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On the semiclassical treatment of Hawking radiation

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Abstract

In the context of the semiclassical treatment of Hawking radiation we prove the universality of the reduced canonical momentum for the system of a massive shell self gravitating in a spherical gravitational field within the Painlevé family of gauges. We show that one can construct modes which are regular on the horizon both by considering as Hamiltonian the exterior boundary term and by using as Hamiltonian the interior boundary term. The late time expansion is given in both approaches and their time Fourier expansion computed to reproduce the self reaction correction to the Hawking spectrum.

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1 Introduction

In paper [1] Kraus and Wilczek introduced a semiclassical treatment of Hawking radiation by considering the mechanics of a thin self-gravitating shell of matter in a spherical gravitational field. The interest of such an approach lies in the fact that contrary to the usual external field treatment, energy conservation is taken into account which allows to compute the self energy correction to the Hawking formula.

In [1, 2] the connection with Hawking radiation was obtained by interpreting the exponential of the classical action as the modes of the system. In this way in the first approximation the well known Hawking result was re-obtained and in the full semiclassical approximation, corrections to the Hawking formula due to the self-energy were computed. In this analysis the key role is played by the imaginary part of the canonical momentum which appears in the reduced Hamiltonian.

Subsequently [3] the results obtained in [1, 2] were given a new interpretation as describing a tunneling phenomenon. Alternative formulae and criticisms were proposed in this context [4, 5, 6, 7, 8, 9, 10, 11, 12]. In the present paper however we shall go back to the computation of the Bogoliubov coefficients as originally done in [1, 2, 13].

In [14, 15] the problem of the dynamics of one thin shell of matter in the framework of [1, 16] was examined critically and a precise definition of the canonical momentum was given through a limit process extending the treatment also to a massive shell of matter. The procedure for deriving the reduced Hamiltonian in [1], which is obtained in implicit form, is usually considered as very complicated. In [17] it was shown that by introducing a proper generating function it is possible to drastically simplify such a derivation and that the procedure works also for massive shells. The method allows also to extend the treatment to any finite number of massive or massless shells which can cross in their time development. Moreover, it was shown that the expression of the canonical momentum obtained in [1, 14] holds in a more general setting and that no limit procedure is necessary for obtaining it.

In this paper we shall examine two aspects of the problem: The first is the universal character of the canonical momentum which appears in the reduced Hamiltonian within the class of the Painlevé gauges. This is done in Sec. 2. Then we shall point out how the shell dynamics can be obtained by considering as Hamiltonian either the exterior mass or an interior mass which appear in the boundary terms. This is done in Sec. 3.

In Sec. 4 we revisit the extraction of the Bogoliubov coefficients from the semiclassical modes. A general treatment along this line was given in the paper [13], but here we shall go back to the explicit late-time development of the modes. We show that in constructing

the semiclassical modes which are regular at the horizon one can use either the exterior mass as Hamiltonian as was done originally in [1, 13] or an “interior” mass. The two Hamiltonians are related to two different times, the exterior time, which is the usual Killing time t at space infinity, and the interior time which we shall denote by t' . Both approaches are consistent and they produce the same result for the absolute value of the ratio of the β to the α Bogoliubov coefficients. In Sec. 5 we give a discussion of the main conclusions. In Appendix A we give some details of the calculations.

2 The reduced action

In this section we recall some features of the reduced action in the Painlevé family of gauges. The spherically symmetric metric is written as

$$ds^2 = -N^2 dt^2 + L(dr + N^r dt)^2 + R^2 d\Omega^2 \quad (1)$$

where the functions N, N^r, L, R can be consistently assumed to be continuous functions of r and t [14]. In [17] the function

$$F = RL\sqrt{\left(\frac{R'}{L}\right)^2 - 1 + \frac{2\mathcal{M}}{R}} + RR' \log\left(\frac{R'}{L} - \sqrt{\left(\frac{R'}{L}\right)^2 - 1 + \frac{2\mathcal{M}}{R}}\right) \quad (2)$$

was introduced which has the remarkable property of generating the conjugate momenta through the variational derivatives

$$\begin{aligned} \pi_L &= \frac{\partial F}{\partial L} - \frac{\partial}{\partial r} \frac{\partial F}{\partial L'} = \frac{\partial F}{\partial L}, \\ \pi_R &= \frac{\partial F}{\partial R} - \frac{\partial}{\partial r} \frac{\partial F}{\partial R'}. \end{aligned}$$

\mathcal{M} denotes the quantity

$$\mathcal{M} = \frac{\pi_L^2}{2R} + \frac{R}{2} - \frac{R(R')^2}{2L^2}$$

which as a consequence of the constraints is constant in r above and below the shell position \hat{r} [16, 1, 17]. The value of \mathcal{M} below and above \hat{r} will be denoted by M and H respectively. The Painlevé family of gauges is identified by the choice $L = 1$ in the metric (1) where the generating function F assumes the form

$$F = RW(R, R', \mathcal{M}) + RR'(\mathcal{L}(R, R', \mathcal{M}) - \mathcal{B}(R, \mathcal{M})) \quad (3)$$

where

$$W(R, R', \mathcal{M}) = \sqrt{R'^2 - 1 + \frac{2\mathcal{M}}{R}}, \quad \mathcal{L}(R, R', \mathcal{M}) = \log(R' - W(R, R', \mathcal{M})) \quad (4)$$

and

$$\mathcal{B}(R, \mathcal{M}) = \sqrt{\frac{2\mathcal{M}}{R}} + \log\left(1 - \sqrt{\frac{2\mathcal{M}}{R}}\right). \quad (5)$$

In going from eq.(2) to eq.(3) we exploited the freedom of adding a total derivative to F with the result that the function (3) has the useful property of vanishing whenever $R' = 1$. One is still left with the freedom to impose a gauge condition on R . In [17] several choices were examined which can be characterized by a deformation function, of bounded support around the shell position \hat{r}

$$R(r) = r + h(r - \hat{r}).$$

We shall call a gauge of the outer type if $h(x) = 0$ for $x \geq 0$; if $h(x) = 0$ for $x \leq 0$ the gauge is called of inner type. There are however other choices in which h is not vanishing in a neighborhood of 0 both for positive and negative argument. The mathematical nature of the deformation $h(x)$ is specified before eq.(7).

It is useful to start from the general form of the action on a bounded region of space-time as given in [18] to which the shell action

$$S_{shell} = \int_{t_i}^{t_f} dt \hat{p} \dot{\hat{r}}$$

is added and using the generating function F , one can derive in a straightforward way [17] the reduced action, boundary terms included

$$\int_{t_i}^{t_f} \left(p_c \dot{\hat{r}} - \dot{M}(t) \int_{r_0}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr + (-N^r \pi_L + NRR')|_{r_0}^{r_m} \right) dt \quad (6)$$

where the outer gauge $g(x) = 0$ for $x \geq 0$ has been adopted in the Painlevé family of gauges. The boundary terms are just the one given in the paper [18] computed for the spherically symmetric problem at hand. They are equivalent to

$$-HN(r_m) + MN(r_0)$$

where as a consequence of the constraints M and H are constant in r except at the position of the shell \hat{r} . M is the interior mass while H denotes the exterior mass. Furthermore as a consequence of the gravitational equations combined with the constraints, M and H are also constant in t .

However, this does not entail one to drop the \dot{M} term appearing in eq.(6), because such a time constancy can be used only after having derived the equations of motion. In deriving (6) the so called outer gauge has been used i.e. in which the radial function $R(r)$ appearing in the metric (1) is equal to r for $r > \hat{r}$, and is deformed below \hat{r} on a finite

tract by a smooth function g of bounded support with $g(x) = 0$ for $x \geq 0$, $g'_{left}(0) = 1$ and [14, 17]

$$R(r, t) = r + \frac{V(t)}{\hat{r}} g(r - \hat{r}(t)) \quad (7)$$

due to the fact that as a consequence of the constraints

$$\Delta R' \equiv R'(\hat{r} + 0) - R'(\hat{r} - 0) = -\frac{V}{R}, \quad V = \sqrt{\hat{p}^2 + m^2} \quad (8)$$

being m the mass of the shell. The imposition of the constraints is the reason why one cannot adopt $R = r$ for all r .

In the inner gauge a term of type \dot{H} will appear while in a generic gauge both \dot{M} and \dot{H} terms will appear. Moreover, the constraints impose

$$\Delta \pi_L = -\hat{p} \quad (9)$$

being

$$\pi_L = R \sqrt{(R')^2 - 1 + \frac{2\mathcal{M}}{R}} \equiv RW(R, R', \mathcal{M})$$

where by \mathcal{M} we denote the mass which is function constant in r for $r > \hat{r}$ and also for $r < \hat{r}$ but discontinuous at \hat{r} .

The general form of p_c is given by [17]

$$p_c = \hat{r}(\Delta \mathcal{L} - \Delta \mathcal{B})$$

with \mathcal{L} and \mathcal{B} given by eqs.(4,5). Using relations (8,9) p_c becomes

$$p_c(\hat{r}) = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \left(\frac{\hat{r} + \sqrt{\hat{p}^2 + m^2} - \hat{p} - \sqrt{2H\hat{r}}}{\hat{r} - \sqrt{2M\hat{r}}} \right) \quad (10)$$

where [14, 17] \hat{p} is given implicitly by the solution of the equation

$$H - M = V + \frac{m^2}{2\hat{r}} - \hat{p} \sqrt{\frac{2H}{\hat{r}}}. \quad (11)$$

We stress that no limit procedure is necessary to obtain (10,11) which hold with any deformation to the left of \hat{r} .

In the inner gauges in which the deformation is taken to the right of \hat{r} one obtains for the canonical momentum p_c

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \left(\frac{\hat{r} - \sqrt{2H\hat{r}}}{\hat{r} - V^i + \hat{p}^i - \sqrt{2M\hat{r}}} \right)$$

where now \hat{p}^i is given by the implicit equation

$$H - M = V^i - \frac{m^2}{2\hat{r}} - \hat{p}^i \sqrt{\frac{2M}{\hat{r}}}, \quad V^i = \sqrt{\hat{p}^i + m^2}$$

which is different from eq.(11). More general gauges can also be considered in which the discontinuity (8) in R' at $r = \hat{r}$ is partly due to a deformation on the right and to a deformation on the left of \hat{r} . It is possible to show that actually p_c is independent from any gauge choices in which $L = 1$. In fact after some algebra given in Appendix A, we find

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \frac{\hat{r} - \sqrt{2H\hat{r}}}{\hat{r} - \sqrt{2M\hat{r}}} - \hat{r} \log \frac{\hat{r} - H - M - \frac{m^2}{2\hat{r}} - \hat{r}\sqrt{C}}{\hat{r} - 2H} \quad (12)$$

being C the discriminant

$$C = \left(\frac{H - M}{\hat{r}}\right)^2 + \frac{m^2}{\hat{r}^2} \left(\frac{H + M}{\hat{r}} - 1\right) + \frac{m^4}{4\hat{r}^4}.$$

Notice that the argument of the second logarithm becomes 1 for $m^2 = 0$ and in this situation the last term in (12) disappears.

The mass m of the shell does not play a really important role in the phenomena we shall examine in the following, the reason being that, as seen from eq.(43) of the Appendix at the horizon $r = 2H$ where p_c is singular \hat{p} diverges and it cancels the term V and we know that Hawking radiation which is a late time phenomenon depends on the behavior of the modes at the horizon. Moreover eq.(16) below holds also for $m \neq 0$.

For $m = 0$

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \left(\frac{\hat{r} - \sqrt{2H\hat{r}}}{\hat{r} - \sqrt{2M\hat{r}}} \right) \quad (13)$$

which is the original result by Kraus and Wilczek [1]. The space-part of the semiclassical mode is given by

$$u(r) = \text{const} \exp(i(\int_{\hat{r}_1}^{\hat{r}} p_c(\hat{r}) d\hat{r} + \text{const})) \quad (14)$$

where \hat{r}_1 is any value of the radius outside the horizon and the additive constant can be taken also to depend on H , M and contributes only to a phase factor in the semiclassical mode. For $m = 0$ the integral appearing in eq.(14) can be easily computed. One finds

$$\int p_c(\hat{r}) d\hat{r} = f(\hat{r}, M) - f(\hat{r}, H) \quad (15)$$

where

$$f(\hat{r}, M) = \frac{\hat{r}^2 - 4M^2}{2} \log(\sqrt{\hat{r}} - \sqrt{2M}) + \frac{\hat{r} - 2M}{2} (\sqrt{2M\hat{r}} + \frac{\hat{r}}{2}).$$

The large \hat{r} behavior of p_c is

$$\lim_{\hat{r} \rightarrow \infty} p_c(\hat{r}) = H - M = \sqrt{\hat{p}^2(+\infty) + m^2} \equiv \omega.$$

The p_c as a function of \hat{r} develops an imaginary part $i\pi\hat{r}$ in the interval $(2M, 2H)$. This is evident from eq.(13) but in [17] it was shown to be true also in presence of mass i.e. for eq.(12) independently of the value of $m < H - M$; the last is the necessary and sufficient condition for a massive shell to reach space infinity. One has as a consequence

$$\text{Im} \int_{r_0}^{r_m} p_c d\hat{r} = 2\pi(H^2 - M^2). \quad (16)$$

Thus we have shown that p_c is independent of the gauge choice, within the family of the Painlevé gauges, and that eq.(16) holds also for $m \neq 0$.

The exponential of minus eq.(16) was given the interpretation of a tunneling amplitude by Parikh and Wilczek [3]. Criticism and alternative proposal for the tunneling amplitudes were given in the literature [4, 5, 6, 7, 8, 9, 10, 11, 12].

Here however, we shall not pursue this line of thought and instead still employing eqs. (10,11,15) we shall go back to the calculation of coefficients of the Bogoliubov transformation.

The expansion in $\omega = H - M$ of eq.(15) to second order in ω is

$$\begin{aligned} & \omega \left(M + 2\sqrt{2M\hat{r}} + \hat{r} + 4M \log\left(\frac{\sqrt{\hat{r}} - \sqrt{2M}}{\sqrt{2M}}\right) \right) + \\ & \frac{\omega^2}{2} \left(1 - \frac{2\sqrt{2M}}{\sqrt{\hat{r}} - \sqrt{2M}} + \frac{\sqrt{2\hat{r}}}{\sqrt{M}} + 4 \log\left(\frac{\sqrt{\hat{r}} - \sqrt{2M}}{\sqrt{2M}}\right) \right) + \dots \end{aligned} \quad (17)$$

What matters in the late time emission is the behavior of the mode at the horizon which is given by

$$4M\omega \log\left(\frac{\hat{r} - 2M}{4M}\right) + 4M\omega^2 \left(\frac{1}{2M} \log\left(\frac{\hat{r} - 2M}{4M}\right) - \frac{1}{\hat{r} - 2M} \right) + \dots$$

We adopt the usual notation [1] for the expansion of the scalar field ϕ

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} (u_\omega(\hat{r}) e^{-i\omega t} a(\omega) + u_\omega^*(\hat{r}) e^{i\omega t} a^+(\omega))$$

where now t is the Painlevé time and $u_\omega(\hat{r})$ is given by eq.(14). The same field ϕ can be expanded in modes which are regular at the horizon

$$\phi = \int_0^\infty \frac{dk}{\sqrt{2k}} (v_k(\hat{r}, t) b(k) + v_k^*(\hat{r}, t) b^+(k)).$$

The behavior of such v modes at the horizon is given by translating the expression [19]

$$v_k = \frac{1}{\sqrt{2\pi}} e^{ik(X-T)}$$

with

$$X = \frac{M}{2}(V - U), \quad T = \frac{M}{2}(V + U), \quad U = -e^{-\frac{u}{4M}}, \quad V = e^{\frac{v}{4M}}$$

into the Painlevé coordinates, being

$$u = t_s - \hat{r}_*, \quad v = t_s + \hat{r}_*, \quad \hat{r}_* = \hat{r} + 2M \log\left(\frac{\hat{r}}{2M} - 1\right)$$

and t_s the Schwarzschild time

$$t_s = t - 2\sqrt{2M\hat{r}} - 2M \log \frac{\sqrt{\hat{r}} - \sqrt{2M}}{\sqrt{\hat{r}} + \sqrt{2M}} + \text{const.}$$

Near the horizon one finds

$$v_k(\hat{r}, t) = \frac{1}{\sqrt{2\pi}} \exp(ik(\hat{r} - 2M)e^{-\frac{t}{4M}}). \quad (18)$$

It is essential in obtaining eq.(18) to adopt the Painlevé time. In fact such regular modes will be subsequently analyzed in terms of the u -mode which live also on the Painlevé background.

The usual technique for extracting the Bogoliubov coefficient [20] is that of projection of the regular v -modes on the u -modes by space integration. Here, the main difference w.r.t. the usual treatment is that the background metric is now the Painlevé metric with the mass M . Despite being the Painlevé metric a non static metric still it describes a static space time and thus due to the invariance of the space integrals the usual formalism for computing the scalar products works and there is no need to adopt the general form for stationary spaces [21]. Such a scalar product is given by

$$-i \int (\psi_2^* \partial_\rho \psi_1 - \psi_1 \partial_\rho \psi_2^*) g^{\rho t} \sqrt{-g} \varepsilon_{tr\theta\phi} d\hat{r} d\theta d\phi \quad (19)$$

where the integration extends to the region outside the horizon. Using $\sqrt{-g} = \hat{r}^2 \sin \theta$ we have for eq.(19)

$$4\pi i \int_{2M}^{\infty} (\psi_2^* \partial_t \psi_1 - \psi_1 \partial_t \psi_2^*) \hat{r}^2 d\hat{r} - 4\pi i \int_{2M}^{\infty} (\psi_2^* \partial_r \psi_1 - \psi_1 \partial_r \psi_2^*) N^r \hat{r}^2 d\hat{r}, \quad N^r = g^{rt} = \sqrt{\frac{2M}{\hat{r}}}.$$

Taking into account that eq.(14,18) are reduced radial modes we find for the most singular term in the integrand giving the scalar product of v_k with u_ω

$$\frac{\sqrt{M}}{\pi} \int_0^\infty e^{-ikx\tau} e^{4i\omega M \log x - i\omega t} \frac{dx}{x}$$

where such term originates from $g^{rt}v_k^*\partial_r u_\omega$ and $\tau = e^{-\frac{t}{4M}}$, $x = \hat{r} - 4M$ and we took into account the normalization of the u_ω .

Integrating in $x = \hat{r} - 2M$ we have

$$\beta_{k\omega}^* = \frac{\sqrt{M}}{\pi} \sqrt{\frac{\omega}{k}} \int_0^\infty e^{ikx\tau} e^{4iM\omega \log x - i\omega t} \frac{dx}{x} = \frac{\sqrt{M}}{\pi} \sqrt{\frac{\omega}{k}} \int_0^\infty e^{ikx\tau} e^{4iM\omega \log(x\tau)} \frac{d(x\tau)}{x\tau}$$

while $\alpha_{k\omega}$ is obtained changing ω in $-\omega$. On the other hand one can extract the Bogoliubov coefficients also performing a time Fourier transform of the regular modes at fixed r

$$\beta_{k\omega}^* = -\frac{1}{4\sqrt{M}\pi} \sqrt{\frac{\omega}{k}} \int_{-\infty}^\infty e^{ikx\tau} e^{4iM\omega \log x - i\omega t} dt = \frac{\sqrt{M}}{\pi} \sqrt{\frac{\omega}{k}} \int_0^\infty e^{ikx\tau} e^{4iM\omega \log(x\tau)} \frac{d(x\tau)}{x\tau}$$

which also shows that the result does not depend on the value of \hat{r} . The independence of the time Fourier transform from \hat{r} can be proved on general grounds for the exact modes. Thus, we have that the first order term reproduces the well-known Hawking integrals and the treatment outlined above is obviously equivalent to the original external field treatment even if we developed it in the Painlevé reference frame.

There are several reasons why one cannot exploit the second order term in the expansion (17). The second order contains a non integrable singularity in \hat{r} ; in addition it would not be proper to compute the scalar product of eq.(17) with the regular mode v eq.(18) as also here one should consider the correction due to the back reaction. In addition one cannot employ any longer the Painlevé background metric characterized by M but one should also consider the corrections to it due to the presence of the shell. For this reasons in Sec. 4 we shall adopt the time Fourier transform technique as done in [1]

3 The equations of motion

We recall that in deriving the equations of motion one can vary H keeping M as a fixed datum or vary M keeping H as a fixed datum [17]. The first procedure is much simpler if one adopts the outer gauge because the term in \dot{M} is zero in eq.(6) and one obtains

$$\dot{\hat{r}} \frac{\partial p_c}{\partial H} - N(r_m) = 0 \quad (20)$$

where $N(r_m)$ does not depend on r_m for $r_m > \hat{r}$ and the usual normalization $N(r_m) = 1$ corresponds to identify t with the Killing time at space infinity. In this case eq.(20) becomes [14, 17]

$$\frac{d\hat{r}}{dt} = \left(\frac{\hat{p}}{V} - \sqrt{\frac{2H}{\hat{r}}} \right). \quad (21)$$

The procedure in which one varies M keeping H fixed is more complicated due to the presence of the term in \dot{M} in action (6) and which cannot be neglected to obtain the correct equations of motion. It was explicitly proven in [17] that one gets the same equations of motion (21) if one maintains the normalization $N(r_m) = 1$. On the other hand if one adopts the normalization $N(r_0) = 1$ one obtains the equations of motion w.r.t. the time which flows at r_0 and which we shall denote by t' . The calculation is most easily performed in the inner gauge obtaining

$$\frac{d\hat{r}}{dt'} \frac{\partial p_c}{\partial M} + N(r_0) = 0, \quad \text{with} \quad \frac{\partial p_c}{\partial M} = \left(\frac{\hat{p}^i}{V^i} - \sqrt{\frac{2M}{\hat{r}}} \right)^{-1} \quad (22)$$

and thus

$$\frac{d\hat{r}}{dt'} = \left(\frac{\hat{p}^i}{V^i} - \sqrt{\frac{2M}{\hat{r}}} \right). \quad (23)$$

Thus, we see that, due to the sign of the boundary term in eq.(6), choosing $N(r_0) = 1$ gives $-M$ as the Hamiltonian in this scheme.

We shall use both calculational schemes in the next section. The values of H and M which on shell are constant of motions give the values of the mass contained in $r < r_m$ and the mass contained in $r < r_0$. Despite being the coordinate r one dimensional, r_0 and r_m do not play an exactly symmetrical role as in deriving the action the Jacobian R^2 plays an essential role.

4 The late time expansion

In this section we give a simplified derivation of the late time expansion of the outgoing modes which are regular at the horizon. In constructing such regular modes one can use the scheme in which H plays the role of the Hamiltonian while M is a given fixed parameter. This is the scheme followed in [1, 13]. One can also construct regular modes by giving H as a fixed parameter and using M as Hamiltonian as was described in Sec. 3 in the second derivation of the equations of motion.

We start from the case in which M is a given constant, while H plays the role of the Hamiltonian. The distinguishing feature of Hawking radiation is that of being a late time effect. Below we shall compute with a simple technique the late time expansion of the regular mode confining ourselves to the $(\omega/M)^2$ corrections to the Hawking result. To that end it is sufficient to compute just the first order correction in the late time expansion. In the expansion the terms are classified in power of $\tau = e^{-\frac{t}{4M}}$ and powers of t . We shall need only the $O(\tau)$ and $O(t\tau^2)$ terms which can be extracted in a very simple manner.

By keeping only the singular terms in p_c and in the time development we have the equations which were used in [1]

$$p_c = -\hat{r} \log \frac{\sqrt{\hat{r}} - \sqrt{2H}}{\sqrt{\hat{r}} - \sqrt{2M}} \quad (24)$$

$$t = 4H \log(\sqrt{\hat{r}} - \sqrt{2H}) - 4H \log(\sqrt{\hat{r}(0)} - \sqrt{2H}). \quad (25)$$

Furthermore, regularity is obtained by imposing that at $t = 0$ $p_c = k$ with $k > 0$. Thus we have the further restriction

$$0 < k = -\hat{r}(0) \log \frac{\sqrt{\hat{r}(0)} - \sqrt{2H}}{\sqrt{\hat{r}(0)} - \sqrt{2M}} \quad (26)$$

The action is given by [1, 13]

$$k\hat{r}(0) + \int_0^t (p_c(\hat{r}(t'), H(t), M, k)\dot{\hat{r}}(t') - H(t))dt' \quad (27)$$

where for once we wrote explicitly the dependence of p_c on time. p_c is given by the solution of the equation of motion with the two conditions $p_c = k$ at $t = 0$ and $\hat{r}(t) = \hat{r}$, being \hat{r} a fixed value of the shell coordinate outside the horizon and t an arbitrary time. In the procedure we shall develop below the only thing we shall need is the function $H(t)$ which we shall compute in the late time expansion. In the previous as in the following equations till eq.(36) H stays for $H(t)$.

From eq.(25) we obtain

$$\sqrt{\hat{r}(0)} - \sqrt{2H} = (\sqrt{\hat{r}} - \sqrt{2H})e^{-\frac{t}{4H}}$$

which substituted in eq.(26) gives the implicit equation for the time evolution of $H(t)$

$$k = (\sqrt{2H} + (\sqrt{\hat{r}} - \sqrt{2H})e^{-\frac{t}{4H}})^2 \log \frac{(\sqrt{\hat{r}} - \sqrt{2H})e^{-\frac{t}{4H}} + \delta_H}{(\sqrt{\hat{r}} - \sqrt{2H})e^{-\frac{t}{4H}}} \quad (28)$$

with

$$\delta_H = \sqrt{2H} - \sqrt{2M}. \quad (29)$$

From eq.(28) we see that t going to $+\infty$, $\sqrt{2H}$ goes over to the finite value $\sqrt{2M}$ because δ_H has to go to zero. To lowest order we have

$$\delta_H = c_H T$$

with $T \equiv e^{-\frac{t}{4H}}$ which substituted in eq.(28) gives for c_H

$$c_H = (e^{\frac{k}{2M}} - 1)L, \quad L \equiv \sqrt{\hat{r}} - \sqrt{2M}.$$

Having determined the constant c_H we can go over to the next term. By simply expanding eq.(28) we obtain for large t

$$\frac{t}{4H(t)} = \frac{t}{4(M + \sqrt{2M}c_H e^{-\frac{t}{4M}})} = \frac{t}{4M} \left(1 - \frac{\sqrt{2M}}{M} c_H \tau + O(\tau^2) \right) \quad (30)$$

giving

$$\delta_H \equiv \sqrt{2H} - \sqrt{2M} = c_H \tau + (c_H \tau)^2 \frac{t}{(2M)^{3/2}} + O(\tau^2)$$

or

$$H(t) = M + \sqrt{2M}c_H \tau + \frac{t}{2M}(c_H \tau)^2 + O(\tau^2) \quad (31)$$

where $\tau = e^{-\frac{t}{4M}}$.

The last obtained relation (31) is necessary and sufficient to compute the ω^2 corrections in the late time framework. In fact, we are interested only in the time dependence of the mode given by the exponential of i times the expression (27) and we have [13]

$$\frac{\partial S}{\partial t} = -H(t)$$

which holds also with the boundary conditions (26) as can be explicitly verified. Thus the time development of S is given by

$$S = f(\hat{r}) - \int_0^t H(t') dt'.$$

Integrating we find

$$\int_0^t H(t') dt' = \text{const} + Mt - 4M\sqrt{2M}\tau_1 - t\tau_1^2 + ..$$

where for notational simplicity we set $\tau_1 = c_H \tau$.

Thus, S behaves at large times as $-Mt$ independently of k . On the other hand the Fourier time analysis of e^{iS} contains frequencies which are above and below such value M and this is the well known fact that the mode of the system which is regular at the horizon does not represent an eigenvalue of the energy as measured by a stationary observer at space infinity. The deviations from the value M represent the positive and negative frequency content of the radiation mode. Thus, after subtracting the background frequency M we have in the time Fourier analysis the exponent

$$i(S - Mt - \omega t) = i(f(r) - \int_0^t H(t') dt' + Mt \pm \omega t). \quad (32)$$

The saddle point $-H(t) + M \pm \omega = 0$ is given by

$$\sqrt{2M}\tau_1 + \frac{t}{2M}\tau_1^2 \mp \omega = 0. \quad (33)$$

The upper sign refers to the calculation of the α coefficients while the lower sign refers to the β coefficients. For the upper sign we see that the saddle point is real thus giving in the saddle approximation the contribution 1 to the modulus of α . For the lower sign, i.e. for the β coefficient, the exponent (32) becomes, using eq.(33)

$$i[4M\sqrt{2M}\tau_1 + t(\tau_1^2 + \omega)] = -i[4M\omega + t\omega(1 + \frac{\omega}{2M})]. \quad (34)$$

Thus to compute $|\beta_{k\omega}/\alpha_{k\omega}|$ to second order in ω we need $\text{Im}t$ to first order in ω .

We have

$$\sqrt{2M}\tau_1 = -\omega(1 + \frac{t}{4M^2}\omega) = -\omega(1 - \frac{\omega}{M} \log(-\frac{\omega}{\sqrt{2M}c_H}))$$

so that to first order in ω we have

$$t = -4M(1 - \frac{\omega}{M}) \log(-\frac{\omega}{\sqrt{2M}c_H})$$

giving

$$\text{Im}t = 4\pi M(1 - \frac{\omega}{M})$$

Combining with eq.(34)

$$\left| \frac{\beta_{k\omega}}{\alpha_{k\omega}} \right| = e^{-4\pi M(1 - \frac{\omega}{2M})}. \quad (35)$$

This is the explicit derivation through the late time expansion of the result obtained in [13] which corrects a previous result in [1]. From eq.(35) through a standard procedure [1] which exploits the Wronskian property of the Bogoliubov coefficients one obtains the spectrum of the radiation

$$F(\omega)d\omega = \frac{d\omega}{2\pi} \frac{1}{e^{8\pi M\omega(1 - \frac{\omega}{2M})} - 1}. \quad (36)$$

In the above treatment the parameter M which characterizes the regular modes has to be identified with the semiclassical mass of the system which emits the radiation because it is the value of the Hamiltonian $H(t)$ at large values of the time t and $H(t)$ is the boundary term at r_m which is related to the total energy of the system, i.e. the exterior mass.

In order to clarify the issue we want now to repeat the analysis from a different view point, i.e. by considering H as a given parameter and M , i.e. the interior mass, as Hamiltonian. Actually from the boundary term in eq.(6) with $N(r_0) = 1$ we see that the Hamiltonian i.e. the generator of the time translations is $-M$ as we discussed at the end of Sec. 3. Thus we have to compute the regular mode of the system where H is a parameter with a given fixed value and the role of the Hamiltonian is played by $-M$. The physical

identification of the parameter H is obtained after eq.(41) from the time development of the regular mode. Taking into account the equation of motion (23) we have now

$$t' = 4M \log(\sqrt{\hat{r}} - \sqrt{2M}) - 4M \log(\sqrt{\hat{r}(0)} - \sqrt{2M}).$$

while p_c has still the form (24). Furthermore, regularity is obtained by imposing that at $t' = 0$, $p_c = k$ with $k > 0$. Thus we have the restriction

$$0 < k = -\hat{r}(0) \log \frac{\sqrt{\hat{r}(0)} - \sqrt{2H}}{\sqrt{\hat{r}(0)} - \sqrt{2M}}$$

where now H is a constant parameter and M is a function of t' . The implicit equation for $M(t')$ we obtain is

$$k = (\sqrt{2M} + T'(\sqrt{\hat{r}} - \sqrt{2M}))^2 \log \frac{(\sqrt{\hat{r}} - \sqrt{2M})T'}{(\sqrt{\hat{r}} - \sqrt{2M})T' - \delta_M} \quad (37)$$

with

$$T' = e^{-\frac{t'}{4M(t')}} \quad \text{and} \quad \delta_M = \sqrt{2H} - \sqrt{2M}.$$

For $t' \rightarrow \infty$ we have $M(t') \rightarrow H$ and thus expanding in the implicit variable T' we have

$$\delta_M = c_M T' + O(T'^2) \quad (38)$$

where c_M is easily computed from eq.(37)

$$c_M = (1 - e^{-\frac{k}{2H}})L', \quad L' = \sqrt{\hat{r}} - \sqrt{2H}.$$

Repeating the same procedure as the one described in eq.(30) and squaring the relation (38) we obtain

$$M(t') = H - \sqrt{2H}c_M \tau' + \frac{t'}{2H}(c_M \tau')^2$$

which is the analogue of eq.(31). Then, the v_k mode, i.e. the mode regular at the horizon is given by

$$e^{i(k\hat{r}(0) + \int_0^{t'} p_c \dot{\hat{r}} dt'' + M(t')t')}.$$

Again we have

$$\frac{\partial(k\hat{r}(0) + \int_0^{t'} p_c \dot{\hat{r}} dt'' + M(t')t')}{\partial t'} = M(t')$$

so that the time dependence of the regular mode for fixed \hat{r} is

$$e^{i(f(\hat{r}) + \int_0^{t'} M(t'') dt'')}.$$

For $t' \rightarrow +\infty$ we have $M(t') \rightarrow H$. Subtracting such background frequency we have the radiation mode regular at the horizon

$$e^{i(f(\hat{r}) + \int_0^{t'} M(t'') dt'' - Ht')}.$$

We have now to analyze such regular mode in terms of the u -modes which are characterized as in the previous treatment by well defined values of M and H . On dealing with $-M$ as Hamiltonian one can use either the outer or the inner gauge with no change in the results in p_c as we have stressed in Sec. 2. By subtracting again the background frequency Ht' from the total mode

$$e^{i(\int_{\hat{r}_1}^{\hat{r}} p_c d\hat{r} + Mt')}$$

we obtain the u - radiation mode

$$e^{i(\int_{\hat{r}_1}^{\hat{r}} p_c d\hat{r} - (H-M)t')} = e^{i(\int_{\hat{r}_1}^{\hat{r}} p_c d\hat{r} - \omega t')}$$

with

$$\omega \equiv H - M = \sqrt{\hat{p}^2(+\infty) + m^2}.$$

Thus we have for the time Fourier analysis at fixed \hat{r} of the regular mode in terms of the u -modes

$$\int_{-\infty}^{+\infty} dt' e^{-iHt' + i \int_0^{t'} M(t'') dt''} e^{\pm i\omega t'}$$

which is the same as eq.(32) with $-i \int_0^t H(t') dt' + iMt$ replaced by $-iHt' + i \int_0^{t'} M(t'') dt''$. Repeating the steps after eq.(32) we obtain for the saddle point referring to the calculation of the β coefficient the relation

$$H - M(t') = \sqrt{2H} \tau'_1 - \frac{t'}{2H} (\tau'_1)^2 = -\omega$$

and for the saddle point value of the exponent

$$-i[4H\omega + t'\omega(1 - \frac{\omega}{2H})]. \quad (39)$$

Again we need t' to first order in ω .

$$t' = -4M(1 + \frac{\omega}{H}) \log(-\frac{\omega}{\sqrt{2H}c_M}), \quad \text{Im}t' = -4M(1 + \frac{\omega}{H}). \quad (40)$$

Combining eq.(39) and (40) we obtain the ratio

$$\left| \frac{\beta_{k\omega}}{\alpha_{k\omega}} \right| = e^{-4\pi\omega H(1 + \frac{\omega}{2H})}.$$

Now the asymptotic behavior of the time development of the regular mode in the t' representation is $e^{iHt'}$ and being t' related to the interior mass we have to identify the parameter H as the mass of the remnant of the black hole after the emission of the quantum of energy ω . This is in agreement with eq.(35) where as stated at the beginning of the present section we have been working consistently to second order in ω/M .

The validity of the late time expansion can be examined by comparing the first order result, in eq.(34) with the second order term. The typical frequency of the Hawking radiation is $\omega \approx l_P^2/M$ with M the mass of the black hole and l_P the Planck length; at such a frequency the ratio of the second order to the first is l_P^2/M^2 apart logarithmic corrections. Thus the expansion holds for black holes with mass of a few Planck masses or higher mass.

5 Conclusions

In this paper we have examined some issues related to the semiclassical treatment of Hawking radiation. The main advantage of this approach is that it allows one to treat the backreaction, i.e. the fact that during the transition the mass of the black hole varies. The reduced Hamiltonian description is a critical component of this approach and such a reduction can be performed rigorously at the classical level. The introduction of the generating function F in Sec. 2 greatly simplifies the reduction process. We proved that the canonical conjugate momentum p_c which appears in the reduced action and which plays the major role in all subsequent developments, is invariant in the class of the Painlevé gauges. We do not expect this to be true for all possible gauges but the Painlevé class of gauges, which are defined by the choice $L \equiv 1$ in the metric (1), gives a good description of an asymptotic observer at space infinity and has the advantage of being nonsingular at the horizon. There is still some gauge freedom in the choice of R but we proved that p_c does not depend on this choice. The classical dynamics can be developed using either the exterior mass or the interior mass as Hamiltonian and these two descriptions are equivalent.

By interpreting the exponential of the reduced semiclassical action as modes of the system, as was done in [1], it is possible to calculate the Bogoliubov coefficients. We gave a simplified treatment of the late time expansion introduced in [1] leading to the backreaction corrections, both using the exterior and the interior mass as Hamiltonians. The late time expansion used in this treatment is expected to hold for black holes whose mass is a few Planck masses or more. In [17] formula (16) was proven also for the emission of two interacting shells (massive or massless) which can cross during their time evolution. This

would point to the absence of correlations among emitted quanta [22]. However, up to now we have no mode interpretation of such a result.

Appendix A

In this appendix we derive the explicit expression for the conjugate canonical momentum p_c in the class of Painlevé gauges, showing its universality within these gauges. The general form of p_c was given in [17]

$$\frac{p_c}{R} = \Delta\mathcal{L} - \Delta\mathcal{B} = \sqrt{\frac{2M}{R}} - \sqrt{\frac{2H}{R}} + \log \left[\frac{R'_+ - W_+}{R'_- - W_-} \cdot \frac{1 - \sqrt{\frac{2M}{R}}}{1 - \sqrt{\frac{2H}{R}}} \right] \quad (41)$$

where R stays for $R(\hat{r})$ which in our scheme equals \hat{r} , R_+ and R_- stay for the right and left derivatives of R at \hat{r} . We solve in terms of R'_+ using the two relations (8,9)

$$R'_- = R'_+ + \frac{V}{R}, \quad W_- = W_+ + \frac{\hat{p}}{R}.$$

By squaring the second equation we obtain

$$R'_+ \frac{V}{R} = A + \frac{\hat{p}}{R} \sqrt{R'^2_+ - 1 + \frac{2H}{R}} \quad (42)$$

with

$$A = \frac{H - M}{R} - \frac{m^2}{2R^2}.$$

Squaring again we obtain for \hat{p} the second order equation

$$\left(1 - \frac{2H}{R}\right) \frac{\hat{p}^2}{R^2} - 2A \sqrt{R'^2_+ - 1 + \frac{2H}{R}} \frac{\hat{p}}{R} - A^2 + m^2 \left(\frac{R'_+}{R}\right)^2 = 0$$

whose discriminant is given by

$$R'^2_+ C = R'^2_+ \left[\left(\frac{H - M}{R}\right)^2 + \frac{m^4}{4R^4} + \frac{m^2}{R^2} \left(\frac{H + M}{R} - 1\right) \right].$$

Then

$$\frac{\hat{p}}{R} = \frac{AW_+ + R'_+ \sqrt{C}}{1 - \frac{2H}{R}} \quad (43)$$

from which we see that \hat{p} is a gauge dependent quantity i.e. a quantity which depends on R'_+ . Using eq.(42) with \hat{p} given by the previous equation substituting R'_- and W_- into

eq.(41) and multiplying both numerator and denominator by $R'_+ + W_+$ we obtain for p_c

$$\begin{aligned} \frac{p_c}{R} &= \sqrt{\frac{2M}{R}} - \sqrt{\frac{2H}{R}} + \log \frac{(1 - \sqrt{\frac{2M}{R}})(1 + \sqrt{\frac{2H}{R}})}{1 - \frac{2H}{R} + \frac{H-M}{R} - \frac{m^2}{2R^2} - \sqrt{C}} = \\ &= \sqrt{\frac{2M}{R}} - \sqrt{\frac{2H}{R}} + \log \frac{1 - \frac{2H}{R} + \frac{H-M}{R} - \frac{m^2}{2R^2} + \sqrt{C}}{(1 + \sqrt{\frac{2M}{R}})(1 - \sqrt{\frac{2H}{R}})} \end{aligned}$$

showing the gauge independence of p_c within the family of Painlevé gauges.

For the discussion of the analytic properties of p_c it is however simpler to use the system eq.(10,11) as done in [17].

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