Subjective expected utility without preferences

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Abstract

This paper proposes a theory of subjective expected utility based on primitives only involving the fact that an act can be judged either "attractive" or "unattractive". We give conditions implying that there are a utility function on the set of consequences and a probability distribution on the set of states such that attractive acts have a subjective expected utility above some threshold. The numerical representation that is obtained has strong uniqueness properties.

Keywords: Decision making under uncertainty, Subjective Expected Utility, Ordered partitions, Conjoint measurement

1. Introduction

In spite of the large amount of experimental evidence showing its limited ability to explain the behavior of many subjects, the classical model of Subjective Expected Utility (henceforth model SEU_c, the subscript c being

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a mnemonic for “classical”) remains the focal point of most works in decision under uncertainty. This is surely due to a rather unique combination of simplicity, analytical tractability and normative appeal.

Four main routes have been followed to obtain behavioral foundations for model SEU_c (see Wakker 1989a). The first one works with a finite set of states and a finite set of consequences and uses separation techniques to ensure that the resulting equalities and inequalities will not be contradictory (see Shapiro 1979). As in the case of conjoint measurement (see Scott 1964 or Krantz, Luce, Suppes, and Tversky 1971, Ch. 9), this technique leads to complex conditions that are not easy to test and interpret. The resulting numerical representation does not have strong uniqueness properties. The second route was opened by Savage (1954). It makes no hypothesis on the set of consequences but requires a rich set of states. It leads to relatively simple conditions. The obtained numerical representation has strong uniqueness properties (for a recent advance along this line, see Abdellaoui and Wakker 2005). The third route is a kind of dual to the second one: it imposes richness on the set of consequences, while working with a finite set of states. Early contributions of this type include Gul (1992), Nakamura (1990) and Wakker (1984, 1989b). Recent advances along this line are surveyed and consolidated in Köbberling and Wakker (2003, 2004) and Wakker and Zank (1999). As the second one, this route leads to simple conditions together with strong uniqueness properties. It uses conditions that are easily compared with the ones used in conjoint measurement (see Krantz et al. 1971, Ch. 6 or Wakker 1989a, Ch. 3) since, in this framework, acts can be viewed as elements of a homogeneous Cartesian product (nevertheless, this approach can be extended to deal with more general set of states, see, e.g., Wakker 1989a, Ch. 5). A fourth route includes “lotteries” using “objective probabilities” in the analysis. It leads to relatively simpler results than the preceding two approaches. The price to pay for this simplicity is a richer framework that is often seen as less “pure” than frameworks refusing the introduction of objective probabilities. This approach was pioneered by Anscombe and Aumann (1963). It leads to simple conditions together with strong uniqueness properties (a recent development along this line is Sarin and Wakker 1997). The last three approaches have also been used to analyze models extending SEU_c, such as Rank Dependent Utility (Gilboa 1987, Wakker 1989a) or Cumulative Prospect Theory (Tversky and Kahneman 1992, Wakker and Tversky 1993). Recent reviews of the field of decision making under uncertainty are Gilboa (2009) and Wakker (2010). This paper will only be concerned with subjective expected utility.

In line with the "revealed preferences" tradition in Economics, in the four approaches considered above, the primitives consist of a (well behaved) preference relation on the set of acts. Given any two acts, the decision maker (DM)[1] is supposed to be in position to compare them in terms of strict preference or indifference. With the last three approaches, i.e., the ones leading to strong uniqueness results, the construction of the numerical representation involves building "standard sequences" (Krantz et al. 1971, Ch. 2). This clearly implies working with several indifference curves (see, e.g., Wakker 1989a, Fig. 3.5.2, p. 54).

The central originality of this paper will be to work with different primitives. For any act, we only expect the DM to be in position to tell us if she finds it "attractive" or "unattractive". Hence, our framework only allows to work with a single indifference curve that implicitly lies at the frontier between attractive and unattractive acts. We work with a finite set of states and a rich set of consequences as in the third route mentioned above. We give conditions implying that the set of attractive acts consists of all acts having a subjective expected utility that is above some threshold. The numerical representation in this new model will have strong uniqueness properties.

This paper is not the first one in decision theory to work with ordered partitions instead of preference relations. The first move in this direction was made by Vind (1991) (see also Vind 2003) in a rather abstract setting that has immediate application to conjoint measurement. This work was later developed in Bouyssou and Marchant (2009, 2010). While these papers were mainly concerned with additive representations, Goldstein (1991) studied decomposable numerical representations on the basis of such primitives. His work was later developed in Bouyssou and Marchant (2007a,b) and Słowiński, Greco, and Matarazzo (2002). In the area of decision making under risk, Nakamura (2004) has analyzed various models using similar primitives. In particular, he gives expected utility à la von Neumann-Morgenstern foundations that are similar to the ones sought here for subjective expected utility.

Since our primitives are non-standard, they deserve to be motivated. Our initial motivation was mainly of a theoretic nature: we wanted to derive a model closely resembling model SEU_c while using primitives that would be

[1] Some of our readers may prefer to use the terms "agent" or "subject".

different from the classical ones.

Moreover, our primitives involve what seems to be a simple cognitive task: the division of the set of acts between the ones that are "attractive" and the ones that are not. These primitives may be seen as more parsimonious than the classical ones consisting in a preference relation on the set of acts. Indeed, given such a preference relation, one can always obtain a partition of the set of acts between attractive and unattractive ones by choosing one act as a reference act and declaring acceptable all acts that are strictly preferred to it. Since this can done for several reference acts, a preference relation may also be understood as several interconnected ordered twofold partitions. We only use one.

A particularly interesting situation occurs when the reference act mentioned above is interpreted a "status quo". In this case, an attractive act can be interpreted as an act that the DM is willing to accept given her current situation, i.e., an attractive act is felt strictly preferable to her "status quo". Bleichrodt (2007, 2009) has forcefully argued that, when the status quo is available in all choice sets presented to the DM, it is unreasonable to suppose that it is possible to derive a preference between acts that are judged strictly less desirable than the status quo since such acts are never chosen [2]. In contrast, Bleichrodt (2007, 2009) supposes that a preference relation can be derived for "attractive" acts. Bleichrodt (2009, Th. 1) studies a model in which the preference relation between attractive acts can be explained by subjective expected utility. Our paper may be viewed as an extension of Bleichrodt (2009) that postulates a more radical form of incompleteness since, in our model, "attractive" acts are not compared in terms of preference. This situation may seem uncommon. For instance, in our setting, the set of attractive acts may contain acts that are dominated by other acceptable acts. Let us simply observe that, besides being parsimonious in terms of information, this situation corresponds to the observation of the behavior of a DM that judges all "attractive" acts "choosable". This may happen, e.g., because she has no constraints and can afford to have them all or because this is the result of first-cut analysis that only aims at discarding "unattractive" acts [3]. Examples of a DM willing to perform a first-cut analysis are easy to

[2] We thank Peter P. Wakker for bringing these papers to our attention.

[3] As pointed out to us by a referee, the distinction between "attractive" and "unattractive" may also be of interest to analyze situations in which acts are only available as the result of a costly search. Indeed, in such cases, many simple heuristics can be devised that

imagine. For an example of a DM choosing all acceptable acts, we may think of a banker receiving credit applications and facing no immediate budget constraint: she is likely to accept all applications that meet the standards defined by the bank.

In the model proposed in Bleichrodt (2009) the status quo is seen as "reference point", in line with models for decision making under uncertainty that deviate from model SEU_c such as Prospect Theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992) and other models (Kőszegi and Rabin 2007, Sugden 2003). Therefore, he also studies what happens when the reference point changes (Bleichrodt 2009, Th. 2). In this paper, we are much closer to a strict Bayesian framework. In our model, subjective expected utility will be used to distinguish between acts "above" and "below" the status quo that is viewed as the current endowment of the DM and is not supposed to vary. In our model, "tastes" and/or "beliefs" are identical above and below the status quo.

The paper is organized as follows. Section 2 introduces our setting, model, and notation. Section 3 presents the conditions used in this paper. Our main result is presented and discussed in Section 4. Section 5 concludes. Most proofs are relegated to Appendix I. A sketch of a possible assessment of the parameters of our model is included in Appendix II.

2. The setting

2.1. Notation

We adopt a classical setting for decision under uncertainty with a finite number of states and we mainly follow the terminology and notation used in Wakker (1989a). Let $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be a set of *consequences*. The set of *states* is $N = \{1, 2, \ldots, n\}$. It is understood that the elements of N are exhaustive and mutually exclusive: one and only one state will turn out to be true. An *act* is a function from N to Γ. The set of all acts is denoted by $\mathcal{X} = \{a, b, c, \ldots\} = \Gamma^N$. It will prove convenient to identify the set of acts with the homogeneous Cartesian product $\prod_{i=1}^n \Gamma_i$, where, for all $i \in N$,

make use of the division between attractive and unattractive acts. Such an interpretation, suggesting a form of bounded rationality, is however not well in line with the model that we study below and the way in which we would like to interpret it. As with the classical model based on a preference relation, our model based a twofold partition is proposed here mainly with a normative interpretation in mind.

$\Gamma_i = \Gamma$. Hence, the set $\mathcal{X}$ can be identified to Γ^n and the act $a \in \mathcal{X}$ will often be written as $(a_1, a_2, \ldots, a_n)$.

Let $E \subseteq N$ and $a, b \in \mathcal{X}$. We denote by (a_E, b_{-E}) the act $c \in \mathcal{X}$ such that $c_i = a_i$, for all $i \in E$, and $c_i = b_i$, for all $i \in N \setminus E$. We will also write that $a_E \in \Gamma_E$ and $b_{-E} \in \Gamma_{-E}$, abusing notation. Similarly (α_E, b_{-E}) will denote the act $d \in \mathcal{X}$ such that $d_i = \alpha \in \Gamma$, for all $i \in E$, and $d_i = b_i$, for all $i \in N \setminus E$. When sets contain few elements, we often omit braces around them and write, e.g., (a_i, b_{-i}), (α_{ij}, b_{-ij}) or (α_i, a_j, b_{-ij}). This should cause no confusion.

2.2. Primitives

The classical primitives in decision making under uncertainty consist of a binary relation[4] $\succsim$ on $\mathcal{X}$. We use here a twofold ordered partition of $\mathcal{X}$. We suppose that acts in $\mathcal{X}$ are presented to a DM. For each of these acts, she will specify whether she finds it "attractive" or "unattractive". This process defines a *twofold ordered partition* $\langle \mathscr{A}, \mathscr{U} \rangle$ of the set $\mathcal{X}$ (note that we abuse terminology here since, at this stage, we do not require each of $\mathscr{A}$ and $\mathscr{U}$ to be nonempty). Acts in $\mathscr{A}$ are judged Attractive. Acts in $\mathscr{U}$ are judged Unattractive. As already noted, a suggestive, but by no means compulsory, interpretation of our setting is that attractive acts are the acts that are judged strictly better than the status quo.

The two categories in $\langle \mathscr{A}, \mathscr{U} \rangle$ are *ordered*. All acts in $\mathscr{A}$ are preferable to all acts in $\mathscr{U}$. Our primitives are completely silent about the comparison of two acts in $\mathscr{A}$. Some of them may be quite attractive, while some others are only slightly better than the status quo. A similar remark holds for $\mathscr{U}$.

We say that a state $i \in N$ is *influential* for $\langle \mathscr{A}, \mathscr{U} \rangle$ if there are $\alpha, \beta \in \Gamma$ and $a \in \mathcal{X}$ such that $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_i, a_{-i}) \in \mathscr{U}$ (this notion of influence of a state is similar to the classical notion of essentialness used in models based on a preference relation; we use a different word since our model uses different primitives). A state that is not influential has no impact on the ordered partition $\langle \mathscr{A}, \mathscr{U} \rangle$ and, thus, may be suppressed from N. Hence, we will suppose that all states are influential. As explained in Bouyssou and Marchant (2009, 2011), the analysis of the case of two states requires

[4] In what follows, whenever the symbol $\succsim$ is used to denote a binary relation, it will be understood that $\succ$ denotes its asymmetric part and $\sim$ its symmetric part. Similar conventions hold when $\succsim$ is subscripted or superscripted. A complete and transitive binary relation will be called a weak order.

techniques that are quite different and much simpler than the ones used here (this case does not lead to strong uniqueness results). This explains that the following assumption is maintained throughout this paper.

Assumption 1. *There are at least three states. All states are influential.*

Observe that Assumption 1 implies that both $\mathscr{A}$ and $\mathscr{U}$ are nonempty.

2.3. Model

We analyze a model in which all attractive acts have a subjective expected utility above some threshold. This model involves a real-valued function u on Γ and nonnegative real numbers $p_1, p_2, \ldots, p_n$ that add up to one. The function u is interpreted as a utility function and the number p_i as the subjective probability of state $i \in N$. Hence $\sum_{i=1}^n p_i u(a_i)$ is interpreted as the subjective expected utility of act $a \in \mathcal{X}$. Our model is such that, for all $a \in \mathcal{X}$,

$$a \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) > 0. \qquad (\mathrm{SEU}_m)$$

In our model, an act is attractive if and only if its subjective expected utility is above a threshold. Although it is based on subjective expected utility, our model is clearly different from model SEU_c in which the preference relation between two acts is represented by a comparison of two subjective expected utilities. We call our model SEU_m in order to distinguish it from model SEU_c based on a preference relation, the subscript "m" is a mnemonic for "manichean" since our primitives only distinguish two different types of acts[5].

It is clear that it is not restrictive to suppose that the threshold separating attractive and unattractive acts is set to 0. When the set Γ is endowed with a topology, we might additionally require that the function u is continuous w.r.t. this topology. We will do so below.

Under Assumption 1, all states are influential. It is easy to see that this implies that, for all $i \in N$, $p_i > 0$ and that u is nonconstant.

[5] We have chosen to define model SEU_m using a strict inequality. This is a matter of convention only. An obvious modification of condition A3 (Openness) defined below, would allow us to deal with the case of a non-strict inequality in the definition of model SEU_m. When acceptable acts are defined with respect to a reference act interpreted as the status quo, our model implies that the status quo is unacceptable. Although this may seem strange, this might facilitate the assessment of the acts in $\mathscr{A}$. An act will belong to $\mathscr{A}$ only if the DM is prepared to pay something to obtain it.

3. Axioms

3.1. Tradeoff consistency

Our main non-technical condition is presented first. It is inspired by the "tradeoff consistency" conditions used in Wakker (1989a) and we have kept this name.

A1 (Tradeoff consistency) *For all* $i, j, k, \in N$, *all* $\alpha, \beta, \gamma, \delta, \lambda, \mu, \tau, \xi \in \Gamma$, *all* $a, b, c, d \in \mathcal{X}$,

$$\left.\begin{array}{rl} (\alpha_j, \lambda_k, a_{-jk}) \in \mathscr{A} & \textit{and} \\ (\gamma_j, \mu_k, b_{-jk}) \in \mathscr{A} & \textit{and} \\ (\delta_i, \tau_k, c_{-ik}) \in \mathscr{A} & \textit{and} \\ (\beta_i, \xi_k, d_{-ik}) \in \mathscr{A} & \end{array}\right\} \Rightarrow \left\{\begin{array}{rl} (\beta_j, \mu_k, a_{-jk}) \in \mathscr{A} & \textit{or} \\ (\delta_j, \lambda_k, b_{-jk}) \in \mathscr{A} & \textit{or} \\ (\gamma_i, \xi_k, c_{-ik}) \in \mathscr{A} & \textit{or} \\ (\alpha_i, \tau_k, d_{-ik}) \in \mathscr{A}. & \end{array}\right.$$

Condition A1 (Tradeoff consistency) is necessary for model (SEU_m). Indeed, suppose it is violated. We easily obtain:

$$\begin{aligned} p_j u(\alpha) - p_j u(\beta) &> p_k u(\mu) - p_k u(\lambda), \\ p_k u(\mu) - p_k u(\lambda) &> p_j u(\delta) - p_j u(\gamma), \\ p_i u(\delta) - p_i u(\gamma) &> p_k u(\xi) - p_k u(\tau), \\ p_k u(\xi) - p_k u(\tau) &> p_i u(\alpha) - p_i u(\beta). \end{aligned}$$

Since $p_i, p_j > 0$, the first two inequalities imply that $u(\alpha) - u(\beta) > u(\delta) - u(\gamma)$ while the last two imply $u(\delta) - u(\gamma) > u(\alpha) - u(\beta)$, a contradiction. The interpretation of the above condition is simple: tradeoffs between consequences should be independent from the state in which they are revealed. Notice that A1 (Tradeoff consistency) involves eight acts, as the original tradeoff consistency condition (see Wakker 1989a, Lemma IV.2.5, p. 80)

Let us now introduce a condition that is a variant of a condition used in Vind (1991) that has been modified to cope with the case of a homogeneous Cartesian product.

B1 (1-Linearity) *For all* $i, j \in N$, *all* $\alpha, \beta \in \Gamma$ *and all* $a, b \in \mathcal{X}$,

$$\left.\begin{array}{c} (\alpha_i, a_{-i}) \in \mathscr{A} \\ \textit{and} \\ (\beta_j, b_{-j}) \in \mathscr{A} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (\beta_i, a_{-i}) \in \mathscr{A} \\ \textit{or} \\ (\alpha_j, b_{-j}) \in \mathscr{A}. \end{array}\right.$$

Condition B1 is necessary for model (SEU_m) since its violation would lead to $p_i u(\alpha) > p_i u(\beta)$ and $p_j u(\beta) > p_j u(\alpha)$, a contradiction.

The intuitive idea behind this condition is that consequences can be ordered. We define the binary relation $\succsim^{\mathscr{A}}$ on Γ letting, for all $\alpha, \beta \in \Gamma$,

$$\alpha \succsim^{\mathscr{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathscr{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathscr{A}, \text{ for all } i \in N \text{ and all } a \in \mathcal{X}].$$

It is simple to check that $\succsim^{\mathscr{A}}$ is always reflexive and transitive. Hence, it will be a weak order as soon as it is complete. The following lemma shows that $\succsim^{\mathscr{A}}$ is complete iff $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies B1 (1-Linearity).

Lemma 1. *The relation $\succsim^{\mathscr{A}}$ is complete iff $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies B1 (1-Linearity).*

Proof. Necessity. Suppose that $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_j, b_{-j}) \in \mathscr{A}$. Because $\succsim^{\mathscr{A}}$ is complete, we have either $\alpha \succsim^{\mathscr{A}} \beta$ or $\beta \succsim^{\mathscr{A}} \alpha$. In the first case $(\beta_j, b_{-j}) \in \mathscr{A}$ and $\alpha \succsim^{\mathscr{A}} \beta$ imply $(\alpha_j, b_{-j}) \in \mathscr{A}$. In the second case, $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $\beta \succsim^{\mathscr{A}} \alpha$ imply $(\beta_i, a_{-i}) \in \mathscr{A}$. Hence, $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies B1 (1-Linearity).

Sufficiency. Suppose that $Not[\alpha \succsim^{\mathscr{A}} \beta]$. This implies that, for some $i \in N$ and some $a \in \mathcal{X}$, $(\beta_i, a_{-i}) \in \mathscr{A}$ and $(\alpha_i, a_{-i}) \in \mathscr{U}$. Similarly, $Not[\beta \succsim^{\mathscr{A}} \alpha]$ implies that, for some $j \in N$ and some $b \in \mathcal{X}$, $(\alpha_j, b_{-j}) \in \mathscr{A}$ and $(\beta_j, b_{-j}) \in \mathscr{U}$. Using B1 (1-Linearity), $(\beta_i, a_{-i}) \in \mathscr{A}$ and $(\alpha_j, b_{-j}) \in \mathscr{A}$ imply either $(\alpha_i, a_{-i}) \in \mathscr{A}$ or $(\beta_j, b_{-j}) \in \mathscr{A}$, a contradiction. $\square$

It will be useful to note the following.

Lemma 2. *If a twofold partition satisfies A1 (Tradeoff consistency) then it satisfies B1 (1-Linearity).*

Proof. Suppose that $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_j, b_{-j}) \in \mathscr{A}$. Take any $k \in N$ such that $k \neq i, j$. Using A1 (Tradeoff consistency),

$$\left.\begin{array}{ll} (\alpha_i, a_k, a_{-ik}) \in \mathscr{A} & \text{and} \\ (\alpha_i, a_k, a_{-ik}) \in \mathscr{A} & \text{and} \\ (\beta_j, b_k, b_{-jk}) \in \mathscr{A} & \text{and} \\ (\beta_j, b_k, b_{-jk}) \in \mathscr{A} & \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} (\beta_i, a_k, a_{-ik}) \in \mathscr{A} & \text{or} \\ (\beta_i, a_k, a_{-ik}) \in \mathscr{A} & \text{or} \\ (\alpha_j, b_k, b_{-jk}) \in \mathscr{A} & \text{or} \\ (\alpha_j, b_k, b_{-jk}) \in \mathscr{A}, & \end{array}\right.$$

which is clearly equivalent to B1 (1-Linearity). $\square$

It will be also useful to note that A1 (Tradeoff consistency) also implies the following condition.

B2 (2-Linearity) *For all $i, j \in N$, all $\alpha, \beta, \gamma, \delta \in \Gamma$ and all $a, b \in \mathcal{X}$,*

$$\left.\begin{array}{c}(\alpha_i, \beta_j, a_{-ij}) \in \mathscr{A} \\ \textit{and} \\ (\gamma_i, \delta_j, b_{-ij}) \in \mathscr{A}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(\gamma_i, \delta_j, a_{-ij}) \in \mathscr{A} \\ \textit{or} \\ (\alpha_i, \beta_j, b_{-ij}) \in \mathscr{A}.\end{array}\right.$$

Lemma 3. *If a twofold partition satisfies A1 (Tradeoff consistency) then it satisfies B2 (2-Linearity).*

Proof. Suppose that $(\alpha_i, \beta_j, a_{-ij}) \in \mathscr{A}$ and $(\gamma_i, \delta_j, b_{-jj}) \in \mathscr{A}$. Using A1 (Tradeoff consistency),

$$\left.\begin{array}{rl}(\alpha_i, \beta_j, a_{-ij}) \in \mathscr{A} & \text{and} \\ (\gamma_i, \delta_j, b_{-ij}) \in \mathscr{A} & \text{and} \\ (\alpha_i, \beta_j, a_{-ij}) \in \mathscr{A} & \text{and} \\ (\gamma_i, \delta_j, b_{-ij}) \in \mathscr{A} & \end{array}\right\} \Rightarrow \left\{\begin{array}{ll}(\gamma_i, \delta_j, a_{-ij}) \in \mathscr{A} & \text{or} \\ (\alpha_i, \beta_j, b_{-ij}) \in \mathscr{A} & \text{or} \\ (\gamma_i, \delta_j, a_{-ij}) \in \mathscr{A} & \text{or} \\ (\alpha_i, \beta_j, b_{-ij}) \in \mathscr{A}, & \end{array}\right.$$

which is clearly equivalent to B2 (2-Linearity). □

3.2. Other conditions

Under A1 (Tradeoff consistency) and, hence, B1 (1-Linearity), we have a weak order $\succsim^{\mathscr{A}}$ on Γ. The set Γ is endowed with the order topology generated by $\succsim^{\mathscr{A}}$. The set $\mathcal{X}$, viewed as Γ^n, is then endowed with the product topology. This will allow us to introduce our main structural assumption, the definition of which presupposes that $\succsim^{\mathscr{A}}$ is a weak order. It is clearly not necessary for model (SEU_m).

A2 (Connectedness) *When $\succsim^{\mathscr{A}}$ is a weak order, the set Γ is connected in the order topology generated by $\succsim^{\mathscr{A}}$.*

Our next condition will be necessary if the function u on Γ is required to be continuous w.r.t. the topology on Γ introduced above.

A3 (Openness) *The set $\mathscr{A}$ is open in the product topology on $\mathcal{X}$.*

Our final condition says that, given any act $a \in \mathcal{X}$ and any state $i \in N$, it is always possible to modify the consequence of act a on state i so as to reach $\mathscr{A}$ and $\mathscr{U}$. This will imply that in our model the function u has range $\mathbb{R}$. This condition is quite strong and, as in Bleichrodt (2009), we only introduce it to keep things simple.

A4 (Unboundedness) *For all $i \in N$ and all $a \in \mathcal{X}$, we have $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_i, a_{-i}) \in \mathscr{U}$, for some $\alpha, \beta \in \Gamma$.*

4. Result and comments

4.1. Result

Our characterization of model SEU_m is as follows.

Theorem 1. *Consider a twofold ordered partition $\langle \mathscr{A}, \mathscr{U} \rangle$ of $\mathcal{X}$ such that Assumption 1 holds. Suppose that $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfies A1 (Tradeoff consistency), A2 (Connectedness), A3 (Openness), and A4 (Unboundedness). Then there are a continuous real-valued function u on Γ with range $\mathbb{R}$ and n strictly positive numbers $p_1, p_2, \ldots, p_n$ adding up to 1 such that* (SEU_m) *holds.*

The numbers $p_1, p_2, \ldots, p_n$ are unique. The function u is unique up to a multiplication by a strictly positive constant.

The proof of Theorem 1 is in Appendix I. It uses techniques from Bleichrodt (2007, 2009) Bouyssou and Marchant (2009, 2010), and Gilboa, Schmeidler, and Wakker (2002). In Appendix II, we sketch an assessment procedure for the parameters of model (SEU_m).

4.2. Interpretation and comments

The purpose of this subsection is twofold. First, we analyze the relations between our uncommon primitives and the classical ones. Second, we analyze the relation between the numerical representation obtained in Theorem 1 and the one obtained in classical results on model SEU_c.

4.2.1. Relation to classical results

Given any weak order $\succsim$ on $\mathcal{X}$ and given any act $r \in \mathcal{X}$, we can build an ordered partition letting $\mathscr{A}_r = \{a \in \mathcal{X} : a \succ r\}$ and $\mathscr{U}_r = \{a \in \mathcal{X} : r \succsim a\}$. Classical results on model SEU_c therefore, implicitly, manipulate several twofold ordered partitions. We only use one. Yet, we obtain a model closely resembling model SEU_c and that has strong uniqueness properties. The reader may therefore think that this is due to the fact that the conditions used in Theorem 1 are quite strong. In order to show that this is not the case, let us first briefly recall a classical result on model SEU_c.

Let $\succsim$ be a binary relation on $\mathcal{X}$. This relation is said to satisfy *CCI* (Cardinal Coordinate Independence. This condition was originally introduced in Wakker (1984). It is often called "tradeoff consistency" in Wakker's later texts, e.g., Wakker 1988b, 1989a) if:

$$\left.\begin{array}{c}(\alpha_i, a_{-i}) \succsim (\beta_i, b_{-i}) \\ \text{and} \\ (\gamma_i, b_{-i}) \succsim (\delta_i, a_{-i}) \\ \text{and} \\ (\delta_j, c_{-j}) \succsim (\gamma_j, d_{-j})\end{array}\right\} \Rightarrow (\alpha_j, c_{-j}) \succsim (\beta_j, d_{-j}),$$

for all $i, j \in N$, all $a, b, c, d \in \mathcal{X}$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$. It is well known that, when $\succsim$ is complete and satisfies CCI, if any of the premises of CCI holds with $\succ$ instead of $\succsim$, the conclusion of CCI must hold with $\succ$. Similarly, when the relation $\succsim$ is complete and satisfies CCI, it also satisfies CI (Coordinate Independence), i.e., for all $i \in N$, all $\alpha, \beta \in \Gamma$ and all $a, b \in \mathcal{X}$,

$$(\alpha_i, a_{-i}) \succsim (\alpha_i, b_{-i}) \Rightarrow (\beta_i, a_{-i}) \succsim (\beta_i, b_{-i}).$$

We say that a state $i \in N$ is *essential* (for $\succsim$) if there are $\alpha, \beta \in \Gamma$ and $a \in \mathcal{X}$ such that $(\alpha_i, a_{-i}) \succ (\beta_i, a_{-i})$. Essentialness plays the same rôle for binary relations as does influence for twofold partitions.

When $\mathcal{X}$ is endowed with a topological structure, we say that a binary relation $\succsim$ on $\mathcal{X}$ is *continuous* if the sets $\{a \in \mathcal{X} : a \succ b\}$ and $\{a \in \mathcal{X} : b \succ a\}$ are open for all $b \in \mathcal{X}$.

We have:

Theorem 2 (Wakker 1989a, Th. IV.2.7, page 83)**.** *Suppose that Γ is a connected topological space and that $\mathcal{X}$ is endowed with the product topology. Suppose furthermore that $n \geq 2$ and that all states are essential.*

If $\succsim$ is a weak order on $\mathcal{X}$ that satisfies CCI and is continuous there are a continuous real valued function v on Γ and n strictly positive numbers $q_1, q_2, \ldots, q_n$ adding up to 1 such that, for all $a, b \in \mathcal{X}$,

$$a \succsim b \Leftrightarrow \sum_{i=1}^{n} q_i v(a_i) \geq \sum_{i=1}^{n} q_i v(b_i). \qquad (\text{SEU}_c)$$

Furthermore, the function u is unique up to scale and location and the numbers q_i are unique.

In the above statement, we have omitted the hypothesis that the topology on Γ is separable, since this is only needed when only one state is essential (see Wakker 1989a, Remark A.3.1, page 163 or Wakker 1988a, Th. 6.2, page 430).

Suppose now that $n \geq 3$ and that we have a binary relation on $\mathcal{X}$ satisfying the conditions of Theorem 2. Take any $r \in \mathcal{X}$. Let us show that the twofold ordered partition $\mathscr{A}_r = \{b \in \mathcal{X} : b \succ r\}$ and $\mathscr{U}_r = \{b \in \mathcal{X} : r \succsim b\}$ will satisfy the conditions of Theorem 1.

Let $i \in N$ and define the binary relation $\succsim_i$ on Γ letting, for all $\alpha, \beta \in \Gamma$, $\alpha \succsim_i \beta \Leftrightarrow [(\alpha_i, a_{-i}) \succsim (\beta_i, a_{-i})$, for all $a \in \mathcal{X}]$. Let us also denote by $\overline{\alpha}$ the constant act giving the outcome $\alpha \in \Gamma$ in all states $i \in N$. Define the relation $\succsim_\Gamma$ on Γ by letting, for all $\alpha, \beta \in \Gamma$, $\alpha \succsim_\Gamma \beta \Leftrightarrow \overline{\alpha} \succsim \overline{\beta}$. Since $\succsim$ is a weak order satisfying CCI, it is easy to check that $\succsim_i = \succsim_j = \succsim_\Gamma$, for all $i, j \in N$, and that $\succsim_\Gamma$ is a weak order. As observed in Wakker (1989a, p. 50), the topology on Γ can always be taken to be the order topology generated by $\succsim_\Gamma$.

By definition, $\alpha \succ^{\mathscr{A}_r} \beta$ implies that $(\alpha_i, a_{-i}) \in \mathscr{A}_r$ and $(\beta_i, a_{-i}) \notin \mathscr{A}_r$, for some $a, b \in \mathcal{X}$ and some $i \in N$. Hence, we have $(\alpha_i, a_{-i}) \succ r$ and $r \succsim (\beta_i, a_{-i})$, so that $(\alpha_i, a_{-i}) \succ (\beta_i, a_{-i})$. This shows that $\alpha \succ^{\mathscr{A}_r} \beta$ implies $\alpha \succ_\Gamma \beta$. Hence, the order topology generated by $\succsim^{\mathscr{A}_r}$ is coarser than the order topology generated by $\succsim_\Gamma$. This shows that A2 (Connectedness) and A3 (Openness) will hold.

Let us show that A1 (Tradeoff consistency) holds. Suppose, in contradiction with the thesis that we have

$$\begin{array}{rr}
(\alpha_j, \lambda_k, a_{-jk}) \in \mathscr{A}_r & (\beta_j, \mu_k, a_{-jk}) \in \mathscr{U}_r \\
(\gamma_j, \mu_k, b_{-jk}) \in \mathscr{A}_r & (\delta_j, \lambda_k, b_{-jk}) \in \mathscr{U}_r \\
(\delta_i, \tau_k, c_{-ik}) \in \mathscr{A}_r & (\gamma_i, \xi_k, c_{-ik}) \in \mathscr{U}_r \\
(\beta_i, \xi_k, d_{-ik}) \in \mathscr{A}_r & (\alpha_i, \tau_k, d_{-ik}) \in \mathscr{U}_r.
\end{array}$$

This implies

$$\begin{array}{rl}
(\alpha_j, \lambda_k, a_{-jk}) \succ r, & r \succsim (\beta_j, \mu_k, a_{-jk}) \\
(\gamma_j, \mu_k, b_{-jk}) \succ r, & r \succsim (\delta_j, \lambda_k, b_{-jk}) \\
(\delta_i, \tau_k, c_{-ik}) \succ r, & r \succsim (\gamma_i, \xi_k, c_{-ik}) \\
(\beta_i, \xi_k, d_{-ik}) \succ r, & r \succsim (\alpha_i, \tau_k, d_{-ik}).
\end{array}$$

Using the fact that $\succsim$ is a weak order, this leads to

$$\begin{aligned}
(\alpha_j, \lambda_k, a_{-jk}) &\succ (\beta_j, \mu_k, a_{-jk}) \\
(\gamma_j, \mu_k, b_{-jk}) &\succ (\delta_j, \lambda_k, b_{-jk}) \\
(\delta_i, \tau_k, c_{-ik}) &\succ (\gamma_i, \xi_k, c_{-ik}) \\
(\beta_i, \xi_k, d_{-ik}) &\succ (\alpha_i, \tau_k, d_{-ik}).
\end{aligned}$$

Using CI, $(\gamma_j, \mu_k, b_{-jk}) \succ (\delta_j, \lambda_k, b_{-jk})$ implies $(\gamma_j, \mu_k, a_{-jk}) \succ (\delta_j, \lambda_k, a_{-jk})$. Using CCI,

$$\left.\begin{aligned}
(\alpha_j, \lambda_k, a_{-jk}) &\succ (\beta_j, \mu_k, a_{-jk}) \\
(\gamma_j, \mu_k, a_{-jk}) &\succ (\delta_j, \lambda_k, a_{-jk}) \\
(\delta_i, \tau_k, c_{-ik}) &\succ (\gamma_i, \xi_k, c_{-ik})
\end{aligned}\right\} \Rightarrow (\alpha_i, \tau_k, c_{-ik}) \succsim (\beta_i, \xi_k, c_{-ik})$$

Using CI, $(\alpha_i, \tau_k, c_{-ik}) \succsim (\beta_i, \xi_k, c_{-ik})$ implies $(\alpha_i, \tau_k, d_{-ik}) \succsim (\beta_i, \xi_k, d_{-ik})$, a contradiction. Hence, A1 (Tradeoff consistency) holds.

Finally, observe that, if the function v in Theorem 2 has range $\mathbb{R}$, condition A4 (Unboundedness) will hold.

Hence, when there are at least three states, the conditions underlying model SEU_m do not seem to be stronger than the conditions usually supposed in classical derivations of model SEU_c. Any twofold partition derived from a preference relation satisfying the conditions of Theorem 2 will satisfy the conditions of Theorem 1 (up to the fact that a utility function obtained in Theorem 2 does not have to have range $\mathbb{R}$).

4.2.2. Relating numerical representations in models SEU_c and SEU_m

It is first clear that, starting with the representation u, $p_1, p_2, \ldots, p_n$ of $\langle \mathscr{A}, \mathscr{U} \rangle$ built in Theorem 1, we can build a preference relation $\succsim$ on $\mathcal{X}$ satisfying all conditions of Theorem 2. Indeed, it suffices to build $\succsim$ so as to reflect the comparison of the subjective expected utility of each act as given by u and $p_1, p_2, \ldots, p_n$. Clearly, such a relation $\succsim$ on $\mathcal{X}$ is unique and satisfies all conditions of Theorem 2. Applying Theorem 2 to this relation will lead to a representation v and $q_1, q_2, \ldots, q_n$. Clearly, we must have that $p_i = q_i$, for all $i \in N$. Moreover, provided that v is appropriately scaled (i.e., scaled in such a way that all acts in the equivalence class that includes the acts belonging to the intersection of $\mathscr{U}$ and the closure of $\mathscr{A}$ have a subjective expected utility of 0), we must have that $u = v$.

Conversely, starting with a relation $\succsim$ on $\mathcal{X}$ satisfying the conditions of Theorem 2, we obtain a representation v and $q_1, q_2, \ldots, q_n$. Whenever v has range $\mathbb{R}$, we know that taking any $r \in \mathcal{X}$, the twofold partition $\langle \mathscr{A}_r, \mathscr{U}_r \rangle$ satisfies the conditions of Theorem 1. Applying Theorem 1 to $\langle \mathscr{A}_r, \mathscr{U}_r \rangle$ leads to representation u, $p_1, p_2, \ldots, p_n$. It is clear that $q_i = p_i$, for all $i \in N$. Moreover, provided that v is scaled in such a way as to give a subjective expected utility of 0 to r, we have $v = u$.

Hence under minimal consistency requirements, the representation built in Theorem 1 must be the same as the representation built in Theorem 2 (a similar observation was made in Vind 2003, Th. 11, p. 42 in the context of conjoint measurement). Obtaining model SEU_m is therefore tantamount to obtaining model SEU_c. Whenever a preference relation is compatible with the twofold partition that has been used to assess model SEU_m, assessing model SEU_c on the basis of this preference relation will lead to the same subjective probabilities and the same utility function as the ones obtained in model SEU_m. Therefore, although model SEU_m that is based on the twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ is silent on the comparison in terms of desirability of the acts in $\mathscr{A}$ and $\mathscr{U}$, there is only one way to compare these acts in terms of preference, if we want this preference relation to be compatible with the ordered partition $\langle \mathscr{A}, \mathscr{U} \rangle$ and to have a representation in model SEU_c. It implies using the numerical representation obtained for model SEU_m as if it were a representation of model SEU_c.

4.3. Relation to classical experiments

Our central condition is A1 (Tradeoff consistency). Although we have not tested it in experiments, we have good reasons to believe that adapting classical experiments to our setting will easily lead to falsify it. As pointed out to us by a referee, the reader may be puzzled by the fact that what follows seems to show that our model will be easily falsifiable. Our view is that model SEU_m, as model SEU_c, is mainly intended for normative purposes.

4.3.1. Ellsberg's problem

Consider first the classical problem presented in Ellsberg (1961) and depicted in Table 1. An urn contains 90 balls that are Red (R), Black (B) or Yellow (Y). It is known that 30 of these balls are Red. The other 60 are either Black or Yellow in unknown proportions. Confronted with the acts x, y, z, and w in Table 1, the modal preferences of subjects in experiments are $x \succ y$ and $w \succ z$. These preferences are easily explained by the desire to bet on an "unambiguous event" (R, when comparing x and y, and [B or Y], when comparing z and w). They are incompatible with model SEU_c (these modal preferences violate CI). In this example, it seems natural to suppose that all

	R	B	Y
x	100	0	0
y	0	100	0
z	100	0	100
w	0	100	100

Table 1: Ellsberg's problem.

acts will be judged acceptable since they all involve a possible gain without any possibility of loosing money. Hence, this problem cannot directly lead to falsify model (SEU$_m$).

Consider now the following variation on Ellsberg's theme. Suppose that the urn contains 100 balls that are Red (R), White (W), Black (B) or Yellow (Y). It is known that 30 balls are Red and that 10 are White. The other 60 are either Black or Yellow. Consider the acts described in Table 2 and let us imagine how the subjects having the modal preference in Ellsberg's problem might react to them. Since $x \succ y$ in the original problem and there are

	R	W	B	Y
x_ϵ	100	$-\epsilon$	0	0
y_ϵ	0	$-\epsilon$	100	0
z_τ	100	$-\tau$	0	100
w_τ	0	$-\tau$	100	100

Table 2: Ellsberg's problem modified.

only 10 White balls, it is likely that these subjects will consider that $x_\epsilon \succ y_\epsilon$. When ϵ is small, it is likely that both x_ϵ and y_ϵ will be considered acceptable. When ϵ increases, the desirability of both acts decreases. We may eventually find a value of ϵ such that, for many of these subjects, $x_\epsilon \in \mathscr{A}$ and $y_\epsilon \in \mathscr{U}$. Using a similar reasoning, it is likely that there will be a value of τ such that, for many of these subjects, $z_\tau \in \mathscr{U}$ and $w_\tau \in \mathscr{A}$. Summarizing, it is likely that there will be subjects stating that:

$$x_\epsilon = (100_R, -\epsilon_W, \quad 0_B, \quad 0_Y) \in \mathscr{A}, \qquad y_\epsilon = (\quad 0_R, -\epsilon_W, 100_B, \quad 0_Y) \in \mathscr{U},$$
$$w_\tau = (\quad 0_R, -\tau_W, 100_B, 100_Y) \in \mathscr{A}, \qquad z_\tau = (100_R, -\tau_W, \quad 0_B, 100_Y) \in \mathscr{U},$$

which will violate B2 (2-Linearity) with $i = R$ and $j = B$. Hence, A1 (Tradeoff consistency) will be violated.

4.3.2. Allais' problem

Tversky and Kahneman (1992) have shown how the classical problem in Allais (1953) can be adapted to cover the case of decisions under uncertainty. Their example involves a bet of the absolute value d of the difference between the closing values of the Dow-Jones between two consecutive days. Given the acts presented in Table 3, the modal preferences of subjects are $x \succ y$ and $w \succ z$, which is incompatible with model (SEU$_c$) (these modal preferences violate CI). Yet, these preferences are easily explained by the attraction of a sure win (comparing x with y) combined with the desire to go for the larger gain when uncertain (comparing w with z).

	$d < 30$	$30 \leq d \leq 35$	$35 < d$
x	25 000	25 000	25 000
y	25 000	0	75 000
z	0	25 000	25 000
w	0	0	75 000

Table 3: Adaptation of Allais' problem in Tversky and Kahneman (1992, p. 303).

Again, all these acts are likely to be judged acceptable since they do not involve losses. Consider now the modified problem presented in Table 4. Using the same reasoning as with Ellsberg's problem, it is likely that there

	$0 \leq d \leq 1$	$1 < d < 30$	$30 \leq d \leq 35$	$35 < d$
x_ϵ	$-\epsilon$	25 000	25 000	25 000
y_ϵ	$-\epsilon$	25 000	0	75 000
z_τ	$-\tau$	0	25 000	25 000
w_τ	$-\tau$	0	0	75 000

Table 4: Allais' problem modified.

are ϵ and τ such that many subjects will state that $x_\epsilon \in \mathscr{A}$, $y_\epsilon \in \mathscr{U}$, $z_\tau \in \mathscr{U}$ and $w_\tau \in \mathscr{A}$. It is simple to check that this will violate B2 (2-Linearity) and, hence, A1 (Tradeoff consistency).

5. Discussion

This paper has analyzed decision making under uncertainty replacing the classical primitives consisting of a preference relation on the set of acts by a twofold ordered partition of this set. A possible interpretation of this setting is that each act is positioned vis-à-vis a status quo and is either "acceptable" or "unacceptable". Within a framework using a finite set of states and a rich set of consequences, we have given conditions on a twofold ordered partition ensuring that it can be represented in such a way that all attractive acts have a subjective expected utility above some threshold. The obtained representation has strong uniqueness properties. We have also seen that any representation in model SEU_c of a preference relation that is compatible with the twofold ordered partition must lead to a similar numerical representation. This gives subjective expected utility alternative behavioral foundations based on primitives that are more parsimonious than the ones classically considered in the literature and using conditions that are reasonably simple. We have shown that our conditions do not seem to be stronger than the classical ones used to characterize model SEU_c. We have also shown that it is likely that simple adaptations of Ellsberg's and Allais's problems will lead to falsify our central condition (Tradeoff consistency).

The line of research consisting in replacing a preference relation by an ordered partition to analyze classical models is still quite open. Let us simply mention here four directions for future research. First, it is clearly interesting to present a result similar to Theorem 1 without the use of A4 (Unboundedness). In view of Bouyssou and Marchant (2009, Cor. 1), this should not raise major problems. Second, an important question would be to study models extending model SEU_c, such as Rank Dependent Utility (Gilboa 1987, Wakker 1989a) or Cumulative Prospect Theory (Tversky and Kahneman 1992, Wakker and Tversky 1993), using primitives such as the ones used here. Our method of proof does not however seem to be well adapted to cover such models. Dealing with them is likely to require a rather different type of analysis. The third question is empirical. On the basis of the numerous empirical studies on the validity of model SEU_c, we have suggested experiments that are likely to lead to a falsification of our model. We have not performed any empirical analysis however. It might be the case that questions phrased in terms of our primitives generate different behavior than questions phrased in terms of classical primitives (for an example of the influence of the questioning mode on behavior, see Tversky, Sattah, and Slovic

1988). Moreover, we have conjectured that dividing acts between the ones that are acceptable and the ones that are not is a simple cognitive task that, in any case, does not seem to be more complex than the comparison of two acts in terms of preference. This remains a conjecture however.

The fourth question is more conceptual. We have abandoned in this paper the usual primitives in decision theory. Although we think to have shown that this was a feasible and interesting avenue, this also comes at a cost. While the foundations of the classical primitives have been investigated in depth (in particular through the literature on "revealed preferences" connecting preferences and choices), we are presently lacking similar foundations for the primitives used here (such foundations do not seem out of reach however). Moreover, our emphasis has been on obtaining a model that we are inclined to motivate on a normative basis. It may be the case that models having a less firm normative basis can also be profitably analyzed with our premises. As already mentioned, a referee has suggested that this might well be the case in situations, such as job search or mate selection, in which acts are not readily available but have to be searched, this search involving a cost. In such situations, stopping the search as soon as an acceptable act has been obtained appears to be a reasonable heuristics that could be analyzed with our primitives. Clearly, this would call for analyses that are rather different from the ones proposed here.

Appendix I: Proofs

We suppose throughout this appendix that all conditions in Theorem 1 hold. Define, for each $i \in N$, the binary relation $\succsim_i^{\mathscr{A}}$ on Γ letting, for all $\alpha, \beta \in \Gamma$,

$$\alpha \succsim_i^{\mathscr{A}} \beta \Leftrightarrow [(\beta_i, a_{-i}) \in \mathscr{A} \Rightarrow (\alpha_i, a_{-i}) \in \mathscr{A}, \text{ for all } a \in \mathcal{X}].$$

It is clear that all relations $\succsim_i^{\mathscr{A}}$ are reflexive and transitive. We obviously have that $\succsim^{\mathscr{A}} = \bigcap_{i=1}^{n} \succsim_i^{\mathscr{A}}$. When B1 (1-Linearity) holds, all relations $\succsim_i^{\mathscr{A}}$ are complete and are compatible (see Lemma 1).

Let $\overline{\mathscr{A}} = \mathrm{Cl}(\mathscr{A})$, $\mathcal{F} = \overline{\mathscr{A}} \setminus \mathscr{A}$, and $\mathscr{U}^\circ = \mathscr{U} \setminus \mathcal{F}$. For all $i \in N$ and all $a \in \mathcal{X}$, define $\mathscr{U}(a_{-i}) = \{\beta \in \Gamma : (\beta_i, a_{-i}) \in \mathscr{U}\}$, $\mathscr{A}(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \mathscr{A}\}$, $\overline{\mathscr{A}}(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \overline{\mathscr{A}}\}$, $\mathscr{U}^\circ(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \mathscr{U}^\circ\}$, and $\mathcal{F}(a_{-i}) = \{\alpha \in \Gamma : (\alpha_i, a_{-i}) \in \mathcal{F}\}$. By construction, we have that $\mathscr{U}(a_{-i}) \cap \mathscr{A}(a_{-i}) = \varnothing$ and $\mathscr{U}(a_{-i}) \cup \mathscr{A}(a_{-i}) = \Gamma$.

A. Preliminary lemmas

Lemma 4. *For all $i \in N$, the relation $\succsim_i^{\mathscr{A}}$ has no maximal or minimal elements. The relation $\succsim^{\mathscr{A}}$ has no maximal or minimal elements.*

Proof. Let $\alpha \in \Gamma$, $a \in \mathcal{X}$ and $i, j \in N$ with $i \neq j$. Using A4 (Unboundedness), we have $(\alpha_i, \gamma_j, a_{-ij}) \in \mathscr{U}$, for some $\gamma \in \Gamma$. Using A4 (Unboundedness), we have $(\beta_i, \gamma_j, a_{-ij}) \in \mathscr{A}$, for some $\beta \in \Gamma$. This shows that $\alpha \in \Gamma$ cannot be maximal for $\succsim_i^{\mathscr{A}}$. The proof that $\succsim_i^{\mathscr{A}}$ cannot have minimal elements is similar. That $\succsim^{\mathscr{A}}$ has no maximal or minimal elements now follows from the fact that $\succsim^{\mathscr{A}} = \bigcap_{i=1}^{n} \succsim_i^{\mathscr{A}}$. □

Lemma 5. *For all $a \in \mathcal{X}$ and all $i \in N$, we have $\mathscr{U}(a_{-i}) \neq \varnothing$ and $\mathscr{A}(a_{-i}) \neq \varnothing$. The set $\mathscr{A}(a_{-i})$ is open. The set $\mathscr{U}(a_{-i})$ is closed.*

Proof. The first part results from A4 (Unboundedness). The second part follows from A3 (Openness) (see Wakker 1989a, Lemma 0.2.1, p. 12). The final part follows from the fact that $\mathscr{U}(a_{-i}) = \Gamma \setminus \mathscr{A}(a_{-i})$. □

Lemma 6. *Let $i \in N$, and $a \in \mathcal{X}$. We have $(\gamma_i, a_{-i}) \in \mathcal{F}$, for some $\gamma \in \Gamma$.*

Proof. We know from Lemma 5 that $\mathscr{U}(a_{-i}) \neq \varnothing$ is closed and that $\mathscr{A}(a_{-i}) \neq \varnothing$ is open. Since $\overline{\mathscr{A}}$ is closed, we know (see Wakker 1989a, Lemma 0.2.1, p. 12) that the set $\overline{\mathscr{A}}(a_{-i})$ is closed. It is nonempty because $\mathscr{A}(a_{-i})$ is nonempty. If $\overline{\mathscr{A}}(a_{-i}) \cap \mathscr{U}(a_{-i}) = \varnothing$, we have separated Γ into two closed nonempty sets, violating A2 (Connectedness). Hence we must have $\overline{\mathscr{A}}(a_{-i}) \cap \mathscr{U}(a_{-i}) \neq \varnothing$. Taking any $\gamma \in \overline{\mathscr{A}}(a_{-i}) \cap \mathscr{U}(a_{-i})$, we obtain $(\gamma_i, a_{-i}) \in \mathcal{F}$. □

Lemma 7. *Let $i \in N$, and $a \in \mathcal{X}$. We have $\mathscr{U}(a_{-i}) = \{\beta \in \Gamma : \iota(a_{-i}) \succsim^{\mathscr{A}} \beta\}$ and $\mathscr{A}(a_{-i}) = \{\beta \in \Gamma : \beta \succ^{\mathscr{A}} \iota(a_{-i})\}$, for some $\iota(a_{-i}) \in \Gamma$.*

Proof. We know from Lemma 5 that the set $\mathscr{U}(a_{-i}) = \{\beta \in \Gamma : (\beta_i, a_{-i}) \in \mathscr{U}\}$ is nonempty and closed. Let $\beta \in \mathscr{U}(a_{-i})$. Since $\beta \succ^{\mathscr{A}} \gamma$ implies $\beta \succsim_i^{\mathscr{A}} \gamma$, it is clear that $\beta \succ^{\mathscr{A}} \gamma$ implies $\gamma \in \mathscr{U}(a_{-i})$. By construction, $\beta \sim^{\mathscr{A}} \gamma$ implies $\beta \sim_i^{\mathscr{A}} \gamma$, for all $i \in N$. Hence, $\beta \sim^{\mathscr{A}} \gamma$ implies $\gamma \in \mathscr{U}(a_{-i})$. We have shown that $\beta \in \mathscr{U}(a_{-i})$ and $\beta \succsim^{\mathscr{A}} \gamma$ imply $\gamma \in \mathscr{U}(a_{-i})$. This implies that the nonempty set $\mathscr{U}(a_{-i})$ is unbounded below. Since it is closed, there is $\iota(a_{-i}) \in \Gamma$ such that, $\mathscr{U}(a_{-i}) = \{\beta \in \Gamma : \iota(a_{-i}) \succsim^{\mathscr{A}} \beta\}$. This implies that $\mathscr{A}(a_{-i}) = \{\beta \in \Gamma : \beta \succ^{\mathscr{A}} \iota(a_{-i})\}$. □

Lemma 8. *Let $i \in N$, $\alpha, \beta \in \Gamma$, and $a \in \mathcal{X}$. If $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $(\beta_i, a_{-i}) \in \mathcal{F}$ then $\alpha \sim^{\mathscr{A}} \beta$.*

Proof. In view of Lemma 7, we have $\overline{\mathscr{A}}(a_{-i}) = \{\delta \in \Gamma : \delta \succsim^{\mathscr{A}} \iota(a_{-i})\}$, so that $\mathcal{F}(a_{-i}) = \overline{\mathscr{A}}(a_{-i}) \cap \mathscr{U}(a_{-i}) = \{\delta \in \Gamma : \delta \sim^{\mathscr{A}} \iota(a_{-i})\}$. Therefore, $(\alpha, a_{-i}) \in \mathcal{F}$ and $(\beta, a_{-i}) \in \mathcal{F}$ imply that $\alpha \sim^{\mathscr{A}} \iota(a_{-i})$ and $\beta \sim^{\mathscr{A}} \iota(a_{-i})$, so that $\alpha \sim^{\mathscr{A}} \beta$. □

Lemma 9. *Let $i \in N$, $\alpha, \beta \in \Gamma$, and $a \in \mathcal{X}$. If $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $\alpha \sim^{\mathscr{A}} \beta$ then $(\beta_i, a_{-i}) \in \mathcal{F}$.*

Proof. Suppose $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $\alpha \sim^{\mathscr{A}} \beta$. Because $\mathcal{F}(a_{-i}) = \{\delta \in \Gamma : \delta \sim^{\mathscr{A}} \iota(a_{-i})\}$, we know that $\alpha \sim^{\mathscr{A}} \iota(a_{-i})$. Since $\alpha \sim^{\mathscr{A}} \beta$, we obtain $\beta \sim^{\mathscr{A}} \iota(a_{-i})$, so that $(\beta_i, a_{-i}) \in \mathcal{F}$. □

Lemma 10. *For all $i, j \in N$, all $a, b \in \mathcal{X}$ and all $\alpha, \beta \in \Gamma$,*

$$\left.\begin{array}{c}(\alpha_i, a_{-i}) \in \mathcal{F}\\ \textit{and}\\ (\beta_i, a_{-i}) \in \mathcal{F}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(\alpha_j, b_{-j}) \in \mathscr{A} \Leftrightarrow (\beta_j, b_{-j}) \in \mathscr{A}\\ (\alpha_j, b_{-j}) \in \mathcal{F} \Leftrightarrow (\beta_j, b_{-j}) \in \mathcal{F},\\ (\alpha_j, b_{-j}) \in \mathscr{U}^\circ \Leftrightarrow (\beta_j, b_{-j}) \in \mathscr{U}^\circ.\end{array}\right.$$

Proof. Since $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $(\beta_i, a_{-i}) \in \mathcal{F}$, Lemma 8 implies that $\alpha \sim^{\mathscr{A}} \beta$.

The first part follows from the definition of $\succsim^{\mathscr{A}}$. If $(\alpha_j, b_{-j}) \in \mathcal{F}$, Lemma 9 implies $(\beta_j, b_{-j}) \in \mathcal{F}$. This proves the second part. The last part follows. □

Lemma 11. *For all $i \in N$, all $\alpha, \beta \in \Gamma$ and all $a \in \mathcal{X}$,*

$$\begin{aligned}
(\alpha_i, a_{-i}) \in \mathscr{A} \textit{ and } \beta \succsim^{\mathscr{A}} \alpha &\Rightarrow (\beta_i, a_{-i}) \in \mathscr{A},\\
(\alpha_i, a_{-i}) \in \mathcal{F} \textit{ and } \beta \succ^{\mathscr{A}} \alpha &\Rightarrow (\beta_i, a_{-i}) \in \mathscr{A},\\
(\alpha_i, a_{-i}) \in \mathcal{F} \textit{ and } \beta \sim^{\mathscr{A}} \alpha &\Rightarrow (\beta_i, a_{-i}) \in \mathcal{F},\\
(\alpha_i, a_{-i}) \in \mathcal{F} \textit{ and } \alpha \succ^{\mathscr{A}} \beta &\Rightarrow (\beta_i, a_{-i}) \in \mathscr{U}^\circ,\\
(\alpha_i, a_{-i}) \in \mathscr{U}^\circ \textit{ and } \alpha \succsim^{\mathscr{A}} \beta &\Rightarrow (\beta_i, a_{-i}) \in \mathscr{U}^\circ.
\end{aligned}$$

Proof. The first part follows from the definition of $\succsim^{\mathscr{A}}$. For the second part, observe, using Lemma 7 that, $(\alpha_i, a_{-i}) \in \mathcal{F}$ implies $\alpha \sim^{\mathscr{A}} \iota(a_{-i})$, so that $\beta \succ^{\mathscr{A}} \alpha$ implies $\beta \succ^{\mathscr{A}} \iota(a_{-i})$. This implies that $\beta \in \mathscr{A}(a_{-i})$, so that $(\beta_i, a_{-i}) \in \mathscr{A}$. The third part follows from Lemma 9. To prove the fourth part, suppose that $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $\alpha \succ^{\mathscr{A}} \beta$. In view of the first two parts, it is impossible that $(\beta_i, a_{-i}) \in \overline{\mathscr{A}}$. Hence, we must have $(\beta_i, a_{-i}) \in \mathscr{U}^\circ$. The last part follows from the fact that, if $(\beta_i, a_{-i}) \in \overline{\mathscr{A}}$, $\alpha \succsim^{\mathscr{A}} \beta$ implies $(\alpha_i, a_{-i}) \in \overline{\mathscr{A}}$, a contradiction. □

B. Additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$

Our aim is to show that there is an additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$, i.e., that there are n real-valued functions $v_1, v_2, v_3, \dots, v_n$ on Γ such that for all $a, b \in \mathcal{X}$,

$$a \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} v_i(a_i) > 0. \tag{1}$$

This additive representation is said to be *continuous* if all functions v_i are continuous.

Let $j \in N$. Define the binary relations $\succsim^{(j)}$ and $\succsim^{\mathscr{A}}_{-j}$ on the set $\prod_{i \neq j} \Gamma_i$ letting

$$a_{-j} \succsim^{(j)} b_{-j} \Leftrightarrow \left[(\alpha_j, a_{-j}) \in \overline{\mathscr{A}} \text{ and } (\alpha_j, b_{-j}) \in \mathscr{U}, \text{ for some } \alpha \in \Gamma\right],$$
$$a_{-j} \succsim^{\mathscr{A}}_{-j} b_{-j} \Leftrightarrow [(\alpha_j, b_{-j}) \in \mathscr{A} \Rightarrow (\alpha_j, a_{-j}) \in \mathscr{A}, \text{ for all } \alpha \in \Gamma].$$

B.1. Preliminary lemmas

Lemma 12. *The relation $\succsim^{\mathscr{A}}_{-j}$ is a weak order.*

Proof. This is an immediate consequence of B1 (1-Linearity). □

Lemma 13. *We have $\succsim^{(j)} = \succsim^{\mathscr{A}}_{-j}$.*

Proof. Suppose that $x_{-j} \succsim^{(j)} y_{-j}$ so that we have $(\alpha_j, x_{-j}) \in \overline{\mathscr{A}}$ and $(\alpha_j, y_{-j}) \in \mathscr{U}$, for some $\alpha \in \Gamma$. In contradiction with the thesis, suppose that $Not[x_{-j} \succsim^{\mathscr{A}}_{-j} y_{-j}]$. This implies that, for some $\beta \in \Gamma$, we have $(\beta_j, y_{-j}) \in \mathscr{A}$ and $(\beta_j, x_{-j}) \in \mathscr{U}$. If $\alpha \succsim^{\mathscr{A}} \beta$, $(\beta_j, y_{-j}) \in \mathscr{A}$ implies $(\alpha_j, y_{-j}) \in \mathscr{A}$, a contradiction. Hence, we must have $\beta \succ^{\mathscr{A}} \alpha$. Using Lemma 11, $\beta \succ^{\mathscr{A}} \alpha$ and $(\alpha_j, x_{-j}) \in \overline{\mathscr{A}}$ imply $(\beta_j, x_{-j}) \in \mathscr{A}$, a contradiction. Hence, $x_{-j} \succsim^{(j)} y_{-j}$ implies $x_{-j} \succsim^{\mathscr{A}}_{-j} y_{-j}$.

Suppose that $x_{-j} \succsim^{\mathscr{A}}_{-j} y_{-j}$. Using Lemma 6, we know that $(\beta_j, x_{-j}) \in \mathcal{F}$, for some $\beta \in \Gamma$. If $Not[x_{-j} \succsim^{(j)} y_{-j}]$, we must have $(\beta_j, y_{-j}) \in \mathscr{A}$. But $x_{-j} \succsim^{\mathscr{A}}_{-j} y_{-j}$ then implies $(\beta_j, x_{-j}) \in \mathscr{A}$, a contradiction. □

Lemma 14. *For all $a, b \in \mathcal{X}$,*

$$a_{-j} \sim^{(j)} b_{-j} \Leftrightarrow \left\{ \begin{array}{l} (\alpha_j, a_{-j}) \in \mathcal{F} \\ (\alpha_j, b_{-j}) \in \mathcal{F} \end{array} \right\} \text{ for some } \alpha \in \Gamma.$$

Proof. The [$\Leftarrow$] part follows from the definition of $\succsim^{(j)}$. Let us prove the [$\Rightarrow$] part. Suppose that $a_{-j} \sim^{(j)} b_{-j}$, so that, for some $\alpha, \beta \in \Gamma$,

$$\begin{aligned}(\alpha_j, a_{-j}) \in \overline{\mathscr{A}} \text{ and } (\alpha_j, b_{-j}) \in \mathscr{U},\\ (\beta_j, b_{-j}) \in \overline{\mathscr{A}} \text{ and } (\beta_j, a_{-j}) \in \mathscr{U}.\end{aligned}$$

Using Lemma 6, we know that there is a $\delta \in \Gamma$ such that $(\delta_j, a_{-j}) \in \mathcal{F}$. If $(\delta_j, b_{-j}) \in \mathcal{F}$, there is nothing to prove.

Suppose that $(\delta_j, b_{-j}) \in \mathscr{U}^{\circ}$. Because $(\beta_j, b_{-j}) \in \overline{\mathscr{A}}$, using Lemma 7 implies that $\beta \succsim^{\mathscr{A}} \iota(b_{-j}) \succ^{\mathscr{A}} \delta$, so that $\beta \succ^{\mathscr{A}} \delta$. Using Lemma 11, $(\delta_j, a_{-j}) \in \mathcal{F}$ and $\beta \succ^{\mathscr{A}} \delta$ imply $(\beta_j, a_{-j}) \in \mathscr{A}$, a contradiction.

Suppose that $(\delta_j, b_{-j}) \in \mathscr{A}$. Since $(\alpha_j, b_{-j}) \in \mathscr{U}$, we must have $\delta \succ^{\mathscr{A}} \alpha$. Using Lemma 11, $(\alpha_j, a_{-j}) \in \overline{\mathscr{A}}$ and $\delta \succ^{\mathscr{A}} \alpha$ imply $(\delta_j, a_{-j}) \in \mathscr{A}$, a contradiction. □

Lemma 15. *For all $a, b \in \mathcal{X}$,*

$$\begin{aligned}a_{-j} \succ^{(j)} b_{-j} &\Leftrightarrow \left\{\begin{matrix}(\alpha_j, a_{-j}) \in \mathscr{A}\\ (\alpha_j, b_{-j}) \in \mathcal{F}\end{matrix}\right\} \textit{ for some } \alpha \in \Gamma,\\ &\Leftrightarrow \left\{\begin{matrix}(\delta_j, a_{-j}) \in \mathcal{F}\\ (\delta_j, b_{-j}) \in \mathscr{U}^{\circ}\end{matrix}\right\} \textit{ for some } \delta \in \Gamma.\end{aligned}$$

Proof. Suppose that $(\alpha_j, a_{-j}) \in \mathscr{A}$ and $(\alpha_j, b_{-j}) \in \mathcal{F}$, for some $\alpha \in \Gamma$. This implies $a_{-j} \succsim^{(j)} b_{-j}$. If $b_{-j} \succsim^{(j)} a_{-j}$, we have $a_{-j} \sim^{(j)} b_{-j}$ and Lemma 14 implies that $(\beta_j, a_{-j}) \in \mathcal{F}$ and $(\beta_j, b_{-j}) \in \mathcal{F}$, for some $\beta \in \Gamma$. Because $(\alpha_j, a_{-j}) \in \mathscr{A}$, we must have that $\alpha \succ^{\mathscr{A}} \beta$. Using $(\alpha_j, b_{-j}) \in \mathcal{F}$, Lemma 11 implies $(\beta_j, b_{-j}) \in \mathscr{U}^{\circ}$, a contradiction.

Suppose now that $a_{-j} \succ^{(j)} b_{-j}$. We know that there is $\alpha \in \Gamma$ such that $(\alpha_j, b_{-j}) \in \mathcal{F}$. If $(\alpha_j, a_{-j}) \in \mathscr{U}$, we obtain $b_{-j} \succsim^{(j)} a_{-j}$, a contradiction. Hence, we must have $(\alpha_j, a_{-j}) \in \mathscr{A}$, as required. This completes the proof of the first equivalence.

Suppose that $(\delta_j, a_{-j}) \in \mathcal{F}$ and $(\delta_j, b_{-j}) \in \mathscr{U}^{\circ}$, for some $\delta \in \Gamma$. This implies $a_{-j} \succsim^{(j)} b_{-j}$. If $b_{-j} \succsim^{(j)} a_{-j}$, we have $a_{-j} \sim^{(j)} b_{-j}$ and Lemma 14 implies that $(\beta_j, a_{-j}) \in \mathcal{F}$ and $(\beta_j, b_{-j}) \in \mathcal{F}$, for some $\beta \in \Gamma$. Because $(\delta_j, b_{-j}) \in \mathscr{U}^{\circ}$, we must have that $\beta \succ^{\mathscr{A}} \delta$. Using $(\beta_j, a_{-j}) \in \mathcal{F}$, Lemma 11 implies $(\delta_j, a_{-j}) \in \mathscr{A}$, a contradiction.

Suppose now that $a_{-j} \succ^{(j)} b_{-j}$. We know that there is $\delta \in \Gamma$ such that $(\delta_j, a_{-j}) \in \mathcal{F}$. If $(\delta_j, b_{-j}) \in \overline{\mathscr{A}}$, we obtain $b_{-j} \succsim^{(j)} a_{-j}$, a contradiction. Hence, we must have $(\delta_j, b_{-j}) \in \mathscr{U}^{\circ}$, as required. □

Lemma 16. *The relation $\succsim^{(j)}$ satisfies CCI, i.e.,*

$$\left.\begin{array}{c}(\alpha_i, a_{-ij}) \succsim^{(j)} (\beta_i, b_{-ij}) \\ \textit{and} \\ (\gamma_i, b_{-ij}) \succsim^{(j)} (\delta_i, a_{-ij}) \\ \textit{and} \\ (\delta_k, c_{-kj}) \succsim^{(j)} (\gamma_k, d_{-kj})\end{array}\right\} \Rightarrow (\alpha_k, c_{-kj}) \succsim^{(j)} (\beta_k, d_{-kj}),$$

for all $i, k \in N \setminus \{j\}$, all $a, b, c, d \in \mathcal{X}$ and all $\alpha, \beta, \gamma, \delta \in \Gamma$.

Proof. Suppose that $(\alpha_i, a_{-ij}) \succsim^{(j)} (\beta_i, b_{-ij})$, $(\gamma_i, b_{-ij}) \succsim^{(j)} (\delta_i, a_{-ij})$, $(\delta_k, c_{-kj}) \succsim^{(j)} (\gamma_k, d_{-kj})$, and, in contradiction with the thesis, that $(\beta_k, d_{-kj}) \succ^{(j)} (\alpha_k, c_{-kj})$.

Using Lemma 15 and the definition of $\succsim^{(j)}$, we know that we have

$$(\tau_j, \alpha_i, a_{-ij}) \in \overline{\mathscr{A}}, \quad \text{(2a)} \qquad (\tau_j, \beta_i, b_{-ij}) \in \mathscr{U}, \quad \text{(2e)}$$
$$(\lambda_j, \gamma_i, b_{-ij}) \in \overline{\mathscr{A}}, \quad \text{(2b)} \qquad (\lambda_j, \delta_i, a_{-ij}) \in \mathscr{U}, \quad \text{(2f)}$$
$$(\zeta_j, \delta_k, c_{-kj}) \in \overline{\mathscr{A}}, \quad \text{(2c)} \qquad (\zeta_j, \gamma_k, d_{-kj}) \in \mathscr{U}, \quad \text{(2g)}$$
$$(\rho_j, \beta_k, d_{-kj}) \in \mathscr{A}, \quad \text{(2d)} \qquad (\rho_j, \alpha_k, c_{-kj}) \in \mathcal{F}, \quad \text{(2h)}$$

for some $\tau, \lambda, \zeta, \rho \in \Gamma$.

From (2d), using A2 (Connectedness), we can find $\rho' \in \Gamma$ such that $\rho \succ^{\mathscr{A}} \rho'$ and $(\rho'_j, \beta_k, d_{-kj}) \in \mathscr{A}$. From (2h), using Lemma 11, we obtain. $(\rho'_j, \alpha_k, c_{-kj}) \in \mathscr{U}^\circ$. Let $\ell \neq j, k$. Using the fact that $\succsim^{\mathscr{A}}$ is unbounded and A2 (Connectedness), we can find $\chi, \psi \in \Gamma$ such that $d_\ell \succ^{\mathscr{A}} \psi$, $\chi \succ^{\mathscr{A}} c_\ell$, and

$$(\rho'_j, \beta_k, \psi_\ell, d_{-k\ell j}) \in \mathscr{A}, \quad \text{(3a)} \qquad (\rho'_j, \alpha_k, \chi_\ell, c_{-k\ell j}) \in \mathscr{U}^\circ. \quad \text{(3b)}$$

Using (2c) and (2g), we have $(\zeta_j, \delta_k, \chi_\ell, c_{-k\ell j}) \in \mathscr{A}$, $(\zeta_j, \gamma_k, \psi_\ell, d_{-k\ell j}) \in \mathscr{U}^\circ$. Using the fact that $\succsim^{\mathscr{A}}$ is unbounded and A2 (Connectedness), we can find $\delta', \gamma' \in \Gamma$ such that $\gamma' \succ^{\mathscr{A}} \gamma$ and $\delta \succ^{\mathscr{A}} \delta'$ and

$$(\zeta_j, \delta'_k, \chi_\ell, c_{-k\ell j}) \in \mathscr{A}, \quad \text{(3c)} \qquad (\zeta_j, \gamma'_k, \psi_\ell, d_{-k\ell j}) \in \mathscr{U}^\circ. \quad \text{(3d)}$$

Using (2b) and (2f), we obtain $(\lambda_j, \gamma'_i, b_{-ij}) \in \mathscr{A}$, $(\lambda_j, \delta'_i, a_{-ij}) \in \mathscr{U}^\circ$. Let $m \neq j, i$. Using the fact that $\succsim^{\mathscr{A}}$ is unbounded and A2 (Connectedness), we can find $\nu, \omega \in \Gamma$ such that $b_m \succ^{\mathscr{A}} \omega$, $\nu \succ^{\mathscr{A}} a_m$, and

$$(\lambda_j, \gamma'_i, \omega_m, b_{-imj}) \in \mathscr{A}, \quad \text{(3e)} \qquad (\lambda_j, \delta'_i, \nu_m, a_{-imj}) \in \mathscr{U}^\circ. \quad \text{(3f)}$$

Using (2a) and (2e), we obtain

$$(\tau_j, \alpha_i, \nu_m, a_{-imj}) \in \mathscr{A}, \quad \text{(3g)} \qquad (\tau_j, \beta_i, \omega_m, b_{-imj}) \in \mathscr{U}^\circ. \quad \text{(3h)}$$

Summarizing, after rearranging the terms, we have:

$$
\begin{array}{rlrl}
(\alpha_i, \tau_j, \nu_m, a_{-imj}) \in \mathscr{A}, & \text{(3g)} & (\delta'_i, \lambda_j, \nu_m, a_{-imj}) \in \mathscr{U}^\circ, & \text{(3f)} \\
(\gamma'_i, \lambda_j, \omega_m, b_{-imj}) \in \mathscr{A}, & \text{(3e)} & (\beta_i, \tau_j, \omega_m, b_{-imj}) \in \mathscr{U}^\circ, & \text{(3h)} \\
(\beta_k, \rho'_j, \psi_\ell, d_{-k\ell j}) \in \mathscr{A}, & \text{(3a)} & (\gamma'_k, \zeta_j, \psi_\ell, d_{-k\ell j}) \in \mathscr{U}^\circ, & \text{(3d)} \\
(\delta'_k, \zeta_j, \chi_\ell, c_{-k\ell j}) \in \mathscr{A}, & \text{(3c)} & (\alpha_k, \rho'_j, \chi_\ell, c_{-k\ell j}) \in \mathscr{U}^\circ, & \text{(3b)}
\end{array}
$$

which violates A1 (Tradeoff consistency). $\square$

Lemma 17. *All states $i \in N \setminus \{j\}$ are essential for the relation $\succsim^{(j)}$.*

Proof. Let $i \in N \setminus \{j\}$. We have to show that there are $a \in \mathcal{X}$ and $\alpha, \beta \in \Gamma$ such that $(\alpha_i, a_{-ij}) \succ^{(j)} (\beta_i, a_{-ij})$.

Let $a \in \mathcal{X}$ and $\alpha \in \Gamma$. Using Lemma 6, we can find $\xi \in \Gamma$ such that $(\xi_j, \alpha_i, a_{-ij}) \in \mathcal{F}$. Using Lemma 4, there is $\beta \in \Gamma$ such that $\alpha \succ^{\mathscr{A}} \beta$. Using Lemma 11, we obtain $(\xi_j, \beta_i, a_{-ij}) \in \mathscr{U}^\circ$. Using Lemma 15, this implies that $(\alpha_i, a_{-ij}) \succ^{(j)} (\beta_i, a_{-ij})$. $\square$

Let $i \neq j$. We define the relation $\succsim_i^{(j)}$ on Γ letting,

$$\alpha \succsim_i^{(j)} \beta \Leftrightarrow [(\alpha_i, a_{-ij}) \succsim^{(j)} (\beta_i, a_{-ij}), \text{ for all } a \in \mathcal{X}].$$

Because $\succsim^{(j)}$ is a weak order satisfying CCI, it satisfies CI. Hence, we have:

$$\alpha \succsim_i^{(j)} \beta \Leftrightarrow [(\alpha_i, a_{-ij}) \succsim^{(j)} (\beta_i, a_{-ij}), \text{ for some } a \in \mathcal{X}].$$

Lemma 18. *For all $i \neq j$, we have $\succsim_i^{(j)} = \succsim^{\mathscr{A}}$.*

Proof. Suppose that $\beta \succ^{\mathscr{A}} \alpha$ and $\alpha \succsim_i^{(j)} \beta$. We know that $(\delta_j, \alpha_i, a_{-ij}) \in \mathcal{F}$, for some $\delta \in \Gamma$. Using $\beta \succ^{\mathscr{A}} \alpha$, we obtain $(\delta_j, \beta_i, a_{-ij}) \in \mathscr{A}$. Using Lemma 15, this implies $(\beta_i, a_{-ij}) \succ^{(j)} (\alpha_j, a_{-ij})$, contradicting $\alpha \succsim_i^{(j)} \beta$.

Suppose now that we have $(\beta_i, a_{-ij}) \succ^{(j)} (\alpha_j, a_{-ij})$, for some $a \in \mathcal{X}$, and $\alpha \succsim^{\mathscr{A}} \beta$. Using Lemma 15, we have $(\delta_j, \beta_i, a_{-ij}) \in \mathscr{A}$ and $(\delta_j, \alpha_i, a_{-ij}) \in \mathcal{F}$, for some $\delta \in \Gamma$. Using $\alpha \succsim^{\mathscr{A}} \beta$ and $(\delta_j, \beta_i, a_{-ij}) \in \mathscr{A}$, we obtain $(\delta_j, \alpha_i, a_{-ij}) \in \mathscr{A}$, a contradiction. $\square$

Lemma 19. *The relation $\succsim^{(j)}$ is continuous, i.e., for all $i \in N \setminus \{j\}$ and all $a \in \mathcal{X}$ the sets $\{b_{-j} \in \prod_{i \neq j} \Gamma_i : a_{-j} \succ^{(j)} b_{-j}\}$ and $\{b_{-j} \in \prod_{i \neq j} \Gamma_i : b_{-j} \succ^{(j)} a_{-j}\}$ are open.*

Proof. We prove that $\{b_{-j} \in \prod_{i \neq j} \Gamma_i : b_{-j} \succ^{(j)} a_{-j}\}$ is open, the other case being similar. Using Lemma 15, we have $b_{-j} \succ^{(j)} a_{-j}$ iff $(\alpha_j, b_{-j}) \in \mathscr{A}$ and $(\alpha_j, a_{-j}) \in \mathcal{F}$, for some $\alpha \in \Gamma$.

Let $k \neq j$. The set $\Delta_k^+(a_{-j}) = \{\delta \in \Gamma : (\alpha_j, \delta_k, b_{-jk}) \in \mathscr{A}\}$ is nonempty (it contains b_k). The set $\Delta_k^-(a_{-j}) = \{\delta \in \Gamma : (\alpha_j, \delta_k, b_{-jk}) \in \mathscr{U}\}$ is nonempty (using Lemma 6). Using A3 (Openness), we know that the set $\Delta_k^+(a_{-j})$ is open.

We claim that there is $\beta \in \Gamma$ such that $b_k \succ^{\mathscr{A}} \beta$ and $(\alpha_j, \beta_k, b_{-jk}) \in \mathscr{A}$. Otherwise, we would have $\Delta_k^+(a_{-j}) = \{\delta \in \Gamma : \delta \succsim^{\mathscr{A}} b_k\}$ which would imply that $\Delta_k^+(a_{-j})$ is closed. This contradicts A2 (Connectedness) since we know that $\Delta_k^+(a_{-j})$ is both closed and open, while being nonempty and different from Γ.

Using $(\alpha_j, \beta_k, b_{-jk}) \in \mathscr{A}$ as a starting point, we can now use the same reasoning on any state $\ell \neq j, k$. It is easy to see that this will lead to find $c \in \mathcal{X}$ such that $(\alpha_j, c_{-j}) \in \mathscr{A}$ and, for all $i \neq j$, $b_i \succ^{\mathscr{A}} c_i$. This implies that each $b_{-j} \in \prod_{i \neq j} \Gamma_i$ such that $b_{-j} \succ^{(j)} a_{-j}$ is contained in an open set. The set $\{b_{-j} \in \prod_{i \neq j} \Gamma_i : b_{-j} \succ^{(j)} a_{-j}\}$ is therefore open. □

B.2. Additive representation of $\succsim^{(1)}$

We start by showing that the relation $\succsim^{(j)}$ has an additive representation. We only show this for $j = 1$.

Lemma 20. *There are $n-1$ continuous functions $v_2, v_3, \ldots, v_n$ such that for all $a, b \in \mathcal{X}$,*

$$a_{-1} \succsim^{(1)} b_{-1} \Leftrightarrow \sum_{i=2}^{n} v_i(a_i) \geq \sum_{i=2}^{n} v_i(b_i). \tag{4}$$

If two sets of functions $\langle v_i \rangle_{i \neq 1}$ and $\langle u_i \rangle_{i \neq 1}$ satisfy (4) *then there are there are real numbers $A, B_2, B_3, \ldots, B_n$ such that $A > 0$ and, for all $i \in N \setminus \{1\}$, $v_i = Au_i + B_i$.*

Proof. The plan is to use Wakker (1989a, Th. III.6.6, p. 70) on $\succsim^{(1)}$ with the following modifications. The hypothesis of topological separability can be omitted when at least two attributes are essential (see Wakker 1989a, Remark A.3.1, page 163 or Wakker 1988a, Th. 6.2, page 430). The topology on Γ_i can be taken to be the order topology generated the induced marginal relations on attribute i (Wakker 1989a, Step 1.2, p. 50).

We know from Lemma 13 that $\succsim^{(1)}$ is a weak order. Using Lemma 17, we know all states are essential and $n - 1 \geq 2$. Lemma 16 has shown that $\succsim^{(1)}$ satisfies CCI, which implies generalized triple cancellation in the sense of Wakker (1989a, Th. III.6.6, p. 70). Because of Lemma 18 and A2 (Connectedness), we know that each Γ_i is connected in the order topology generated by $\succsim_i^{(1)} = \succsim^{\mathscr{A}}$. Using Lemma 19, we know that $\succsim^{(1)}$ is continuous. Hence, applying Wakker (1989a, Th. III.6.6, p. 70) together with Wakker (1989a, Observation III.6.6′, p. 71) gives the desired result. □

B.3. Additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$

Lemma 21. *If $\succsim^{(1)}$ has a continuous additive representation then there is an additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$.*

Proof. Suppose that $\langle v_i \rangle_{i \neq 1}$ is a continuous additive representation of $\succsim^{(1)}$. Let $a \in \mathcal{X}$. Using Lemma 6, we can find $\alpha \in \Gamma$ such that $(\alpha_1, a_{-1}) \in \mathcal{F}$. Conversely, given any $\alpha \in \Gamma$, we have $(\alpha_1, a_{-1}) \in \mathcal{F}$, for some $a \in \mathcal{X}$.

Now, define v_1 letting, for all $\alpha \in \Gamma$,

$$v_1(\alpha) = -\sum_{i \neq 1} v_i(a_i) \text{ if } (\alpha_1, a_{-1}) \in \mathcal{F}.$$

It is easy to see that v_1 is well-defined. Indeed if $(\alpha_1, a_{-1}) \in \mathcal{F}$ and $(\alpha_1, b_{-1}) \in \mathcal{F}$, we know from Lemma 14 that $a_{-1} \sim^{(1)} b_{-1}$, so that:

$$\sum_{i \neq 1} v_i(a_i) = \sum_{i \neq 1} v_i(b_i).$$

Let us now show that such a function v_1 together with the functions $\langle v_i \rangle_{i \neq 1}$ give an additive representation for $\langle \mathscr{A}, \mathscr{U} \rangle$.

If $(a_1, a_{-1}) \in \mathcal{F}$, then, by construction, we have $v_1(a_1) + \sum_{i \neq 1} v_i(a_i) = 0$.

Suppose that $(a_1, a_{-1}) \in \mathscr{A}$. Using Lemma 6 on any state k other than 1, we know that $(a_1, \alpha_k, a_{-1k}) \in \mathcal{F}$, for some $\alpha \in \Gamma$. Hence, we have:

$$v_1(a_1) = -\left[v_k(\alpha) + \sum_{i \neq 1, k} v_i(a_i) \right].$$

Using Lemma 15, $(a_1, a_{-1}) \in \mathscr{A}$ and $(a_1, \alpha_k, a_{-1k}) \in \mathcal{F}$ imply $a_{-1} \succ^{(j)} (\alpha_k, a_{-1k})$, so that:

$$\sum_{i \neq 1} v_i(a_i) > v_k(\alpha) + \sum_{i \neq 1, k} v_i(a_i),$$

which implies

$$v_1(a_1) + \sum_{i \neq 1} v_i(a_i) > 0.$$

That $(a_1, a_{-1}) \in \mathscr{U}^\circ$ implies $v_1(a_1) + \sum_{i \neq 1} v_i(a_i) < 0$ is shown similarly. Hence we have built an additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$. Observe that in this representation, we know that $v_2, v_3, \ldots, v_n$ are continuous functions. $\square$

Lemma 22. *If there is an additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ then, for all $i \in N$,*

$$\alpha \succ^{\mathscr{A}} \beta \Rightarrow v_i(\alpha) > v_i(\beta).$$

Proof. Suppose that $\alpha \succ^{\mathscr{A}} \beta$. We have, for some $a \in \mathcal{X}$, $(\alpha_i, a_{-i}) \in \mathscr{A}$ and $(\beta_i, a_{-i}) \notin \mathscr{A}$. This implies that $v_i(\alpha_i) + \sum_{j \neq 1} v_j(a_i) > 0$ and $v_i(\beta_i) + \sum_{j \neq 1} v_j(a_i) \leq 0$, so that $v_i(\alpha) > v_i(\beta)$. $\square$

Lemma 23. *In the additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ built in Lemma 21 the range of each v_i is $\mathbb{R}$.*

Proof. We first show that this is the case for all $i \neq 1$. Take $\alpha, \beta \in \Gamma$ such that $\alpha \succ^{\mathscr{A}} \beta$. Using Lemma 22, we know that $v_i(\alpha) > v_i(\beta)$. Let $v_i(\alpha) - v_i(\beta) = K > 0$. Let us show that we can find $\gamma, \delta \in \Gamma$ such that $v_i(\gamma) - v_i(\alpha) = v_i(\alpha) - v_i(\beta) = v_i(\beta) - v_i(\delta) = K > 0$, which will complete the proof since we know that v_i is continuous.

Take $j \neq i, 1$. We can find $\lambda \in \Gamma$ such that $(\alpha_i, \lambda_j, a_{-ij}) \in \mathcal{F}$, for some $a \in \mathcal{X}$. We can find $\mu \in \Gamma$ and $b \in \mathcal{X}$ such that $(\beta_i, \mu_j, a_{-ij}) \in \mathcal{F}$ and $(\beta_i, \lambda_j, b_{-ij}) \in \mathcal{F}$. Now, there is $\delta \in \Gamma$ such that $(\delta_i, \mu_j, b_{-ij}) \in \mathcal{F}$. This implies $v_i(\alpha) - v_i(\beta) = v_i(\beta) - v_i(\delta)$. The other part of the proof is similar.

Consider now the case of v_1. For all $a \in \mathcal{X}$, we know that there is $\alpha \in \Gamma$ such that $(\alpha_1, a_{-1}) \in \mathcal{F}$. This implies $v_1(\alpha) = -\sum_{i=2}^n v_i(a_i)$. Hence, v_1 has the same range as $-\sum_{i=2}^n v_i$, i.e., $\mathbb{R}$. $\square$

Lemma 24. *In the additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ built in Lemma 21 each function v_i is continuous.*

Proof. We only have to show that v_1 is continuous. Let us show that v_1 on the set $\Gamma/{\sim^{\mathscr{A}}}$ is continuous. If $\alpha \succ^{\mathscr{A}} \beta$, we know that $v_1(\alpha) > v_1(\beta)$. Hence the function v_1 goes from $\Gamma/{\sim^{\mathscr{A}}}$ endowed with the order topology generated by $\succsim^{\mathscr{A}}$ to $\mathbb{R}$ endowed with the standard topology. This function is order preserving. It is bijective since the range of v_1 is $\mathbb{R}$. Hence, it is a homeomorphism (Munkres 1975, p. 111). The proof is complete since we can extend v_1 to Γ making it constant on each equivalence class of $\sim^{\mathscr{A}}$, in view of the construction of v_1 in Lemma 21 and Lemma 11. $\square$

Lemma 25. *If there is a continuous additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$, i.e.,* (1) *holds, then, for all $a \in \mathcal{X}$,*

$$a \in \mathcal{F} \Leftrightarrow \sum_{i=1}^{n} v_i(a_i) = 0.$$

Proof. Suppose that $a \in \mathcal{F}$. Because $a \in \mathcal{F}$ implies $a \in \mathscr{U}$, we know that $\sum_{i=1}^{n} v_i(a_i) \leq 0$. Suppose that $\sum_{i=1}^{n} v_i(a_i) < 0$.

Let $j \in N$. Because v_j is continuous and has range $\mathbb{R}$, there is $\alpha \in \Gamma$ such that $v_j(\alpha) + \sum_{i \neq j} v_i(a_i) < 0$. and $v_j(\alpha) > v_j(a_j)$.

Let $k \neq j$. Because v_k is continuous and has range $\mathbb{R}$, there is $\beta \in \Gamma$ such that $v_j(\alpha) + v_k(\beta) + \sum_{i \neq j,k} v_i(a_i) > 0$, and $v_j(a_j) + v_k(\beta) + \sum_{i \neq j,k} v_i(a_i) < 0$. This implies $(\alpha_j, \beta_k, a_{-jk}) \in \mathscr{A}$ and $(a_j, \beta_k, a_{-jk}) \in \mathscr{U}$, so that $\alpha \succ^{\mathscr{A}} a_j$. Because $a = (a_j, a_{-j}) \in \mathcal{F}$, Lemma 11 implies $(\alpha_j, a_{-i}) \in \mathscr{A}$, so that $v_j(\alpha) + \sum_{i \neq j} v_i(a_i) > 0$, a contradiction.

Conversely, suppose that $\sum_{i=1}^{n} v_i(a_i) = 0$. By construction, it is impossible that $a \in \mathscr{A}$. Suppose that $a \in \mathscr{U}^{\circ}$. Let $j \in N$. Because $\mathscr{U}^{\circ}$ is open, we can find $\alpha \in \Gamma$ such that $\alpha \succ^{\mathscr{A}} a_j$, and $(\alpha_j, a_{-j}) \in \mathscr{U}^{\circ}$. Using Lemma 22, we have $v_j(\alpha) > v_j(a_j)$, so that $v_j(\alpha) + \sum_{i \neq j} v_i(a_i) > 0$. This implies $(\alpha_j, a_{-j}) \in \mathscr{A}$, a contradiction. $\square$

Lemma 26. *If there is a continuous additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$, i.e.,* (1) *holds, then each function v_i represents $\succsim^{\mathscr{A}}$.*

Proof. In view of Lemma 22, we only have to prove that $\alpha \sim^{\mathscr{A}} \beta$ implies $v_i(\alpha) = v_i(\beta)$ Using Lemma 14, $\alpha \sim^{\mathscr{A}} \beta$ implies that for some $a \in \mathcal{X}$, $(\alpha_i, a_{-i}) \in \mathcal{F}$ and $(\beta_i, a_{-i}) \in \mathcal{F}$. Using Lemma 25, this implies that $v_i(\alpha_i) + \sum_{j \neq 1} v_j(a_i) = 0$ and $v_i(\beta_i) + \sum_{j \neq 1} v_j(a_i) = 0$, so that $v_i(\alpha) = v_i(\beta)$. $\square$

Lemma 27. *There are n continuous functions $v_1, v_2, v_3, \ldots, v_n$ such that for all $a, b \in \mathcal{X}$,*

$$a \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} v_i(a_i) > 0.$$

The range of each function v_i is $\mathbb{R}$.

If $\langle v_i \rangle_{i\in N}$ and $\langle u_i \rangle_{i\in N}$ are two sets of functions giving an additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ then there are real numbers $A, B_1, B_2, \ldots, B_n$ such that $A > 0$ and $\sum_{i=1}^n B_i = 0$ such that, for all $i \in N$, $v_i = Au_i + B_i$.

Proof. The existence part follows from combining Lemma 20 with Lemma 21. The statement about the range of v_i follows from Lemma 23. The continuity of each v_i follows from Lemma 24.

Let us prove the uniqueness part. It is first clear that multiplying each function v_i by the same positive constant gives another valid representation. Adding to each v_i a constant B_i is also possible, provided that $\sum_{i=1}^n B_i = 0$. Let us show that only such transformations are possible.

Let $\langle v_i \rangle_{i\in \mathbb{N}}$ be any additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$. Let us show that $\langle v_i \rangle_{i\neq 1}$ must be an additive representation of $\succsim^{(1)}$. Suppose that $a_{-1} \sim^{(1)} b_{-1}$. Using Lemmas 14 and 25, we must have $\sum_{i\neq 1} v_i(a_i) = \sum_{i\neq 1} v_i(b_i)$. Similarly, using Lemmas 15 and 25, $a_{-1} \succ^{(1)} b_{-1}$ implies $\sum_{i\neq 1} v_i(a_i) > \sum_{i\neq 1} v_i(b_i)$. Hence, any additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ must also be an additive representation of $\succsim^{(1)}$. Conversely, Lemma 21 has shown that given any additive representation for $\succsim^{(1)}$, we can obtain an additive representation for $\langle \mathscr{A}, \mathscr{U} \rangle$ that uses the same functions for $i \neq 1$.

Because $\succsim^{(1)}$ satisfies all conditions of Wakker (1989a, Th. III.6.6, p. 70), we know that we know that any two additive representations of $\succsim^{(1)}$, $\langle u_i \rangle_{i\neq 1}$ and $\langle v_i \rangle_{i\neq 1}$, must be such that $v_i(a_i) = Au_i(a_i) + B_i$, with $A > 0$.

Let $a \in \mathcal{X}$. Using Lemma 6 on any state k distinct from 1, we have $(a_1, b_{-1}) \in \mathcal{F}$, for some $b \in \mathcal{X}$. This implies that if $\langle u_i \rangle_{i\in \mathbb{N}}$ and $\langle v_i \rangle_{i\in \mathbb{N}}$ are two representations of $\langle \mathscr{A}, \mathscr{U} \rangle$, for all $a \in \mathcal{X}$, we have

$$u_1(a_1) = -\sum_{i\neq 1} u_i(b_i),$$

$$v_1(a_1) = -\sum_{i\neq 1} v_i(b_i) = -\sum_{i\neq 1} [Au_i(b_i) + B_i],$$

where $b \in \mathcal{X}$ is such that $(a_1, b_{-1}) \in \mathcal{F}$. Therefore, we obtain $v_1 = Au_1 - \sum_{i\neq 1} B_i$. Hence, the two sets of functions will be such that, for all $i \in N$, $v_i = Au_i + B_i$ with $A > 0$ and $\sum_{i=1}^n B_i = 0$. □

C. Subjective Expected Utility representation of $\langle \mathscr{A}, \mathscr{U} \rangle$

We now show that the continuous additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ can be modified in such a way as to give a continuous representation of $\langle \mathscr{A}, \mathscr{U} \rangle$

in model SEU_m.

Lemma 28. *Consider the continuous additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ built in Lemma 27. Take any two states $i, j \in N$ with $i \neq j$. The function v_i is a positive affine transformation of the function v_j, i.e., there are $A, B \in \mathbb{R}$ with $A > 0$ such that $v_i = Av_j + B$.*

Proof. Let us show that, for all $\alpha, \beta, \gamma, \delta \in \Gamma$, we have

$$v_j(\alpha) - v_j(\beta) = v_j(\gamma) - v_j(\delta) \Rightarrow v_i(\alpha) - v_i(\beta) = v_i(\gamma) - v_i(\delta).$$

Because both v_i and v_j are continuous and have range $\mathbb{R}$, this implies that v_i is an affine transformation of v_j. Because both functions represent the nontrivial relation $\succsim^{\mathscr{A}}$, this affine transformation must be strictly positive.

Suppose now that $v_j(\alpha) - v_j(\beta) = v_j(\gamma) - v_j(\delta)$. Take any $k \neq i, j$. We can find $\lambda, \mu \in \Gamma$ and $a, b \in \mathcal{X}$ such that:

$$(\alpha_j, \lambda_k, a_{-jk}) \in \mathcal{F}, \quad (\beta_j, \mu_k, a_{-jk}) \in \mathcal{F},$$
$$(\gamma_j, \lambda_k, b_{-jk}) \in \mathcal{F}.$$

Because $v_j(\alpha) - v_j(\beta) = v_j(\gamma) - v_j(\delta)$, we must have $(\delta_j, \mu_k, b_{-jk}) \in \mathcal{F}$.

We can find $\lambda', \mu' \in \Gamma$ and $c, d \in \mathcal{X}$ such that

$$(\alpha_i, \lambda'_k, c_{-ik}) \in \mathcal{F}, \quad (\beta_i, \mu'_k, c_{-ik}) \in \mathcal{F},$$
$$(\gamma_i, \lambda'_k, d_{-ik}) \in \mathcal{F}.$$

Because $(\alpha_j, \lambda_k, a_{-jk}) \in \mathcal{F}$ and $(\gamma_j, \lambda_k, b_{-jk}) \in \mathcal{F}$, Lemma 14 implies $(\alpha_j, a_{-jk}) \sim^{(k)} (\gamma_j, b_{-jk})$. Similarly, $(\beta_j, \mu_k, a_{-jk}) \in \mathcal{F}$ and $(\delta_j, \mu_k, b_{-jk}) \in \mathcal{F}$ imply $(\beta_j, a_{-jk}) \sim^{(k)} (\delta_j, b_{-jk})$. Finally $(\alpha_i, \lambda'_k, c_{-ik}) \in \mathcal{F}$ and $(\gamma_i, \lambda'_k, d_{-ik}) \in \mathcal{F}$ imply $(\alpha_i, c_{-ik}) \sim^{(k)} (\gamma_i, d_{-ik})$.

We have $(\delta_j, b_{-jk}) \succsim^{(k)} (\beta_j, a_{-jk})$, $(\alpha_j, a_{-jk}) \succsim^{(k)} (\gamma_j, b_{-jk})$, $(\gamma_i, d_{-ik}) \succsim^{(k)} (\alpha_i, c_{-ik})$. Using Lemma 16, we know that $\succsim^{(k)}$ satisfies CCI. Hence, we obtain $(\delta_i, d_{-ik}) \succsim^{(k)} (\beta_i, c_{-ik})$.

Similarly, we have $(\beta_j, a_{-jk}) \succsim^{(k)} (\delta_j, b_{-jk})$, $(\gamma_j, b_{-jk}) \succsim^{(k)} (\alpha_j, a_{-jk})$, $(\alpha_i, c_{-ik}) \succsim^{(k)} (\gamma_i, d_{-ik})$. Using CCI, we obtain $(\beta_i, c_{-ik}) \succsim^{(k)} (\delta_i, d_{-ik})$.

Hence, we have $(\beta_i, c_{-ik}) \sim^{(k)} (\delta_i, d_{-ik})$. Since we know that $(\beta_i, \mu'_k, c_{-ik}) \in \mathcal{F}$, using Lemma 11, we obtain $(\delta_i, \mu'_k, d_{-ik}) \in \mathcal{F}$. Hence, we have

$$(\alpha_i, \lambda'_k, c_{-ik}) \in \mathcal{F}, \quad (\beta_i, \mu'_k, c_{-ik}) \in \mathcal{F},$$
$$(\gamma_i, \lambda'_k, d_{-ik}) \in \mathcal{F}, \quad (\delta_i, \mu'_k, d_{-ik}) \in \mathcal{F},$$

which implies that $v_i(\alpha) - v_i(\beta) = v_i(\gamma) - v_i(\delta)$. $\square$

Proof of Theorem 1. Existence. Using Lemma 27, we know that there is a continuous additive representation of $\langle \mathscr{A}, \mathscr{U} \rangle$. The function $V = \sum_{i=1}^{n} v_i$ on Γ is continuous and has range $\mathbb{R}$. Hence, there is $\alpha \in \Gamma$ such that $\overline{\alpha} \in \mathcal{F}$. Using the B_i in the uniqueness results of the functions v_i, we can always ensure that, for all $i \in N$, $v_i(\alpha) = 0$. Take any $\beta \succ^{\mathscr{A}} \alpha$. Take any $j \in N$. Using the A in the uniqueness result of the functions v_i, we can always ensure that, $v_j(\beta) = 1$. For all $i \in N$, let $v_i(\beta) = \lambda_i > 0$.

Define $u = v_j$ and, for all $i \in N$ let $p_i = \lambda_i / \sum_{k=1}^{n} \lambda_k$. With such definitions, all terms $p_i u$ are proportional to the v_i, so that for all $a \in \mathcal{X}$,

$$a \in \mathscr{A} \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) > 0.$$

Because the range of v_j is $\mathbb{R}$, the range of u is $\mathbb{R}$. The continuity of u follows from the continuity of v_j. This completes the proof of the existence part.

Uniqueness. It is clear that multiplying u by a positive constant leads to another continuous representation of $\langle \mathscr{A}, \mathscr{U} \rangle$ in model SEU$_m$. The only arbitrary choices made above were the choice of a particular j to set $u = v_j$ and the choice of a particular $\beta \in \Gamma$ with $\beta \succ^{\mathscr{A}} \alpha$ to set $v_j(\beta) = 1$. Indeed, the existence of a continuous representation in model SEU$_m$ implies that there is an element $\alpha \in \Gamma$ such that $\overline{\alpha} \in \mathcal{F}$. For this element, it is necessary to have $u(\alpha) = 0$. Since we have to rescale the functions v_i in such a way that for all $i \in N$, $v_i(\alpha) = 0$, all functions v_i become identical up to the multiplication by a positive constant. Now choosing $k \in N$ with $k \neq j$ to set $u = v_k$ and $\gamma \in \Gamma$ different from β and such that $\gamma \succ^{\mathscr{A}} \alpha$ to set $v_k(\gamma) = 1$, will only result in the multiplication of u by a positive constant. Finally, observe that the choice of the p_i is the only possible one to ensure that $\sum_{i=1}^{n} p_i = 1$ and that, for all $i \in N$, $p_i u$ is proportional to v_i. It is clear that multiplying each v_i by a positive constant leaves all p_i unchanged. □

Appendix II: Assessment

The assessment of the parameters of model SEU$_c$ have been discussed at length in the literature (see, e.g., Wakker 2010, Ch. 4). The purpose of this appendix is to sketch how the parameters of model SEU$_m$ could be assessed on the basis of a twofold partition $\langle \mathscr{A}, \mathscr{U} \rangle$ satisfying the conditions of Theorem 1.

For the assessment of model SEU_m, it is crucial to find acts that belong to the "frontier" between acceptable and unacceptable acts, i.e., to the set $\mathcal{F} = \overline{\mathscr{A}} \setminus \mathscr{A} = \overline{\mathscr{A}} \cap \mathscr{U}$. Indeed[6], the subjective expected utility of such acts in model SEU_m must be 0.

Although the frontier $\mathcal{F}$ is a derived concept, it seems feasible to build acts belonging to $\mathcal{F}$. Yet, obtaining acts in $\mathcal{F}$ is not as easy as obtaining two acts that are indifferent: this is the price to pay for working with our primitives.

To obtain an act in $\mathcal{F}$, we may proceed as follows. Let $a \in \mathcal{X}$ and $i \in N$. We know that there are $\alpha, \beta \in \Gamma$ such that $(\beta_i, a_{-i}) \in \mathscr{A}$ and $(\alpha_i, a_{-i}) \in \mathscr{U}$. Consider now the set $\{\gamma \in \Gamma : (\gamma_i, a_{-i}) \in \mathscr{U}\}$. We know that it is closed, nonempty and different from Γ. Taking the supremum, w.r.t. to $\succsim^{\mathscr{A}}$, of the elements of this set will lead to a unique (up to $\sim^{\mathscr{A}}$) element in Γ. This supremum clearly corresponds to an element in $\mathcal{F}$. We suppose below that acts in $\mathcal{F}$ can be assessed in such a way.

Practically, we start from an act (γ_i, a_{-i}) that belongs to $\mathscr{U}$ and we progressively increase γ (w.r.t. $\succsim^{\mathscr{A}}$) up to the point such that the act becomes equivalent to the status quo, in the sense that any further increase in γ will lead to an act in $\mathscr{A}$, i.e., strictly better than the status quo.

Alternatively, if the ordered partition $\langle \mathscr{A}, \mathscr{U} \rangle$ has been obtained w.r.t. to a clearly defined reference act $r \in \mathcal{X}$, one may simply try to obtain acts that are judged "equivalent" to the reference point r. Although our primitives do not involve any notion of equivalence, we may interpret it here as saying that an act $a \in \mathcal{X}$ is "equivalent" to r if improving a by any amount and on any state will result in an act in $\mathscr{A}$, i.e., an act strictly better than r.

D. Assessing the utility function

In order to assess u, let us show how we can find points in Γ that are equally spaced in terms of u. Using standard arguments, this will lead to an assessment procedure for u.

[6]In terms of assessment, it would have therefore been easier to work with a threefold ordered partition, as done in Bouyssou and Marchant (2009, 2010), with the understanding that the intermediate category plays the rôle of the frontier between acceptable and unacceptable acts. Including this frontier in the primitives however leads to more complex result than the one presented here.

Take any $\rho, \tau \in \Gamma$. Using Theorem 1, it is clear that $(\alpha_i, \rho_j, a_{-ij}) \in \mathcal{F}$ and $(\beta_i, \tau_j, a_{-ij}) \in \mathcal{F}$ imply that $p_i(u(\alpha) - u(\beta)) = p_j(u(\tau) - u(\rho))$. Under the conditions of Theorem 1, such $a \in \mathcal{X}$ and $\alpha, \beta \in \Gamma$ can always be found.

Let us now find $b \in \mathcal{X}$ and $\gamma \in \Gamma$ such that $(\beta_i, \rho_j, b_{-ij}) \in \mathcal{F}$ and $(\gamma_i, \tau_j, b_{-ij}) \in \mathcal{F}$. This will imply that $p_i(u(\beta) - u(\gamma)) = p_j(u(\tau) - u(\rho))$. Under the conditions of Theorem 1, such $b \in \mathcal{X}$ and $\gamma \in \Gamma$ can always be found.

We have therefore assessed three elements in $\alpha, \beta, \gamma \in \Gamma$ that are such that $u(\alpha) - u(\beta) = u(\beta) - u(\gamma)$. This is the beginning of a standard sequence. This standard sequence may be extended and refined using classical techniques.

The proper scaling of u can be ensured noticing that there is $\delta \in \Gamma$ such that $\overline{\delta} \in \mathcal{F}$, which implies $u(\delta) = 0$. We finally take any element $\lambda \in \Gamma$ such that $\lambda \succ^{\mathscr{A}} \delta$ and we set $u(\lambda) = 1$.

E. Assessing the subjective probabilities

We now suppose that u has been assessed and scaled in such a way that $u(\delta) = 0$ and $u(\lambda) = 1$.

Take $i, j \in N$ with $i \neq j$. We can find $\alpha, \beta \in \Gamma$ such that $(\lambda_i, \alpha_j, \delta_{-ij}) \in \mathcal{F}$ and $(\beta_i, \lambda_j, \delta_{-ij}) \in \mathcal{F}$.

Combining these two relations easily implies that: $p_i(1 - u(\beta)) = p_j(1 - u(\alpha))$. Moreover, observe that model SEU_m implies that $u(\beta) < 0$ and $u(\alpha) < 0$, so that $1 - u(\beta) > 0$ and $1 - u(\alpha) > 0$. Since we have assessed u, this gives a linear equation linking p_i and p_j, with p_j being a positive multiple of p_i.

We can use a similar technique to obtain a linear equation linking p_i to any p_k, with $k \neq i$. This gives $n - 1$ linear equations that are clearly independent. Adding the constraint that $\sum_{i=1}^{n} p_i = 1$ will lead to determine a unique value for the n numbers $p_1, p_2, \ldots, p_n$.

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