$$

After plugging the expression of ω_q into this equation and differentiating, we get an equality between 3-forms. They both decompose uniquely in terms of $\omega_i \wedge \omega_j \wedge \omega_k$, $\omega_i \wedge \omega_j \wedge df_k$ and $\omega_i \wedge df_j \wedge df_k$, where the subscripts of the 1-forms ω_i range over $0, \ldots, q-1$ and the subscripts of the functions f_i range over $1, \ldots, p$. The term $\omega_0 \wedge \omega_2 \wedge \omega_{q-1}$ does not occur on the left hand side, and Lemma 8.6 implies that only the terms

$$0 = \cdots + (q-2) d\omega_1 \wedge \omega_{q-1} + \cdots - (q-4)\omega_2 \wedge d\omega_{q-2} + \cdots$$

contribute on the right hand side. Thus we arrive at the identity

$$0 = q(q-3)\omega_0 \wedge \omega_2 \wedge \omega_{q-1}$$

which contradicts the integrability conditions if $q > 3$. □

8.4. **Proof of Theorem 8.4.** Suppose first that $\operatorname{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 3$. Let us denote by V the K-vector space $K\langle\omega_0, \omega_1, \omega_2, df_1, \ldots, df_n\rangle$, W the K-subspace generated by the df_i's and $\underline{V}$ the quotient. By assumption, $\underline{V}$ has dimension 3. We note that both V and W are "closed under exterior differential" in the sense that $dV = V \wedge V$ and $dW = W \wedge W$; it follows that the exterior derivative is well-defined on the quotient $\underline{V}$, which is itself closed as well. It is therefore classical that there exists a basis η_0, η_1, η_2 for $\underline{V}$ satisfying the structure equations of $\mathfrak{sl}(2, \mathbb{C})$ in $\underline{V}$:

$$\begin{cases} d\eta_0 &= \eta_0 \wedge \eta_1 \,, \\ d\eta_1 &= 2\eta_0 \wedge \eta_2 \,, \\ d\eta_2 &= \eta_1 \wedge \eta_2 \,. \end{cases} \quad \text{modulo} \quad W \,.$$

We claim that we can choose $\eta_i \in V$ satisfying the same equations in V instead of $\underline{V}$. To prove this claim, let us start by expanding $\frac{\Omega}{p(z)}$ in a power series

$$dz + \frac{\omega_0 + z\omega_1 + z^2\omega_2 + \cdots + z^{n+2}\omega_{n+2}}{a_0 + a_1 z + a_2 z^2 + \cdots + z^n} = dz + \eta_0 + z\eta_1 + z^2\eta_2 + z^3\eta_3 + \cdots$$

Then the classes of η_0, η_1, η_2 in $\underline{V}$ clearly defines a new basis for it. The integrability conditions in V then become

$$\begin{aligned} d\eta_0 &= \eta_0 \wedge \eta_1 \,, \\ d\eta_1 &= 2\eta_0 \wedge \eta_2 \,, \\ d\eta_2 &= 3\eta_0 \wedge \eta_3 + \eta_1 \wedge \eta_2 \,, \\ d\eta_3 &= 4\eta_0 \wedge \eta_4 + 2\eta_1 \wedge \eta_3 \,, \\ &\vdots \end{aligned} \tag{14}$$

If $\eta_0 \wedge \eta_3 = 0$ then we are done. Otherwise Lemma 8.8 allows us to write

$$\eta_3 = a\eta_0 + b\eta_1 + c\eta_2 + \sum_{i=1}^{p} g_i df_i$$

with $a, b, c, g_i \in K$. If we take the exterior derivative of the third line of (14) we get

$$0 = d(d\eta_2) = 6c \cdot \eta_0 \wedge \eta_1 \wedge \eta_2 \mod W \wedge V \wedge V$$

which implies $c = 0$. If $f \in K$ and we replace z by $z + fz^3$ in the differential form $dz + \eta_0 + z\eta_1 + z^2\eta_2 + z^3\eta_3 + \cdots$ then we modify the sequence $\eta_2, \eta_3, \eta_4, \ldots$ without modifying the integrability equations (14). To wit, η_2 is changed to $\eta_2 - 3f \cdot \eta_0$ and η_3 to $\eta_3 - 3f\eta_1 + df$. We use this operation to obtain a new η_3 which takes the form $\eta_3 = a\eta_0 + \sum_{i=1}^{p} g_i df_i \in W$. Finally, if we take the wedge product of η_0 with the exterior derivative the fourth line of (14) then we get

$$\eta_0 \wedge d\eta_3 = \eta_0 \wedge \eta_1 \wedge \left(\sum_{i=1}^{p} g_i df_i \right) .$$

If we see V as the direct sum $\underline{V} \oplus W$ then the lefthand side of the equality above lies in $\underline{V} \wedge \bigwedge^2 W$, while the righthand side lies in $\bigwedge^2 \underline{V} \wedge W$. It follows that $\eta_3 = a\eta_0$, and the claim is proved.

Now, we can write

$$\tilde{\Omega} = \frac{\Omega}{p(z)} = dz + a(z)\eta_0 + b(z)\eta_1 + c(z)\eta_2 + \sum_{i=1}^{p} g_i(z) df_i$$

where $a, b, c, g_i \in K(z)$. If we expand $\tilde{\Omega} \wedge d\tilde{\Omega}$ then we get

$$(a + a'b - ab')\eta_0 \wedge \eta_1 \wedge dz + (2b + b'c - bc')\eta_0 \wedge \eta_2 \wedge dz + (c + b'c - bc')\eta_1 \wedge \eta_2 \wedge dz + \beta$$

where β is a 3-form with monomials involving at most one of the 1-forms η_0, η_1, η_2. Therefore

$$0 = (a + a'b - ab') = (2b + b'c - bc') = (c + b'c - bc')\,.$$

The first equality can be rewritten as $\frac{b'}{b} - \frac{a'}{a} = \frac{1}{b}$. After setting $\alpha(z) = \frac{b}{a}$, we get $\frac{\alpha}{\alpha'} = \frac{1}{b}$ and thus

$$a = \frac{1}{\alpha'} \quad \text{and} \quad b = \frac{\alpha}{\alpha'}.$$

A combination of the 3 equations yields $b^2 = ac$ and we thus get $c = \frac{\alpha^2}{\alpha'}$. Now, we can rewrite

$$\alpha'\tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1 + \alpha^2\eta_2 + \sum_{i=1}^{p} h_i(z)df_i$$

for suitable $h_i \in K(z)$. Examining again the integrability condition, we obtain

$$0 = \eta_0 \wedge \eta_1 \wedge \left(\sum_{i=1}^{p} h_i(z)df_i\right) + \beta$$

where β is, as before, a 3-form with monomials involving at most one of the 1-forms η_0, η_1, η_2. Thus the functions h_i vanish identically and

$$\alpha'\tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1 + \alpha^2\eta_2.$$

This is sufficient to conclude that $\mathcal{G}(\overline{L})$ is defined by the pull-back of $dz + \eta_0 + z\eta_1 + z^2\eta_2$ under the rational map $(z, x) \mapsto (\alpha(z), x)$.

Let us now turn to the case $\mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 2$. As the arguments are very similar to the previous case, we will just sketch them. In this case there are η_0, η_1 satisfying

$$\begin{aligned} d\eta_0 &= \eta_0 \wedge \eta_1\,, \\ d\eta_1 &= 2\eta_0 \wedge \eta_2\,, \end{aligned}$$

with $\eta_2 = a\eta_1 + \sum_{i=1}^{p} g_i df_i$ with $a, g_i \in K$. After killing the coefficient a by adding to η_1 a convenient K-multiple of η_0, the closedness of $d\eta_1$ gives $g_i = 0$, and thus $d\eta_1 = 0$. If we write $\tilde{\Omega} = p(z)^{-1}\Omega = dz + a(z)\eta_0 + b(z)\eta_1 + \sum_{i=1}^{p} h_i(z)df_i$ then the integrability of $\tilde{\Omega}$ implies again $\frac{b'}{b} - \frac{a'}{a} = \frac{1}{b}$ so that we can set $a = \frac{1}{\alpha'}$ and $b = \frac{\alpha}{\alpha'}$ as before. Rewriting the integrability condition for $\alpha'\tilde{\Omega}$ gives $h_i = 0$. Therefore

$$\alpha'\tilde{\Omega} = d\alpha + \eta_0 + \alpha\eta_1$$

and $\mathcal{G}(\overline{L})$ is defined by the pull-back of $dz + \eta_0 + z\eta_1$ under the rational map $(z, x) \mapsto (\alpha(z), x)$.

When $\mathrm{codim}[\overline{\mathcal{F}}_{tang} : \mathcal{F}_{tang}] = 1$, integrability conditions write

$$d\eta_0 = \eta_0 \wedge \eta_1$$

with $\eta_1 = \sum_{i=1}^{p} g_i df_i$. Closedness of $d\eta_0$ gives $d\eta_1 = 0$. We can therefore proceed analogously to the previous case, proving that $\mathcal{G}(\overline{L})$ is defined by the pull-back of $dz + \eta_0 + z\eta_1$ under the rational map $(z, x) \mapsto (\alpha(z), x)$.

Finally, when $\overline{\mathcal{F}}_{tang} = \mathcal{F}_{tang}$, it suffices to notice that the leaves of $\mathcal{F}_{tang}$ lift to leaves of $\mathcal{G}$, and define a subfoliation of $\mathcal{G}$ by algebraic leaves of the same dimension as $\mathcal{F}_{tang}$. □

Remark 8.10. A weaker version of Theorem 8.4 can be deduced in an easier way as a corollary of [17, Theorem 1.1]. Indeed, write

$$\Omega_0 := \frac{\Omega}{p(z)} = dz + g_1(z)\omega_1 + \cdots + g_n(z)\omega_n$$

where $\Theta = \omega_1 \wedge \cdots \wedge \omega_n \neq 0$. We can assume that $W := \text{Wronskian}(g_1, \ldots, g_n) \neq 0$, otherwise we could write Ω_0 with less summands. Then a Godbillon-Vey sequence for $\mathcal{G}$ (see [17]) is given by

$$\begin{aligned}
\Omega_1 &= g_1'\omega_1 + \cdots + g_n'\omega_n \ , \\
\Omega_2 &= g_1''\omega_1 + \cdots + g_n''\omega_n \ , \\
\Omega_3 &= g_1'''\omega_1 + \cdots + g_n'''\omega_n \ , \\
&\vdots
\end{aligned}$$

Since $\Omega_0 \wedge \Omega_1 \wedge \cdots \wedge \Omega_n = W \cdot dz \wedge \Theta$ and $\Omega_0 \wedge \Omega_1 \wedge \cdots \wedge \Omega_n \wedge \Omega_{n+1} \neq 0$ we can apply [17, Theorem 1.1] to deduce that $\mathcal{G}$ is either tranversely projective, or pull-back as in the statement of Theorem 8.4. Actually, our proof above is somehow dual to that one of [17]: there we dealt with vector fields instead of differential forms. Notice that here we obtain a stronger result, since we obtain that $\mathcal{G}$ is a pull-back of a Riccati equation in the transversely projective case. In particular, this excludes the possibility of $\mathcal{F}$ being a transversely hyperbolic foliation like the *Hilbert modular foliations*, see [17, Section 5.5].

8.5. **Variation of projective structure.** Let $\mathcal{F}$ be a transversely projective on a projective manifold X. Recall that attached to $\mathcal{F}$ we have the datum $(P, \mathcal{H}, \sigma)$ where $p : P \to X$ is $\mathbb{P}^1$-bundle over X, $\mathcal{H}$ is a Riccati foliation on P, and $\sigma : X \dashrightarrow P$ is a rational section. We will denote by Δ the polar divisor of $\mathcal{H}$.

If $f : \mathbb{P}^1 \to X$ is a free morphism which is generically transverse to X then $\mathcal{F}$, or rather its first integrals, define a (singular) projective structure on X. If we consider the irreducible component of $\text{Mor}(\mathbb{P}^1, X)$ containing $[f]$, we get a map from an open subset of M to the space of rational 1-forms with poles of order at most $\deg(f^*\Delta)$. Equations (7) and (8) from Section 7 shows that this map is algebraic, and as such defines a foliation on M with algebraic leaves which we will denote by $\mathcal{P}$. The foliation $\mathcal{F}_{tang}$ is clearly tangent to the foliation $\mathcal{P}$.

Theorem 8.11. *If $\dim \mathcal{P} > \dim \mathcal{F}_{tang}$ then for a generic $[f] \in M$ the Riccati foliation $f^*\mathcal{H}$ defined on f^*P is defined by a closed rational 1-form. Moreover, if f is an embedding then there exists a neighborhood $U \subset X$ of $f(\mathbb{P}^1)$ in the metric topology where $\mathcal{F}$ is defined by a closed meromorphic 1-form.*

Proof. Fix a germ of curve $\gamma : (\mathbb{C}, 0) \to M$ contained in a leaf of $\mathcal{P}$ and transverse to the leaf of $\mathcal{F}_{tang}$ through $\gamma(0) = [f]$. Let $F_\gamma = F \circ \gamma : \mathbb{P}^1 \times (\mathbb{C}, 0) \to X$ the composition of the evaluation morphism with γ.

Pulling back $\mathcal{H}$ using F_γ we obtain a Riccati foliation $\widetilde{\mathcal{H}}$ in $\mathbb{P}^1 \times (\mathbb{C}, 0) \times \mathbb{P}^1$ which is defined by

$$dz + \omega_0 z^2 + \omega_1 z + \omega_2$$

where $\omega_0, \omega_1, \omega_2$ are 1-forms in the variables $(x, t) \in \mathbb{P}^1 \times (\mathbb{C}, 0)$ of the basis; and the section σ defined by $z = \infty$.

We can apply a bimeromorphic transformation of the form $z \mapsto \alpha(x,t)z+\beta(x,t)$ in order to write

$$\Omega = dz + (z^2 + \phi(x,t))dx + (a(x,t)z^2 + b(x,t)z + c(x,t))dt\,,$$

as a 1-form defining $\widetilde{\mathcal{H}}$. Since we are in a leaf of $\mathcal{P}$ we indeed get that $\partial_t\phi(x,t) = 0$, i.e. $\phi(x,t) = \phi(x)$ does not depend on t. Therefore

$$\Omega = dz + (dx + a(x,t)dt)z^2 + (b(x,t)dt)z + (\phi(x)dx + c(x,t)dt).$$

The restriction of $\widetilde{\mathcal{H}}$ to $\{z = \infty\}$ is defined by the 1-form $dx + a(x,t)dt$. It also coincides with the foliation $\mathcal{G}(T)$ on $\mathbb{P}^1 \times T$, where $\mathcal{G} = F^*\mathcal{F}$ is, as before, the pull-back of $\mathcal{F}$ by the evaluation morphism and $\mathcal{G}(T)$ is the restriction of $\mathcal{G}$ to $\mathbb{P}^1 \times T$. Since $\gamma : (\mathbb{C}, 0) \to M$ is not contained in a leaf of $\mathcal{F}_{tang}$, it follows that $\mathcal{G}(T)$ does not coincide with the foliation defined by dx. Thus the function $a(x,t)$ is not identically zero. Since $[f] \in M$ is generic, we can assume that $a(x,0)$ is not identically zero.

Notice that $\Omega = i_v i_w dt \wedge dx \wedge dz$ where

$$v = -\partial_t + (a(x,t)z^2 + b(x,t)z + c(x,t))\partial_z \quad \text{and} \quad w = -\partial_x + (z^2 + \phi(x))\partial_z\,.$$

The involutiveness of the $T\widetilde{\mathcal{H}}$ is equivalent to $[v,w] = 0$. As w does not depend on t, it commutes with ∂_t. Therefore it also commutes with $(a(x,0)z^2 + b(x,0)z + c(x,0))\partial_z$, and this vector field is a infinitesimal automorphism of the Riccati foliation $f^*\mathcal{H}$. Consequently the 1-form

$$\widetilde{\Omega} = \frac{dz + (z^2 + \phi(x))dx}{a(x,0)z^2 + b(x,0)z + c(x,0)}$$

is a closed rational 1-form defining $f^*\mathcal{H}$.

If f is an embedding then $C = f(\mathbb{P}^1) \subset X$ is smooth curve. Since f is free we can assume that C intersects the polar divisor of $\mathcal{H}$ generically and that it is disjoint from the indeterminacy locus of σ, see [43, Proposition II.3.7]. Thus [48, Lemma 4.1] implies that for every point $p \in C$ the germ of $\mathcal{H}$ at $\pi^{-1}(p)$ admits a local product structure. That is, there is a local system of coordinates $(x_1, \ldots, x_n, y)$ on $(X,p) \times \mathbb{P}^1$ where the natural projection to X is the morphism forgetting y, C is defined by $x_2 = \cdots = x_n = 0$ and the foliation $\mathcal{H}$ is defined by $\Omega_p = a(x_1,y)dx_1 + b(x_1,y)dy$. The restriction of Ω_p to $\pi^{-1}(C)$ is a multiple of $\widetilde{\Omega}$ and therefore we can write $\widetilde{\Omega} = h(x_1,y)\Omega_p$. It follows that $h(x_1,y)\Omega_p$ is an extension of $\widetilde{\Omega}$ at the neighborhood of p and it still defines $\mathcal{H}$. Notice that such extension is unique. Therefore, the 1-form $\widetilde{\Omega}$ extends as a closed 1-form defining $\mathcal{H}$ at the neighborhood of $\pi^{-1}(C)$. It suffices to pull-back $\widetilde{\Omega}$ using σ to obtain a closed meromorphic 1-form defining $\mathcal{F}$ at a neighborhood of C. □

8.6. **Graphic neighborhood.** The deformations of a morphism $f : \mathbb{P}^1 \to X$ tangent to a foliation $\mathcal{F}$ can be interpreted as deformations of another morphism $\Gamma_f : \mathbb{P}^1 \to \mathbb{P}^1 \times X$ – the graph of f – which are contained in an analytic subvariety Y of $\mathbb{P}^1 \times X$. To keep things simple suppose that the image of f is disjoint from the singular set of $\mathcal{F}$. On each fiber of $\mathbb{P}^1 \times X \to \mathbb{P}^1$ put an unaltered copy of the foliation $\mathcal{F}$ to form a foliation $\hat{\mathcal{F}}$. Let Δ be the graph of f in $\mathbb{P}^1 \times X$, and U an arbitrarily small tubular neighborhood of Δ in the metric topology. If we saturate Δ by the leaves of $\hat{\mathcal{F}}_{|U}$, we obtain a smooth analytic subvariety of U of dimension

$\dim \mathcal{F} + 1$ which we will call Y. The nice thing about Y is that the normal bundle of Δ in Y coincides with the tangent sheaf of $\mathcal{F}$. More precisely,

$$\Gamma_f^* N_Y \Delta = f^* T\mathcal{F}.$$

Bogomolov-McQuillan explored this fact to establish the algebraicity of the leaves in the proof of Theorem 2.4.

Let $\pi_1 : Y \to \mathbb{P}^1$ and $\pi_2 : Y \to X$ be the natural projections. By construction, $\pi_2^* \mathcal{F}$ coincides with the foliation on Y defined by the fibers of π_1.

Notice that a sufficiently small deformations of Δ in Y are graphs of morphisms of $\mathbb{P}^1$ which are tangent to $\mathcal{F}$, and reciprocally a deformation of $f : \mathbb{P}^1 \to X$ along $\mathcal{F}$ gives rise to a family of curves on Y.

Proposition 8.12. *Let $[g] \in M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be a generic element, and k be the number of summands of $g^*T\mathcal{F}$ having strictly positive degree. If $x \in X$ is a generic point then there exists a quasi-projective variety V_x of dimension at least k passing through x and contained in the leaf of $\mathcal{F}$ through x.*

Proof. Let $Y \subset \mathbb{P}^1 \times X$ be a graphic neighborhood of g, and Δ be the graph of g in Y. If L is the leaf of $\mathcal{F}_{tang}$ through $[g]$ and $\varphi : \mathbb{P}^1 \times L \to Y \subset \mathbb{P}^1 \times X$ is the morphism defined by $\varphi(z, [g]) = (z, g(z))$ then the differential

$$d\varphi : T\mathbb{P}^1 \times TL \longrightarrow \varphi^*(T\mathbb{P}^1 \times TX)$$

of φ at $(z, [g])$ is given by

$$d\varphi(z, [g]) = dg(z) + \phi(z, [g])$$

where $\phi(z, g) : H^0(\mathbb{P}^1, g^*T\mathcal{F}) \to g^*TX \otimes \mathcal{O}_X / \mathfrak{m}_z \mathcal{O}_X$ is the evaluation morphism, see [43, page 114], and dg is the differential of g. If $z \in \mathbb{P}^1$ and $[g]$ are generic enough then on the one hand $d\varphi(z, [g])$ has rank equal to the dimension of Y. On the other hand the kernel of $d\varphi(z, [g])$ has dimension at least equal to k, the number of positive summands of $g^*T\mathcal{F}$. Therefore $F = \varphi^{-1}(\varphi(z, [g])$ has dimension at least k. The morphisms parametrized by the projection of $F \subset \mathbb{P}^1 \times L$ to L have image contained in Y and containing $(z, g(z))$. Thus we have a k-dimensional analytic family of curves T contained in the $\mathrm{Hilb}(\mathbb{P}^1 \times X)$, all of them contained in Y and containing $(z, g(z))$. Every subscheme $S \in \mathrm{Hilb}(\mathbb{P}^1 \times X)$ in the Zariski closure $\overline{T}$ of T in $\mathrm{Hilb}(\mathbb{P}^1 \times X)$ will certainly contain $p = (z, g(z))$ and will have defining ideal $\mathcal{I}_S$ such that $\mathcal{I}_{Y,p} \subset I_{S,p}$, as both conditions are closed.

If $p : \mathcal{U}_{\overline{T}} \to \overline{T}$ is the universal family over $\overline{T}$ and $\varrho : \mathcal{U}_{\overline{T}} \to \mathbb{P}^1 \times X$ is the evaluation morphism then for a generic $y \in \mathbb{P}^1$ the analytic variety $\pi_2(\varrho(p^{-1}(T)) \cap \{y\} \times X)$ will be contained in the leaf of $\mathcal{F}$ through $g(y)$. Recall that $\pi_2 : Y \to X$ is nothing but the projection of $Y \subset \mathbb{P}^1 \times X$ to the second factor. Therefore

$$V_{g(y)} = \pi_2(\varrho(p^{-1}(\overline{T})) \cap \{y\} \times X) \cap (X - \mathrm{sing}(\mathcal{F}))$$

makes sense, and is a quasi-projective variety contained in the leaf of $\mathcal{F}$ through $g(y)$. The proposition follows. □

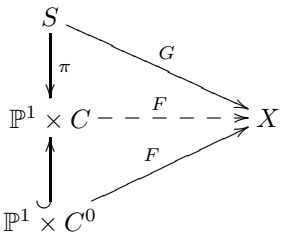
We do not know how to control the geometry of the quasi-projective varieties V_x constructed by the previous proposition. It is conceivable that a variation of Mori's bend-and-break argument would allow us to construct a k-dimensional rationally connected subvariety tangent to $\mathcal{F}$ through a generic point $x \in X$. So far we can only prove the existence of one rational curve tangent to $\mathcal{F}$ through a generic point of X.

Proposition 8.13. *Let $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be a irreducible component containing free morphisms and $[g] \in M$ be a generic element. Suppose $g^*T\mathcal{F}$ has at least one summand having strictly positive degree. If $x \in X$ is a generic point then there exists rational curve through x and contained in the leaf of $\mathcal{F}$ through x.*

Proof. The proof is similar to the one of [52, Lemma 5.2, Lecture I]. Let $x \in X$ be a generic point. The existence of a positive summand in the decomposition of $g^*T\mathcal{F}$ implies that we can algebraically deform g along $\mathcal{F}$ in such a way that the point $p_0 = g^{-1}(g(x)) \in \mathbb{P}^1$ is mapped by the deformations to x. More precisely, there exists a smooth quasiprojective curve $C^0 \subset M$ contained in a leaf of $\mathcal{F}_{tang}$ and such that every $[g] \in C^0$ maps p_0 to x. We can also assume that every $[g] \in C^0$ is generically transverse to $\mathcal{F}$ otherwise we would have already a rational curve through x and contained in a leaf of $\mathcal{F}$.

Let C be a smooth projective curve containing C^0 as an open subset. The evaluation morphism $F : \mathbb{P}^1 \times C^0 \to X$ extends to a rational map $F : \mathbb{P}^1 \times C \dashrightarrow X$. Generically F must have rank two, as otherwise the deformation would have to move points along the image $g(\mathbb{P}^1)$ of one of its member and this is only possible if $g(\mathbb{P}^1)$ is tangent to $\mathcal{F}$. Notice also that $F^*\mathcal{F}$ is nothing but the foliation on $\mathbb{P}^1 \times C$ defined by the projection $\mathbb{P}^1 \times C \to \mathbb{P}^1$.

Since the curve $C_0 = \{p_0\} \times C$ has self-intersection zero in $\mathbb{P}^1 \times C$ and F has image of dimension two, there must exist an indeterminacy point of F on C_0. By resolving the indeterminacies of F we obtain a surface S together with a morphism $G : S \to X$ fitting into the diagram



where $\pi : S \to \mathbb{P}^1 \times C$ is a birational morphism. Moreover, there is a curve $E \subset S$ contracted by π into a point of C_0 whose image under G is a rational curve on X passing through x. Since the foliation $F^*\mathcal{F}$ is a smooth foliation on $\mathbb{P}^1 \times C$, every exceptional divisor of π is also invariant by $(F \circ \pi)^*\mathcal{F} = G^*\mathcal{F}$. Therefore $G(E)$ is the sought rational curve tangent to $\mathcal{F}$ passing through x. □

8.7. **Proof of Theorem 7.** If the generic leaf of $\mathcal{F}_{tang}$ is not algebraic then Theorem 8.4 implies $\mathcal{F}$ is transversely projective. If instead every leaf of $\mathcal{F}_{tang}$ is algebraic then we can conclude applying the next proposition.

Proposition 8.14. *Let X be a uniruled projective manifold, $M \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be an irreducible component containing free morphisms, and $\mathcal{F}$ be a codimension one foliation on X. If all the leaves of the foliation $\mathcal{F}_{tang}$ defined M are algebraic then the foliation $\mathcal{F}$ is the pull-back by a rational map of a foliation $\mathcal{G}$ on a projective manifold of dimension smaller than or equal to $n - \delta_0 + \delta_{-1}$, where $\delta_0 = h^0(\mathbb{P}^1, f^*T\mathcal{F})$, and $\delta_{-1} = h^0(\mathbb{P}^1, f^*T\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$. Moreover, if $\delta_{-1} > 0$ then $\mathcal{F}$ is uniruled.*

Proof. Let $F : \mathbb{P}^1 \times M \to X$ be the evaluation morphism, and $\mathcal{H}_{tang}$ be the foliation defined by intersection of $F^*\mathcal{F}$ and the pull-back of $\mathcal{F}_{tang}$ under the projection

$\mathbb{P}^1 \times M \to M$. Notice that $\mathcal{H}_{tang}$ has the same dimension as $\mathcal{F}_{tang}$ and, if the leaves of $\mathcal{F}_{tang}$ are all algebraic then the same holds true for the leaves of $\mathcal{H}_{tang}$. Thus under our hypothesis $\mathcal{H}_{tang}$ provides a family of algebraic subvarieties tangent to $F^*\mathcal{F}$.

The dimension of $\mathcal{H}_{tang}$ is exactly δ_0, and if we restrict the evaluation map to a generic leaf of $\mathcal{H}_{tang}$ it follows that its rank is exactly $\delta_0 - \delta_{-1}$. Therefore through a generic point of X there exists an algebraic subvariety of dimension $\delta_0 - \delta_{-1}$ tangent to $\mathcal{F}$. Lemma 2.7 implies that $\mathcal{F}$ is a pull-back from a variety having dimension at most $n - \delta_0 + \delta_{-1}$. To produce rational curves tangent to $\mathcal{F}$ when $\delta_{-1} > 0$ it suffices to apply Proposition 8.13. □

Suppose now that X is rationally connected and $f : \mathbb{P}^1 \to X$ is an embedding with ample normal bundle. Theorem 8.11 implies that $\mathcal{F}$ is defined by a closed meromorphic 1-form ω at a neighborhood of $f(\mathbb{P}^1)$. We can apply [35, Theorem 6.7] to extend ω to an algebraic (perhaps multi-valued) 1-form on all of X. It follows that $\mathcal{F}$ is transversely affine. Indeed the algebraicity of ω implies the existence of a projective manifold Y together with a generically finite morphism $p : Y \to X$ such that p^*F is defined by a closed rational 1-form. □

8.8. **Proof of Corollary 9.** If the generic leaf of $\mathcal{F}_{tang}$ is not algebraic then we can use the same argument as in the proof of Corollary 8, replacing the use of [44] by the standard Lefschetz's Theorem, to conclude that $\mathcal{F}$ is defined by a closed rational 1-form.

If all the leaves of $\mathcal{F}_{tang}$ are algebraic then we want to control $\delta_0 - \delta_{-1}$ for a generic linear immersion $f : \mathbb{P}^1 \to \mathbb{P}^n$, as $\mathcal{F}$ will be a pull-back from a projective manifold of dimension at most $n - \delta_0 + \delta_{-1}$ according to Theorem 7. Write $f^*T\mathcal{F}$ as

$$\mathcal{O}_{\mathbb{P}^1}(1)^r \oplus \mathcal{O}_{\mathbb{P}^1}^s \oplus \bigoplus_{i=1}^{(n-1)-r-s} \mathcal{O}_{\mathbb{P}^1}(b_i)$$

where $b_i < 0$. Therefore $\delta_0 - \delta_{-1} = r + s$ and $\deg(f^*T\mathcal{F}) = (n-1) - d = r + \sum b_i \leq r - ((n-1) - (r+s)) = 2r + s - (n-1)$. Thus

$$\delta_0 - \delta_{-1} = r + s \geq \frac{1}{2}(2r + s) \geq n - 1 - \frac{d}{2},$$

which implies that $n - \delta_0 + \delta_{-1} \leq d/2 + 1$ as wanted. □

9. Foliations with numerically trivial canonical bundle

In this section we will conclude the proof of

Theorem 9.1 (Theorem 1 of the Introduction)**.** *Let $\mathcal{F}$ be a codimension one foliation with numerically trivial canonical bundle on a projective manifold X. Then at least one of following assertions hold true.*

(a) *The foliation $\mathcal{F}$ is defined by a closed meromorphic 1-form with coefficients in a torsion line bundle and without divisorial components in its zero set.*
(b) *All the leaves of $\mathcal{F}$ are algebraic.*
(c) *The foliation $\mathcal{F}$ is uniruled.*

Moreover, if $\mathcal{F}$ is not uniruled then $K\mathcal{F}$ is a torsion line bundle.

Let us briefly recall what we already know. On the one hand, if $\mathcal{F}$ does not satisfy none of the conclusions of Theorem 9.1 then Corollary 3.9 and Corollary 6.9 imply that the singular set of $\mathcal{F}$ has at least one component of codimension two, and that the generic point of every component of $\mathrm{sing}(\mathcal{F})$ the foliation admits a local holomorphic first integral. On the other hand, Theorem 7 implies that $\mathcal{F}$ is transversely projective. We will use this information to proof the existence of a divisor satisfying the hypothesis of the Lemma below.

9.1. **Logarithmic division implies smoothness or uniruledness.**

Lemma 9.2. *Let $\mathcal{F}$ be a codimension one foliation on a projective manifold X defined by a twisted 1-form $\omega \in H^0(X, \Omega^1_X \otimes N\mathcal{F})$. Suppose there exists a closed analytic subset $R \subset X$ of codimension at least 3, a $\mathbb{R}$-divisor $D = \sum \lambda_i H_i$ with $\lambda_i > -1$ and H_i irreducible $\mathcal{F}$-invariant hypersurface such that for every $x \in X - R$ we can write locally*

$$\omega \wedge \left(\sum \lambda_i \frac{dh_i}{h_i} \right) = d\omega \tag{15}$$

where $h_1, \ldots, h_k$ are local equations for $H_1, \ldots, H_k$. Then $H^1(X, N^*\mathcal{F}) \neq 0$

Proof. One can assume for simplicity that $R = \emptyset$. Indeed, if (15) holds on $X \setminus R$, it holds also on the whole X by extension properties of $H^1(X \setminus R, N^*\mathcal{F})$ through analytic susets of codimension ≥ 3. Let $\mathcal{U} = \{U_\alpha\}$ be a sufficiently fine open covering of X. If $\{g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})\}$ is a cocycle representing $N\mathcal{F}$ then, according to our hypothesis, the collection of holomorphic 1-forms $\{\theta_{\alpha\beta} \in \Omega^1_X(U_{\alpha\beta})\}$ defined by

$$\frac{dg_{\alpha\beta}}{g_{\alpha\beta}} - \left(\sum \lambda_i \frac{dh_{i,\beta}}{h_{i,\beta}} - \sum \lambda_i \frac{dh_{i,\alpha}}{h_{i,\alpha}} \right)$$

vanishes along the leaves of $\mathcal{F}$. Thus we have an induced class in $H^1(X, N^*\mathcal{F})$. If non-zero there is nothing else to prove. Otherwise, we deduce that $N^*\mathcal{F}$ is numerically equivalent to D.

By assumption, there exists near every point $x \in X$ a function φ expressed locally as $\sum \varphi_i$ with φ_i plurisubharmonic and $\frac{i}{\pi}\partial\overline{\partial}\varphi_i = \lambda_i[H_i]$ as currents. Moreover, adding if necessary a pluriharmonic function, we get the following equalities

$$\omega \wedge \partial\varphi = d\omega \quad \text{and} \quad \partial\varphi = \sum \lambda_i \frac{dh_i}{h_i}\,. \tag{16}$$

Since $N^*\mathcal{F}$ is numerically equivalent to D, we can interpret φ as a local weight of a (singular) metric on $N^*\mathcal{F}$. The expression

$$T = ie^{2\varphi}\omega \wedge \overline{\omega}$$

gives rise to a closed positive $(1,1)$-current on $U = X \setminus \mathrm{sing}(\mathcal{F})$. Indeed, $e^{2\varphi}$ is locally integrable on U since $\lambda_i > -1$, and T is closed thanks to (16). Beware that $e^{2\varphi}$ may fail to be integrable near some point of the singular locus. Nevertheless, T extends uniquely to a closed positive current on X denoted again T by a slight abuse of notations. Remark that T splits on U as $-\eta \wedge \omega$ where $\eta = ie^{2\varphi}\overline{\omega}$ is a well defined $(0,1)$ $\overline{\partial}$-closed current with values in $N\mathcal{F}^*$.

Now, pick a point $p \in \mathrm{sing}(\mathcal{F})$ and denote by B_p an open ball centered at p. On B_p, there exists a plurisubharmonic potential ψ for T:

$$i\partial\overline{\partial}\psi = T.$$

From $i\overline{\partial}(\partial\psi \wedge \omega) = 0 = -T \wedge \omega = 0$, we can deduce the existence of a distribution δ, defined at least on $B_p \setminus \mathrm{sing}(\mathcal{F})$ such that

$$\partial\psi = \delta\omega.$$

Consequently we have, up to a multiplicative constant, the identity

$$e^{2\varphi}\overline{\omega} = \overline{\partial}\delta.$$

Therefore η is $\overline{\partial}$-exact on $B_p \setminus \mathrm{sing}(\mathcal{F})$. By Mayer-Vietoris, the class $\{\eta\}$ defined in $H^1(U, N^*\mathcal{F})$ extends to a class in $H^1(X, N^*\mathcal{F})$. The positivity of T implies that this class is non–trivial and the lemma follows. □

9.2. **Transversely projective structure, Schwartz derivative, and invariant divisors.** Suppose $\mathcal{F}$ is a codimension one foliation on a projective manifold X with numerically trivial canonical bundle. We will also assume that at the generic point of every irreducible component of the singular set of $\mathcal{F}$ having codimension two, the foliation is defined by

$$pxdy + qydx$$

with p, q relatively prime positive integers. Moreover, we can also assume that for at least one of the irreducible codimension two components of $\mathrm{sing}(\mathcal{F})$ the integers p and q are distinct thanks to Corollary 6.9. We will now make use of the transversely projective structure given by Theorem 7 to produce a divisor satisfying the hypothesis of Lemma 9.2. The degeneracy locus of such a structure is of the form $\mathrm{sing}(\mathcal{F}) \cup \Sigma$ where Σ is a finite union of $\mathcal{F}$-invariant hypersurfaces. Outside this set, the foliation is defined by local submersions with values in $\mathbb{P}^1$ and transition functions in $\mathrm{Aut}(\mathbb{P}^1)$. We emphasize that the transverse structure gives distinguished first integrals for the foliation $\mathcal{F}$ outside the degeneracy locus of the projective structure. We will denote the sheaf of such first integrals by $\mathcal{I}$.

Consider a regular point $p \in X - \mathrm{sing}(\mathcal{F})$ where the foliation is locally given by a submersion z, $z(p) = 0$. We can select an open neighborhood U of p and a section f (possibly multi-valued) of $\mathcal{I}$ (which depends only on the z variable) such that the Schwartz derivative of f with respect to z, $\{f, z\}$, is a well defined meromorphic function on the whole open set U. Hence, we can expand the Schwartz derivative of f with respect to z as

$$\{f, z\} = \sum_{i \geq i_0} a_i z^i$$

with $i_0 \in \mathbb{Z}$ and $a_{i_0} \neq 0$, unless $\{f, z\}$ vanishes identically. The following facts can be easily verified.

(1) The first integral f is a submersion if and only if $i_0 \geq 0$. In particular, if $i_0 < 0$ then the local invariant hypersurface $\{z = 0\}$ actually belongs to an algebraic hypersurface in Σ.
(2) If $i_0 \leq -1$ then it is independent of the choice of the local coordinate z. Consequently, i_0 is constant along the irreducible hypersurfaces in Σ. If H is one of such hypersurfaces then we will denote by $i_0(H)$ the value of i_0 along it. Moreover, if $i_0 \geq -2$ then the coefficient of $\frac{1}{z^2}$ is independent of the coordinate and we define $a(H) = a_{-2}$.
(3) the function $f(z) - \log z$ is holomorphic if and only if $i_0 = -2$ and $a_{-2} = \frac{1}{2}$.

We will say that H is an irregular singularity of the projective structure if and only if $i_0(H) < -2$. Otherwise, if $i_0(H) \in \{-2, -1\}$ we will say that H is a regular singularity.

9.2.1. *Passing through corners.* Let $\omega = pydx + qxdy$ be a germ of 1-form at the origin of $\mathbb{C}^n$, with p, q relatively prime positive integers. Suppose that the foliation $\mathcal{F}$ induced by ω is endowed with a projective structure. Let f be a multi-valued section of $\mathcal{I}$ defined on the complement of $\{xy = 0\}$. Let $r = \frac{q}{p}$. Set $i_x = i_0(\{x = 0\})$ and $i_y = i_0(\{y = 0\})$. On the transversals $\{y = 1\}$ and $\{x = 1\}$, we get respectively

$$\{f, x\} = \sum_{i \geq i_x} a_i x^i \quad \text{and} \quad \{f, y\} = \sum_{i \geq i_y} b_i y^i \, .$$

These two restrictions are related by the so called Dulac's transform, a (multi-valued) holonomy transformation between the two transversals, which is explicitly given by $x = h(y) = y^r$.

The composition rule for the Schwartz derivative

$$\{f \circ h, z\} = \{f, h(z)\} h'^2(z) + \{h, z\}$$

applied to $x = h(y) = y^r$, together with the fact that

$$\{h, y\} = \frac{1 - r^2}{y^2}$$

implies the next lemma.

Lemma 9.3.

(1) *If the singularity on* $\{x = 0\}$ *is irregular, i.e.* $i_x < -2$, *then* $i_y = r(i_x+2)-2$ *and* $b_{i_y} = a_{i_x} r^2 (\neq 0)$. *Therefore,* $i_y < -2$ *and the singularity on* $\{y = 0\}$ *is also irregular.*

(2) *If the singularity on* $\{x = 0\}$ *is regular with* $i_x = -2$ *then* $i_y \geq -2$ *and* $b_{-2} = r^2(a_{i_x} - \frac{1}{2}) + \frac{1}{2}$.

(3) *If* $i_x \geq -1$ *and* $r \neq 1$ *then* $i_y = -2$.

It follows that the projective structure determines a *canonical* logarithmic 1-form η on $\{xy = 0\}$ satisfying $d\omega = \eta \wedge \omega$ as follows.

(1) In case of irregular singularities: $\eta = (-i_x - 3)\frac{dx}{x} + (-i_y - 3)\frac{dy}{y}$.

(2) In case of regular singularities: $\eta = (|2a_{-2} - 1|^{\frac{1}{2}} - 1)\frac{dx}{x} + (|2b_{-2} - 1|^{\frac{1}{2}} - 1)\frac{dy}{y}$.

Notice that in both cases the residues are real and strictly greater than -1.

9.3. **Proof of Theorem 9.1 (Theorem 1 of the Introduction).** Let $\mathcal{F}$ be a transversely projective foliation which is of the form $pxdx + qydx$ (p, q relatively prime positive integers) at the generic point of every codimension two irreducible component S of $\mathrm{sing}(\mathcal{F})$. We will now construct a divisor D with support on Σ (the singular set of the transversely projective structure) satisfying the hypothesis of Lemma 9.2.

Write $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_\ell$ as the union of its connected components. Fix an irreducible component Σ_j and pick a point $p \in \mathrm{sing}(\mathcal{F}) \cup \Sigma_j$ in a hypersurface $H_p \subset \Sigma$.

Assume that $i_0(H_p) < -2$. Then every other hypersurface H in Σ_j must satisfy $i(H) < -2$ according to Lemma 9.3. Then we set

$$D_j = \sum_{H \in \Sigma_j} (-i(H) - 3)H.$$

Notice that D_j satisfies the hypothesis of Lemma 9.2 in a neighborhood of Σ_j.

Assume that $i_0(H_p) = -2$ and $a(H_p) = \frac{1}{2}$. Lemma 9.3 implies the same holds true for every $H \in \Sigma_j$. Thus over a generic point of Σ_j we get a logarithmic first integral (induced by the projective structure) which gives rise to a well defined local section η of $d\mathcal{I}$. Indeed, these local sections are logarithmic 1-form with poles on Σ_j which are unique up to a multiplicative constant. Using Mayer-Vietoris sequence, we deduce the existence of a global logarithmic form η_j on a neighborhood V of Σ_j. Moreover, one can choose its residues positive real numbers. In this case we set

$$D_j = \sum_{H \in \Sigma_j} (\mathrm{res}_H(\eta))H.$$

Assume that $i_0(H_p) = -2$ et $a(H_p) \neq \frac{1}{2}$. In this case, using again lemma 9.3, if we set

$$D_j = \sum_{H \in \Sigma_j} (|2a(H) - 1|^{\frac{1}{2}} - 1)H$$

then D_j satisfies the hypothesis of Lemma 9.2 in a neighborhood of Σ_j.

If we sum up the divisors D_j for all the connected components of Σ we obtain a divisor D satisfying the hypothesis of Lemma 9.2, and consequently we get the non-vanishing of $H^1(X, N^*\mathcal{F})$. Proposition 6.5 implies $\mathcal{F}$ is smooth or uniruled. When $\mathcal{F}$ is smooth the explicit description given in [63] (see §1.1) implies that $\mathcal{F}$ is defined by a closed holomorphic 1-form with coefficients in a torsion line bundle. Therefore to conclude the proof of Theorem 9.1 it remains to show that closed 1-forms given by Theorem 2 have coefficients in a torsion line bundle.

9.3.1. *Flat implies torsion.* Let $\mathcal{F}$ be a non-uniruled foliation with $c_1(K\mathcal{F}) = 0$, given by closed rational form ω without zeroes divisor and with coefficients in a flat line bundle L. Assume $\mathrm{sing}(\mathcal{F}) \neq \emptyset$ and write $(\omega)_\infty = \sum \lambda_D D$ as a sum of irreducible divisors with positive integers coefficients.

Assume first that $\lambda_D > 1$ for every D, i.e. the 1-form ω is locally the differential of a meromorphic function. We can argue as in Theorem 3.8 to deduce that $\mathcal{F}$ is uniruled, or has a rational first integral (with generic leaf rationally connected), or every codimension two component of the singular set of $\mathcal{F}$ admits a local holomorphic first integral of the type $x^p y^q$. In the latter case we can apply the arguments of this section to deduce that $\mathcal{F}$ is uniruled or smooth. Therefore Theorem 9.1 is proved when $\lambda_D > 1$ for every D.

Recall that rationally connected manifolds are simply-connected, and consequently L is trivial in these manifolds. Thus we have only to deal with X uniruled, with rational quotient R_X not reduced to a point, and ω has at least one logarithmic pole, i.e. there is a divisor D in the support $(\omega)_\infty$ with $\lambda_D = 1$.

Let us call $\mathcal{F}_{rat}$ the codimension $q = \dim\ R_X$ foliation with algebraic leaves induced by the rationnally connected meromorphic fibration

$$R : X \dashrightarrow R_X.$$

We know [32] that R_X is not uniruled. Therefore [10] implies that $\mathcal{F}_{rat}$ is given by an holomorphic q-form on X without zeroes in codimension 1 and with coefficients in a line bundle E such that E^* is pseudo-effective. The restriction of such q-form on the leaves defines a non trivial section σ of $\Omega^q_{\mathcal{F}} \otimes E$, where $\Omega^q_{\mathcal{F}}$ denotes the q-th wedge power of the cotangent sheaf of $\mathcal{F}$. If $\mathcal{F}$ is not uniruled then $T\mathcal{F}$ is semi-stable with respect to any polarization of X. In this case the section σ has no zeroes in codimension 1. This property forces $R_{|D}$ to be dominant over R_X.

Notice, for later use, that the semi-stability of $T\mathcal{F}$ and the pseudo-effectiveness of E^* implies that E is flat.

Let us denote by $U \subset R_X$ the open Zariski subset such that the fibration R over U is a regular one. Let us pick a small open ball B on U. Over $R^{-1}(B)$ our flat line bundle L is trivial since the fibers of R are rationally-connected. Therefore we can represent ω in $R^{-1}(U)$ by a meromorphic 1-form normalized in such a way that its residue along a branch of D is equal to 1. Since there are only finitely many choices involved, this enables us to conclude that L is torsion. The proof of Theorem 9.1 (Theorem 1 of the Introduction) is concluded. □

10. Toward a more precise structure theorem

In this section we refine the description of non uniruled codimension one foliations with numerically trivial canonical bundle when the ambient manifold X admits a semi-positive smooth $(1,1)$-form representing $c_1(X)$. For a structure theorem for this class of manifolds see [21].

Theorem 10.1. *Let X be a projective manifold carrying a smooth semi-positive closed $(1,1)$-form which represents $c_1(X)$. If $\mathcal{F}$ is a codimension one foliation on X with $c_1(K\mathcal{F}) = 0$ then at least one of the following assertions holds true.*

1. *The foliation $\mathcal{F}$ is uniruled.*
2. *Up to a finite etale covering, the maximally rationally connected fibration of X is a locally trivial smooth fibration over a manifold B with zero canonical class and $\mathcal{F}$ is obtained as a suspension of a representation of $\pi_1(B)$ on the automorphism group of a codimension one foliation $\mathcal{G}$ defined on the rationally connected fiber F and such that $c_1(K\mathcal{G}) = 0$ on F.*
3. *Up to a finite etale covering, X is the product of a projective manifold B with trivial canonical class with a rationally connected projective manifold F. The foliation $\mathcal{F}$ is defined by a closed rational 1-form of the form*

$$\omega = \alpha + \beta$$

where α is a closed rational form without divisorial zeroes defining a foliation $\mathcal{G}$ on F and β is a closed holomorphic 1-form on B.

We conjecture that the result above holds without the hypothesis on $c_1(X)$.

Notice that (3) implies (2) whenever α admits a non trivial infinitesimal symmetry; i.e, there exists on F a holomorphic vector field such that $\alpha(X) = 1$. We do not know if this holds true in general.

10.1. Smoothness of the rationally connected fibration.

The lemma below can be easily deduced from the structure theorem of [21]. We include a proof here for the sake of completeness.

Lemma 10.2. *Under the assumptions of Theorem 10.1, the rational quotient R_X is smooth with vanishing c_1. Moreover, the ambient manifold X is obtained as a suspension of a representation from $\pi_1(R_X)$ to $\mathrm{Aut}(F)$ where F is a fiber of $X \to R_X$.*

Proof. The arguments laid down in §9.3.1 imply that the maximal rationally connected fibration is defined, as a foliation, by a holomorphic q-form ξ with values in a flat line bundle E, where $q = \dim R_X$.

Let us equip E with a metric h such that (E,h) is unitary flat. By assumption there exists a closed semi-positive $(1,1)$ form α which represents $c_1(X)$. By Yau's theorem ([68]), α turns out to be the Ricci form of a Kähler metric g on the whole manifold X.

Let us endow $\Omega^q(X)$ with the induced metric g_q from the Kähler metric g on X, and the vector bundle $F = \Omega^q(X) \otimes E$ with the metric $g_q \otimes h$. In this way, F becomes a Hermitian vector bundle. If we apply Hopf's maximum principle and the standard Bochner identity to the Laplacian of the function $|\alpha|^2$ on X, the curvature condition implies that α is parallel and nowhere vanishing. In particular, the rationally connected fibration is smooth and the subbundle S of TX defined by local holomorphic vector fields tangent to the fibers is indeed parallel. Thus we get, with respect to the metric g, an holomorphic splitting

$$TX = S \oplus S^{\perp}$$

with each member of the summand being an integrable subbundle. Furthermore, since the fibers of the rationally connected fibration are simply-connected, this fibration has no multiple fibers. The lemma follows. □

Using the description of Kähler manifolds with vanishing first Chern class ([4]), we may assume, up to a finite etale covering, that $X = Y \times V$, where V is Calabi-Yau ($KV = \mathcal{O}_V$ and $\pi_1(V) = 0$) and Y is a locally trivial rationally connected fibration over a complex torus T of dimension s. Moreover, the splitting $S \oplus S^{\perp}$ of TX endows Y with a locally free $\mathbb{C}^s$-action transverse to the rationally connected fibration.

Lemma 10.3. *If $H \subset TX$ is the relative tangent bundle of the fibration $X \to Y$ with Calabi-Yau fibers then $H \subset T_{\mathcal{F}}$.*

Proof. Assume the inclusion does not hold. Then, on a generic fiber $\mathcal{F}$ induces a codimension one foliation $\mathcal{F}_V$. Adjunction implies that $N^*\mathcal{F}_{|V} = KX_{|V} = KV$ is indeed trivial and therefore $N^*\mathcal{F}_V$ is effective. This contradicts the vanishing of $H^1(V,\mathbb{C})$. □

It follows that $\mathcal{F}$ is the pull-back of a foliation defined on Y. It is sufficient to restrict our attention to the case $X = Y$ in order to prove Theorem 10.1.

10.2. **Automorphism group of X.** Keeping the notation of Section 9.3.1, consider the non-trivial effective divisor

$$\Delta = (\omega)_{\infty} = \sum \lambda_D D$$

Let us call $\mathrm{Aut}_0(X)$ the connected component of the identity of $\mathrm{Aut}(X)$, the group of biholomorphisms of X. We have a natural action of $\mathrm{Aut}(X)$ on $Div(X)$ compatible with the linear equivalence.

Lemma 10.4. *For every $g \in Aut_0(X)$, $g(\Delta)$ is linearly equivalent to Δ.*

Proof. Following [6], there is a well defined a groups morphism

$$Aut_0(X) \to \mathrm{Pic}^0(X)$$

sending g to the class of $g(\Delta) - \Delta$.

In our context, there exists by assumption a flat line bundle L, arising from a morphism $\rho : \pi_1(X) \to \mathbb{C}^*$ and a holomorphic section σ of $\det(TX) \otimes L$ with Δ as zero divisor. This section lifts to a holomorphic section $\tilde{\sigma}$ of the anticanonical bundle

of the universal covering $\tilde{X}$ such that $h_*\tilde{\sigma} = \rho(h)\tilde{\sigma}$ for every covering transformation h. Now, let us consider the lift $\tilde{v}$ of any holomorphic vector field v on X. For every $g_t = e^{tv}$ in the one parameter subgroup generated by v, we get that $\tilde{g}_t := e^{t\tilde{v}}$ commutes with every covering transformation h. Therefore

$$\tilde{g}_{t*}h_*\tilde{\sigma} = h_*\tilde{g}_{t*}\tilde{\sigma} = \rho(h)\tilde{\sigma}.$$

The meromorphic function $F = \frac{\tilde{\sigma}}{\tilde{g}_{t*}\tilde{\sigma}}$ descends to X as a meromorphic function f such that $(f)_0 = \Delta$ and $(f)_\infty = g_t(\Delta)$. □

The $\mathbb{C}^s$-action induced by the splitting $TX = S \oplus S^\perp$ is determined by s commuting holomorphic vector fields $v_1, \ldots, v_s$ on X linearly independent everywhere. Let us call $G \in Aut_0(X)$ the (non necessarily closed) subgroup determined by the abelian Lie algebra spanned by $v_1, ..., v_s$ and $\overline{G}$ be its Zariski closure in $\mathrm{Aut}_0(X)$.

According to [45], there is an exact sequence of groups

$$1 \to H \to \overline{G} \to Aut_0(T)(= T) \to 1$$

where H is a linear algebraic group with Lie algebra $\mathfrak{h}$ contained in the ideal of holomorphic vectors field tangent to the rationally connected fibration.

Lemma 10.5. *Up to replacing $v_1, ..., v_s$ by $v_1+\xi_1, ..., v_s+\xi_s$ for suitable $\xi_1, ..., \xi_s \in \mathfrak{h}$, one may assume that the natural action of $\overline{G}$ on $\mathbb{P}(|\mathcal{O}(\Delta)|)$ is trivial.*

Proof. Let ρ be the natural group morphism $\overline{G} \to \mathrm{Aut}\,\mathbb{P}(|\mathcal{O}(\Delta)|)$. Remark that ρ is well defined by virtue of lemma 10.4. We call ρ_* the induced morphism at the Lie algebra level.

Since T is a torus, every morphism $T \to \rho(\tilde{G})/\rho(H)$ is trivial. It follows that for every $1 \leq i \leq s$, there exists $\xi_i \in \mathfrak{h}$ such that $\rho_*(v_i) + \rho_*(\xi_i) = 0$. The lemma follows. □

Taking again a finite etale covering if necessary, one can assume that $N\mathcal{F}$ is linearly equivalent to Δ, so the foliation is defined by a rational closed one form ω such that $(\omega)_0 = (0)$ and $(\omega)_\infty = \Delta$. With the same arguments as previously, one can prove the following lemma.

Lemma 10.6. *One can moreover choose $\xi_1, ..., \xi_s$ in the previous lemma such that $\overline{G}$ acts trivially on $H^0(X, \Omega^1 \otimes \mathcal{O}(\Delta))$ (that is, the space of one rational form η with $(\eta)_\infty \leq \Delta$).*

10.3. **Proof of Theorem 10.1.** Using the previous lemmata, we can assert that ω is invariant under the action of $\overline{G}$. Since ω is closed, for every $v \in \mathfrak{g} := Lie(\overline{G})$ we have $\omega(v)$ constant. Taking suitable linear combinations, one can assume that $\omega(v_i) = 0$ for $i = 1, ..., s-1$. There are two possibilities:

(1) $\mathfrak{h}$ is non trivial, so there exists $v \in \mathfrak{h}$ such that $\omega(v) = 1$ (otherwise, $\mathcal{F}$ would be uniruled) and we obtain (2), replacing v_s by $v_s - \omega(v_s)v$;
(2) $\mathfrak{h} = \{0\}$ and in this case we obtain (3).

Theorem 10.1 follows. □

10.4. **The canonical bundle of $\mathcal{F}$ is torsion.** It is natural to enquire if the flatness of $K\mathcal{F}$ implies that it is torsion. We are not aware of any example where this is not the case, and we can prove this is the case under the assumptions of Theorem 10.1.

Corollary 10.7. *Assume that the foliation $\mathcal{F}$ fulfills the hypothesis of Theorem 10.1. Then $KX \simeq \mathcal{O}(-\Delta)$ modulo torsion, so $K\mathcal{F}$ is actually a torsion line bundle.*

Proof. First, observe that we can restrict to the case $X = Y$ in the notation used in §10.1. The universal covering $\pi : \tilde{Y} \to Y$ has the form $\tilde{Y} = \mathbb{C}^s \times F$ and covering transformations act as $h(x, y) = (h_1(x), h_2(y))$, where h_1 is a translation and $h_2 \in \mathrm{Aut}(F)$. We can also assume that h_2 lies in the connected component of the identity (see [6]).

Noticing that Δ is $\overline{G}$ invariant (after a suitable modification described in the previous lemmata), we can conclude that there exists on F an effective divisor Δ' such that $\pi^{-1}(\Delta) = \Delta' \times \mathbb{C}^s$, $KF \simeq \mathcal{O}(-\Delta')$.

Let η a rational meromorphic volume form, expressed as a non trivial section of $KF \otimes \mathcal{O}(\Delta')$. Therefore

$$h_2^*\eta = \lambda_{h_2}\eta,$$

where $\lambda_{h_2} \in \mathbb{C}$. If we take the Zariski closure in $\mathrm{Aut}(F)$ of the group generated by the second component of the deck transformations then, in case $K\mathcal{F}$ is not torsion, we produce a holomorphic vector field in F such that $L_v\eta = \lambda\eta$ for a suitable $\lambda \in \mathbb{C}^*$. Since η has at least one pole of order 1, the corollary will be a consequence of the lemma below. □

Lemma 10.8. *Let M be a complex compact manifold of dimension n, v an holomorphic vector field and η a meromorphic section of the canonical bundle KM with at least a pole of order one. Assume moreover there exists $\lambda \in \mathbb{C}$ such that $L_v\eta = \lambda\eta$. Then, $\lambda = 0$ (equivalently, $(e^{tv})^*\eta = \eta$ for every $t \in \mathbb{C}$).*

Proof. Let be D a simple pole of η and assume for a moment that D and only intersects with normal crossings $(\eta)_\infty$ along simple poles. Then, there is a well defined non zero residue α, in this context a $n-1$ holomorphic form on D (see for instance [3]). Let λ_t such that $(e^{tv})^*\eta = \lambda_t\eta$; we get also

$$(e^{tv})^*\alpha = \lambda_t\alpha$$

with the first term of the equality above making sense because v is necessarily tangent to D. Using now that $\int_D \alpha \wedge \overline{\alpha}$ is invariant under small biholomorphisms of D, we obtain that $|\lambda_t| = 1$, for all $t \in \mathbb{C}$, hence $\lambda_t = 1$.

The general case can be reduced to the one treated above by taking a suitable composition of blowing ups with smooth centers contained in the singular set of $(\eta)_\infty$. □

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