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# Spherical structures on torus knots and links \*

Alexander Kolpakov, Alexander Mednykh

## Abstract

The present paper considers two infinite families of cone-manifolds endowed with spherical metric. The singular strata is either the torus knot  $t(2n+1, 2)$  or the torus link  $t(2n, 2)$ . Domains of existence for a spherical metric are found in terms of cone angles and volume formulæ are presented.

**Key words:** Spherical geometry, cone-manifold, knot, link.

## 1 Introduction

A three-dimensional cone-manifold is a metric space obtained from a collection of disjoint simplices in the space of constant sectional curvature  $k$  by isometric identification of their faces in such a combinatorial fashion that the resulting topological space is a manifold (also called the underlying space for a given cone-manifold).

Such the metric space inherits the metric of sectional curvature  $k$  on the union of its 2- and 3-dimensional cells. In case  $k = +1$  the corresponding cone-manifold is called spherical (or admits a spherical structure). By analogy, one defines euclidean ( $k = 0$ ) and hyperbolic ( $k = -1$ ) cone-manifolds.

The metric structure around each 1-cell is determined by a cone angle that is the sum of dihedral angles of corresponding simplices sharing the 1-cell under identification. The singular locus of a cone-manifold is the closure of all its 1-cells with cone angle different from  $2\pi$ . For the further account we suppose that every component of the singular locus is an embedded circle with constant cone angle along it.

A particular case of cone-manifold is an orbifold with cone angles  $2\pi/m$ , where  $m$  is an integer (cf. [1]).

The present paper considers two infinite families of cone-manifolds with underlying space the three-dimensional sphere  $\mathbb{S}^3$ . The first family consists of cone-manifolds with singular locus the torus knot  $t(2n+1, 2)$  with  $n \geq 1$ . In the rational census [2] these knots are denoted by  $(2n+1)/1$ . The second family of cone-manifolds consists of those with singular locus a two-component torus link  $t(2n, 2)$  with  $n \geq 2$ . These links are two-bridge and correspond to the links

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$2n/1$  in the rational census. The simplest examples of such the knots and links are the trefoil knot  $3/1$  and the link  $4/1$ . In the Rolfsen table [2] one finds them as the knot  $3_1$  and the link  $4_1^2$ .

By the Theorem of W. Thurston [3], the manifold  $\mathbb{S}^3 \setminus 3_1$  does not admit a hyperbolic structure. However, it admits two other geometric structures [4]:  $\mathbb{H}^2 \times \mathbb{R}$  and  $\widetilde{\text{PSL}}(2, \mathbb{R})$ . It follows from the paper [5] that the spherical dodecahedron space (i.e. Poincaré homology sphere) is a cyclic 5-fold covering of  $\mathbb{S}^3$  branched over  $3_1$ . Thus, the orbifold  $3_1(\frac{2\pi}{5})$  with singular locus the trefoil knot and cone angle  $\frac{2\pi}{5}$  is spherical. Due to the Dunbar's census [6], orbifold  $3_1(\frac{2\pi}{n})$  is spherical if  $n \leq 5$ , Nil-orbifold if  $n = 6$  and  $\widetilde{\text{PSL}}(2, \mathbb{R})$ -orbifold if  $n \geq 7$ . Spherical structure on the cone-manifold  $3_1(\alpha)$  with underlying space the three-dimensional sphere  $\mathbb{S}^3$  is studied in [7].

The consideration of two-bridge torus links is carried out starting with the simplest one possessing non-abelian fundamental group, namely  $4_1^2$ .

The previous investigation on spherical structures for cone-manifolds is carried out mainly in the papers [8, 9, 10]. The present paper develops a method to analyse existence of a spherical metric for two-bridge torus knot and link cone-manifolds. Also, the lengths of singular geodesics are calculated and the volume formulæ are obtained (cf. Theorem 1 and Theorem 2).

## 2 Projective model $\mathbb{S}_\lambda^3$

The purpose of the present section is to construct the projective model  $\mathbb{S}_\lambda^3$  that one can use to study geometric properties of two-bridge torus knots and links and to build up holonomy representation for the corresponding cone-manifolds. Other projective models for homogeneous geometries are described in [11].

Consider the set  $\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$  as a four-dimensional vector space over  $\mathbb{R}$ . We denote it by  $\mathbb{C}_\mathbb{R}^2$  and equip with Hermitian product

$$\langle (z_1, z_2), (w_1, w_2) \rangle_{\mathbb{H}} = (z_1, z_2) \mathcal{H} \overline{(w_1, w_2)}^T,$$

where

$$\mathcal{H} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

is a symmetric matrix with  $-1 < \lambda < +1$ .

The natural inner product is associated to the Hermitian form above:

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \text{Re} \langle (z_1, z_2), (w_1, w_2) \rangle_{\mathbb{H}}$$

and the respective norm is

$$\|(z_1, z_2)\| = |z_1|^2 + |z_2|^2 + \lambda(z_1 \bar{z}_2 + \bar{z}_1 z_2).$$

Call two elements  $(z_1, z_2)$  and  $(w_1, w_2)$  in  $\overset{\circ}{\mathbb{C}}_\mathbb{R}^2 = \mathbb{C}_\mathbb{R}^2 \setminus (0, 0)$  equivalent if there is  $\mu > 0$  such that  $(z_1, z_2) = (\mu w_1, \mu w_2)$ . We denote this equivalence relation as  $(z_1, z_2) \sim (w_1, w_2)$ .

Identify the factor-space  $\overset{\circ}{\mathbb{C}}_{\mathbb{R}}^2 / \sim$  with the three-dimensional sphere

$$\mathbb{S}_{\lambda}^3 = \{(z_1, z_2) \in \mathbb{C}_{\mathbb{R}}^2 : \|(z_1, z_2)\| = 1\},$$

endowed with the Riemannian metric

$$ds_{\lambda}^2 = |dz_1|^2 + |dz_2|^2 + \lambda(dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2).$$

By means of equality

$$ds_{\lambda}^2 = \frac{1+\lambda}{2} |dz_1 + dz_2|^2 + \frac{1-\lambda}{2} |dz_1 - dz_2|^2,$$

the linear transformation

$$\xi_1 = \sqrt{\frac{1+\lambda}{2}} (z_1 + z_2), \quad \xi_2 = \sqrt{\frac{1-\lambda}{2}} (z_1 - z_2)$$

provides an isometry between  $(\mathbb{S}_{\lambda}^3, ds_{\lambda}^2)$  and  $(\mathbb{S}^3, ds^2)$ , where  $ds^2 = |d\xi_1|^2 + |d\xi_2|^2$  is the standard metric of sectional curvature +1 on the unit sphere  $\mathbb{S}^3 = \{(\xi_1, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1\}$ .

Let  $P, Q$  be two points in  $\mathbb{S}_{\lambda}^3$ . The spherical distance between  $P$  and  $Q$  is a real number  $d_{\lambda}(P, Q)$  that is uniquely determined by the conditions  $0 \leq d_{\lambda}(P, Q) \leq \pi$  and  $\cos d_{\lambda}(P, Q) = \langle P, Q \rangle$ .

### 3 Torus knots $\mathbb{T}_n$

Let  $\mathbb{T}_n, n \geq 1$  be the torus knot  $t(2n+1, 2)$  embedded in  $\mathbb{S}^3$ . The knot  $\mathbb{T}_n$  is the two-bridge knot  $(2n+1)/1$  in the rational census (Fig. 1). Let  $\mathbb{T}_n(\alpha)$  denote a cone-manifold with singular locus  $\mathbb{T}_n$  and the cone angle  $\alpha$  along it.

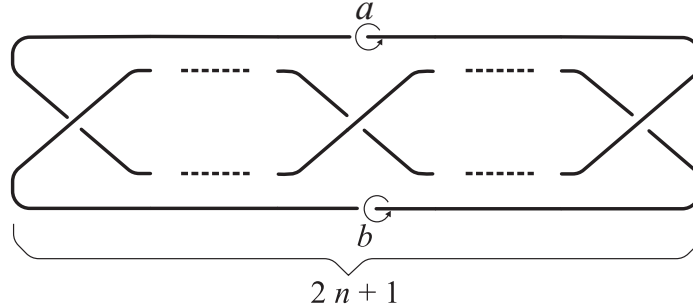


Figure 1: Knot  $(2n+1)/1$

The aim of the present section is to investigate cone-manifolds  $\mathbb{T}_n(\alpha)$ ,  $n \geq 1$  to find out the domain of sphericity in terms of the cone angle and to derive the volume formulæ.

Two lemmas precede the further exposition:

**Lemma 1** For every  $0 < \alpha < 2\pi$  and  $-1 < \lambda < +1$  the linear transformations

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\alpha} & -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} \\ 0 & 1 \end{pmatrix}$$

are isometries of  $\mathbb{S}_\lambda^3$ .

**Proof.** For the further account let us assume that the multiplication of vectors by matrices is to the right. A linear transformation  $L$  of the space  $\mathbb{C}_\mathbb{R}^2$  preserves the corresponding Hermitian form if and only if for every pair of vectors  $P, Q \in \mathbb{C}_\mathbb{R}^2$  it holds that

$$\langle P, Q \rangle_{\mathbb{H}} = P\mathcal{H}\bar{Q}^T = PL\mathcal{H}\bar{L}^T\bar{Q}^T = \langle PL, QL \rangle_{\mathbb{H}}.$$

The condition above is equivalent to

$$\mathcal{H} = L\mathcal{H}\bar{L}^T.$$

In particular,

$$\cos d_\lambda(P, Q) = \langle P, Q \rangle = \langle PL, QL \rangle = \cos d_\lambda(PL, QL),$$

that means  $L$  preserves the spherical distance between  $P$  and  $Q$ .

Let  $L = A$  and  $L = B$  in series, one verifies that  $A$  and  $B$  preserve the Hermitian norm on  $\mathbb{C}_\mathbb{R}^2$  and, consequently, the spherical distance on  $\mathbb{S}_\lambda^3$ .  $\square$

**Lemma 2** Let  $A$  and  $B$  be the same matrices as in the affirmation of Lemma 1. Then for all integer  $n \geq 1$  one has

$$(AB)^n A - B(AB)^n = 2U_{2n}(\Lambda) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M,$$

where  $M$  is a non-zero  $2 \times 2$ -matrix and  $U_{2n}(\Lambda)$  is the second kind Chebyshev polynomial of power  $2n$  in variable  $\Lambda = \lambda \sin \frac{\alpha}{2}$ .

**Proof.** As far as  $-1 < \lambda < +1$ , one obtains

$$-1 < \Lambda = \lambda \sin \frac{\alpha}{2} < +1.$$

Substitute

$$\Lambda = \cos \theta,$$

with the unique  $0 < \theta < \pi$ .

Then matrices  $A$  and  $B$  are rewritten in the form

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \cos \theta & e^{i\alpha} \end{pmatrix},$$

$$B = \begin{pmatrix} e^{i\alpha} & -2i e^{i\frac{\alpha}{2}} \cos \theta \\ 0 & 1 \end{pmatrix}.$$

On purpose to diagonalize the matrix  $AB$ , use

$$V = \begin{pmatrix} i e^{-i\frac{\alpha}{2}} e^{-i\theta} & i e^{-i\frac{\alpha}{2}} e^{i\theta} \\ 1 & 1 \end{pmatrix},$$

and obtain

$$D = V^{-1}(AB)V = \begin{pmatrix} -e^{i\alpha} e^{2i\theta} & 0 \\ 0 & -e^{i\alpha} e^{-2i\theta} \end{pmatrix}.$$

Note, that  $V$  might be not an isometry, but it is utile for computation.

Thus

$$\begin{aligned} (AB)^n A - B(AB)^n &= (V D^n V^{-1})A - B(V D^n V^{-1}) = \\ &= 2 \frac{\sin(2n+1)\theta}{\sin \theta} e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} \begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix} = \\ &= 2 U_{2n}(\cos \theta) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M = 2 U_{2n}(\Lambda) e^{i\frac{(2n+1)(\pi+\alpha)}{2}} \sin \frac{\alpha}{2} M, \end{aligned}$$

with the matrix

$$M = \begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix}$$

as the present Lemma claims.  $\square$

The main theorem of the section follows:

**Theorem 1** *The cone-manifold  $\mathbb{T}_n(\alpha)$ ,  $n \geq 1$  is spherical if*

$$\frac{2n-1}{2n+1} \pi < \alpha < 2\pi - \frac{2n-1}{2n+1} \pi.$$

*The length of its singular geodesic (i.e. the length of the knot  $\mathbb{T}_n$ ) equals*

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

*The volume of  $\mathbb{T}_n(\alpha)$  is*

$$\text{Vol } \mathbb{T}_n(\alpha) = \frac{1}{2n+1} \left( \frac{2n+1}{2} \alpha - \frac{2n-1}{2} \pi \right)^2.$$

**Proof.** The fundamental group of the knot  $\mathbb{T}_n$  is presented as

$$\pi_1(\mathbb{S}^3 \setminus \mathbb{T}_n) = \langle a, b | (ab)^n a = b(ab)^n \rangle,$$

with generators  $a$  and  $b$  as at Fig. 1.

Since the cone-manifold  $\mathbb{T}_n(\alpha)$  admits a spherical structure, then there exists a holonomy mapping [1], that is a homomorphism

$$h : \pi_1(\mathbb{S}^3 \setminus \mathbb{T}_n) \mapsto \text{Isom } \mathbb{S}_\lambda^3.$$

We will choose  $h$  in respect with geometric construction of the cone-manifold. All the further computations to find the length of the knot  $\mathbb{T}_n$  and the volume of the cone-manifold  $\mathbb{T}_n(\alpha)$  are performed making use of the corresponding fundamental polyhedron  $\mathcal{P}_n$  (Fig. 2). The construction algorithm for the polyhedron is given in [12].

The combinatorial polyhedron  $\mathcal{P}_n$  has vertices  $P_i$ ,  $i \in \{1, \dots, 4n+2\}$  and edges  $P_i P_{i+1}$ ,  $i \in \{1, \dots, 4n+2\}$ , with  $P_{4n+3} = P_1$ , also  $P_1 P_{2n+2}$  and  $P_2 P_{2n+3}$ . Let  $N$ ,  $S$  denote the middle points (the North and the South poles of  $\mathcal{P}_n$ ) on the edges  $P_1 P_{2n+2}$  and  $P_2 P_{2n+3}$ , respectively. Then, consider also edges  $NP_i$ ,  $SP_i$ ,  $i \in \{1, \dots, 4n+2\}$ .

Without loss in generality, choose the holonomy representation such that

$$h(a) = A, \quad h(b) = B,$$

where  $A$  and  $B$  are matrices from Lemma 1.

The generators of the fundamental group for  $\mathbb{T}_n$  under the holonomy mapping  $h$  correspond to isometries acting on  $\mathcal{P}_n$ . These isometries identify its faces by means of rotation about the edge  $P_1 P_{2n+2}$  for the top ‘‘cupola’’ of  $\mathcal{P}_n$  and rotation about  $P_2 P_{2n+3}$  for the bottom one (see, Fig. 2). Then the edges  $P_1 P_{2n+2}$  and  $P_2 P_{2n+3}$  knot itself to produce  $\mathbb{T}_n$  (cf. [12, 13]).

In order to construct the polyhedron  $\mathcal{P}_n$  assume that its edge  $P_1 P_2$  is given by

$$P_1 = (1, 0), \quad P_2 = (0, 1).$$

Then one has

$$\cos d_\lambda(P_1, P_2) = \langle P_1, P_2 \rangle = \lambda,$$

i.e. the spherical distance between the points  $P_1$  and  $P_2$  can vary from 0 to  $\pi$ . Thus, prescribing certain coordinates to the end-points of the edge  $P_1 P_2$  we do not loss in generality of the consideration.

Note, that the axis of the isometry  $A$  from Lemma 1 contains  $P_1$  and the axis of  $B$  contains  $P_2$ . The aim of the construction for the polyhedron  $\mathcal{P}_n$  is to bring its edges  $P_1 P_{2n+2}$  and  $P_2 P_{2n+3}$  to be axes of the respective isometries  $A$  and  $B$ . The other vertices  $P_i$  has to be images of  $P_1$  and  $P_2$  under action of  $A$  and  $B$ . The polyhedron  $\mathcal{P}_n$  is said to be proper if

- (a) inner dihedral angles along  $P_1 P_{2n+2}$  and  $P_2 P_{2n+3}$  are equal to  $\alpha$ ;
- (b) the following curvilinear faces are identified by  $A$  and  $B$ :

$$A : NP_1 P_2 \dots P_{2n+2} \rightarrow NP_1 P_{4n+2} \dots P_{2n+3} P_{2n+2},$$

$$B : SP_2 P_1 P_{4n+2} \dots P_{2n+3} \rightarrow SP_2 P_3 \dots P_{2n+3};$$

- (c) sum of the inner dihedral angles  $\psi_i$  along  $P_i P_{i+1}$ ,  $i \in \{1, \dots, 4n+1\}$  equals  $2\pi$ ;
- (d) sum of the dihedral angles  $\phi_i$  for corresponding tetrahedra  $NSP_i P_{i+1}$ ,  $i \in \{1, \dots, 4n+1\}$  at their common edge  $NS$  is  $2\pi$ ;

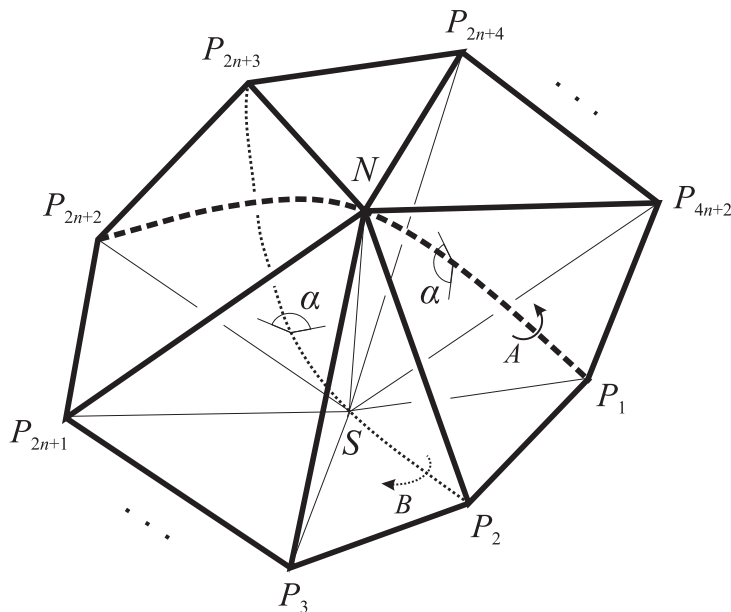


Figure 2: Fundamental polyhedron  $\mathcal{P}_n$  for  $\mathbb{T}_n(\alpha)$

(e) all the tetrahedra  $NSP_iP_{i+1}$  with  $i \in \{1, \dots, 4n+2\}$ ,  $P_{4n+3} = P_1$  are non-degenerated and coherently oriented.

By the orientation of a tetrahedron  $NSP_iP_{i+1}$  one means the sign of the Gram determinant  $\det(S, N, P_i, P_{i+1})$  for corresponding quadruple  $S, N, P_i, P_{i+1} \in \mathbb{C}_{\mathbb{R}}^2$ , where  $i \in \{1, \dots, 4n+2\}$ ,  $P_{4n+3} = P_1$ . A tetrahedron is non-degenerated if  $\det(S, N, P_i, P_{i+1}) \neq 0$ . Thus, claim (e) is satisfied if all the Gram determinants are non-zero and of the same sign.

If  $\alpha = \frac{2\pi}{m}$ ,  $m \in \mathbb{N}$ , then due to the Poincaré Theorem [14, Theorem 13.5.3] claims (a) – (e) imply that the group generated by the isometries  $A$  and  $B$  is discrete and its presentation is

$$\Gamma = \langle A, B | (AB)^n A = B(AB)^n, A^m = B^m = \text{id} \rangle.$$

The metric space  $\mathbb{S}_\lambda^3/\Gamma \cong \mathbb{T}_n(\frac{2\pi}{m})$  is a spherical orbifold, and  $\mathcal{P}_n$  is its fundamental polyhedron. If  $m \notin \mathbb{N}$  then the group generated by  $A$  and  $B$  might be non-discrete. However, the identification for the faces of  $\mathcal{P}_n$  is of the same fashion as if it were  $m \in \mathbb{N}$  and as the result one obtains the cone-manifold  $\mathbb{T}_n(\alpha)$ . By means of Lemma 1 and construction of  $\mathcal{P}_n$  claims (a) and (b) are satisfied. For the holonomy mapping  $h$  to exist the following relation should be satisfied:

$$h((ab)^n a) - h(b(ab)^n) = (AB)^n A - B(AB)^n = 0.$$

By Lemma 2, the condition above is satisfied if and only if

$$U_{2n}(\Lambda) = 0,$$



where  $\Lambda = \lambda \sin \frac{\alpha}{2}$ .

Thus, the parameter  $\lambda$  of the metric  $ds_\lambda^2$  is determined completely by a root of the polynomial  $U_{2n}(\Lambda)$ . From the above formula,  $\lambda$  is related to the cone angle  $\alpha$  by means of the equality

$$\lambda = \frac{\Lambda}{\sin \frac{\alpha}{2}}.$$

The roots of  $U_{2n}(\Lambda)$  are given by the following formula:

$$\Lambda_k = \cos \frac{k\pi}{2n+1},$$

with  $k \in \{1, \dots, 2n\}$ .

The parameter  $\lambda$  for the metric  $ds_\lambda^2$  has to be chosen in order the polyhedron  $\mathcal{P}_n$  be proper and the metric itself be spherical.

Note, that the edges  $P_i P_{i+1}$ ,  $i \in \{1, \dots, 4n+2\}$ ,  $P_{4n+3} = P_1$  are equivalent under action of the group  $\Gamma = \langle A, B \rangle$ . Thus, the relation  $(AB)^n A = B(AB)^n$  implies the equality

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi,$$

where  $k$  is an integer.

Show that one can choose  $\lambda$  for the equality  $k = 1$  to hold for all  $\alpha$  in the affirmation of the Theorem. Due to the paper [15], every two-bridge knot cone-manifold with cone angle  $\pi$  is a spherical orbifold. In this case all the vertices  $P_i$  of the fundamental polyhedron belong to the same circle and all the dihedral angles  $\psi_i$  and  $\phi_i$  are equal to each other [12]:

$$\phi_i = \psi_i = \frac{\pi}{2n+1}.$$

As far as  $\cos d_\lambda(N, S) = \cos d_\lambda(P_i, P_{i+1}) = \lambda$ , then in case  $\alpha = \pi$  one obtains

$$\lambda = \frac{\Lambda_k}{\sin \frac{\pi}{2}} = \cos \theta$$

for certain  $k \in \{1, \dots, 2n\}$  and then

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2(2n+1)\theta.$$

Using the formula for the roots of  $U_{2n}(\Lambda)$  obtain that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi$$

if  $\alpha = \pi$ . Thus, claim (c) for the polyhedron  $\mathcal{P}_n$  with  $\alpha = \pi$  is satisfied if  $k = 1$ . As far as the parameter  $\alpha$  varies continuously and sum of the angles  $\psi_i$

represents a multiple of  $2\pi$ , one has that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2\pi$$

for all  $\alpha$ .

By analogy, show that with

$$\lambda = \frac{\Lambda_1}{\sin \frac{\alpha}{2}}$$

the equality

$$\sum_{i=1}^{2(2n+1)} \phi_i = 2\pi$$

holds, that means claim (d) is also satisfied.

Verify that under conditions of the Theorem the metric  $ds_\lambda^2$  is spherical. This claim is equivalent to the inequality

$$-1 < \lambda < +1.$$

Note, that for

$$\frac{2n-1}{2n+1} \pi < \alpha < 2\pi - \frac{2n-1}{2n+1} \pi$$

it follows

$$\sin \frac{\alpha}{2} > \sin \frac{(2n-1)\pi}{2(2n+1)}.$$

As far as  $\sin \frac{\alpha}{2} > 0$  and  $\Lambda_1 = \sin \frac{(2n-1)\pi}{2(2n+1)} > 0$ , one has

$$0 < \lambda < 1.$$

By analogy with Lemma 1 verify that

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an isometry of  $ds_\lambda^2$ .

Fixed point sets of  $A$  and  $B$  in  $\mathbb{S}_\lambda^3$  are circles

$$\text{Fix } A = \{(z_1, 0) : z_1 \in \mathbb{C}, |z_1| = 1\}$$

and

$$\text{Fix } B = \{(0, z_2) : z_2 \in \mathbb{C}, |z_2| = 1\},$$

correspondingly. The geometric meaning of  $C$  is that it maps the first fixed circle to the other. Thus, the relation  $B = CAC^{-1}$  holds.

The following equalities

$$P_{2k+1} = P_1(AB)^k, \quad k \in \{0, \dots, n\},$$

$$P_{2k} = P_2(AB)^{k-1}, k \in \{1, \dots, n+1\};$$

and

$$\begin{aligned} P_{2k+1} &= P_1(BA)^{2n-k+1}, k \in \{n+1, \dots, 2n\}, \\ P_{2k} &= P_2(BA)^{2n-k+2}, k \in \{n+2, \dots, 2n+1\}, \end{aligned}$$

follow from the identification scheme of the edges of  $\mathcal{P}_n$ .  
Define the auxiliary function

$$\varepsilon(m) = \frac{m}{2} \alpha - \frac{4n-m}{2} \pi.$$

By analogy with the proof of Lemma 2 it follows that

$$\begin{aligned} (AB)^k &= C(BA)^k C^{-1} = \\ &= \begin{pmatrix} -\frac{\sin(2k-1)\theta}{\sin\theta} e^{i\varepsilon(2k)} & -\frac{\sin 2k\theta}{\sin\theta} e^{i\varepsilon(2k-1)} \\ \frac{\sin 2k\theta}{\sin\theta} e^{i\varepsilon(2k+1)} & \frac{\sin(2k+1)\theta}{\sin\theta} e^{i\varepsilon(2k)} \end{pmatrix}, \end{aligned}$$

where  $\theta = \frac{\pi}{2n+1}$ .

Suppose  $N$  and  $S$  to be middle-points of the edges  $P_1P_{2n+2}$  and  $P_2P_{2n+3}$ , respectively. Then

$$N = (e^{i\frac{\varepsilon(2n+1)}{2}}, 0), S = (0, e^{i\frac{\varepsilon(2n+1)}{2}}).$$

For the lengths  $l_\alpha$  of the singular geodesic one has

$$\cos \frac{l_\alpha}{4} = \langle P_1, N \rangle = \langle P_1C, NC \rangle = \langle P_2, S \rangle.$$

Thus

$$\cos \frac{l_\alpha}{4} = \cos \frac{(2n+1)\alpha - (2n-1)\pi}{4}.$$

By construction of the polyhedron  $\mathcal{P}_n$ , the inequality  $0 < l_\alpha < 4\pi$  holds. Then it follows

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

Given the coordinates of the vertices  $P_i$  and the poles  $N$  and  $S$  of the polyhedron  $\mathcal{P}_n$ , verify claim (e).

For every four points  $A, B, C, D \in \mathbb{C}_{\mathbb{R}}^2$ , where

$$A = (A_1, A_2), B = (B_1, B_2), C = (C_1, C_2), D = (D_1, D_2),$$

their Gram determinant is

$$\det(A, B, C, D) := \det \begin{pmatrix} \operatorname{Re} A_1 & \operatorname{Im} A_1 & \operatorname{Re} A_2 & \operatorname{Im} A_2 \\ \operatorname{Re} B_1 & \operatorname{Im} B_1 & \operatorname{Re} B_2 & \operatorname{Im} B_2 \\ \operatorname{Re} C_1 & \operatorname{Im} C_1 & \operatorname{Re} C_2 & \operatorname{Im} C_2 \\ \operatorname{Re} D_1 & \operatorname{Im} D_1 & \operatorname{Re} D_2 & \operatorname{Im} D_2 \end{pmatrix}.$$

Each tetrahedron  $NSP_iP_{i+1}$  with  $i \in \{1, \dots, 2n+1\}$  is isometric to  $NSP_{2n+i+1}P_{2n+i+2}$ ,  $i \in \{1, \dots, 2n+1\}$ ,  $P_{4n+3} = P_1$  by means of the isometry  $C$  defined above. Thus, we consider only the tetrahedra  $NSP_iP_{i+1}$  with  $i \in \{1, \dots, 2n+1\}$ . Split them into two groups: the tetrahedra  $NSP_{2k+1}P_{2k+2}$  with  $k \in \{0, \dots, n\}$  and the tetrahedra  $NSP_{2k}P_{2k+1}$  with  $k \in \{1, \dots, n\}$ . Substitute  $\alpha = \beta + \pi$  and proceed with straightforward calculations:

$$\begin{aligned}\Delta_k^{(1)}(\beta) &= \det(S, N, P_{2k+1}, P_{2k+2}) = \cos^2 \frac{L_1 \beta}{4} - U_{2k-1}^2(\cos \theta) \sin^2 \frac{\beta}{2} = \\ &= T_{L_1}^2(\cos \frac{\beta}{4}) - U_{2k-1}^2(\cos \theta) \sin^2 \frac{\beta}{2},\end{aligned}$$

where  $k \in \{0, \dots, n\}$ ,  $L_1 = |2n - 4k + 1|$ ,  $\theta = \frac{\pi}{2n+1}$ ,  $\beta \in [-2\theta, 2\theta]$ ;

$$\begin{aligned}\Delta_k^{(2)}(\beta) &= \det(S, N, P_{2k}, P_{2k+1}) = \cos^2 \frac{L_2 \beta}{4} - U_{2k-2}^2(\cos \theta) \sin^2 \frac{\beta}{2} = \\ &= T_{L_2}^2(\cos \frac{\beta}{4}) - U_{2k-2}^2(\cos \theta) \sin^2 \frac{\beta}{2},\end{aligned}$$

where  $k \in \{1, \dots, n\}$ ,  $L_2 = |2n - 4k + 3|$ ,  $\theta$  and  $\beta$  the same as above. The first kind Chebyshev polynomial of degree  $k \geq 0$  is denoted by  $T_k$ . Assume that

$$U_{-1}(\cos \theta) = 0, \quad U_0(\cos \theta) = 1$$

for the sake of brevity.

All the functions  $\Delta_k^{(j)}(\beta)$ ,  $j \in \{1, 2\}$  are even on the interval  $[-2\theta, 2\theta]$ . Then one considers them only on the interval  $[0, 2\theta]$ . Note, that the polynomial  $T_{L_j}^2(\cos \beta)$  monotonously decreases and the function  $\sin^2 \frac{\beta}{2}$  monotonously increases with  $\beta \in [0, 2\theta]$ . Moreover,  $T_{L_j}^2(\cos 0) = T_{L_j}^2(1) = 1$ . Then it follows that  $\Delta_k^{(j)}(\beta) > 0$  with  $\beta \in (-2\theta, 2\theta)$ . Also, one has  $\Delta_k^{(j)}(\pm 2\theta) = 0$ . Then for all  $\beta \in (-2\theta, 2\theta)$  (i.e. for all  $\alpha$  in the affirmation of the Theorem)

$$\det(S, N, P_i, P_{i+1}) > 0$$

where  $i \in \{1, \dots, 4n+2\}$ ,  $P_{4n+3} = P_1$ . Thus, claim (e) for the polyhedron  $\mathcal{P}_n$  is satisfied.

Use the Schläfli formula [16] to obtain the volume formula for  $\mathbb{T}_n(\alpha)$ . One has

$$d\text{Vol } \mathbb{T}_n(\alpha) = \frac{l_\alpha}{2} d\alpha = \frac{(2n+1)\alpha - (2n-1)\pi}{2} d\alpha.$$

Note, that  $\text{Vol } \mathbb{T}_n(\alpha) \rightarrow 0$  with  $\alpha \rightarrow \frac{2n-1}{2n+1}\pi$ . In this case  $d_\lambda(P_i, P_{i+1}) \rightarrow 0$ , where  $i \in \{1, \dots, 4n+2\}$ ,  $P_{4n+3} = P_1$  and the fundamental polyhedron collapses to a point. Thus

$$\text{Vol } \mathbb{T}_n(\alpha) = \frac{1}{2n+1} \left( \frac{2n+1}{2} \alpha - \frac{2n-1}{2} \pi \right)^2.$$

□

**Remark 1** *The domain of the spherical metric existence in Theorem 1 was indicated before in [10, Proposition 2.1].*

## 4 Torus links $\mathbb{L}_n$

Let  $\mathbb{L}_n, n \geq 2$  be a torus link  $t(2n, 2)$  with two components. The corresponding link in the rational census is  $2n/1$  (Fig. 3). The fundamental group of  $\mathbb{L}_n$  is presented as

$$\pi_1(\mathbb{S}^3 \setminus \mathbb{L}_n) = \langle a, b | (ab)^n = (ba)^n \rangle.$$

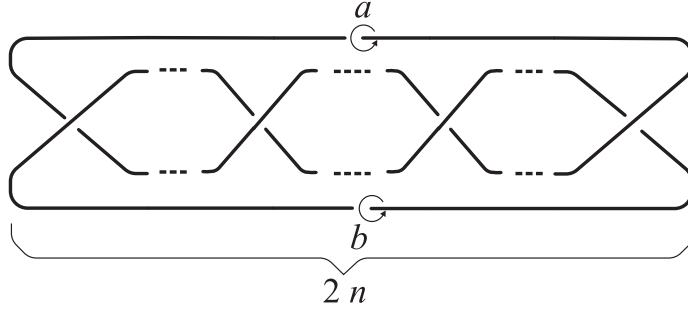


Figure 3: Link  $2n/1$

Let  $\mathbb{L}_n(\alpha, \beta)$  denote a cone-manifold with singular locus the link  $\mathbb{L}_n$  and the cone angles  $\alpha, \beta$  along its components.

For every  $\alpha, \beta \in (0, 2\pi)$  and  $\lambda \in (-1, +1)$ , we denote

$$A = \begin{pmatrix} 1 & 0 \\ -2i e^{i\frac{\alpha}{2}} \lambda \sin \frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\beta} & -2i e^{i\frac{\beta}{2}} \lambda \sin \frac{\beta}{2} \\ 0 & 1 \end{pmatrix}.$$

By Lemma 1, linear transformations  $A$  and  $B$  are isometries of  $\mathbb{S}_\lambda^3$ .

**Lemma 3** *For every integer  $n \geq 2$  the following equality holds*

$$(AB)^n - (BA)^n = 4U_{n-1}(\Lambda) \lambda e^{i(\frac{\alpha+\beta}{2} + \pi)n} \sin \frac{\alpha}{2} \sin \frac{\beta}{2} M,$$

where  $M$  is a non-zero  $2 \times 2$  matrix and  $U_{n-1}(\Lambda)$  is the second kind Chebyshev polynomial of degree  $n - 1$  in variable

$$\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}.$$

**Proof.** By analogy with Lemma 2.  $\square$

With Lemma 3 the main theorem of the section follows:

**Theorem 2** *The cone-manifold  $\mathbb{L}_n(\alpha, \beta)$ ,  $n \geq 2$  is spherical if*

$$\begin{aligned} -2\pi \left(1 - \frac{1}{n}\right) &< \alpha - \beta < 2\pi \left(1 - \frac{1}{n}\right), \\ 2\pi \left(1 - \frac{1}{n}\right) &< \alpha + \beta < 2\pi \left(1 + \frac{1}{n}\right). \end{aligned}$$

*The lengths  $l_\alpha, l_\beta$  of its singular geodesics (i.e. lengths of the components for  $\mathbb{L}_n$ ) are equal to each other and*

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi(n - 1).$$

*The volume of  $\mathbb{L}_n(\alpha, \beta)$  is*

$$\text{Vol } \mathbb{L}_n(\alpha, \beta) = \frac{1}{2n} \left( \frac{\alpha + \beta}{2} n - (n - 1)\pi \right)^2.$$

**Proof.** One continues the proof by analogy with Theorem 1. Suppose that  $\mathbb{L}_n(\alpha, \beta)$  is spherical. Then there exists a holonomy mapping [1]:

$$\begin{aligned} h : \pi_1(\mathbb{S}^3 \setminus \mathbb{L}_n) &\longmapsto \text{Isom } \mathbb{S}_\lambda^3, \\ h(a) &= A, \quad h(b) = B. \end{aligned}$$

Also,

$$h((ab)^n) - h((ba)^n) = (AB)^n - (BA)^n = 0.$$

By means of Lemma 3 the equality above holds either if  $\lambda = 0$ , or if

$$\Lambda = (1 - \lambda^2) \cos \frac{\alpha - \beta}{2} + \lambda^2 \cos \frac{\alpha + \beta}{2}$$

is a root of the equation  $U_{n-1}(\Lambda) = 0$ .

In case  $\lambda = 0$  the image of  $h$  is abelian, because of the additional relation  $AB = BA$ . With  $n \geq 2$  this leads to a degenerate geometric structure. Thus, one has to choose the parameter  $\lambda$  for the metric  $ds_\lambda^2$  using roots of the Chebyshev polynomial  $U_{n-1}(\Lambda)$ .

The fundamental polyhedron  $\mathcal{F}_n$  for the cone-manifold  $\mathbb{L}_n(\alpha, \beta)$  is depicted at Fig. 4. Suppose its vertices  $P_1$  and  $P_2$  to be

$$P_1 = (1, 0), \quad P_2 = (0, 1).$$

The axes of isometries  $A$  and  $B$  correspond to the edges  $P_1P_{2n+1}$  and  $P_2P_{2n+2}$ . Points  $N$  and  $S$  are respective middles of the edges  $P_1P_{2n+1}$  and  $P_2P_{2n+2}$ . Those are called North and South poles of the polyhedron. The polyhedron  $\mathcal{F}_n$  is said to be proper if

- (a) respective inner dihedral angles along the edges  $P_1P_{2n+1}$  and  $P_2P_{2n+2}$  are equal to  $\alpha$  and  $\beta$ ;

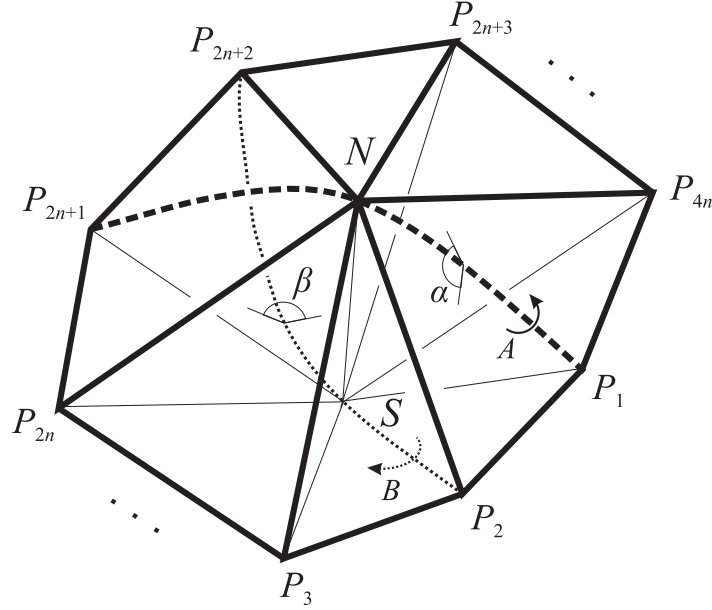


Figure 4: The fundamental polyhedron  $\mathcal{F}_n$  for  $\mathbb{L}_n(\alpha, \beta)$

(b) curvilinear faces of the polyhedron are identified by  $A$  and  $B$ :

$$A : NP_1P_2 \dots P_{2n+1} \rightarrow NP_1P_{4n} \dots P_{2n+2}P_{2n+1},$$

$$B : SP_2P_1P_{4n} \dots P_{2n+2} \rightarrow SP_2P_3 \dots P_{2n+2};$$

(c) sum of the inner dihedral angles  $\psi_i$  along the edges  $P_iP_{i+1}$ ,  $i \in \{1, \dots, 4n-1\}$  equals  $2\pi$ ;

(d) sum of the dihedral angles  $\phi_i$  for tetrahedra  $NSP_iP_{i+1}$ ,  $i \in \{1, \dots, 4n-1\}$  at their common edge  $NS$  equals  $2\pi$ ;

(e) all the tetrahedra  $NSP_iP_{i+1}$  with  $i \in \{1, \dots, 4n\}$ ,  $P_{4n+1} = P_1$  are non-degenerated and coherently oriented.

In order to choose the parameter  $\lambda$  for the corresponding metric consider the fundamental polyhedron  $\mathcal{F}_n$  with  $\alpha = \beta = \pi$ . Then all its vertices belong to the same circle and all the dihedral angles  $\psi_i$  of the tetrahedra  $NSP_iP_{i+1}$  along the edges  $P_iP_{i+1}$  are equal to  $\psi = \frac{\pi}{2n}$  [12]. Also the dihedral angles  $\phi_i$  of the tetrahedra  $NSP_iP_{i+1}$  along their common edge  $NS$  are equal to each other:

$$\phi_i = \phi = \frac{\pi}{2n}.$$

In this case  $\lambda = \langle P_1, P_2 \rangle = \cos \phi$  and

$$\Lambda = -\cos 2\phi = \cos \frac{(n-1)\pi}{n}.$$

All the roots of  $U_{n-1}(\Lambda)$  are given by the formula

$$\Lambda_k = \cos \frac{k\pi}{n}, \quad k \in \{1, \dots, n-1\},$$

so one choose the root  $\Lambda_k$  with  $k = n-1$ . Then, by analogy with Theorem 1, equalities

$$\sum_{i=1}^{4n} \psi_i = 2\pi$$

and

$$\sum_{i=1}^{4n} \phi_i = 2\pi$$

are satisfied at the point  $\alpha = \beta = \pi$  of the domain

$$\mathcal{D} = \left\{ (\alpha, \beta) : |\alpha - \beta| < 2\pi \left(1 - \frac{1}{n}\right), |\alpha + \beta - 2\pi| < \frac{2\pi}{n} \right\},$$

depicted at Fig. 5.

In terms of the parameter  $\lambda$ , that defines the metric  $ds_\lambda^2$ , one has

$$\lambda^2 = \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\pi}{n}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}}.$$

As for all  $(\alpha, \beta) \in \mathcal{D}$  the inequality  $0 < \lambda^2 < 1$  is satisfied, the metric  $ds_\lambda^2$  is spherical regarding the corresponding domain. By analogy with Theorem 1 one can show that claims (a) – (d) for the polyhedron  $\mathcal{F}_n$  are satisfied in the interior of  $\mathcal{D}$ .

The lengths  $l_\alpha$  and  $l_\beta$  of singular geodesics for the cone-manifold  $\mathbb{L}_n(\alpha, \beta)$  meet the relations

$$\begin{aligned} \cos \frac{l_\alpha}{2} &= \langle P_1, N \rangle, \\ \cos \frac{l_\beta}{2} &= \langle P_2, S \rangle. \end{aligned}$$

By analogy with the proof of Theorem 1 one obtains

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2} n - \pi(n-1).$$

Given the coordinates of the vertices for the fundamental polyhedron verify claim (e) for all  $(\alpha, \beta)$  in the domain  $\mathcal{D}$ .

Make use of the Schläfli formula [16] to obtain the volume of  $\mathbb{L}_n(\alpha, \beta)$ :

$$d \text{Vol } \mathbb{L}_n(\alpha, \beta) = \frac{l_\alpha}{2} d\alpha + \frac{l_\beta}{2} d\beta = \left( \frac{\alpha + \beta}{2} n - \pi(n-1) \right) d \left( \frac{\alpha + \beta}{2} \right).$$

Note, that with

$$\alpha = \beta \rightarrow \pi \frac{n-1}{n}$$

the fundamental polyhedron  $\mathcal{F}_n$  collapses to a point (i.e. the volume tends to 0). The last affirmation of the Theorem follows.  $\square$



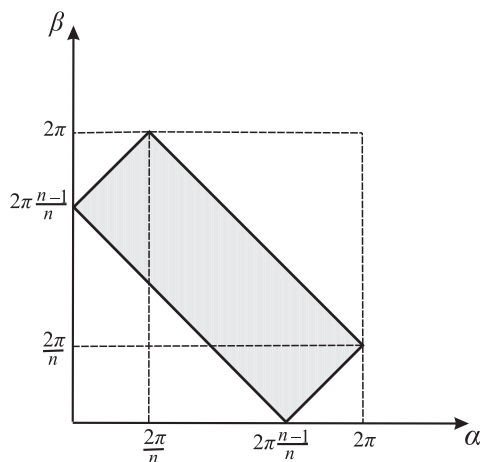


Figure 5: The domain  $\mathcal{D}$  of sphericity for  $\mathbb{L}_n(\alpha, \beta)$

**Remark 2** Under condition  $\alpha = \beta$  the inequality from the affirmation of Theorem 2 coincides with the inequality from [10, Proposition 2.2].

**Remark 3** Note, that the lengths of the singular geodesics for  $\mathbb{L}_n(\alpha, \beta)$  are equal even if  $\alpha \neq \beta$ .

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