Infection load structured SI model with exponential velocity and external source of contamination

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***Abstract*—A mathematical SI model is developed for the dynamics of a contagious disease in a closed population with an external source of contamination. We prove existence and uniqueness of a non-negative mild solution of the problem using semigroup theory. We finally illustrate the model with numerical simulations.**

***Index Terms*—Epidemiology, SI model, nonlinear PDE, transport equation, semigroup theory**

I. Introduction and notations

In this article is considered the following SI model, described par a system of integro-partial differential equations of reaction-transport type,

$$\begin{cases} \dot{S}(t) = (b-\mu_0-\alpha)S(t) - \beta S(t)\displaystyle\int_{i^-}^{i^+} I(t,i)\ di, \quad t\geq 0, \\ \dfrac{\partial I(t,i)}{\partial t} = -\dfrac{\partial(\nu i I)(t,i)}{\partial i} + (b-\mu(i))I(t,i) \\ \qquad +\Phi(i)\beta S(t)\displaystyle\int_{i^-}^{i^+} I(t,i)\ di, \quad t\geq 0,\ i\in(i^-,i^+), \\ I(t,i^-) = \alpha S(t), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(i^-,i^+), \end{cases} \tag{1}$$

with the following assumptions:

(i) $0 < i^- < i^+ < +\infty$,
(ii) i satisfies the evolution equation $\frac{di}{dt} = \nu i$,
(iii) $b, \alpha, \beta, \mu_0, \nu > 0$,
(iv) function $\mu \in L^\infty(i^-,i^+)$ is such that $\mu(i) \geq \mu_0$ for almost every (f.a.e) $i \in (i^-,i^+)$,
(v) function $\Phi \in C^\infty[i^-,i^+]$ is a non negative function such that $\Phi(i^-) = \Phi(i^+) = 0$ and $\int_{i^-}^{i^+}\Phi = 1$.

This mathematical model is a variation of an epidemiological model of scrapie [6] that has been studied in [4]. See [5] for a review of SI models described by transport equations. Problem (1) describes the dynamics of a contagious disease in a closed flock with an external source of contamination, which is the main contribution of this article. This incorporates infection load structure of the infected population, denoted $i \in (i^-,i^+)$, where i^-, respectively i^+, is the minimal, respectively maximal infection load at which infected individuals die from the disease or are withdrawn of the flock population. It is supposed that i increases exponentially with respect to time t, as stated in assumption (ii). The model also incorporates a constant mortality rate μ_0 for the susceptible S and an infection load dependent mortality rate $\mu(i)$ for the infected I. It is assumed that there is total vertical transmission so the reproductive rate b is the same for S and I. The horizontal transmission, with rate β, is modeled with variable initial load of the infectious agent at the contamination, which is assigned using the function Φ. The external contamination affects the susceptible with a constant rate α, attributing the minimal initial infection load i^-. This is stated with a loopback boundary condition $I(t,i^-) = \alpha S(t)$.

This article investigates the existence and the uniqueness of a non-negative mild solution of Problem (1) using a semigroup approach. To achieve that goal, is proved in Section II the existence of a strongly continuous semigroup for the linearized problem, incorporating the loopback boundary condition in the domain of a densely defined differential operator. Section III is then dedicated to the study of the nonlinear part of Problem (1), proving that it satisfies a Lipschitz regularity. In Section IV is stated the main result of the article. The existence and the uniqueness of a non-negative solution is proved on finite time horizon periods, an then is extended to the time horizon $[0,+\infty[$. Finally, in Section V, we illustrate the model with numerical simulations.

In all that follows, Δ denotes the set

$$\Delta = \{\lambda \in \mathbb{R},\ \lambda > b - \mu_0\},$$

$(X, \|\cdot\|_X)$ is the Banach space with product norm given by

$$X = \mathbb{R} \times L^1(i^-,i^+),$$

and X_+ is the non-negative cone of X.
For every constant $R > 0$, B_R denotes the ball of X

$$B_R = \{x \in X, \|x\|_X \leq R\}.$$

II. The linear problem

Related to Problem (1), we consider the differential operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \{(x,\varphi) \in X,\ (i\varphi) \in W^{1,1}(i^-,i^+) \text{ and } \varphi(i^-) = \alpha x\},$$
$$A = \begin{pmatrix} b-\mu_0-\alpha & 0 \\ 0 & L \end{pmatrix},$$

with

$$L\varphi = -\frac{d}{di}(\nu i\varphi) + (b-\mu)\varphi.$$

The aim of this section is to prove that $(A, D(A))$ generates a positive C_0 semigroup.

Proposition 1. *The domain $D(A)$ is a dense subset of X, and the resolvent set $\rho(A)$ contains Δ. Moreover, the resolvent R_λ is given for every $\lambda \in \Delta$ by*

$$R_\lambda(y,g) = \begin{pmatrix} R_{1,\lambda}(y) \\ R_{2,\lambda}(y,g) \end{pmatrix}, \tag{2}$$

where

$$R_{1,\lambda}(y) = \frac{1}{\lambda+\mu_0+\alpha-b}y,$$
$$R_{2,\lambda}(y,g) = \frac{\alpha}{\nu i}R_{1,\lambda}(y)e^{-\int_{i^-}^{i}\frac{\lambda+\mu(r)-b}{\nu r}dr} + \frac{1}{\nu i}\int_{i^-}^{i} e^{-\int_s^i \frac{\lambda+\mu(r)-b}{\nu r}dr} g(s)\ ds.$$

Proof: Consider for every $x \in \mathbb{R}$ the dense subset D_x of $L^1(i^+,i^-)$ given by

$$D_x = \{g \in C_c[i^-,i^+],\ g(i^-) = \alpha x\},$$

where $C_c[i^-,i^+]$ denotes the set of continuous functions with compact support. We clearly have $\bigcup_{x\in\mathbb{R}} \{x\} \times D_x \subset D(A)$, and since $\overline{\cup_{x\in\mathbb{R}} \{x\} \times D_x} = X$, we deduce that $D(A)$ is dense in X.

For $(y,g) \in X$, let us look for $(x,\varphi) \in D(A)$ such that $(\lambda I - A)(x,\varphi) = (y,g)$. This is clearly equivalent to

$$x = \frac{1}{\lambda+\mu_0+\alpha-b}y, \tag{3}$$
$$\frac{d}{di}(\nu i\varphi) + \frac{(\lambda+\mu-b)}{\nu i}(\nu i\varphi) = g.$$

An integration of the previous equality gives for $\iota \in (i^-,i^+)$ and $i \geq \iota$,

$$\nu i\varphi(i) = \varphi(\iota)e^{-\int_\iota^i \frac{\lambda+\mu(r)-b}{\nu r}dr} + \int_\iota^i e^{-\int_s^i \frac{\lambda+\mu(r)-b}{\nu r}dr} g(s)\ ds.$$

Since we want $(x,\varphi) \in D(A)$, when ι goes to i^- one gets

$$\varphi(i) = \frac{\alpha x}{\nu i}e^{-\int_{i^-}^{i}\frac{\lambda+\mu(r)-b}{\nu r}dr} + \frac{1}{\nu i}\int_{i^-}^{i} e^{-\int_s^i \frac{\lambda+\mu(r)-b}{\nu r}dr} g(s)\ ds. \tag{4}$$

We now check that $(x,\varphi) \in D(A)$. Indeed, using the expression of φ given in (4) and assumption (iv) on μ, classical majorations and Fubini's theorem imply for $\lambda \in \Delta$,

$$\int_{i^-}^{i^+} |\varphi(i)|\ di \leq \frac{\alpha x}{\lambda+\mu_0-b} + \int_{i^-}^{i^+}\left(\int_s^{i^+} \frac{1}{\nu i}e^{-\int_s^i \frac{\lambda+\mu(r)-b}{\nu r}dr}\ di\right)|g(s)|\ ds. \tag{5}$$

For $\lambda \in \Delta$ equation (3) implies $\alpha|x| \leq |y|$ and we deduce from (5)

$$\int_{i^-}^{i^+} |\varphi(i)|\ di \leq \frac{1}{\lambda+\mu_0-b}(|y| + \|g\|_{L^1}). \tag{6}$$

This finally proves that $(x,\varphi) \in X$ and consequently to (3),

$$\|(x,\varphi)\|_X \leq \frac{2}{\lambda+\mu_0-b}\|(y,g)\|_X.$$

We now prove that $(i\varphi) \in W^{1,1}(i^-,i^+)$.
Let $\psi \in C_c[i^-,i^+]$. For $\lambda \in \Delta$ straightforward calculations give

$$\int_{i^-}^{i^+} |i\varphi(i)\psi(i)|\ di \leq \frac{1}{\nu}(\alpha x + \|g\|_{L^1})\|\psi\|_{L^1},$$

and the majoration above applied to $\psi \equiv 1$ gives

$$\int_{i^-}^{i^+} |i\varphi(i)|\ di \leq \frac{i^+-i^-}{\nu}(\alpha x + \|g\|_{L^1}),$$

and consequently $(i\varphi) \in W^{1,1}(i^-,i^+)$. So $(x,\varphi) \in D(A)$ and the expression (2) of R_λ follows from (4). ∎

Corollary 1. *The resolvent R_λ satisfies*

$$\|R_\lambda^n\| \leq \frac{2}{(\lambda+\mu_0-b)^n}, \quad \forall\lambda \in \Delta, \quad \forall n \in \mathbb{N}^*. \tag{7}$$

Proof: Let us denote $R_\lambda^n = (R_{1,\lambda}^n, R_{2,\lambda}^n)$ for every $n \in \mathbb{N}$. Using equation (3) and the arguments we used to get (6), an induction proves that for every $n \in \mathbb{N}^*$ and every $(y,g) \in X$,

$$|R_{1,\lambda}^n(y)| \leq \frac{1}{(\lambda+\mu_0-b)^n}|y|,$$
$$\int_{i^-}^{i^+} |R_{2,\lambda}^n(x,y)|\ di \leq \frac{1}{(\lambda+\mu_0-b)^n}(|y| + \|g\|_{L^1}),$$

and (7) directly yields. ∎

Theorem 1. *The differential operator $(A, D(A))$ is an infinitesimal generator of a strongly continuous positive semigroup $\{T_A(t)\}_{t\geq 0}$ on X that satisfies*

$$\|T_A(t)\| \leq 2\,e^{(b-\mu_0)t} \quad \forall t \geq 0. \tag{8}$$

Proof: The existence of $\{T_A(t)\}_{t\geq 0}$ and the majoration are direct consequences of Corollary 1 and the Hille-Yosida theorem [2]. Moreover, the resolvent R_λ is positive and as it is proved in [1], $\{T_A(t)\}_{t\geq 0}$ is also positive. ∎

III. The non-linear problem

In this section, we tackle the non-linearity in Problem 1 proving it satisfies a Lipschitz condition. To this goal, we check that Problem 1 rewrites as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = A\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} + f(S(t), I(t)), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(i^-, i^+), \end{cases} \tag{9}$$

where function $f : X \to X$ is given by

$$f(u,v) = \begin{pmatrix} -\beta u \int_{i^-}^{i^+} v \\ \beta\Phi\, u \int_{i^-}^{i^+} v \end{pmatrix}. \tag{10}$$

Proposition 2. *The function $f : X \to X$ given in* (10) *satisfies the following properties :*

1) $\exists \Lambda > 0$, $\forall M > 0$, $\forall ((u_1, v, 1), (u_2, v_2)) \in B^2_M$,

$$\|f(u_1, v_1), (u_2, v_2)\|_X \leq \Lambda M \|(u_1, v_1) - (u_2, v_2)\|_X,$$

2) $\forall m > 0, \exists \lambda_m > 0$,

$$(u,v) \in B_m \cap X_+ \Rightarrow f(u,v) + \lambda_m(u,v) \in X_+. \tag{11}$$

Proof: Let $M > 0$ and $((u_1, v_1), (u_2, v_2)) \in B^2_M$. Straightforward computations give

$$\left| u_1 \int_{i^-}^{i^+} v_1(i)di - u_2 \int_{i^-}^{i^+} v_2(i)di \right| \leq M\|(u_1, v_1) - (u_2, v_2)\|_X.$$

Hypothesis (v) on Φ and the previous inequalities imply

$$\|f(u_1, v_1) - f(u_2, v_2)\|_X \leq \Lambda M \|(u_1, v_2) - (u_2, v_2)\|_X,$$

where $\Lambda = 2\beta$ is a positive constant. Moreover, given $m > 0$, (11) is satisfied for every $\lambda_m \geq \beta m$. ∎

IV. Existence and uniqueness of the solution

A. Existence on finite time horizon

Proposition 3. *There exists $t_{max} \leq +\infty$ such that Problem* (1) *has a unique mild solution $(S, I) \in C([0, T], X_+)$ for every $T < t_{max}$.*

Proof: We prove the theorem with a fixed point method, adapting the ideas of [7].
Let $m > 0$. Consider, for λ_m that satisfies (11), the operator $A_m = A - \lambda_m I$ and the function $f_{\lambda_m} = f + \lambda_m I$. A consequence of Theorem 1 is that A_m is an infinitesimal generator on X of a positive C_0 semigroup $\{T_{A_m}(t)\}_{t\geq 0}$ that satisfies

$$\|T_{A_m}(t)\| \leq 2e^{(b-\mu_0-\lambda_m)t}, \quad \forall t \geq 0,$$

so one can consider $m > 0$ big enough such that $r_m > 0$ given by

$$r_m = 2\|(S_0, I_0)\|_X \sup_{t\in[0,1]} \|T_{A_m}(t)\|.$$

satisfies

$$r_m \leq m.$$

In all that follows, let us denote $X^{r_m}_+$ the subset of X given by

$$X^{r_m}_+ = X_+ \cap B_{r_m}.$$

Since $r_m \leq m$ we have

$$X^{r_m}_+ \subset B_m. \tag{12}$$

Let $\tau > 0$ be such that

$$\tau \leq \min\left(1, \frac{\|(S_0, I_0)\|_X}{r_m(\Lambda r_m + \lambda_m)}\right), \tag{13}$$

where Λ is given in Proposition 2.
Consider the mapping $F : C([0,\tau], X) \to C([0,\tau], X)$ defined by

$$\begin{aligned} F(u(s), v(s)) =& T_{A_m}(t)(S_0, I_0) \\ &+ \int_0^t T_{A_m}(t-s) f_{\lambda_m}(u(s), v(s))\, ds. \end{aligned}$$

Since $f_{\lambda_m}(0) = 0$ in X, Proposition 2 implies that for $t \in [0,\tau]$ and $(u, v) \in C([0,\tau], B_{r_m})$,

$$\begin{aligned} \|F(u(t), v(t))\|_X \leq \sup_{s\in[0,t]} \|T_{A_m}(s)\|(\|S_0, I_0)\|_X \\ + t r_m(\Lambda r_m + \lambda_m)), \end{aligned}$$

and consequently to (13) the mapping F preserves $C([0,\tau], B_{r_m})$. Moreover, equations (11) and (12) imply that F preserves $C([0,\tau], X^{r_m}_+)$ for $(S_0, I_0) \in X_+$ since the semigroup $\{T_{A_m}(t)\}_{t\geq 0}$ is positive.
Similar calculations prove that F is a contraction mapping of $C([0,\tau], X)$ with Lipschitz constant $\frac{1}{2}$.
Consequently, F is a contraction of $C([0,\tau], X^{r_m}_+)$ and the Banach fixed point theorem implies the existence and the uniqueness of $(\bar{u}, \bar{v}) \in C([0,\tau], X^{r_m}_+)$ such that $F(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$ in $C([0,\tau], X)$. As it is proved in [3], the solution can be extended on $[0, t_{max}[$ with $t_{max} \leq +\infty$.
Finally, every mild solution of Problem (1) is a mild solution of the following problem,

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = A_m\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} + f_{\lambda_m}(S(t), I(t)), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(i^-, i^+), \end{cases}$$

so the unique fixed point $(\bar{u}, \bar{v})$ of F is also the unique mild solution of Problem (9). ∎

B. Global existence

Theorem 2. *The Problem* (1) *has a unique mild solution $(S, I) \in C([0, +\infty[, X_+)$.*

Proof: We suppose by contradiction that $t_{max} < +\infty$. Then Proposition 3 implies that $(S, I) \in C([0, t_{max}[, X_+)$ and satisfies

$$\lim_{t\to t_{max}} \|(S(t), I(t))\|_X = +\infty. \tag{14}$$

Consider the change of variables $\phi = (\phi_1, \phi_2)$ given by

$$\phi : (t, \xi) \mapsto (t, i) = (t, i^- e^{\nu(t-\xi)}).$$

Since I is a non-negative function on $[0, t_{max}[$, we can easily check that $I \circ \phi$ satisfies the following sub-linear differential equation,

$$\frac{\partial(I \circ \phi(t,\xi))}{\partial t} \leq (b - \mu \circ \phi_2(t,\xi))\, I \circ \phi(t,\xi), \qquad (15)$$

which implies that $I \circ \phi(t,\cdot)$ is a bounded function on $[0, t_{max}[$ and so is $I(t,\cdot)$. Moreover, I being non-negative, Problem (1) also implies that

$$S(t) \leq e^{(b-\mu_0-\alpha)t},$$

and S is a bounded function on $[0, t_{max}[$. The previous inequality and (15) yield a contradiction with (14), and necessarily $t_{max=+\infty}$. ■

V. Numerical simulations

In this section, we illustrate the model through three simulation scenarios. All scenarios suppose that the initial population does not contain infected, stated by $I_0 = 0$.

TABLE I
PARAMETER VALUES USED FOR THE SIMULATIONS.

Parameter definition	symbol	value
initial susceptible population size	S_0	100 indiv.
initial infected population size	I_0	0 indiv.
susceptible mortality rate	μ_0	0.1 year^{-1}
infected mortality rate	μ	0.15 year^{-1}
reproductive rate	b	0.12 year^{-1}
infection load growth rate	ν	10^{-3} year^{-1}
Scenario 1 specific parameters		
contamination rate	α	$2.0\,10^{-2}$ year^{-1}
horizontal transmission rate	β	0 (indiv. year)$^{-1}$
Scenario 2 specific parameters		
contamination rate	α	$8.5\,10^{-2}$ year^{-1}
horizontal transmission rate	β	0 (indiv. year)$^{-1}$
Scenario 3 specific parameters		
contamination rate	α	$9.5\,10^{-3}$ year^{-1}
horizontal transmission rate	β	$3\,10^{-3}$ (indiv. year)$^{-1}$

In the simulations, $k(I)(t)$ denotes the total infected population at time t given by $k(I)(t) = \int_{i^-}^{i^+} I(t,i)\ di$.

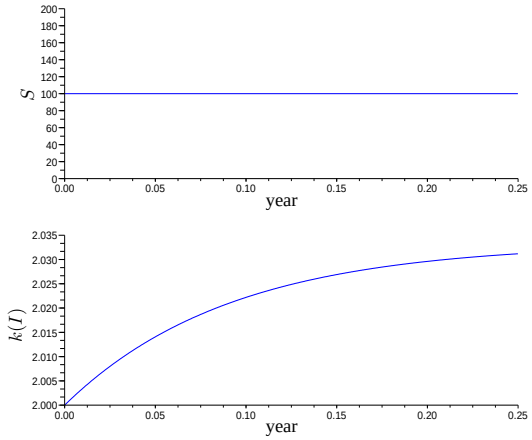

Fig. 1. Scenario 1, $S(t)$ and $k(I)(t)$ on a quarter of year

Scenario 1 corresponds to the case where S is constant, under the assumption $b - \mu_0 - \alpha = 0$, and no horizontal transmission ($\beta = 0$). In this case, we have $b - \mu_0 > 0$ and the boundary condition is constant and positive. Figure 1 represents the curves of susceptibles and total infected over time on a period of a quarter of year. It shows that the total infected population increases with time as we could expect.

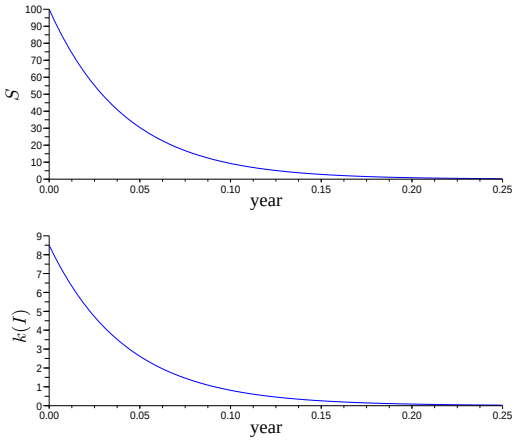

Fig. 2. Scenario 2, $S(t)$ and $k(I)(t)$ on a quarter of year

Scenario 2 represents the case of a high level of contamination, with $b - \mu_0 - \alpha < 0$, and no horizontal transmission. This implies that the susceptible population S decreases with time, therefore also does the boundary condition αS. The rate $b - \mu < 0$ for infected implies a fast decrease of the total population. Figure 2 illsutrates this scenario on a time period of a quarter of year, and shows that the total population converges to 0 with time.

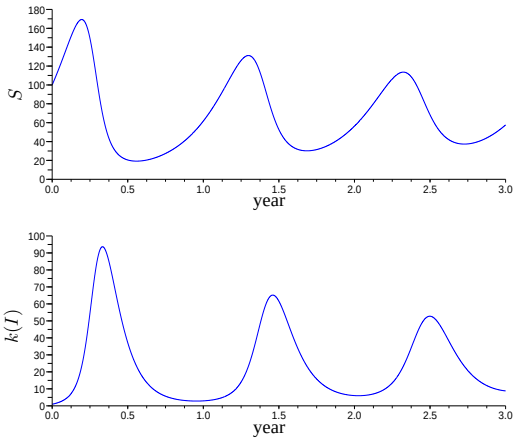

Fig. 3. Scenario 3, $S(t)$ and $k(I)(t)$ on 3 years

Scenario 3 finally represents the case where the transmission rate is small enough to imply that S would increase without horizontal transmission ($\beta = 0$). But one can check on Figure 3 that a positive value of β implies a time oscillation of susceptibles and infected, with a delay of the curve of infected

compared to the curve of suscetibles. Specifically, Figure 4, which represents the phase-plane $(S, k(I))$ on a time horizon of 10 years, shows a convegence to an equilibrium point of the model.

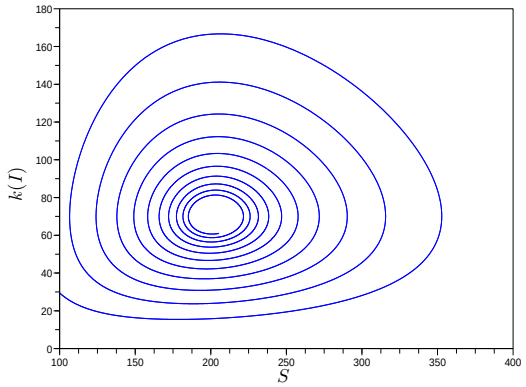

Fig. 4. Scenario 3, phase-plane $(S, k(I))$ on 10 years

VI. Conclusion

In this article, we have proved the existence and the uniqueness of a non negative mild solution for a SI model that describes the evolution of a disease in a closed population. This disease is characterized by an exponential velocity of the infection load, a contagious process between individuals, and an external source of contamination. This last is supposed to be proportional to the susceptible population and is modeled with a loopback boundary condition. The simulations made show that, accordingly to the values of fundamental biological parameters, the total population density converges with time to equlibrium points of the dynamical system.

References

[1] W. Arendt, *Resolvent positive operators*, Proc. London Math. Soc. 54:321-349, 1987

[2] K.J Engel and R. Nagel, *A short course on operator semigroups*, Universitext, Springer, 2006

[3] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences 44, Springer, 1983

[4] A. Perasso and B. Laroche, *Well-posedness of an epidemiological model described by an evolution PDE*, ESAIM:PROC. 25:29-43, 2008

[5] B. Perthame, *Transport Equations in Biology*, Birkhäuser, 2010

[6] S.M. Stringer, N. Hunter and M.E.J. Woolhouse, *A mathematical model of the dynamics of scrapie in a sheep flock*, Math. Biosci. 153(2):79-98, 1998

[7] I. Segal. *Non-linear semi-groups*, Annals of Mathematics, 78:339364, 1963.