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Spherical structures on torus knots and links *

Alexander Kolpakov, Alexander Mednykh

Abstract

The present paper considers two infinite families of cone-manifolds endowed with spherical metric. The singular strata is either the torus knot $t(2n+1, 2)$ or the torus link $t(2n, 2)$. Domains of existence for a spherical metric are found in terms of cone angles and volume formulæ are presented.

Key words: Spherical geometry, cone-manifold, knot, link.

1 Introduction

A three-dimensional cone-manifold is a metric space obtained from a collection of disjoint simplices in the space of constant sectional curvature k by isometric identification of their faces in such a combinatorial fashion that the resulting topological space is a manifold (also called the underlying space for a given cone-manifold).

Such the metric space inherits the metric of sectional curvature k on the union of its 2- and 3-dimensional cells. In case $k = +1$ the corresponding cone-manifold is called spherical (or admits a spherical structure). By analogy, one defines euclidean ($k = 0$) and hyperbolic ($k = -1$) cone-manifolds.

The metric structure around each 1-cell is determined by a cone angle that is the sum of dihedral angles of corresponding simplices sharing the 1-cell under identification. The singular locus of a cone-manifold is the closure of all its 1-cells with cone angle different from 2π . For the further account we suppose that every component of the singular locus is an embedded circle with constant cone angle along it.

A particular case of cone-manifold is an orbifold with cone angles $2\pi/m$, where m is an integer (cf. [1]).

The present paper considers two infinite families of cone-manifolds with underlying space the three-dimensional sphere \mathbb{S}^3 . The first family consists of cone-manifolds with singular locus the torus knot $t(2n+1, 2)$ with $n \geq 1$. In the rational census [2] these knots are denoted by $(2n+1)/1$. The second family of cone-manifolds consists of those with singular locus a two-component torus link $t(2n, 2)$ with $n \geq 2$. These links are two-bridge and correspond to the links

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$2n/1$ in the rational census. The simplest examples of such the knots and links are the trefoil knot $3/1$ and the link $4/1$. In the Rolfsen table [2] one finds them as the knot 3_1 and the link 4_1^2 .

By the Theorem of W. Thurston [3], the manifold $\mathbb{S}^3 \setminus 3_1$ does not admit a hyperbolic structure. However, it admits two other geometric structures [4]: $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$. It follows from the paper [5] that the spherical dodecahedron space (i.e. Poincaré homology sphere) is a cyclic 5-fold covering of \mathbb{S}^3 branched over 3_1 . Thus, the orbifold $3_1(\frac{2\pi}{5})$ with singular locus the trefoil knot and cone angle $\frac{2\pi}{5}$ is spherical. Due to the Dunbar's census [6], orbifold $3_1(\frac{2\pi}{n})$ is spherical if $n \leq 5$, Nil-orbifold if $n = 6$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$ -orbifold if $n \geq 7$. Spherical structure on the cone-manifold $3_1(\alpha)$ with underlying space the three-dimensional sphere \mathbb{S}^3 is studied in [7].

The consideration of two-bridge torus links is carried out starting with the simplest one possessing non-abelian fundamental group, namely 4_1^2 .

The previous investigation on spherical structures for cone-manifolds is carried out mainly in the papers [8, 9, 10]. The present paper develops a method to analyse existence of a spherical metric for two-bridge torus knot and link cone-manifolds. Also, the lengths of singular geodesics are calculated and the volume formulæ are obtained (cf. Theorem 1 and Theorem 2).

2 Projective model \mathbb{S}_λ^3

The purpose of the present section is to construct the projective model \mathbb{S}_λ^3 that one can use to study geometric properties of two-bridge torus knots and links and to build up holonomy representation for the corresponding cone-manifolds. Other projective models for homogeneous geometries are described in [11].

Consider the set $\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ as a four-dimensional vector space over \mathbb{R} . We denote it by $\mathbb{C}_\mathbb{R}^2$ and equip with Hermitian product

$$\langle (z_1, z_2), (w_1, w_2) \rangle_{\mathbb{H}} = (z_1, z_2) \mathcal{H} \overline{(w_1, w_2)}^T,$$

where

$$\mathcal{H} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

is a symmetric matrix with $-1 < \lambda < +1$.

The natural inner product is associated to the Hermitian form above:

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \text{Re} \langle (z_1, z_2), (w_1, w_2) \rangle_{\mathbb{H}}$$

and the respective norm is

$$\|(z_1, z_2)\| = |z_1|^2 + |z_2|^2 + \lambda(z_1 \bar{z}_2 + \bar{z}_1 z_2).$$

Call two elements (z_1, z_2) and (w_1, w_2) in $\overset{\circ}{\mathbb{C}}_\mathbb{R}^2 = \mathbb{C}_\mathbb{R}^2 \setminus (0, 0)$ equivalent if there is $\mu > 0$ such that $(z_1, z_2) = (\mu w_1, \mu w_2)$. We denote this equivalence relation as $(z_1, z_2) \sim (w_1, w_2)$.

Identify the factor-space $\overset{\circ}{\mathbb{C}}_{\mathbb{R}}^2 / \sim$ with the three-dimensional sphere

$$\mathbb{S}_{\lambda}^3 = \{(z_1, z_2) \in \mathbb{C}_{\mathbb{R}}^2 : \|(z_1, z_2)\| = 1\},$$

endowed with the Riemannian metric

$$ds_{\lambda}^2 = |dz_1|^2 + |dz_2|^2 + \lambda(dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2).$$

By means of equality

$$ds_{\lambda}^2 = \frac{1+\lambda}{2} |dz_1 + dz_2|^2 + \frac{1-\lambda}{2} |dz_1 - dz_2|^2,$$

the linear transformation

$$\xi_1 = \sqrt{\frac{1+\lambda}{2}} (z_1 + z_2), \quad \xi_2 = \sqrt{\frac{1-\lambda}{2}} (z_1 - z_2)$$

provides an isometry between $(\mathbb{S}_{\lambda}^3, ds_{\lambda}^2)$ and (\mathbb{S}^3, ds^2) , where $ds^2 = |d\xi_1|^2 + |d\xi_2|^2$ is the standard metric of sectional curvature +1 on the unit sphere $\mathbb{S}^3 = \{(\xi_1, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1\}$.

Let P, Q be two points in \mathbb{S}_{λ}^3 . The spherical distance between P and Q is a real number $d_{\lambda}(P, Q)$ that is uniquely determined by the conditions $0 \leq d_{\lambda}(P, Q) \leq \pi$ and $\cos d_{\lambda}(P, Q) = \langle P, Q \rangle$.

3 Torus knots \mathbb{T}_n

Let $\mathbb{T}_n, n \geq 1$ be the torus knot $t(2n+1, 2)$ embedded in \mathbb{S}^3 . The knot \mathbb{T}_n is the two-bridge knot $(2n+1)/1$ in the rational census (Fig. 1). Let $\mathbb{T}_n(\alpha)$ denote a cone-manifold with singular locus \mathbb{T}_n and the cone angle α along it.

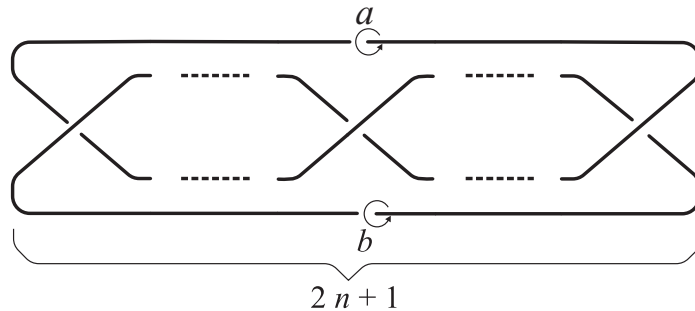


Figure 1: Knot $(2n+1)/1$

The aim of the present section is to investigate cone-manifolds $\mathbb{T}_n(\alpha)$, $n \geq 1$ to find out the domain of sphericity in terms of the cone angle and to derive the volume formulæ.

Two lemmas precede the further exposition:

Lemma 1 For every $0 < \alpha < 2\pi$ and $-1 < \lambda < +1$ the linear transformations

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\alpha} & -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} \\ 0 & 1 \end{pmatrix}$$

are isometries of \mathbb{S}_λ^3 .

Proof. For the further account let us assume that the multiplication of vectors by matrices is to the right. A linear transformation L of the space $\mathbb{C}_\mathbb{R}^2$ preserves the corresponding Hermitian form if and only if for every pair of vectors $P, Q \in \mathbb{C}_\mathbb{R}^2$ it holds that

$$\langle P, Q \rangle_{\mathbb{H}} = P\mathcal{H}\bar{Q}^T = PL\mathcal{H}\bar{L}^T\bar{Q}^T = \langle PL, QL \rangle_{\mathbb{H}}.$$

The condition above is equivalent to

$$\mathcal{H} = L\mathcal{H}\bar{L}^T.$$

In particular,

$$\cos d_\lambda(P, Q) = \langle P, Q \rangle = \langle PL, QL \rangle = \cos d_\lambda(PL, QL),$$

that means L preserves the spherical distance between P and Q .

Let $L = A$ and $L = B$ in series, one verifies that A and B preserve the Hermitian norm on $\mathbb{C}_\mathbb{R}^2$ and, consequently, the spherical distance on \mathbb{S}_λ^3 . \square

Lemma 2 Let A and B be the same matrices as in the affirmation of Lemma 1. Then for all integer $n \geq 1$ one has

$$(AB)^n A - B(AB)^n = 2U_{2n}(\Lambda) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M,$$

where M is a non-zero 2×2 -matrix and $U_{2n}(\Lambda)$ is the second kind Chebyshev polynomial of power $2n$ in variable $\Lambda = \lambda \sin \frac{\alpha}{2}$.

Proof. As far as $-1 < \lambda < +1$, one obtains

$$-1 < \Lambda = \lambda \sin \frac{\alpha}{2} < +1.$$

Substitute

$$\Lambda = \cos \theta,$$

with the unique $0 < \theta < \pi$.

Then matrices A and B are rewritten in the form

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \cos \theta & e^{i\alpha} \end{pmatrix},$$

$$B = \begin{pmatrix} e^{i\alpha} & -2i e^{i\frac{\alpha}{2}} \cos \theta \\ 0 & 1 \end{pmatrix}.$$

On purpose to diagonalize the matrix AB , use

$$V = \begin{pmatrix} i e^{-i\frac{\alpha}{2}} e^{-i\theta} & i e^{-i\frac{\alpha}{2}} e^{i\theta} \\ 1 & 1 \end{pmatrix},$$

and obtain

$$D = V^{-1}(AB)V = \begin{pmatrix} -e^{i\alpha} e^{2i\theta} & 0 \\ 0 & -e^{i\alpha} e^{-2i\theta} \end{pmatrix}.$$

Note, that V might be not an isometry, but it is utile for computation.

Thus

$$\begin{aligned} (AB)^n A - B(AB)^n &= (V D^n V^{-1})A - B(V D^n V^{-1}) = \\ &= 2 \frac{\sin(2n+1)\theta}{\sin \theta} e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} \begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix} = \\ &= 2 U_{2n}(\cos \theta) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M = 2 U_{2n}(\Lambda) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M, \end{aligned}$$

with the matrix

$$M = \begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix}$$

as the present Lemma claims. \square

The main theorem of the section follows:

Theorem 1 *The cone-manifold $\mathbb{T}_n(\alpha)$, $n \geq 1$ is spherical if*

$$\frac{2n-1}{2n+1} \pi < \alpha < 2\pi - \frac{2n-1}{2n+1} \pi.$$

The length of its singular geodesic (i.e. the length of the knot \mathbb{T}_n) equals

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

The volume of $\mathbb{T}_n(\alpha)$ is

$$\text{Vol } \mathbb{T}_n(\alpha) = \frac{1}{2n+1} \left(\frac{2n+1}{2} \alpha - \frac{2n-1}{2} \pi \right)^2.$$

Proof. The fundamental group of the knot \mathbb{T}_n is presented as

$$\pi_1(\mathbb{S}^3 \setminus \mathbb{T}_n) = \langle a, b | (ab)^n a = b(ab)^n \rangle,$$

with generators a and b as at Fig. 1.

Since the cone-manifold $\mathbb{T}_n(\alpha)$ admits a spherical structure, then there exists a holonomy mapping [1], that is a homomorphism

$$h : \pi_1(\mathbb{S}^3 \setminus \mathbb{T}_n) \mapsto \text{Isom } \mathbb{S}_\lambda^3.$$

We will choose h in respect with geometric construction of the cone-manifold. All the further computations to find the length of the knot \mathbb{T}_n and the volume of the cone-manifold $\mathbb{T}_n(\alpha)$ are performed making use of the corresponding fundamental polyhedron \mathcal{P}_n (Fig. 2). The construction algorithm for the polyhedron is given in [12].

The combinatorial polyhedron \mathcal{P}_n has vertices P_i , $i \in \{1, \dots, 4n+2\}$ and edges $P_i P_{i+1}$, $i \in \{1, \dots, 4n+2\}$, with $P_{4n+3} = P_1$, also $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$. Let N , S denote the middle points (the North and the South poles of \mathcal{P}_n) on the edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$, respectively. Then, consider also edges NP_i , SP_i , $i \in \{1, \dots, 4n+2\}$.

Without loss in generality, choose the holonomy representation such that

$$h(a) = A, \quad h(b) = B,$$

where A and B are matrices from Lemma 1.

The generators of the fundamental group for \mathbb{T}_n under the holonomy mapping h correspond to isometries acting on \mathcal{P}_n . These isometries identify its faces by means of rotation about the edge $P_1 P_{2n+2}$ for the top ‘‘cupola’’ of \mathcal{P}_n and rotation about $P_2 P_{2n+3}$ for the bottom one (see, Fig. 2). Then the edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ knot itself to produce \mathbb{T}_n (cf. [12, 13]).

In order to construct the polyhedron \mathcal{P}_n assume that its edge $P_1 P_2$ is given by

$$P_1 = (1, 0), \quad P_2 = (0, 1).$$

Then one has

$$\cos d_\lambda(P_1, P_2) = \langle P_1, P_2 \rangle = \lambda,$$

i.e. the spherical distance between the points P_1 and P_2 can vary from 0 to π . Thus, prescribing certain coordinates to the end-points of the edge $P_1 P_2$ we do not loss in generality of the consideration.

Note, that the axis of the isometry A from Lemma 1 contains P_1 and the axis of B contains P_2 . The aim of the construction for the polyhedron \mathcal{P}_n is to bring its edges $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ to be axes of the respective isometries A and B . The other vertices P_i has to be images of P_1 and P_2 under action of A and B . The polyhedron \mathcal{P}_n is said to be proper if

- (a) inner dihedral angles along $P_1 P_{2n+2}$ and $P_2 P_{2n+3}$ are equal to α ;
- (b) the following curvilinear faces are identified by A and B :

$$A : NP_1 P_2 \dots P_{2n+2} \rightarrow NP_1 P_{4n+2} \dots P_{2n+3} P_{2n+2},$$

$$B : SP_2 P_1 P_{4n+2} \dots P_{2n+3} \rightarrow SP_2 P_3 \dots P_{2n+3};$$

- (c) sum of the inner dihedral angles ψ_i along $P_i P_{i+1}$, $i \in \{1, \dots, 4n+1\}$ equals 2π ;
- (d) sum of the dihedral angles ϕ_i for corresponding tetrahedra $NSP_i P_{i+1}$, $i \in \{1, \dots, 4n+1\}$ at their common edge NS is 2π ;

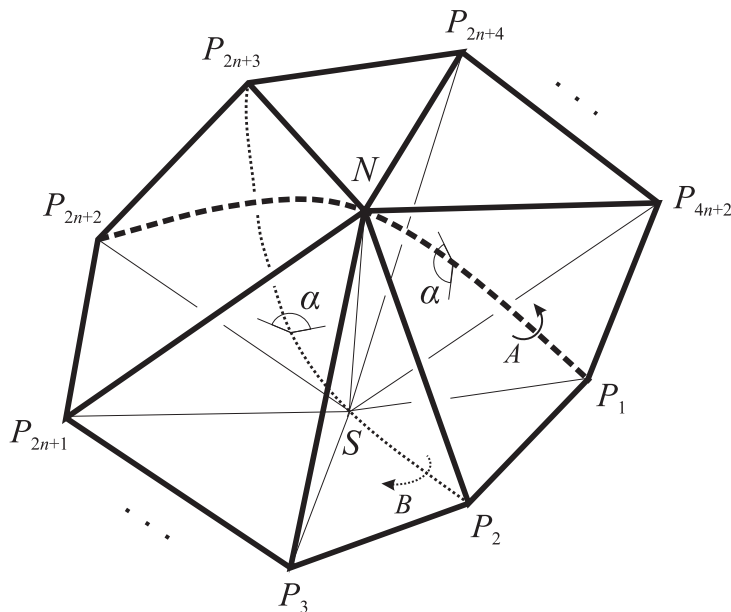


Figure 2: Fundamental polyhedron \mathcal{P}_n for $\mathbb{T}_n(\alpha)$

(e) all the tetrahedra NSP_iP_{i+1} with $i \in \{1, \dots, 4n+2\}$, $P_{4n+3} = P_1$ are non-degenerated and coherently oriented.

By the orientation of a tetrahedron NSP_iP_{i+1} one means the sign of the Gram determinant $\det(S, N, P_i, P_{i+1})$ for corresponding quadruple $S, N, P_i, P_{i+1} \in \mathbb{C}_{\mathbb{R}}^2$, where $i \in \{1, \dots, 4n+2\}$, $P_{4n+3} = P_1$. A tetrahedron is non-degenerated if $\det(S, N, P_i, P_{i+1}) \neq 0$. Thus, claim (e) is satisfied if all the Gram determinants are non-zero and of the same sign.

If $\alpha = \frac{2\pi}{m}$, $m \in \mathbb{N}$, then due to the Poincaré Theorem [14, Theorem 13.5.3] claims (a) – (e) imply that the group generated by the isometries A and B is discrete and its presentation is

$$\Gamma = \langle A, B | (AB)^n A = B(AB)^n, A^m = B^m = \text{id} \rangle.$$

The metric space $\mathbb{S}_\lambda^3/\Gamma \cong \mathbb{T}_n(\frac{2\pi}{m})$ is a spherical orbifold, and \mathcal{P}_n is its fundamental polyhedron. If $m \notin \mathbb{N}$ then the group generated by A and B might be non-discrete. However, the identification for the faces of \mathcal{P}_n is of the same fashion as if it were $m \in \mathbb{N}$ and as the result one obtains the cone-manifold $\mathbb{T}_n(\alpha)$. By means of Lemma 1 and construction of \mathcal{P}_n claims (a) and (b) are satisfied. For the holonomy mapping h to exist the following relation should be satisfied:

$$h((ab)^n a) - h(b(ab)^n) = (AB)^n A - B(AB)^n = 0.$$

By Lemma 2, the condition above is satisfied if and only if

$$U_{2n}(\Lambda) = 0,$$

where $\Lambda = \lambda \sin \frac{\alpha}{2}$.

Thus, the parameter λ of the metric ds_λ^2 is determined completely by a root of the polynomial $U_{2n}(\Lambda)$. From the above formula, λ is related to the cone angle α by means of the equality

$$\lambda = \frac{\Lambda}{\sin \frac{\alpha}{2}}.$$

The roots of $U_{2n}(\Lambda)$ are given by the following formula:

$$\Lambda_k = \cos \frac{k\pi}{2n+1},$$

with $k \in \{1, \dots, 2n\}$.

The parameter λ for the metric ds_λ^2 has to be chosen in order the polyhedron \mathcal{P}_n be proper and the metric itself be spherical.

Note, that the edges $P_i P_{i+1}$, $i \in \{1, \dots, 4n+2\}$, $P_{4n+3} = P_1$ are equivalent under action of the group $\Gamma = \langle A, B \rangle$. Thus, the relation $(AB)^n A = B(AB)^n$ implies the equality

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi,$$

where k is an integer.

Show that one can choose λ for the equality $k = 1$ to hold for all α in the affirmation of the Theorem. Due to the paper [15], every two-bridge knot cone-manifold with cone angle π is a spherical orbifold. In this case all the vertices P_i of the fundamental polyhedron belong to the same circle and all the dihedral angles ψ_i and ϕ_i are equal to each other [12]:

$$\phi_i = \psi_i = \frac{\pi}{2n+1}.$$

As far as $\cos d_\lambda(N, S) = \cos d_\lambda(P_i, P_{i+1}) = \lambda$, then in case $\alpha = \pi$ one obtains

$$\lambda = \frac{\Lambda_k}{\sin \frac{\pi}{2}} = \cos \theta$$

for certain $k \in \{1, \dots, 2n\}$ and then

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2(2n+1)\theta.$$

Using the formula for the roots of $U_{2n}(\Lambda)$ obtain that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi$$

if $\alpha = \pi$. Thus, claim (c) for the polyhedron \mathcal{P}_n with $\alpha = \pi$ is satisfied if $k = 1$. As far as the parameter α varies continuously and sum of the angles ψ_i

represents a multiple of 2π , one has that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2\pi$$

for all α .

By analogy, show that with

$$\lambda = \frac{\Lambda_1}{\sin \frac{\alpha}{2}}$$

the equality

$$\sum_{i=1}^{2(2n+1)} \phi_i = 2\pi$$

holds, that means claim (d) is also satisfied.

Verify that under conditions of the Theorem the metric ds_λ^2 is spherical. This claim is equivalent to the inequality

$$-1 < \lambda < +1.$$

Note, that for

$$\frac{2n-1}{2n+1} \pi < \alpha < 2\pi - \frac{2n-1}{2n+1} \pi$$

it follows

$$\sin \frac{\alpha}{2} > \sin \frac{(2n-1)\pi}{2(2n+1)}.$$

As far as $\sin \frac{\alpha}{2} > 0$ and $\Lambda_1 = \sin \frac{(2n-1)\pi}{2(2n+1)} > 0$, one has

$$0 < \lambda < 1.$$

By analogy with Lemma 1 verify that

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an isometry of ds_λ^2 .

Fixed point sets of A and B in \mathbb{S}_λ^3 are circles

$$\text{Fix } A = \{(z_1, 0) : z_1 \in \mathbb{C}, |z_1| = 1\}$$

and

$$\text{Fix } B = \{(0, z_2) : z_2 \in \mathbb{C}, |z_2| = 1\},$$

correspondingly. The geometric meaning of C is that it maps the first fixed circle to the other. Thus, the relation $B = CAC^{-1}$ holds.

The following equalities

$$P_{2k+1} = P_1(AB)^k, \quad k \in \{0, \dots, n\},$$

$$P_{2k} = P_2(AB)^{k-1}, k \in \{1, \dots, n+1\};$$

and

$$\begin{aligned} P_{2k+1} &= P_1(BA)^{2n-k+1}, k \in \{n+1, \dots, 2n\}, \\ P_{2k} &= P_2(BA)^{2n-k+2}, k \in \{n+2, \dots, 2n+1\}, \end{aligned}$$

follow from the identification scheme of the edges of \mathcal{P}_n .
Define the auxiliary function

$$\varepsilon(m) = \frac{m}{2} \alpha - \frac{4n-m}{2} \pi.$$

By analogy with the proof of Lemma 2 it follows that

$$\begin{aligned} (AB)^k &= C(BA)^k C^{-1} = \\ &= \begin{pmatrix} -\frac{\sin(2k-1)\theta}{\sin\theta} e^{i\varepsilon(2k)} & -\frac{\sin 2k\theta}{\sin\theta} e^{i\varepsilon(2k-1)} \\ \frac{\sin 2k\theta}{\sin\theta} e^{i\varepsilon(2k+1)} & \frac{\sin(2k+1)\theta}{\sin\theta} e^{i\varepsilon(2k)} \end{pmatrix}, \end{aligned}$$

where $\theta = \frac{\pi}{2n+1}$.

Suppose N and S to be middle-points of the edges P_1P_{2n+2} and P_2P_{2n+3} , respectively. Then

$$N = (e^{i\frac{\varepsilon(2n+1)}{2}}, 0), S = (0, e^{i\frac{\varepsilon(2n+1)}{2}}).$$

For the lengths l_α of the singular geodesic one has

$$\cos \frac{l_\alpha}{4} = \langle P_1, N \rangle = \langle P_1 C, N C \rangle = \langle P_2, S \rangle.$$

Thus

$$\cos \frac{l_\alpha}{4} = \cos \frac{(2n+1)\alpha - (2n-1)\pi}{4}.$$

By construction of the polyhedron \mathcal{P}_n , the inequality $0 < l_\alpha < 4\pi$ holds. Then it follows

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

Given the coordinates of the vertices P_i and the poles N and S of the polyhedron \mathcal{P}_n , verify claim (e).

For every four points $A, B, C, D \in \mathbb{C}_{\mathbb{R}}^2$, where

$$A = (A_1, A_2), B = (B_1, B_2), C = (C_1, C_2), D = (D_1, D_2),$$

their Gram determinant is

$$\det(A, B, C, D) := \det \begin{pmatrix} \operatorname{Re} A_1 & \operatorname{Im} A_1 & \operatorname{Re} A_2 & \operatorname{Im} A_2 \\ \operatorname{Re} B_1 & \operatorname{Im} B_1 & \operatorname{Re} B_2 & \operatorname{Im} B_2 \\ \operatorname{Re} C_1 & \operatorname{Im} C_1 & \operatorname{Re} C_2 & \operatorname{Im} C_2 \\ \operatorname{Re} D_1 & \operatorname{Im} D_1 & \operatorname{Re} D_2 & \operatorname{Im} D_2 \end{pmatrix}.$$

Each tetrahedron NSP_iP_{i+1} with $i \in \{1, \dots, 2n+1\}$ is isometric to $NSP_{2n+i+1}P_{2n+i+2}$, $i \in \{1, \dots, 2n+1\}$, $P_{4n+3} = P_1$ by means of the isometry C defined above. Thus, we consider only the tetrahedra NSP_iP_{i+1} with $i \in \{1, \dots, 2n+1\}$. Split them into two groups: the tetrahedra $NSP_{2k+1}P_{2k+2}$ with $k \in \{0, \dots, n\}$ and the tetrahedra $NSP_{2k}P_{2k+1}$ with $k \in \{1, \dots, n\}$. Substitute $\alpha = \beta + \pi$ and proceed with straightforward calculations:

$$\begin{aligned}\Delta_k^{(1)}(\beta) &= \det(S, N, P_{2k+1}, P_{2k+2}) = \cos^2 \frac{L_1 \beta}{4} - U_{2k-1}^2(\cos \theta) \sin^2 \frac{\beta}{2} = \\ &= T_{L_1}^2(\cos \frac{\beta}{4}) - U_{2k-1}^2(\cos \theta) \sin^2 \frac{\beta}{2},\end{aligned}$$

where $k \in \{0, \dots, n\}$, $L_1 = |2n - 4k + 1|$, $\theta = \frac{\pi}{2n+1}$, $\beta \in [-2\theta, 2\theta]$;

$$\begin{aligned}\Delta_k^{(2)}(\beta) &= \det(S, N, P_{2k}, P_{2k+1}) = \cos^2 \frac{L_2 \beta}{4} - U_{2k-2}^2(\cos \theta) \sin^2 \frac{\beta}{2} = \\ &= T_{L_2}^2(\cos \frac{\beta}{4}) - U_{2k-2}^2(\cos \theta) \sin^2 \frac{\beta}{2},\end{aligned}$$

where $k \in \{1, \dots, n\}$, $L_2 = |2n - 4k + 3|$, θ and β the same as above. The first kind Chebyshev polynomial of degree $k \geq 0$ is denoted by T_k . Assume that

$$U_{-1}(\cos \theta) = 0, \quad U_0(\cos \theta) = 1$$

for the sake of brevity.

All the functions $\Delta_k^{(j)}(\beta)$, $j \in \{1, 2\}$ are even on the interval $[-2\theta, 2\theta]$. Then one considers them only on the interval $[0, 2\theta]$. Note, that the polynomial $T_{L_j}^2(\cos \beta)$ monotonously decreases and the function $\sin^2 \frac{\beta}{2}$ monotonously increases with $\beta \in [0, 2\theta]$. Moreover, $T_{L_j}^2(\cos 0) = T_{L_j}^2(1) = 1$. Then it follows that $\Delta_k^{(j)}(\beta) > 0$ with $\beta \in (-2\theta, 2\theta)$. Also, one has $\Delta_k^{(j)}(\pm 2\theta) = 0$. Then for all $\beta \in (-2\theta, 2\theta)$ (i.e. for all α in the affirmation of the Theorem)

$$\det(S, N, P_i, P_{i+1}) > 0$$

where $i \in \{1, \dots, 4n+2\}$, $P_{4n+3} = P_1$. Thus, claim (e) for the polyhedron \mathcal{P}_n is satisfied.

Use the Schläfli formula [16] to obtain the volume formula for $\mathbb{T}_n(\alpha)$. One has

$$d\text{Vol } \mathbb{T}_n(\alpha) = \frac{l_\alpha}{2} d\alpha = \frac{(2n+1)\alpha - (2n-1)\pi}{2} d\alpha.$$

Note, that $\text{Vol } \mathbb{T}_n(\alpha) \rightarrow 0$ with $\alpha \rightarrow \frac{2n-1}{2n+1}\pi$. In this case $d_\lambda(P_i, P_{i+1}) \rightarrow 0$, where $i \in \{1, \dots, 4n+2\}$, $P_{4n+3} = P_1$ and the fundamental polyhedron collapses to a point. Thus

$$\text{Vol } \mathbb{T}_n(\alpha) = \frac{1}{2n+1} \left(\frac{2n+1}{2} \alpha - \frac{2n-1}{2} \pi \right)^2.$$

□

Remark 1 *The domain of the spherical metric existence in Theorem 1 was indicated before in [10, Proposition 2.1].*

4 Torus links \mathbb{L}_n

Let $\mathbb{L}_n, n \geq 2$ be a torus link $t(2n, 2)$ with two components. The corresponding link in the rational census is $2n/1$ (Fig. 3). The fundamental group of \mathbb{L}_n is presented as

$$\pi_1(\mathbb{S}^3 \setminus \mathbb{L}_n) = \langle a, b | (ab)^n = (ba)^n \rangle.$$

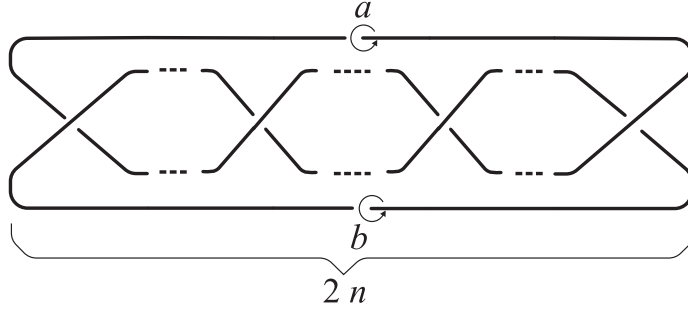


Figure 3: Link $2n/1$

Let $\mathbb{L}_n(\alpha, \beta)$ denote a cone-manifold with singular locus the link \mathbb{L}_n and the cone angles α, β along its components.

For every $\alpha, \beta \in (0, 2\pi)$ and $\lambda \in (-1, +1)$, we denote

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\beta} & -2i e^{i\frac{\beta}{2}} \lambda \sin \frac{\beta}{2} \\ 0 & 1 \end{pmatrix}.$$

By Lemma 1, linear transformations A and B are isometries of \mathbb{S}_λ^3 .

Lemma 3 *For every integer $n \geq 2$ the following equality holds*

$$(AB)^n - (BA)^n = 4U_{n-1}(\Lambda) \lambda e^{i(\frac{\alpha+\beta}{2}+\pi)n} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} M,$$

where M is a non-zero 2×2 matrix and $U_{n-1}(\Lambda)$ is the second kind Chebyshev polynomial of degree $n - 1$ in variable

$$\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}.$$

Proof. By analogy with Lemma 2. \square

With Lemma 3 the main theorem of the section follows:

Theorem 2 *The cone-manifold $\mathbb{L}_n(\alpha, \beta)$, $n \geq 2$ is spherical if*

$$\begin{aligned} -2\pi \left(1 - \frac{1}{n}\right) &< \alpha - \beta < 2\pi \left(1 - \frac{1}{n}\right), \\ 2\pi \left(1 - \frac{1}{n}\right) &< \alpha + \beta < 2\pi \left(1 + \frac{1}{n}\right). \end{aligned}$$

The lengths l_α, l_β of its singular geodesics (i.e. lengths of the components for \mathbb{L}_n) are equal to each other and

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi(n - 1).$$

The volume of $\mathbb{L}_n(\alpha, \beta)$ is

$$\text{Vol } \mathbb{L}_n(\alpha, \beta) = \frac{1}{2n} \left(\frac{\alpha + \beta}{2} n - (n - 1)\pi \right)^2.$$

Proof. One continues the proof by analogy with Theorem 1. Suppose that $\mathbb{L}_n(\alpha, \beta)$ is spherical. Then there exists a holonomy mapping [1]:

$$\begin{aligned} h : \pi_1(\mathbb{S}^3 \setminus \mathbb{L}_n) &\longmapsto \text{Isom } \mathbb{S}_\lambda^3, \\ h(a) &= A, \quad h(b) = B. \end{aligned}$$

Also,

$$h((ab)^n) - h((ba)^n) = (AB)^n - (BA)^n = 0.$$

By means of Lemma 3 the equality above holds either if $\lambda = 0$, or if

$$\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}$$

is a root of the equation $U_{n-1}(\Lambda) = 0$.

In case $\lambda = 0$ the image of h is abelian, because of the additional relation $AB = BA$. With $n \geq 2$ this leads to a degenerate geometric structure. Thus, one has to choose the parameter λ for the metric ds_λ^2 using roots of the Chebyshev polynomial $U_{n-1}(\Lambda)$.

The fundamental polyhedron \mathcal{F}_n for the cone-manifold $\mathbb{L}_n(\alpha, \beta)$ is depicted at Fig. 4. Suppose its vertices P_1 and P_2 to be

$$P_1 = (1, 0), \quad P_2 = (0, 1).$$

The axes of isometries A and B correspond to the edges P_1P_{2n+1} and P_2P_{2n+2} . Points N and S are respective middles of the edges P_1P_{2n+1} and P_2P_{2n+2} . Those are called North and South poles of the polyhedron. The polyhedron \mathcal{F}_n is said to be proper if

- (a) respective inner dihedral angles along the edges P_1P_{2n+1} and P_2P_{2n+2} are equal to α and β ;

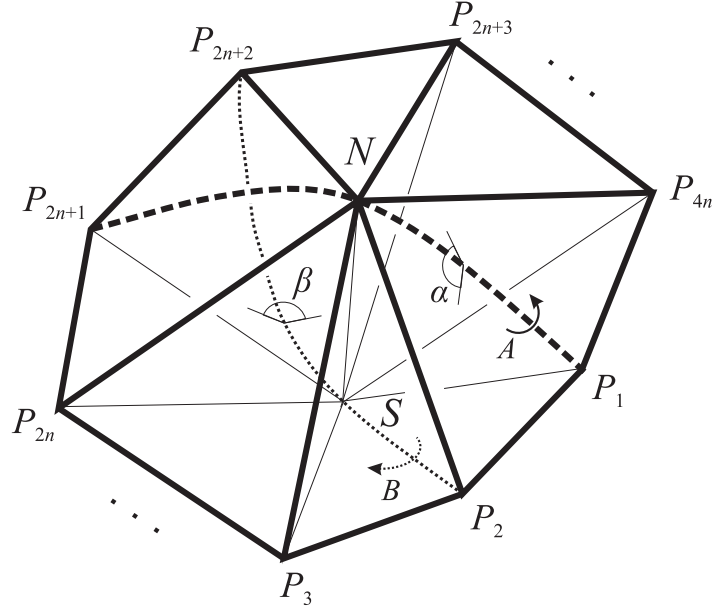


Figure 4: The fundamental polyhedron \mathcal{F}_n for $\mathbb{L}_n(\alpha, \beta)$

(b) curvilinear faces of the polyhedron are identified by A and B :

$$A : NP_1P_2 \dots P_{2n+1} \rightarrow NP_1P_{4n} \dots P_{2n+2}P_{2n+1},$$

$$B : SP_2P_1P_{4n} \dots P_{2n+2} \rightarrow SP_2P_3 \dots P_{2n+2};$$

(c) sum of the inner dihedral angles ψ_i along the edges P_iP_{i+1} , $i \in \{1, \dots, 4n-1\}$ equals 2π ;

(d) sum of the dihedral angles ϕ_i for tetrahedra NSP_iP_{i+1} , $i \in \{1, \dots, 4n-1\}$ at their common edge NS equals 2π ;

(e) all the tetrahedra NSP_iP_{i+1} with $i \in \{1, \dots, 4n\}$, $P_{4n+1} = P_1$ are non-degenerated and coherently oriented.

In order to choose the parameter λ for the corresponding metric consider the fundamental polyhedron \mathcal{F}_n with $\alpha = \beta = \pi$. Then all its vertices belong to the same circle and all the dihedral angles ψ_i of the tetrahedra NSP_iP_{i+1} along the edges P_iP_{i+1} are equal to $\psi = \frac{\pi}{2n}$ [12]. Also the dihedral angles ϕ_i of the tetrahedra NSP_iP_{i+1} along their common edge NS are equal to each other:

$$\phi_i = \phi = \frac{\pi}{2n}.$$

In this case $\lambda = \langle P_1, P_2 \rangle = \cos \phi$ and

$$\Lambda = -\cos 2\phi = \cos \frac{(n-1)\pi}{n}.$$

All the roots of $U_{n-1}(\Lambda)$ are given by the formula

$$\Lambda_k = \cos \frac{k\pi}{n}, \quad k \in \{1, \dots, n-1\},$$

so one choose the root Λ_k with $k = n-1$. Then, by analogy with Theorem 1, equalities

$$\sum_{i=1}^{4n} \psi_i = 2\pi$$

and

$$\sum_{i=1}^{4n} \phi_i = 2\pi$$

are satisfied at the point $\alpha = \beta = \pi$ of the domain

$$\mathcal{D} = \left\{ (\alpha, \beta) : |\alpha - \beta| < 2\pi \left(1 - \frac{1}{n}\right), |\alpha + \beta - 2\pi| < \frac{2\pi}{n} \right\},$$

depicted at Fig. 5.

In terms of the parameter λ , that defines the metric ds_λ^2 , one has

$$\lambda^2 = \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\pi}{n}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}}.$$

As for all $(\alpha, \beta) \in \mathcal{D}$ the inequality $0 < \lambda^2 < 1$ is satisfied, the metric ds_λ^2 is spherical regarding the corresponding domain. By analogy with Theorem 1 one can show that claims (a) – (d) for the polyhedron \mathcal{F}_n are satisfied in the interior of \mathcal{D} .

The lengths l_α and l_β of singular geodesics for the cone-manifold $\mathbb{L}_n(\alpha, \beta)$ meet the relations

$$\begin{aligned} \cos \frac{l_\alpha}{2} &= \langle P_1, N \rangle, \\ \cos \frac{l_\beta}{2} &= \langle P_2, S \rangle. \end{aligned}$$

By analogy with the proof of Theorem 1 one obtains

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi(n-1).$$

Given the coordinates of the vertices for the fundamental polyhedron verify claim (e) for all (α, β) in the domain \mathcal{D} .

Make use of the Schläfli formula [16] to obtain the volume of $\mathbb{L}_n(\alpha, \beta)$:

$$d \text{Vol } \mathbb{L}_n(\alpha, \beta) = \frac{l_\alpha}{2} d\alpha + \frac{l_\beta}{2} d\beta = \left(\frac{\alpha + \beta}{2} n - \pi(n-1) \right) d \left(\frac{\alpha + \beta}{2} \right).$$

Note, that with

$$\alpha = \beta \rightarrow \pi \frac{n-1}{n}$$

the fundamental polyhedron \mathcal{F}_n collapses to a point (i.e. the volume tends to 0). The last affirmation of the Theorem follows. \square

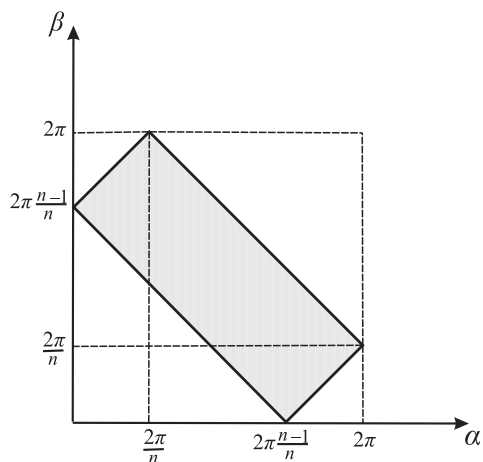


Figure 5: The domain \mathcal{D} of sphericity for $\mathbb{L}_n(\alpha, \beta)$

Remark 2 Under condition $\alpha = \beta$ the inequality from the affirmation of Theorem 2 coincides with the inequality from [10, Proposition 2.2].

Remark 3 Note, that the lengths of the singular geodesics for $\mathbb{L}_n(\alpha, \beta)$ are equal even if $\alpha \neq \beta$.

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