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KAZHDAN'S PROPERTY (T) WITH RESPECT TO NON-COMMUTATIVE L_p -SPACES

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Abstract

We show that a group with Kazhdan's property (T) has property (T_B) for B the Haagerup non-commutative $L_p(\mathcal{M})$ -space associated with a von Neumann algebra \mathcal{M} , $1 < p < \infty$. We deduce that higher rank groups have property $F_{L_p(\mathcal{M})}$.

Keywords: Haagerup non-commutative $L_p(\mathcal{M})$ -spaces, property (T) , group representations, Mazur map, property $F_{L_p(\mathcal{M})}$.

1 INTRODUCTION

Kazhdan's property (T) of a topological group G is an important rigidity property, defined in terms of the unitary representations of G on Hilbert spaces. We recall the precise definition :

Definition 1.1. A pair (G, H) of topological groups, where H is a closed subgroup of G , is said to have relative property (T) if there exist a compact subset Q of G and $\epsilon > 0$ such that : whenever a unitary representation π of G on a Hilbert space \mathcal{H} has a (Q, ϵ) -invariant vector, that is a vector $\xi \in \mathcal{H}$ such that

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| < \epsilon \|\xi\|$$

then π has a non-zero $\pi(H)$ -invariant vector. The pair (Q, ϵ) is called a Kazhdan pair.

A topological group G is said to have property (T) if the pair (G, G) has relative property (T) .

For more details on property (T) , see the monography [2]. The following variant of this property for Banach spaces was recently introduced by Bader, Furman, Gelander and Monod in [1]. Let B be a Banach space and

$O(B)$ the orthogonal group of B , that is, the group of linear bijective isometries of B . Recall that an orthogonal representation of a topological group G on a Banach space B is a homomorphism $\rho : G \rightarrow O(B)$ such that the map $g \mapsto \rho(g)x$ is continuous for every $x \in B$. If $\rho : G \rightarrow O(B)$ is an orthogonal representation of a group G , we denote the subspace of $\rho(G)$ -invariant vectors by

$$B^{\rho(G)} = \{x \in B \mid \rho(g)x = x \text{ for all } g \in G \}.$$

Observe that $B^{\rho(G)}$ is invariant under G . The representation ρ is said to almost have invariant vectors if it has (Q, ϵ) -invariant vector for every compact subset Q of G and $\epsilon > 0$.

Definition 1.2. Let G be a topological group and H be a closed *normal* subgroup of G . The pair (G, H) has relative property (T_B) for a Banach space B if, for any orthogonal representation $\rho : G \rightarrow O(B)$, the quotient representation $\rho' : G \rightarrow O(B/B^{\rho(H)})$ does not almost have $\rho'(G)$ -invariant vectors.

A topological group G has property (T_B) if the pair (G, G) has relative property (T_B) .

The authors of [1] studied the case where B is a superreflexive Banach space, and among other things, they showed that a group which has property (T) has property $(T_{L^p(\mu)})$ for μ a σ -finite measure on a standard Borel space (X, \mathcal{B}) and $1 < p < \infty$. We will extend this result to the non-commutative setting.

Non-commutative L_p -spaces were introduced by Dixmier [3] and studied by various authors, among them Yeadon [13] and Haagerup [4] (for a survey on these spaces, see Pisier and Xu [6]). Apart from the standard $L^p(\mu)$ -spaces, common examples are the p -Schatten ideals

$$S_p = \{x \in \mathcal{B}(\mathcal{H}) \mid \text{tr}(|x|^p) < \infty \}$$

where \mathcal{H} is a separable Hilbert space.

We review below (in Section 2) Haagerup's definition of these non-commutative L_p -spaces. Here is our main result :

Theorem 1.3. *Let G be a topological group and H a closed normal subgroup of G . Assume that the pair (G, H) has relative property (T) . For every von Neumann algebra \mathcal{M} , the pair (G, H) has relative property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$.*

In particular, if G has property (T) , then G has property $(T_{L_p(\mathcal{M})})$ for $1 < p < \infty$. Property (T_B) has a stronger version which is a fixed point property for affine actions.

Definition 1.4. Let B a Banach space. A topological group G has property (F_B) if every continuous action of G by affine isometries on B has a G -fixed point.

The authors of [1] showed that higher rank groups and their lattices have property $(F_{L^p(\mu)})$.

Definition 1.5. For $1 \leq i \leq m$, let k_i be local fields and $\mathbb{G}_i(k_i)$ be the k_i -points of connected simple k_i -algebraic groups \mathbb{G}_i . Assume that each simple factor \mathbb{G}_i has k_i -rank ≥ 2 . The group $G = \prod_{i=1}^m \mathbb{G}_i(k_i)$ is called a higher rank group.

Our next result shows that Theorem B in [1] remains true for non-commutative L_p -spaces.

Theorem 1.6. *Let G be a higher rank group and \mathcal{M} a von Neumann algebra. Then G , as well as every lattice in G , has property $F_{L_p(\mathcal{M})}$ for $1 < p < \infty$.*

Theorem 1.6 was proved by Puschnigg in [7] in the case $L_p(\mathcal{M}) = S_p$. The strategy of the proof of Theorem 1.3 (as in [7]) follows the one from [1]. To achieve the result, we will need some results on the Mazur map and the description of the surjective isometries of $L_p(\mathcal{M})$ given by Sherman in [9].

The paper is organized as follows. In Section 2, useful properties of the Mazur map are established. Group representations on $L_p(\mathcal{M})$ are studied in Section 3. The proof of Theorem 1.3 is given in Section 4. In Section 5, we show how Theorem 1.6 can be obtained from a variant of Theorem 1.3.

2 SOME PROPERTIES OF THE MAZUR MAP

Let \mathcal{M} be a von Neumann algebra, acting on a Hilbert space \mathcal{H} , and equipped with a normal semi-finite weight φ_0 . Let $t \mapsto \sigma_t^{\varphi_0}$ be the one-parameter group of modular automorphisms of \mathcal{M} with respect to φ_0 . We denote by $\mathcal{N}_{\varphi_0} = \mathcal{M} \rtimes_{\varphi_0} \mathbb{R}$ the crossed product von Neumann algebra, which is a von Neumann algebra acting on $L^2(\mathbb{R}, \mathcal{H})$, and generated by the operators $\pi_{\varphi_0}(x)$, $x \in \mathcal{M}$, and λ_s , $s \in \mathbb{R}$, defined by

$$\begin{aligned} \pi_{\varphi_0}(x)(\xi)(t) &= \sigma_{-t}^{\varphi_0}(x)\xi(t) \\ \lambda_s(\xi)(t) &= \xi(t-s) \quad \text{for any } \xi \in L^2(\mathbb{R}, \mathcal{H}) \text{ and } t \in \mathbb{R}. \end{aligned}$$

There is a dual action $s \mapsto \theta_s$ of \mathbb{R} on \mathcal{N}_{φ_0} . Then let τ_{φ_0} be the semi-finite normal trace on \mathcal{N}_{φ_0} satisfying

$$\tau_{\varphi_0} \circ \theta_s = e^{-s} \tau_{\varphi_0} \text{ for all } s \in \mathbb{R}.$$

We denote by $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ the *-algebra of τ_{φ_0} -measurable operators affiliated with \mathcal{N}_{φ_0} . For $1 \leq p \leq \infty$, the Haagerup non-commutative L_p -space associated with \mathcal{M} is defined by

$$L_p(\mathcal{M}) = \{ x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \mid \theta_s(x) = e^{-s/p} x \text{ for all } s \in \mathbb{R} \}.$$

It is known that this space is independent of the weight φ_0 up to isomorphism. The space $L_1(\mathcal{M})$ is isomorphic to \mathcal{M}_* . The identification goes as follows : there

exists a normal faithful semi-finite operator valued weight from \mathcal{N}_{φ_0} to \mathcal{M} defined by

$$\Phi_{\varphi_0}(x) = \pi_{\varphi_0}^{-1}\left(\int_{\mathbb{R}} \theta_s(x) ds\right), \text{ for } x \in \mathcal{N}_{\varphi_0}.$$

Now, if $\varphi \in \mathcal{M}_*^+$, and $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{\mathcal{M}}^+$, the extended positive part of \mathcal{M} , we then put

$$\tilde{\varphi}^{\varphi_0} = \hat{\varphi} \circ \Phi_{\varphi_0}.$$

We associate to φ the Radon-Nikodym derivative $\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}$ of $\tilde{\varphi}^{\varphi_0}$ with respect to the trace τ_{φ_0} . This isomorphism between \mathcal{M}_*^+ and $L_1(\mathcal{M})^+$ extends to the whole spaces by linearity.

If $x \in L_1(\mathcal{M})$, and φ_x is the element of \mathcal{M}_*^+ associated to x , we define a linear functional Tr by

$$\text{Tr}(x) = \varphi_x(1)$$

and we have, p' being the conjugate exponent of p ,

$$\text{Tr}(xy) = \text{Tr}(yx) \text{ for } x \in L_p(\mathcal{M}), y \in L_{p'}(\mathcal{M})$$

For $1 \leq p < \infty$, if $x = u|x|$ is the polar decomposition of $x \in L_p(\mathcal{M})$, we define

$$\|x\|_p = \text{Tr}(|x|^p)^{1/p}.$$

Equipped with $\|\cdot\|_p$, $L_p(\mathcal{M})$ is a Banach space. For $1 < p < \infty$, the dual space of $L_p(\mathcal{M})$ is $L_{p'}(\mathcal{M})$ and $L_p(\mathcal{M}, \tau)$ is known to be superreflexive.

We now introduce the Mazur map and establish some of its properties.

Definition 2.1. Let $1 \leq p, q < \infty$. For an operator a , let $\alpha|a|$ be its polar decomposition. The map

$$\begin{aligned} M_{p,q} : L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) &\rightarrow L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0}) \\ x = \alpha|a| &\mapsto \alpha|a|^{\frac{p}{q}} \end{aligned}$$

is called the Mazur map.

We will need the following lemma.

Lemma 2.2. Let $1 \leq p, q, r < \infty$. Then $M_{r,q} \circ M_{p,r} = M_{p,q}$.

Proof. Let $\alpha|x|$ be the polar decomposition of $x \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. Let $\beta > 0$, and set $y = \alpha|x|^\beta$. We claim that the polar decomposition of y is given by α and $|x|^\beta$. To show this, it suffices to prove that $\overline{\text{Im}}(|x|^\beta) = \overline{\text{Im}}(|x|)$.

By taking orthogonals, we have to show that $\text{Ker}(|x|) = \text{Ker}(|x|^\beta)$ for all $\beta > 0$. Recall that the domain $D(|x|^\beta)$ of $|x|^\beta$ is

$$D(|x|^\beta) = \{\xi \mid \int_0^\infty \lambda^{2\beta} d\mu_\xi(\lambda) < \infty\}.$$

If $\xi \in \text{Ker}(|x|)$, we have for all $\eta \in L^2(\mathbb{R}, \mathcal{H})$

$$\langle |x|\xi, \eta \rangle = \int_0^\infty \lambda d\mu_{\xi, \eta}(\lambda) = 0.$$

In particular, $\mu_\xi(]0, \infty[) = 0$. So $\xi \in D(|x|^\beta)$ and $\xi \in \text{Ker}(|x|^\beta)$ thanks to

$$\langle |x|^\beta \xi, \eta \rangle = \int_0^\infty \lambda^\beta d\mu_{\xi, \eta}(\lambda) = 0.$$

By exchanging the role of $|x|$ and $|x|^\beta$, we get the equality.

Let $1 \leq p, q, r < \infty$, and $\beta = p/r$; then $M_{p,r}(x) = \alpha|x|^\beta$. It follows from what we have just seen that $M_{r,q}(M_{p,r}(x)) = \alpha|x|^{\frac{p}{q}} = M_{p,q}(x)$. \square

Proposition 2.3. *Let $1 \leq p, q < \infty$, and $a \in L_p(\mathcal{M})$. Then*

$$\|M_{p,q}(a)\|_q^q = \|a\|_p^p.$$

Proof. We denote again by $\alpha|a|$ the polar decomposition of a . We have already seen that $|M_{p,q}(a)| = |\alpha|a|^{\frac{p}{q}}$. So we have

$$\text{Tr}(|M_{p,q}(a)|^q) = \text{Tr}(|a|^p).$$

\square

Proposition 2.4. *Let $p, q \in]1, \infty[$ be conjugate. The map*

$$\begin{aligned} L_p(\mathcal{M}) &\rightarrow L_q(\mathcal{M}) \\ x &\mapsto M_{p,q}(x)^* \end{aligned}$$

is the duality map from $L_p(\mathcal{M})$ to $L_q(\mathcal{M})$.

Proof. We first notice that $M_{p,q}$ sends $L_p(\mathcal{M})$ into $L_q(\mathcal{M})$. Let $x = \alpha|x| \in L_p(\mathcal{M})$ and $s \in \mathbb{R}$. By uniqueness in the polar decomposition, we have $\theta_s(\alpha) = \alpha$ and $\theta_s(|x|) = e^{-s/p}|x|$, and then

$$\begin{aligned} \theta_s(M_{p,q}(x)) &= \theta_s(\alpha)\theta_s(|x|^{\frac{p}{q}}) \\ &= \alpha(\theta_s(|x|)^{\frac{p}{q}}) \\ &= e^{-s/q}M_{p,q}(x). \end{aligned}$$

Thanks to the uniqueness of the duality map in superreflexive spaces, we just have to check that $\text{Tr}(M_{p,q}(a)^*a) = 1$ for a in the unit sphere $S(L_p(\mathcal{M}))$ of $L_p(\mathcal{M})$. Let $a = \alpha|a| \in S(L_p(\mathcal{M}))$; then $M_{p,q}(a) = \alpha|a|^{\frac{p}{q}}$. Since $\alpha^*\alpha|a| = |a|$, it follows that

$$\text{Tr}(|a|^{\frac{p}{q}}\alpha^*\alpha|a|) = \text{Tr}(|a|^{\frac{p}{q}}|a|) = \text{Tr}(|a|^p) = 1.$$

□

Proposition 2.5. *If $a, b \in L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and if e, f are two central projections in \mathcal{N}_{φ_0} such that $ef = 0$, then $M_{p,q}(ae + bf) = M_{p,q}(ae) + M_{p,q}(bf)$.*

Proof. As is easily checked, we have

$$|ae + bf| = |a|e + |b|f.$$

Let γ be the partial isometry occuring in the polar decomposition of $ae + bf$, and let $a = \alpha|a|$, $b = \beta|b|$ be the polar decompositions of a and b . We claim that $\gamma = \alpha e + \beta f$. Indeed, we have

$$\begin{aligned} ae + bf &= \gamma|ae + bf| \\ \text{and } ae + bf &= (\alpha e)(|a|e) + (\beta f)(|b|f) = (\alpha e + \beta f)|ae + bf|. \end{aligned}$$

Since αe is zero on $\text{Ker}(|a|e)$ and βf is zero on $\text{Ker}(|b|f)$, $\alpha e + \beta f$ is zero on $\text{Im}(|ae + bf|)^\perp = \text{Ker}(|ae + bf|) = \text{Ker}(|a|e) \cap \text{Ker}(|b|f)$ ($ef = 0$).

Using again the fact that $ef = 0$ and that e, f are central elements, we deduce that

$$\begin{aligned} M_{p,q}(ae + bf) &= (\alpha e + \beta f)|ae + bf|^{\frac{p}{q}} \\ &= (\alpha e + \beta f)(e|a|^{\frac{p}{q}} + f|b|^{\frac{p}{q}}) \\ &= M_{p,q}(ae) + M_{p,q}(bf). \end{aligned}$$

□

Proposition 2.6. *Let J be a Jordan-isomorphism of \mathcal{N}_{φ_0} , and let $1 \leq p, q < \infty$. Then we have*

$$J(x) = M_{p,q} \circ J \circ M_{q,p}(x) \text{ for all } x \in \mathcal{N}_{\varphi_0}.$$

Proof. By Lemma 3.2 in [10], we have a decomposition $J = J_1 + J_2$ with the following properties : J_1 is a *-homomorphism, J_2 is a *-anti-homomorphism and $J_1(x) = J(x)e$, $J_2(x) = J(x)f$ for all $x \in \mathcal{M}$, with e, f two orthogonal and central projections such that $e + f = I$.

Observe first that, for $a \in \mathcal{N}_{\varphi_0}$ with $a \geq 0$ and a positive real number r , we have

$$J_1(a^r) = J_1(a)^r$$

and the same is true for J_2 .

If α is a partial isometry, then $J_1(\alpha)$ and $J_2(\alpha)$ are partial isometries with initial

supports $J_1(\alpha^*\alpha)$ and $J_2(\alpha\alpha^*)$, and final supports $J_1(\alpha\alpha^*)$ and $J_2(\alpha^*\alpha)$ respectively.

Let $x = \alpha|x| \in \mathcal{N}_{\varphi_0}$. Since the supports of J_1 and J_2 are orthogonal, it follows from Proposition 2.5 that

$$\begin{aligned} M_{p,q} \circ J \circ M_{q,p}(x) &= M_{p,q}(J_1(M_{q,p}(x)) + J_2(M_{q,p}(x))) \\ &= M_{p,q}(J_1(M_{q,p}(x))) + M_{p,q}(J_2(M_{q,p}(x))). \end{aligned}$$

Moreover, we have

$$\begin{aligned} M_{p,q}(J_1(M_{q,p}(x))) &= M_{p,q}(J_1(\alpha|x|^{\frac{2}{p}})) \\ &= M_{p,q}(J_1(\alpha)J_1(|x|^{\frac{2}{p}})) \\ &= J_1(x) \end{aligned}$$

and

$$\begin{aligned} M_{p,q}(J_2(M_{q,p}(x))) &= M_{p,q}(J_2(\alpha|x|^{\frac{2}{p}}\alpha^*)) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|^{\frac{2}{p}}\alpha^*)) \\ &= M_{p,q}(J_2(\alpha)J_2((\alpha|x|\alpha^*)^{\frac{2}{p}})) \\ &= M_{p,q}(J_2(\alpha)J_2(\alpha|x|\alpha^*)^{\frac{2}{p}}) \\ &= J_2(x). \end{aligned}$$

□

An essential tool for the proof of Theorem 1.3 is the following result about the local uniform continuity of $M_{p,q}$, which is proved in Lemma 3.2 of [8] (for an independant proof in the case $L_p(\mathcal{M}, \tau) = S_p$, see [7]).

Proposition 2.7. [8] *For $1 \leq p, q < \infty$, the Mazur map $M_{p,q}$ is uniformly continuous on the unit sphere $S(L_p(\mathcal{M}))$.*

3 GROUP REPRESENTATIONS ON $L_p(\mathcal{M})$

Sherman's description of the surjective isometries of $L_p(\mathcal{M})$ in [9] is a crucial tool in the following result (non surjective isometries in the semi-finite case, and 2-isometries in the general case are described in [14] and [5] respectively). This will allow us to transfer a representation of a group G on $L_p(\mathcal{M})$ to a representation of G on $L_2(\mathcal{M})$.

Proposition 3.1. *For $p > 2$, and $U \in O(L_p(\mathcal{M}))$, the map $V = M_{p,2} \circ U \circ M_{2,p}$ belongs to $O(L_2(\mathcal{M}))$.*

Proof. The fact that $\|V(x)\|_2 = \|x\|_2$ for all $x \in L_2(\mathcal{M})$ follows from Proposition 2.3, and V is bijective by Lemma 2.2. We have to prove that V is linear on $L_2(\mathcal{M})$. By Theorem 1.2 in [9], there exist a Jordan-isomorphism J of \mathcal{M} and a unitary $w \in \mathcal{M}$ such that

$$U(\varphi^{1/p}) = w(\varphi \circ J^{-1})^{1/p} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

It was shown in [12] that J extends to a Jordan- $*$ -isomorphism \tilde{J} between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$; moreover, \tilde{J} is an extension of an isomorphism between \mathcal{N}_{φ_0} and $\mathcal{N}_{\varphi_0 \circ J^{-1}}$ as well as a homeomorphism for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$. The isomorphism \tilde{J} satisfies the relations

$$\begin{aligned} \tau_{\varphi_0} \circ \tilde{J}^{-1} &= \tau_{\varphi_0 \circ J^{-1}} \\ J^{-1} \circ \Phi_{\varphi_0 \circ J^{-1}} &= \Phi_{\varphi_0} \circ \tilde{J}^{-1} \end{aligned}$$

Lemma 3.2. *For $\varphi \in \mathcal{M}_*^+$, we have*

$$\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}} = \tilde{J}^{-1} \left(\frac{d\varphi \circ \tilde{J}^{-1 \varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \right).$$

Proof. For all $\varphi \in \mathcal{M}_*^+$, we have

$$\begin{aligned} \tau_{\varphi_0} \left(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}} \cdot \right) &= \varphi \circ \Phi_{\varphi_0} \\ &= \varphi \circ J^{-1} \circ \Phi_{\varphi_0 \circ J^{-1}} \circ \tilde{J} \\ &= \tau_{\varphi_0 \circ J^{-1}} \left(\frac{d\varphi \circ \tilde{J}^{-1 \varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \tilde{J}(\cdot) \right) \\ &= \tau_{\varphi_0} \circ \tilde{J}^{-1} \left(\frac{d\varphi \circ \tilde{J}^{-1 \varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \tilde{J}(\cdot) \right) \\ &= \tau_{\varphi_0} \left(\tilde{J}^{-1} \left(\frac{d\varphi \circ \tilde{J}^{-1 \varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \right) \cdot \right), \end{aligned}$$

where in the last equality we used the fact that \tilde{J} is Jordan homomorphism. \square

In Lemma 2.1 in [11], it is shown that there exists a topological $*$ -isomorphism $\tilde{\mathcal{K}}$ between $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ and $L_0(\mathcal{N}_{\varphi_0 \circ J^{-1}}, \tau_{\varphi_0 \circ J^{-1}})$ which satisfies the following relation on the Radon-Nikodym derivatives :

$$\tilde{\mathcal{K}} \left(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}} \right) = \frac{d\tilde{\varphi}^{\varphi_0 \circ J^{-1}}}{d\tau_{\varphi_0 \circ J^{-1}}} \text{ for all } \varphi \in \mathcal{M}_*^+.$$

From Lemma 3.2, we obtain

$$\frac{d\varphi \circ \tilde{J}^{-1\varphi_0}}{d\tau_{\varphi_0}} = \tilde{\mathcal{K}}^{-1} \circ \tilde{J}\left(\frac{d\tilde{\varphi}^{\varphi_0}}{d\tau_{\varphi_0}}\right) \text{ for all } \varphi \in \mathcal{M}_*^+.$$

As a consequence, the linear and bijective isometry U of $L_p(\mathcal{M})$ is given by the following relation on positive elements :

$$U(x) = w(\tilde{\mathcal{K}}^{-1} \circ \tilde{J}(x)) \text{ for all } x \in L_p(\mathcal{M})^+.$$

This relation extends by linearity to the whole $L_p(\mathcal{M})$.

Now notice that $\tilde{\mathcal{K}}^{-1} \circ \tilde{J}$ is a Jordan-isomorphism on \mathcal{N}_{φ_0} and a topological isomorphism (for the measure topology) on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. By Proposition 2.6, for $x \in \mathcal{N}_{\varphi_0}$, we have

$$\begin{aligned} V(x) &= M_{p,2} \circ U \circ M_{2,p}(x) \\ &= w(M_{p,2} \circ \tilde{\mathcal{K}}^{-1} \circ \tilde{J} \circ M_{2,p}(x)) \\ &= w(\tilde{\mathcal{K}}^{-1} \circ \tilde{J}(x)). \end{aligned}$$

Recall from [8] that the Mazur map is continuous for the measure topology on $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$. So by density of \mathcal{N}_{φ_0} in $L_0(\mathcal{N}_{\varphi_0}, \tau_{\varphi_0})$ for the measure topology, we have

$$V(x) = w(\tilde{\mathcal{K}}^{-1} \circ \tilde{J}(x)) \text{ for all } x \in L_2(\mathcal{M})$$

which gives the linearity of V on $L_2(\mathcal{M})$. \square

Remark 3.3. The proof of the linearity of the map V in Proposition 3.1 is simpler in the case where \mathcal{M} is a von Neumann algebra equipped with a faithful semi-finite normal trace τ . Indeed, by Theorem 2 in [14], there exist a Jordan-isomorphism J , a positive operator B commuting with $J(\mathcal{M})$, and a partial isometry W in \mathcal{M} with the property that W^*W is the support of B , such that

$$U(x) = WBJ(x) \text{ for all } x \in \mathcal{M} \cap L_p(\mathcal{M}, \tau).$$

Using the fact that B commutes with $J(\mathcal{M})$, and as in the proof of Proposition 2.6, for all $x = \alpha|x| \in \mathcal{M} \cap L_p(\mathcal{M}, \tau)$, we have

$$\begin{aligned} V(x) &= WM_{p,2}(BJ_1(\alpha|x|^{\frac{p}{2}}) + BJ_2(\alpha|x|^{\frac{p}{2}})) \\ &= WM_{p,2}(BJ_1(\alpha|x|^{\frac{p}{2}})) + WM_{p,2}(BJ_2(\alpha|x|^{\frac{p}{2}})) \\ &= WJ_1(\alpha)B^{\frac{p}{2}}J_1(|x|) + WJ_2(\alpha)B^{\frac{p}{2}}J_2(\alpha|x|\alpha^*) \\ &= WB^{\frac{p}{2}}J(x). \end{aligned}$$

The linearity on the whole $L_p(\mathcal{M}, \tau)$ follows from the density of $\mathcal{M} \cap L_p(\mathcal{M}, \tau)$ in $L_p(\mathcal{M}, \tau)$.

Corollary 3.4. *Let G be a topological group, $p \geq 2$, and $U : G \rightarrow O(L_p(\mathcal{M}))$ be a representation on $L_p(\mathcal{M})$. For $g \in G$, define $V(g) : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M})$ by*

$$V(g) = M_{p,2} \circ U(g) \circ M_{2,p}.$$

Then V is a representation of G on $L_2(\mathcal{M})$.

Proof. By the previous proposition, $V(g) \in O(L_2(\mathcal{M}))$ for every g in G . Moreover, the map $g \mapsto V(g)x$ is continuous, since $g \mapsto U(g)M_{2,p}(x)$ is continuous and since $M_{p,2} : L_p(\mathcal{M}) \rightarrow L_2(\mathcal{M})$ is continuous.

It remains to check that V is a homomorphism. For this, let $g_1, g_2 \in G$. Then, by Lemma 2.2,

$$\begin{aligned} V(g_1)V(g_2) &= M_{p,2} \circ U(g_1) \circ M_{2,p} \circ M_{p,2} \circ U(g_2) \circ M_{2,p} \\ &= M_{p,2} \circ U(g_1) \circ U(g_2) \circ M_{2,p} \\ &= M_{p,2} \circ U(g_1g_2) \circ M_{2,p} \\ &= V(g_1g_2). \end{aligned}$$

□

Let U be a representation of a topological group G on $L_p(\mathcal{M})$ and let

$$L_p(\mathcal{M})^{U(G)} = \{x \in L_p(\mathcal{M}) \mid U(g)x = x \text{ for all } g \in G\}$$

be the space of $U(G)$ -invariant vectors in $L_p(\mathcal{M})$. Let p' be the conjugate of p and U^* the contragredient representation of U on the dual space $L_{p'}(\mathcal{M})$ of $L_p(\mathcal{M})$. Since $L_p(\mathcal{M})$ is superreflexive, there exists a complement $L_p(\mathcal{M})'$ for $L_p(\mathcal{M})^{U(G)}$ (see Proposition 2.6 in [1]) and we have

$$L_p(\mathcal{M})' = \{v \in L_p(\mathcal{M}) \mid \text{Tr}(vc) = 0 \text{ for all } c \in L_{p'}(\mathcal{M})^{U^*(G)}\}.$$

Proposition 3.5. *Let $v \in S(L_p(\mathcal{M})')$, then*

$$d(v, L_p(\mathcal{M})^{U(G)}) \geq \frac{1}{2}.$$

Proof. Assume, by contradiction, that there exists $b \in L_p(\mathcal{M})^{U(G)}$ such that

$$\|v - b\|_p < \frac{1}{2}.$$

Then $\frac{1}{2} \leq \|b\|_p \leq \frac{3}{2}$. Setting $c = \frac{b}{\|b\|_p}$, we have $\|b - c\|_p \leq \frac{1}{2}$.

Since $c \in L_p(\mathcal{M})^{U(G)}$, it is easily checked that $M_{p,p'}(c)^* \in L_{p'}(\mathcal{M})^{U^*(G)}$; hence

$$\text{Tr}((c - v)M_{p,p'}(c)^*) = \text{Tr}(cM_{p,p'}(c)^*) = \|c\|_p^p = 1.$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned}
1 &= \text{Tr}((c - v)M_{p,p'}(c)^*) \\
&\leq \|c - v\|_p \|M_{p,p'}(c)^*\|_{p'} \\
&= \|c - v\|_p \|c\|_p^{\frac{p}{p'}} \\
&= \|c - v\|_p.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|v - b\|_p &\geq \|v - c\|_p - \|c - b\|_p \\
&\geq \frac{1}{2}
\end{aligned}$$

and this is a contradiction. \square

4 PROOF OF THEOREM 1.3

We follow the strategy of the proof of Theorem A in [1]. Let $p \in]1, \infty[$ and let U be a representation on $L_p(\mathcal{M})$ of a group G . Let H be a closed subgroup of G such that the pair (G, H) has property (T) . We claim that the representation U' of G on the complement $L_p(\mathcal{M})'$ of $L_p(\mathcal{M})^{U(H)}$ has no almost $U'(G)$ -invariant vectors. This will prove Theorem 1.3.

Let Q be a compact subset in G , and take $\epsilon > 0$. Assume by contradiction that there exists almost $U(G)$ -invariant vectors in $L_p(\mathcal{M})'$. Then, we can find, for every n , a unit vector v_n such that

$$\sup_{g \in Q} \|U(g)v_n - v_n\|_p < \frac{1}{n}.$$

By Corollary 3.4, $V = M_{p,2} \circ U \circ M_{2,p}$ defines a representation of G on $L_2(\mathcal{M})$. Let w_n be the orthogonal projection of $M_{p,2}(v_n)$ on the orthogonal complement $L_2(\mathcal{M})'$ of $L_2(\mathcal{M})^{V(H)}$. We claim that w_n is (Q, ϵ) -invariant for V for n sufficiently large. This will contradict property (T) for the pair (G, H) .

We first show that there exists $\delta' > 0$ such that

$$d(M_{p,2}(v_n), L_2(\mathcal{M})^{V(H)}) \geq \delta' \text{ for all } n.$$

Indeed, otherwise for some n , there exists $a_k \in L_2(\mathcal{M})^{V(H)}$ such that

$$\|M_{p,2}(v_n) - a_k\|_2 \xrightarrow{k \rightarrow \infty} 0.$$

By Proposition 2.3, we have

$$\|M_{p,2}(v_n)\|_2 = \|v_n\|_p^{\frac{p}{2}} = 1.$$

Since $\|a_k\|_2 \xrightarrow[k \rightarrow \infty]{} \|M_{p,2}(v_n)\|_2 = 1$, we can assume that $\|a_k\|_2 = 1$. Notice that

$$M_{2,p}(L_2(\mathcal{M})^{V(H)}) = L_p(\mathcal{M})^{U(H)}.$$

Hence, $M_{2,p}(a_k)$ belongs to $L_p(\mathcal{M})^{U(H)}$ for every k . Moreover

$$\|v_n - M_{2,p}(a_k)\|_p \xrightarrow[k \rightarrow \infty]{} 0$$

by the uniform continuity of $M_{2,p}$ on the unit sphere (see Proposition 2.4). This is a contradiction to Proposition 3.5.

In particular, we have

$$\|w_n\|_2 = d(M_{p,2}(v_n), L_2(\mathcal{M})^{V(H)}) \geq \delta'.$$

For $g \in Q$, we have

$$\begin{aligned} \|V(g)w_n - w_n\|_2 &\leq \|V(g)M_{p,2}(v_n) - M_{p,2}(v_n)\|_2 \\ &= \|M_{p,2}(U(g)v_n) - M_{p,2}(v_n)\|_2. \end{aligned}$$

Recall that $\|v_n\|_p^{\frac{p}{2}} = 1$ and that

$$\sup_{g \in Q} \|U(g)v_n - v_n\|_p < \frac{1}{n}.$$

Hence, by the uniform continuity of $M_{p,2}$ on $S(L_2(\mathcal{M}))$, there exists an integer N (depending only on (Q, ϵ)) such that

$$\sup_{g \in Q} \|V(g)w_n - w_n\|_2 < \epsilon \delta' \text{ for } n \geq N.$$

Since $\|w_n\|_2 \geq \delta'$, it follows that

$$\sup_{g \in Q} \|V(g)w_n - w_n\|_2 < \epsilon \|w_n\|_2 \text{ for } n \geq N.$$

This shows that w_n is (Q, ϵ) -invariant for U when $n \geq N$. This finishes the proof of Theorem 1.3.

5 PROPERTY $(F_{L_p(\mathcal{M})})$ FOR HIGHER RANK GROUPS

Let H be a closed normal subgroup of G and let L be a closed group of G . Assume that $G = L \ltimes H$. The following strong relative property (T_B) was considered in [1] :

Definition 5.1. A pair $(L \ltimes H, H)$ has property (T_B) if, for any orthogonal representation $\rho : L \ltimes H \rightarrow O(B)$, the quotient representation $\rho' : L \rightarrow O(B/B^{\rho(H)})$ does not almost have $\rho'(L)$ -invariant vectors.

A straightforward modification of our proof of Theorem 1.3 shows that we also have the following result :

Theorem 5.2. *Let $(L \rtimes H, H)$ be a pair with strong relative property (T) . Then $(L \rtimes H, H)$ has strong relative property $(T_{L^p(\mathcal{M})})$ for $1 < p < \infty$.*

Let G be a higher rank group as defined in the introduction. Using an analogue of Howe-Moore's theorem on vanishing of matrix coefficients, the authors of [1] showed that G has property (F_B) whenever B is a superreflexive Banach space and a certain pair $(L \rtimes H, H)$ of subgroups, which has property (T) , has also (T_B) . The property $(F_{L^p(\mathcal{M})})$ for higher rank groups in Theorem 1.6 is then a consequence of Theorem 5.2. Moreover, the result for lattices in higher rank groups is obtained by an induction process exactly as in the Proposition 8.8 of [1].

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