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Saber Amdouni, Patrick Hild, Vanessa Lleras, Maher Moakher, Yves Renard. A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2012, 46, pp.813-839. 10.1051/m2an/2011072 . hal-00606313

**HAL Id: hal-00606313**

**<https://hal.science/hal-00606313>**

Submitted on 6 Jul 2011

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# A stabilized Lagrange multiplier method for the enriched finite-element approximation of contact problems of cracked elastic bodies

S. AMDOUNI <sup>1</sup>, P. HILD <sup>2</sup>, V. LLERAS <sup>3</sup>, M. MOAKHER <sup>4</sup>, Y. RENARD <sup>5</sup>

## Abstract

The purpose of this paper is to provide a priori error estimates on the approximation of contact conditions in the framework of the eXtended Finite-Element Method (XFEM). This method allows to perform finite-element computations on cracked domains by using meshes of the non-cracked domain. We consider a stabilized Lagrange multiplier method whose particularity is that no discrete inf-sup condition is needed in the convergence analysis. The contact condition is prescribed on the crack with a discrete multiplier which is the trace on the crack of a finite-element method on the non-cracked domain, avoiding the definition of a specific mesh of the crack. Additionally, we present numerical experiments which confirm the efficiency of the proposed method.

## 1 Introduction

With the aim of gaining flexibility in the finite-element method, Moës, Dolbow and Belytschko [33] introduced in 1999 the XFEM (eXtended Finite-Element Method) which allows to perform finite-element computations on cracked domains by using meshes of the non-cracked domain. The main feature of this method is the ability to take into account the discontinuity across the crack and the asymptotic displacement at the crack tip by addition of special functions into the finite-element space. These special functions include both non-smooth functions representing the singularities at the reentrant corners (as in the singular enrichment method introduced in [40]) and also step functions (of Heaviside type) along the crack.

In the original method, the asymptotic displacement is incorporated into the finite-element space multiplied by the shape function of a background Lagrange finite-element method. In this paper, we deal with a variant, introduced in [13], where the asymptotic displacement is multiplied by a cut-off function. After numerous numerical works developed in various contexts of mechanics, the first a priori error estimate results for XFEM (in linear elasticity) were recently obtained in [13] and [35]: in the convergence analysis, a difficulty consists in evaluating the local error in the elements cut by the crack by using appropriate extension operators and specific estimates. In the latter references, the authors obtained an optimal error estimate of order  $h$  ( $h$  being the discretization parameter) for an affine finite-element method under  $H^2$  regularity of the regular part of the solution (keeping in mind that the solution is only  $H^{\frac{3}{2}-\varepsilon}$  regular).

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Let us remark that some convergence analysis results have been performed on a posteriori error estimation for XFEM. A simple derivative recovery technique and its associated a posteriori error estimator have been proposed in [10, 11, 12, 38]. These recovery based a posteriori error estimations outperform the super-convergent patch recovery technique (SPR) introduced by Zienkiewicz and Zhu. In [24], an error estimator of residual type for the elasticity system in two space dimensions is proposed.

Concerning a priori error estimates for the contact problem of linearly elastic bodies approximated by a standard affine finite-element method, a rate of convergence of order  $h^{3/4}$  can be obtained for most methods (see [8, 23, 32] for instance). An optimal order of  $h$  (resp.  $h\sqrt[4]{|\log(h)|}$  and  $h\sqrt{|\log(h)|}$ ) has been obtained in [27] (resp. [8] and [7]) for the direct approximation of the variational inequality and with the additional assumption that the number of transition between contact and non contact is finite on the contact boundary. However, for stabilized Lagrange multiplier methods and with the only assumption that the solution is in  $H^2(\Omega)$ , the best a priori error estimates proven is of order  $h^{3/4}$  (see [26]). This limitation may be only due to technical reasons since it has never been found on the numerical experiments. It affects the a priori error estimates we present in this paper.

Only a few works have been devoted to contact and XFEM, and they mainly use two methods to formulate contact problems: penalty method and Lagrange multiplier method. In penalty method, the penetration between two contacting boundaries is introduced and the normal contact force is related to the penetration by a penalty parameter [31]. Khoei et al. [29, 30] give the formulation with the penalization for plasticity problems. Contrary to penalization techniques, in the method of Lagrangian multipliers, the stability is improved without compromising the consistency of the method. Dolbow et al. [15] propose a formulation of the problem of a crack with frictional contact in 2D with an augmented Lagrangian method. Géniaut et al. [16, 17] choose an XFEM approach with frictional contact in the three dimensional case. They use a hybrid and continuous formulation close to the augmented Lagrangian method introduced by Ben Dhia [9]. Pierres et al. in [36] introduced a method with a three-fields description of the contact problem, the interface being seen as an autonomous entity with its own discretization.

In all the works cited above, a uniform discrete inf-sup condition is theoretically required between the finite-element space for the displacement and the one for the multiplier in order to obtain a good approximation of the solution. However, a uniform inf-sup condition is difficult to obtain on the crack since it does not coincide with element edges. Consequently, we consider a stabilization method which avoids the need of such an inf-sup condition. This method, which provides stability of the multiplier by adding supplementary terms in the weak formulation, has been originally introduced and analyzed by Barbosa and Hughes in [3, 4]. The great advantage is that the finite-element spaces on the primal and dual variables can be chosen independently. Note that, in [39], the connection was made between this method and the former one of Nitsche [39]. The studies in [3, 4] were generalized to a variational inequality framework in [5] (Signorini-type problems among others). This method has also been extended to interface problems on non-matching meshes in [6, 19] and more recently for bilateral (linear) contact problems in [22] and for contact problems in elastostatics [26].

None of the previous works treats the error estimates for contact problems approximated by the XFEM method. The rapid uptake of the XFEM method by industry requires adequate error estimation tools to be available to the analysts. Our purpose in this paper is to extend the work done in [26] to the enriched finite-element approximation of contact problems of cracked elastic bodies.

The paper is organized as follows. In Section 2, we introduce the formulation of the contact problem on a crack of an elastic structure. In Section 3, we present the elasticity problem approximated by both the enrichment strategy introduced in [13] and the stabilized Lagrange multiplier method of Barbosa-Hughes. A subsection is devoted to a priori error estimates following three different discrete contact conditions. Finally, in Section 4, we present some numerical experiments on a very simple situation. We compare the stabilized and the non-stabilized cases for different finite-element approximations. Optimal rates of convergence are observed for the stabilized case. The influence of the stabilization parameter is also investigated.

## 2 Formulation of the continuous problem

We introduce some useful notations and several functional spaces. In what follows, bold letters like  $\mathbf{u}, \mathbf{v}$ , indicate vector-valued quantities, while the capital ones (e.g.,  $\mathbf{V}, \mathbf{K}, \dots$ ) represent functional sets involving vector fields. As usual, we denote by  $(L^2(\cdot))^d$  and by  $(H^s(\cdot))^d$ ,  $s \geq 0$ ,  $d = 1, 2$  the Lebesgue and Sobolev spaces in  $d$ -dimensional space (see [1]). The usual norm of  $(H^s(D))^d$  is denoted by  $\|\cdot\|_{s,D}$  and we keep the same notation when  $d = 1$  or  $d = 2$ . For shortness, the  $(L^2(D))^d$ -norm will be denoted by  $\|\cdot\|_D$  when  $d = 1$  or  $d = 2$ . In the sequel the symbol  $|\cdot|$  will denote either the Euclidean norm in  $\mathbb{R}^2$ , or the length of a line segment, or the area of a planar domain.

We consider a cracked elastic body occupying a domain  $\Omega$  in  $\mathbb{R}^2$ . The boundary  $\partial\Omega$  of  $\Omega$ , which is assumed to be polygonal for simplicity, is composed of three non-overlapping parts  $\Gamma_D$ ,  $\Gamma_N$  and  $\Gamma_C$  with  $\text{meas}(\Gamma_D) > 0$  and  $\text{meas}(\Gamma_C) > 0$ . A Dirichlet and a Neumann conditions are prescribed on  $\Gamma_D$  and  $\Gamma_N$ , respectively. The boundary part  $\Gamma_C$  represents also the crack location which, for the sake of simplicity, is assumed to be a straight line segment. In order to deal with the contact between the two sides of the crack as a contact between two elastic bodies, we denote by  $\Gamma_{C+}$  and  $\Gamma_{C-}$  each of the two sides of the crack (see Fig. 1). Of course, in the initial configuration, both  $\Gamma_{C+}$  and  $\Gamma_{C-}$  coincide. Let  $\mathbf{n} = \mathbf{n}^+ = -\mathbf{n}^-$  denote the normal unit outward vector on  $\Gamma_{C+}$ .

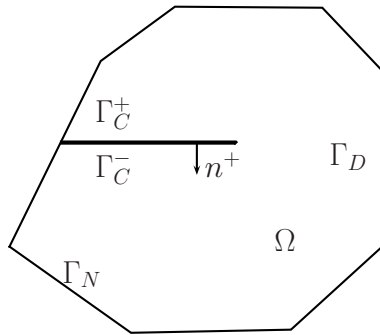


Figure 1: A cracked domain.

We assume that the body is subjected to volume forces  $\mathbf{f} = (f_1, f_2) \in (L^2(\Omega))^2$  and to surface loads  $\mathbf{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$ . Then, under planar small strain assumptions, the problem of homogeneous isotropic linear elasticity consists in finding the displacement field  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$

satisfying

$$\begin{aligned}
(1) \quad & \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} && \text{in } \Omega, \\
(2) \quad & \boldsymbol{\sigma}(\mathbf{u}) = \lambda_L \operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u}) I + 2\mu_L \boldsymbol{\varepsilon}(\mathbf{u}), && \text{in } \Omega, \\
(3) \quad & \mathbf{u} = \mathbf{0} && \text{on } \Gamma_D, \\
(4) \quad & \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} && \text{on } \Gamma_N,
\end{aligned}$$

where  $\boldsymbol{\sigma} = (\sigma_{ij})$ ,  $1 \leq i, j \leq 2$ , stands for the stress tensor field,  $\boldsymbol{\varepsilon}(\mathbf{v}) = (\nabla \mathbf{v} + \nabla \mathbf{v}^T)/2$  represents the linearized strain tensor field,  $\lambda_L \geq 0$ ,  $\mu_L > 0$  are the Lamé coefficients, and  $I$  denotes the identity tensor. For a displacement field  $\mathbf{v}$  and a density of surface forces  $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$  defined on  $\partial\Omega$ , we adopt the following notations:

$$\mathbf{v}^+ = v_n^+ \mathbf{n}^+ + v_t^+ \mathbf{t}, \quad \mathbf{v}^- = v_n^- \mathbf{n}^- + v_t^- \mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_t(\mathbf{v})\mathbf{t},$$

where  $\mathbf{t}$  is a unit tangent vector on  $\Gamma_C$ ,  $\mathbf{v}^+$  (resp.  $\mathbf{v}^-$ ) is the trace of displacement on  $\Gamma_C$  on the  $\Gamma_C^+$  side (resp. on the  $\Gamma_C^-$  side). The conditions describing the frictionless unilateral contact on  $\Gamma_C$  are:

$$(5) \quad \llbracket u_n \rrbracket = u_n^+ + u_n^- \leq 0, \quad \sigma_n(\mathbf{u}) \leq 0, \quad \sigma_n(\mathbf{u}) \cdot \llbracket u_n \rrbracket = 0, \quad \sigma_t(\mathbf{u}) = \mathbf{0},$$

where  $\llbracket u_n \rrbracket$  is the jump of the normal displacement across the crack  $\Gamma_C$ .

We present now some classical weak formulation of Problem (1)–(5). We introduce the following Hilbert spaces:

$$\mathbf{V} = \left\{ \mathbf{v} \in (H^1(\Omega))^2 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}, \quad W = \left\{ v_n|_{\Gamma_C} : \mathbf{v} \in \mathbf{V} \right\},$$

and their topological dual spaces  $\mathbf{V}'$ ,  $W'$ , endowed with their usual norms. Since  $\Gamma_C$  is a straight line segment, we have  $H_0^{1/2}(\Gamma_C) \subset W \subset H^{1/2}(\Gamma_C)$  which implies  $W' \subset H^{-1/2}(\Gamma_C) = (H_0^{1/2}(\Gamma_C))'$ . We also introduce the following convex cone of multipliers on  $\Gamma_C$ :

$$M^- = \left\{ \mu \in W' : \langle \mu, \psi \rangle_{W',W} \geq 0 \text{ for all } \psi \in W, \psi \leq 0 \text{ a.e. on } \Gamma_C \right\},$$

where the notation  $\langle \cdot, \cdot \rangle_{W',W}$  stands for the duality pairing between  $W'$  and  $W$ . Finally, for  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbf{V}$  and  $\mu$  in  $W'$  we define the following forms

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, & b(\mu, \mathbf{v}) &= \langle \mu, \llbracket v_n \rrbracket \rangle_{W',W} \\
L(\mathbf{v}) &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, d\Gamma.
\end{aligned}$$

The mixed formulation of the unilateral contact problem (1)–(5) consists then in finding  $\mathbf{u} \in \mathbf{V}$  and  $\lambda \in M^-$  such that

$$(6) \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) - b(\lambda, \mathbf{v}) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mu - \lambda, \mathbf{u}) \geq 0, & \forall \mu \in M^-. \end{cases}$$

An equivalent formulation of (6) consists in finding  $(\mathbf{u}, \lambda) \in \mathbf{V} \times M^-$  satisfying

$$\mathcal{L}(\mathbf{u}, \mu) \leq \mathcal{L}(\mathbf{u}, \lambda) \leq \mathcal{L}(\mathbf{v}, \lambda), \quad \forall \mathbf{v} \in \mathbf{V}, \forall \mu \in M^-,$$

where  $\mathcal{L}(\cdot, \cdot)$  is the classical Lagrangian of the system defined as

$$\mathcal{L}(\mathbf{v}, \mu) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) - b(\mu, \mathbf{v}).$$

Another classical weak formulation of problem (1)–(5) is given by the following variational inequality: find  $\mathbf{u} \in \mathbf{K}$  such that

$$(7) \quad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K},$$

where  $\mathbf{K}$  denotes the closed convex cone of admissible displacement fields satisfying the non-interpenetration condition

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} : \llbracket v_n \rrbracket \leq 0 \text{ on } \Gamma_C\}.$$

The existence and uniqueness of  $(\mathbf{u}, \lambda)$  solution to (6) has been established in [20]. Moreover, the first argument  $\mathbf{u}$  solution to (6) is also the unique solution of problem (7) and one has  $\lambda = \sigma_n(\mathbf{u})$  is in  $W'$ .

### 3 Discretization with the stabilized Lagrange multiplier method

#### 3.1 The discrete problem

We will denote by  $\mathbf{V}^h \subset \mathbf{V}$  a family of enriched finite-dimensional vector spaces indexed by  $h$  coming from a family  $\mathcal{T}^h$  of triangulations of the uncracked domain  $\bar{\Omega}$  (here  $h = \max_{T \in \mathcal{T}^h} h_T$  where  $h_T$  is the diameter of the triangle  $T$ ). The family of triangulations is assumed to be regular, i.e., there exists  $\beta > 0$  such that  $\forall T \in \mathcal{T}^h$ ,  $h_T/\rho_T \leq \beta$  where  $\rho_T$  denotes the radius of the inscribed circle in  $T$  (see [14]). We consider the variant, called the cut-off XFEM, introduced in [13] in which the whole area around the crack tip is enriched by using a cut-off function denoted by  $\chi(\cdot)$ . In this variant, the enriched finite-element space  $\mathbf{V}^h$  is defined as

$$\mathbf{V}^h = \left\{ \mathbf{v}^h \in (\mathcal{C}(\bar{\Omega}))^2 : \mathbf{v}^h = \sum_{i \in \mathcal{N}_h} \mathbf{a}_i \varphi_i + \sum_{i \in \mathcal{N}_h^H} \mathbf{b}_i H \varphi_i + \chi \sum_{j=1}^4 \mathbf{c}_j F_j, \quad \mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_j \in \mathbb{R}^2 \right\} \subset \mathbf{V}.$$

Here  $(\mathcal{C}(\bar{\Omega}))^2$  is the space of continuous vector fields over  $\bar{\Omega}$ ,  $H(\cdot)$  is the Heaviside-like function used to represent the discontinuity across the straight crack and defined by

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } (\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{n}^+ \geq 0, \\ -1 & \text{otherwise,} \end{cases}$$

where  $\mathbf{x}^*$  denotes the position of the crack tip. The notation  $\varphi_i$  represents the scalar-valued shape functions associated with the classical degree one finite-element method at the node of index  $i$ ,  $\mathcal{N}_h$  denotes the set of all node indices, and  $\mathcal{N}_h^H$  denotes the set of nodes indices enriched by the function  $H(\cdot)$ , i.e., nodes indices for which the support of the corresponding shape function is completely cut by the crack (see Fig. 2). The cut-off function is a third order polynomial on  $[r_0, r_1]$  such that:

$$\begin{cases} \chi(r) = 1 & \text{if } r < r_0, \\ \chi(r) \in (0, 1) & \text{if } r_0 < r < r_1, \\ \chi(r) = 0 & \text{if } r > r_1. \end{cases}$$

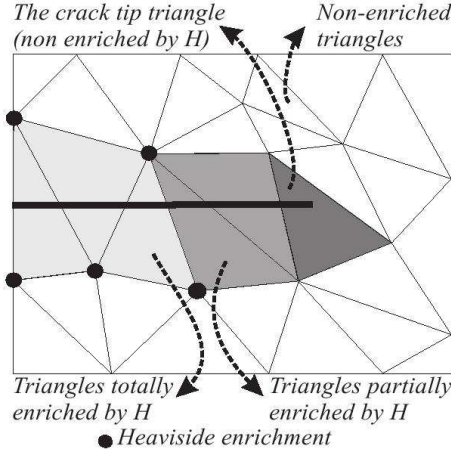


Figure 2: A cracked domain.

The functions  $\{F_j(\mathbf{x})\}_{1 \leq j \leq 4}$  are defined in polar coordinates located at the crack tip by

$$(8) \quad \{F_j(\mathbf{x}), 1 \leq j \leq 4\} = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos \frac{\theta}{2} \sin \theta \right\}.$$

An important point of the approximation is whether the contact pressure  $\sigma_n$  is regular or not at the crack tip. If it were singular, it should be taken into account by the discretization of the multiplier. Nevertheless, it seems, that this is not the case in homogeneous isotropic linear elasticity. This results has not been proved yet, and seems to be a difficult issue. However, if we consider the formulation (6) and if we assume that there is a finite number of transition points between contact and non contact zones near the crack tip, then we are able to prove (see Lemma A.1 in Appendix A) that the contact stress  $\sigma_n$  is in  $H^{1/2}(\Gamma_C)$ .

Now, concerning the discretization of the multiplier, let  $\mathbf{x}_0, \dots, \mathbf{x}_N$  be given distinct points lying in  $\overline{\Gamma_C}$  (note that we can choose these nodes to coincide with the intersection between  $\mathcal{T}^h$  and  $\overline{\Gamma_C}$ ). These nodes form a one-dimensional family of meshes of  $\Gamma_C$  denoted by  $T^H$ . We set  $H = \max_{0 \leq i \leq N-1} |\mathbf{x}_{i+1} - \mathbf{x}_i|$ . The mesh  $T^H$  allows us to define a finite-dimensional space  $W^H$  approximating  $W'$  and a nonempty closed convex set  $M^{H-} \subset W^H$  approximating  $M^-$ :

$$M^{H-} = \{ \mu^H \in W^H : \mu^H \text{ satisfy a "nonpositivity condition" on } \Gamma_C \}.$$

Following [26], we consider two possible elementary choices of  $W^H$ :

$$W_0^H = \left\{ \mu^H \in L^2(\Gamma_C) : \mu^H|_{(\mathbf{x}_i, \mathbf{x}_{i+1})} \in P_0(\mathbf{x}_i, \mathbf{x}_{i+1}), \forall 0 \leq i \leq N-1 \right\},$$

$$W_1^H = \left\{ \mu^H \in \mathcal{C}(\Gamma_C) : \mu^H|_{(\mathbf{x}_i, \mathbf{x}_{i+1})} \in P_1(\mathbf{x}_i, \mathbf{x}_{i+1}), \forall 0 \leq i \leq N-1 \right\},$$

where  $P_k(E)$  denotes the space of polynomials of degree less or equal to  $k$  on  $E$ . This allows to provide the following three elementary definitions of  $M^{H-}$ :

$$(9) \quad M_0^{H-} = \{ \mu^H \in W_0^H : \mu^H \leq 0 \text{ on } \Gamma_C \},$$

$$(10) \quad M_1^{H-} = \{ \mu^H \in W_1^H : \mu^H \leq 0 \text{ on } \Gamma_C \},$$

$$(11) \quad M_{1,*}^{H-} = \left\{ \mu^H \in W_1^H : \int_{\Gamma_C} \mu^H \psi^H d\Gamma \geq 0, \forall \psi^H \in M_1^{H-} \right\}.$$

Now we divide the domain  $\Omega$  into  $\Omega_1$  and  $\Omega_2$  according to the crack and a straight extension of the crack (see Fig. 3) such that the value of  $H(\cdot)$  is  $(-1)^k$  on  $\Omega_k$ ,  $k = 1, 2$ . Now, let  $R_h$  be an operator from  $\mathbf{V}^h$  onto  $L^2(\Gamma_C)$  which approaches the normal component of the stress vector on  $\Gamma_C$  defined for all  $T \in \mathcal{T}^h$  with  $T \cap \Gamma_C \neq \emptyset$  as

$$R_h(\mathbf{v}^h)|_{T \cap \Gamma_C} = \begin{cases} \sigma_n(\mathbf{v}_1^h), & \text{if } |T \cap \Omega_1| \geq \frac{|T|}{2}, \\ \sigma_n(\mathbf{v}_2^h), & \text{if } |T \cap \Omega_2| > \frac{|T|}{2}, \end{cases}$$

where  $\mathbf{v}_1^h = \mathbf{v}^h|_{\Omega_1}$  and  $\mathbf{v}_2^h = \mathbf{v}^h|_{\Omega_2}$ .

This allow us to define the following stabilized discrete approximation of Problem (6): find  $\mathbf{u}^h \in \mathbf{V}^h$  and  $\lambda^H \in M^{H-}$  such that

$$(12) \quad \begin{cases} a(\mathbf{u}^h, \mathbf{v}^h) - b(\lambda^H, \mathbf{v}^h) + \int_{\Gamma_C} \gamma(\lambda^H - R_h(\mathbf{u}^h))R_h(\mathbf{v}^h)d\Gamma = L(\mathbf{v}^h), & \forall \mathbf{v}^h \in \mathbf{V}^h, \\ b(\mu^H - \lambda^H, \mathbf{u}^h) + \int_{\Gamma_C} \gamma(\mu^H - \lambda^H)(\lambda^H - R_h(\mathbf{u}^h))d\Gamma \geq 0, & \forall \mu^H \in M^{H-}, \end{cases}$$

where  $\gamma$  is defined to be constant on each element  $T$  as  $\gamma = \gamma_0 h_T$  where  $\gamma_0 > 0$  is a given constant independent of  $h$  and  $H$ . Problem (12) represents the optimality conditions for the Lagrangian

$$\mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = \frac{1}{2}a(\mathbf{v}^h, \mathbf{v}^h) - L(\mathbf{v}^h) - b(\mu^H, \mathbf{v}^h) - \frac{1}{2} \int_{\Gamma_C} \gamma(\mu^H - R_h(\mathbf{v}^h))^2 d\Gamma.$$

We note that, without loss of generality, we can assume that  $\Gamma_C$  is a straight line segment parallel to the  $x$ -axis. Let  $T \in \mathcal{T}^h$  and  $E = T \cap \Gamma_C$ . Then, for any  $\mathbf{v}^h \in \mathbf{V}^h$  and since  $\sigma_n(\mathbf{v}_i^h)$  is a constant over each element, we have

$$\begin{aligned} \|R_h(\mathbf{v}^h)\|_{0,E} &= \|\sigma_n(\mathbf{v}_i^h)\|_{0,E}, \quad \text{with } i \text{ such that } |T \cap \Omega_i| \geq \frac{|T|}{2}, \\ &= \|\sigma_{yy}(\mathbf{v}_i^h)\|_{0,E}, \\ &= \frac{|E|^{1/2}}{|T \cap \Omega_i|^{1/2}} \|\sigma_{yy}(\mathbf{v}_i^h)\|_{0,T \cap \Omega_i}, \\ &\lesssim h_T^{-\frac{1}{2}} \|\sigma_{yy}(\mathbf{v}_i^h)\|_{0,T \cap \Omega_i}, \\ &= \left( \frac{\gamma}{\gamma_0} \right)^{-\frac{1}{2}} \|\sigma_{yy}(\mathbf{v}_i^h)\|_{0,T \cap \Omega_i}. \end{aligned}$$

Here and throughout the paper, we use the notation  $a \lesssim b$  to signify that there exists a constant  $C > 0$ , independent of the mesh parameters  $(h, H)$ , the solution and the position of the crack-tip, such that  $a \leq Cb$ .

By summation over all the edges  $E \subset \Gamma_C$  we get

$$(13) \quad \|\gamma^{\frac{1}{2}} R_h(\mathbf{v}^h)\|_{0,\Gamma_C}^2 \lesssim \gamma_0 \|\sigma_{yy}(\mathbf{v}^h)\|_{0,\Omega}^2 \lesssim \gamma_0 \|\mathbf{v}^h\|_{1,\Omega}^2.$$



Hence, when  $\gamma_0$  is small enough, it follows from Korn's inequality and (13), that there exists  $C > 0$  such that for any  $\mathbf{v}^h \in \mathbf{V}^h$

$$a(\mathbf{v}^h, \mathbf{v}^h) - \int_{\Gamma_C} \gamma (R_h(\mathbf{v}^h))^2 d\Gamma \geq C \|\mathbf{v}^h\|_{1,\Omega}^2.$$

The existence of a unique solution to Problem (12) when  $\gamma_0$  is small enough follows from the fact that  $\mathbf{V}^h$  and  $M^{H-}$  are two nonempty closed convex sets,  $\mathcal{L}_\gamma(\cdot, \cdot)$  is continuous on  $\mathbf{V}^h \times W^H$ ,  $\mathcal{L}_\gamma(\mathbf{v}^h, \cdot)$  (resp.  $\mathcal{L}_\gamma(\cdot, \mu^H)$ ) is strictly concave (resp. strictly convex) for any  $\mathbf{v}^h \in \mathbf{V}^h$  (resp. for any  $\mu^H \in M^{H-}$ ) and  $\lim_{\mathbf{v}^h \in \mathbf{V}^h, \|\mathbf{v}^h\|_{\mathbf{V}^h} \rightarrow \infty} \mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = +\infty$  for any  $\mu^H \in M^{H-}$  (resp.  $\lim_{\mu^H \in M^{H-}, \|\mu^H\|_{W^H} \rightarrow \infty} \mathcal{L}_\gamma(\mathbf{v}^h, \mu^H) = -\infty$  for any  $\mathbf{v}^h \in \mathbf{V}^h$ ), see [20, pp. 338-339].

### 3.2 Convergence analysis

First, let us define for any  $\mathbf{v} \in (H^1(\Omega))^2$  and any  $\mu \in L^2(\Gamma_C)$  the following norms:

$$\|\mathbf{v}\| = a(\mathbf{v}, \mathbf{v})^{1/2},$$

$$\|(\mathbf{v}, \mu)\| = \left( \|\mathbf{v}\|^2 + \|\gamma^{1/2} \mu\|_{0,\Gamma_C}^2 \right)^{1/2}.$$

In order to study the convergence error, we recall the definition of the XFEM interpolation operator  $\Pi^h$  introduced in [35].

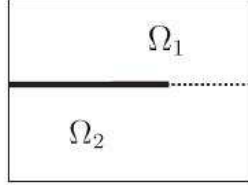


Figure 3: Decomposition of  $\Omega$  into  $\Omega_1$  and  $\Omega_2$ .

We assume that the displacement has the regularity  $(H^2(\Omega))^2$  except in the vicinity of the crack-tip where the singular part of the displacement is a linear combination of the functions  $\{F_j(\mathbf{x})\}_{1 \leq j \leq 4}$  given by (8) (see [18] for a justification). Let us denote by  $\mathbf{u}_s$  the singular part of  $\mathbf{u}$ ,  $\mathbf{u}_r = \mathbf{u} - \chi \mathbf{u}_s$  the regular part of  $\mathbf{u}$ , and  $\mathbf{u}_r^k$  the restriction of  $\mathbf{u}_r$  to  $\Omega_k$ ,  $k \in \{1, 2\}$ . Then, for  $k \in \{1, 2\}$ , there exists an extension  $\tilde{\mathbf{u}}_r^k \in (H^2(\bar{\Omega}))^2$  of  $\mathbf{u}_r^k$  to  $\bar{\Omega}$  such that (see [1])

$$\|\tilde{\mathbf{u}}_r^1\|_{2,\bar{\Omega}} \lesssim \|\mathbf{u}_r^1\|_{2,\Omega_1},$$

$$\|\tilde{\mathbf{u}}_r^2\|_{2,\bar{\Omega}} \lesssim \|\mathbf{u}_r^2\|_{2,\Omega_2}.$$

**Definition 3.1** ([35]). *Given a displacement field  $\mathbf{u}$  satisfying  $\mathbf{u} - \mathbf{u}_s \in H^2(\Omega)$ , and two extensions  $\tilde{\mathbf{u}}_r^1$  and  $\tilde{\mathbf{u}}_r^2$  in  $H^2(\bar{\Omega})$  of  $\mathbf{u}_r^1$  and  $\mathbf{u}_r^2$ , respectively, we define  $\Pi^h \mathbf{u}$  as the element of  $\mathbf{V}^h$  such that*

$$\Pi^h \mathbf{u} = \sum_{i \in \mathcal{N}_h} \mathbf{a}_i \varphi_i + \sum_{i \in \mathcal{N}_h^H} \mathbf{b}_i H \varphi_i + \chi \mathbf{u}_s,$$

where  $\mathbf{a}_i, \mathbf{b}_i$  are given as follows for  $\mathbf{y}_i$  the finite-element node associated to  $\varphi_i$ :

$$\text{if } i \in \{\mathcal{N}_h \setminus \mathcal{N}_h^H\} \text{ then } \mathbf{a}_i = \mathbf{u}_r(\mathbf{y}_i),$$

$$\text{if } i \in \mathcal{N}_h^H \text{ and } \mathbf{y}_i \in \bar{\Omega}_k \text{ for } k \in \{1, 2\} \text{ then for } l = 3 - k :$$

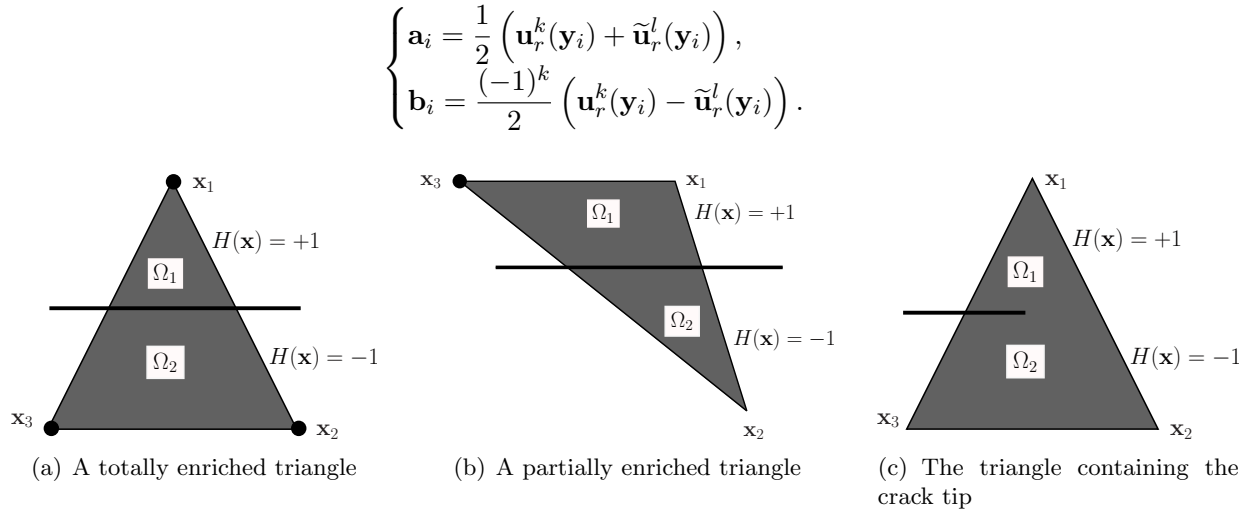


Figure 4: The different types of enriched triangles. The enrichment with the Heaviside function is marked with a bullet.

From this definition, we can distinguish three different kinds of triangle enriched with the Heaviside-like function  $H$ . This is illustrated in Fig. 2 and in Fig. 4. A totally enriched triangle is a triangle whose finite-element shape functions have their supports completely cut by the crack. A partially enriched triangle is a triangle having one or two shape functions whose supports are completely cut by the crack. Finally, the triangle containing the crack tip is a special triangle which is in fact not enriched by the Heaviside-like function. In [35], the following lemma is proved:

**Lemma 3.2.** *The function  $\Pi^h \mathbf{u}$  satisfies*

(i)  $\Pi^h \mathbf{u} = I^h \mathbf{u}_r + \chi \mathbf{u}_s$  over a triangle non-enriched by  $H$ ,

(ii)  $\Pi^h \mathbf{u}|_{T \cap \Omega_k} = I^h \tilde{\mathbf{u}}_r^k + \chi \mathbf{u}_s$  over a triangle  $T$  totally enriched by  $H$ ,

where  $I^h$  denotes the classical Lagrange interpolation operator for the associated finite-element method.

It is also proved in [35] that this XFEM interpolation operator satisfies the following interpolation error estimate:

$$(14) \quad \|\mathbf{u} - \Pi^h \mathbf{u}\| \lesssim h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega},$$

For a triangle  $T$  cut by the crack, we denote by  $E_T^i \mathbf{u}_r$  the polynomial extension of  $\Pi^h \mathbf{u}_r|_{T \cap \Omega_i}$  on  $T$  (i.e. the polynomial  $\Pi^h \mathbf{u}_r|_{T \cap \Omega_i}$  extended to  $T$ ). We will need the following result which gives an interpolation error estimate on the enriched triangles:

**Lemma 3.3.** *Let  $T$  an element such that  $T \cap \Gamma_C \neq \emptyset$ , then for  $i \in \{1, 2\}$  then the following estimates hold:*

$$\begin{aligned} \|\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r\|_{0,T} &\lesssim h_T^2 \left( \|\tilde{\mathbf{u}}_r^i\|_{2,T} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right), \\ \|\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r\|_{1,T} &\lesssim h_T \left( \|\tilde{\mathbf{u}}_r^i\|_{2,T} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right), \end{aligned}$$

where  $h_T$  is the size of triangle  $T$  and  $B(\mathbf{x}^*, h_T)$  is the ball centered at the crack tip  $\mathbf{x}^*$  and with radius  $h_T$ .

The proof of this lemma can be found in Appendix B. Let us now give an abstract error estimate for the discrete contact problem (12).

**Proposition 3.4.** *Assume that the solution  $(\mathbf{u}, \lambda)$  to Problem (6) is such that  $\lambda \in L^2(\Gamma_C)$ . Let  $\gamma_0$  be small enough. Then, the solution  $(\mathbf{u}^h, \lambda^H)$  to Problem (12) satisfies the following estimate*

$$\begin{aligned} \left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\|^2 &\lesssim \left[ \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left( \left\| \left( \mathbf{u} - \mathbf{v}^h, \sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h) \right) \right\|^2 + \|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket v_n^h \rrbracket)\|_{0, \Gamma_C}^2 \right) \right. \\ &\quad + \inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma \\ &\quad \left. + \inf_{\mu^H \in M^{H-}} \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \right]. \end{aligned}$$

**Proof.** (This proof is a straightforward adaptation of the proof in [26]) We have

$$\|\gamma^{1/2}(\lambda - \lambda^H)\|_{0, \Gamma_C}^2 = \int_{\Gamma_C} \gamma \lambda^2 d\Gamma - 2 \int_{\Gamma_C} \gamma \lambda \lambda^H d\Gamma + \int_{\Gamma_C} \gamma (\lambda^H)^2 d\Gamma.$$

From (6) and (12) we obtain

$$\int_{\Gamma_C} \gamma \lambda^2 d\Gamma \leq \int_{\Gamma_C} \gamma \lambda \mu d\Gamma + \int_{\Gamma_C} (\mu - \lambda) \llbracket u_n \rrbracket d\Gamma - \int_{\Gamma_C} \gamma (\mu - \lambda) \sigma_n(\mathbf{u}) d\Gamma, \quad \forall \mu \in M^-,$$

$$\int_{\Gamma_C} \gamma (\lambda^H)^2 d\Gamma \leq \int_{\Gamma_C} \gamma \lambda^H \mu^H d\Gamma + \int_{\Gamma_C} (\mu^H - \lambda^H) \llbracket u_n^h \rrbracket d\Gamma - \int_{\Gamma_C} \gamma (\mu^H - \lambda^H) R_h(\mathbf{u}^h) d\Gamma, \quad \forall \mu^H \in M^{H-},$$

which gives

$$\begin{aligned} \|\gamma^{1/2}(\lambda - \lambda^H)\|_{0, \Gamma_C}^2 &\leq \int_{\Gamma_C} \gamma (\mu - \lambda^H) \lambda d\Gamma + \int_{\Gamma_C} \gamma (\mu^H - \lambda) \lambda^H d\Gamma + \int_{\Gamma_C} (\mu - \lambda) \llbracket u_n \rrbracket d\Gamma \\ &\quad - \int_{\Gamma_C} \gamma (\mu - \lambda) \sigma_n(\mathbf{u}) d\Gamma + \int_{\Gamma_C} (\mu^H - \lambda^H) \llbracket u_n^h \rrbracket d\Gamma - \int_{\Gamma_C} \gamma (\mu^H - \lambda^H) R_h(\mathbf{u}^h) d\Gamma \\ &= \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma + \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ &\quad - \int_{\Gamma_C} \gamma (\lambda^H - \lambda) (\sigma_n(\mathbf{u}) - R_h(\mathbf{u}^h)) d\Gamma \\ (15) \quad &\quad + \int_{\Gamma_C} (\lambda^H - \lambda) (\llbracket u_n \rrbracket - \llbracket u_n^h \rrbracket) d\Gamma, \quad \forall \mu \in M^-, \forall \mu^H \in M^{H-}. \end{aligned}$$

According to (12) for any  $\mathbf{v}^h \in \mathbf{V}^h$  we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^h\|^2 &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{u}^h) \\ &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + a(\mathbf{u} - \mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h) \\ &= a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + \int_{\Gamma_C} (\lambda - \lambda^H) (\llbracket v_n^h \rrbracket - \llbracket u_n^h \rrbracket) d\Gamma \\ (16) \quad &\quad + \int_{\Gamma_C} \gamma (\lambda^H - R_h(\mathbf{u}^h)) R_h(\mathbf{v}^h - \mathbf{u}^h) d\Gamma. \end{aligned}$$

From the addition of (15) and (16), we deduce

$$\begin{aligned}
\left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\|^2 &\leq a(\mathbf{u} - \mathbf{u}^h, \mathbf{u} - \mathbf{v}^h) + \int_{\Gamma_C} (\lambda - \lambda^H) (\llbracket v_n^h \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma \\
&\quad + \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\
(17) \quad &\quad + \int_{\Gamma_C} \gamma(\lambda - \lambda^H) (\sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h)) d\Gamma + \int_{\Gamma_C} \gamma(\lambda - R_h(\mathbf{u}^h)) R_h(\mathbf{v}^h - \mathbf{u}^h) d\Gamma,
\end{aligned}$$

for all  $\mathbf{v}^h \in \mathbf{V}^h$ ,  $\mu \in M^-$  and  $\mu^H \in M^{H-}$ . The last term in the previous inequality is estimated by using (13) and recalling that  $\lambda = \sigma_n(\mathbf{u})$  as follows

$$\begin{aligned}
&\int_{\Gamma_C} \gamma(\lambda - R_h(\mathbf{u}^h)) R_h(\mathbf{v}^h - \mathbf{u}^h) d\Gamma \\
&\leq \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\mathbf{u}^h))\|_{0,\Gamma_C} \gamma_0^{1/2} \|h^{1/2}(R_h(\mathbf{v}^h - \mathbf{u}^h))\|_{0,\Gamma_C} \\
&\lesssim \gamma_0^{1/2} \|\mathbf{v}^h - \mathbf{u}^h\| \left( \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h))\|_{0,\Gamma_C} + \gamma_0^{1/2} \|h^{1/2}(R_h(\mathbf{v}^h - \mathbf{u}^h))\|_{0,\Gamma_C} \right) \\
&\lesssim \left( \gamma_0 \|\mathbf{v}^h - \mathbf{u}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h))\|_{0,\Gamma_C}^2 \right) \\
(18) \quad &\lesssim \left( \gamma_0 \|\mathbf{u} - \mathbf{u}^h\|^2 + \gamma_0 \|\mathbf{u} - \mathbf{v}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h))\|_{0,\Gamma_C}^2 \right).
\end{aligned}$$

By combining (17) and (18), and using Young's inequality we come to the conclusion that if  $\gamma_0$  is sufficiently small then

$$\begin{aligned}
&\left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\|^2 \\
&\lesssim \left[ \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left( \|\mathbf{u} - \mathbf{v}^h\|^2 + \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h))\|_{0,\Gamma_C}^2 + \|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket v_n^h \rrbracket)\|_{0,\Gamma_C}^2 \right) \right. \\
&\quad \left. + \inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma + \inf_{\mu^H \in M^{H-}} \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \right],
\end{aligned}$$

and hence the result follows.  $\square$

In order to estimate the first infimum of the latter proposition, we first recall the following scaled trace inequality (see [21]) for  $T \in \mathcal{T}^h$  and  $E = T \cap \Gamma_C$ :

$$(19) \quad \|v\|_{0,E} \lesssim \left( h_T^{-1/2} \|v\|_{0,T} + h_T^{1/2} \|\nabla v\|_{0,T} \right), \quad \forall v \in H^1(T).$$

We can deduce the following estimate :

$$\begin{aligned}
\|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket (\Pi^h \mathbf{u}) \cdot \mathbf{n} \rrbracket)\|_{0,E} &\leq \|\gamma^{-1/2}(\llbracket \mathbf{u} \rrbracket - \llbracket \Pi^h \mathbf{u} \rrbracket)\|_{0,E}, \\
&\leq \|\gamma^{-1/2}(\mathbf{u}_1 - \Pi^h \mathbf{u}|_{\Omega_1})\|_{0,E} + \|\gamma^{-1/2}(\mathbf{u}_2 - \Pi^h \mathbf{u}|_{\Omega_2})\|_{0,E}, \\
&\leq \|\gamma^{-1/2}(\tilde{\mathbf{u}}_r^1 - \Pi^h \mathbf{u}|_{\Omega_1})\|_{0,E} + \|\gamma^{-1/2}(\tilde{\mathbf{u}}_r^2 - \Pi^h \mathbf{u}|_{\Omega_2})\|_{0,E}, \\
&\lesssim h_T^{-1/2} h_T^{-1/2} \|\tilde{\mathbf{u}}_r^1 - E_T^1 \mathbf{u}_r\|_{0,T} + h_T^{-1/2} h_T^{1/2} \|\nabla \tilde{\mathbf{u}}_r^1 - \nabla E_T^1 \mathbf{u}_r\|_{0,T}, \\
&\quad + h_T^{-1/2} h_T^{-1/2} \|\tilde{\mathbf{u}}_r^2 - E_T^2 \mathbf{u}_r\|_{0,T} + h_T^{-1/2} h_T^{1/2} \|\nabla \tilde{\mathbf{u}}_r^2 - \nabla E_T^2 \mathbf{u}_r\|_{0,T},
\end{aligned}$$

and by using Lemma 3.3 (see Appendix B) we have:

$$\|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket (\Pi^h \mathbf{u}) \cdot \mathbf{n} \rrbracket)\|_{0,E} \lesssim h_T (\|\tilde{\mathbf{u}}_r^1\|_{2,T} + \|\tilde{\mathbf{u}}_r^2\|_{2,T} + \|\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2\|_{2,B(0,h_K)}).$$

By summation over all the edges we obtain

$$(20) \quad \|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket (\Pi^h \mathbf{u}) \cdot \mathbf{n} \rrbracket)\|_{0,\Gamma_C} \lesssim h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}.$$

It remains then to estimate  $\|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u}))\|_{0,\Gamma_C}$ . Still for  $T \in \mathcal{T}^h$  and  $E = T \cap \Gamma_C$ , assuming, without loss of generality, that  $\Gamma_C$  is parallel to the  $x$ -axis and by using the trace inequality (19) we have

$$\begin{aligned} \|\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u})\|_{0,E} &= \|\sigma_n(\mathbf{u}_r) - \sigma_n(\Pi^h \mathbf{u}_r|_{T \cap \Omega_i})\|_{0,E}, \quad \text{with } i \text{ such that } |T \cap \Omega_i| \geq \frac{|T|}{2}, \\ &= \|\sigma_n(\tilde{\mathbf{u}}_r^i - \Pi^h \mathbf{u}_r|_{T \cap \Omega_i})\|_{0,E}, \\ &\lesssim \left( h_T^{-\frac{1}{2}} \|\sigma_{yy}(\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r)\|_{0,T} + h_T^{\frac{1}{2}} \|\nabla \sigma_{yy}(\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r)\|_{0,T} \right), \\ &= \left( h_T^{-\frac{1}{2}} \|\sigma_{yy}(\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r)\|_{0,T} + h_T^{\frac{1}{2}} \|\nabla \sigma_{yy}(\tilde{\mathbf{u}}_r^i)\|_{0,T} \right), \\ &\lesssim \left( h_T^{-\frac{1}{2}} \|\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r\|_{1,T} + h_T^{\frac{1}{2}} \|\tilde{\mathbf{u}}_r^i\|_{2,T} \right). \end{aligned}$$

Then, by summation over all the edges and using again Lemma 3.3 the following estimate holds

$$(21) \quad \|\gamma^{1/2}(\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u}))\|_{0,\Gamma_C} \lesssim h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}.$$

Putting together the previous bounds (14), (20) and (21) we deduce that

$$(22) \quad \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left( \left\| \left( \mathbf{u} - \mathbf{v}^h, \sigma_n(\mathbf{u}) - R_h(\mathbf{v}^h) \right) \right\|^2 + \|\gamma^{-1/2}(\llbracket u_n \rrbracket - \llbracket v_n^h \rrbracket)\|_{0,\Gamma_C}^2 \right) \lesssim h^2 \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}^2.$$

Finally, we have to estimate the error terms in Proposition 3.4 coming from the contact approximation:

$$(23) \quad \inf_{\mu^H \in M^{H-}} \int_{\Gamma_C} (\mu^H - \lambda)(\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma$$

and

$$(24) \quad \inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma.$$

In order to estimate these terms, we need to distinguish the different contact conditions (i.e., we must specify the definition of  $M^{H-}$ ). We consider hereafter three different standard discrete contact conditions.

### 3.2.1 First contact condition: $M^{H-} = M_0^{H-}$

We first consider the case of nonpositive discontinuous piecewise constant multipliers where  $M^{H-}$  is defined by (9). The error estimate is given next.

**Theorem 3.5.** *Let  $(\mathbf{u}, \lambda)$  be the solution to Problem (6). Assume that  $\mathbf{u}_r \in (H^2(\Omega))^2$ . Let  $\gamma_0$  be small enough and let  $(\mathbf{u}^h, \lambda^H)$  be the solution to the discrete problem (12) where  $M^{H-} = M_0^{H-}$ . Then, for any  $\eta > 0$  we have*

$$\left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\| \lesssim (h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega} + h^{1/2} H^{1/2} \|\lambda\|_{1/2,\Gamma_C} + H^{3/4-\eta/2} (\|\mathbf{u}\|_{3/2-\eta,\Omega} + \|\lambda\|_{1/2,\Gamma_C})).$$

**Proof.** Choosing  $\mu = 0$  in (24) yields

$$\inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma \leq - \int_{\Gamma_C} \lambda^H \llbracket u_n \rrbracket d\Gamma \leq 0.$$

In (23) we choose  $\mu^H = \pi_0^H \lambda$  where  $\pi_0^H$  denotes the  $L^2(\Gamma_C)$ -projection onto  $W_0^H$ . We recall that the operator  $\pi_0^H$  is defined for any  $v \in L^2(\Gamma_C)$  by

$$\pi_0^H v \in W_0^H, \quad \int_{\Gamma_C} (v - \pi_0^H v) \mu d\Gamma = 0, \quad \forall \mu \in W_0^H,$$

and satisfies the following error estimates for any  $0 \leq r \leq 1$  (see [8])

$$(25) \quad H^{-1/2} \|v - \pi_0^H v\|_{-1/2, \Gamma_C} + \|v - \pi_0^H v\|_{0, \Gamma_C} \lesssim H^r \|v\|_{r, \Gamma_C}.$$

Clearly,  $\pi_0^H \lambda \in M_0^{H^-}$  and

$$(26) \quad \begin{aligned} \inf_{\mu^H \in M_0^{H^-}} \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma &\leq \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma \\ &+ \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda) (\lambda^H - R_h(\mathbf{u}^h)) d\Gamma. \end{aligned}$$

The first integral term in (26) is estimated using (25) as follows

$$\begin{aligned} \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma &= \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) (\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) \llbracket u_n \rrbracket d\Gamma \\ &= \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) (\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} (\pi_0^H \lambda - \lambda) (\llbracket u_n \rrbracket - \pi_0^H \llbracket u_n \rrbracket) d\Gamma \\ &\leq \|\pi_0^H \lambda - \lambda\|_{-1/2, \Gamma_C} \|\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket\|_{1/2, \Gamma_C} + \|\pi_0^H \lambda - \lambda\|_{0, \Gamma_C} \|\llbracket u_n \rrbracket - \pi_0^H \llbracket u_n \rrbracket\|_{0, \Gamma_C} \\ &\lesssim H \|\lambda\|_{1/2, \Gamma_C} \|\mathbf{u} - \mathbf{u}^h\| + H^{3/2-\eta} \|\lambda\|_{1/2, \Gamma_C} \|\llbracket u_n \rrbracket\|_{1-\eta, \Gamma_C} \end{aligned}$$

Therefore, for any  $\alpha > 0$  we have

$$(27) \quad \begin{aligned} &\int_{\Gamma_C} (\pi_0^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma \\ &\lesssim \alpha \|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha^{-1} H^2 \|\lambda\|_{1/2, \Gamma_C}^2 + \alpha^{-1} H^{3/2-\eta} \|\lambda\|_{1/2, \Gamma_C}^2 + \alpha H^{3/2-\eta} \|\mathbf{u}\|_{3/2-\eta, \Omega}^2. \end{aligned}$$

For the second integral term in (26), by using the estimates (25), (13), (21), we have

$$\begin{aligned} \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda) (\lambda^H - R_h(\mathbf{u}^h)) d\Gamma &= \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda) (\lambda^H - \lambda) d\Gamma \\ &+ \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda) (\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u})) d\Gamma \\ &+ \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda) (R_h(\Pi^h \mathbf{u}) - R_h(\mathbf{u}^h)) d\Gamma \\ &\lesssim \gamma_0^{1/2} h^{1/2} \|\pi_0^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0, \Gamma_C} \\ &+ \gamma_0^{1/2} h^{1/2} \|\pi_0^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} (\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u}))\|_{0, \Gamma_C} \\ &+ \gamma_0^{1/2} h^{1/2} \|\pi_0^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} R_h(\Pi^h \mathbf{u} - \mathbf{u}^h)\|_{0, \Gamma_C} \\ &\lesssim \gamma_0^{1/2} h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0, \Gamma_C} \\ &+ \gamma_0^{1/2} h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2, \Omega} \\ &+ \gamma_0 h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} \|\mathbf{u}^h - \Pi^h \mathbf{u}\|. \end{aligned}$$

Since  $\|\mathbf{u}^h - \Pi^h \mathbf{u}\| \lesssim \|\mathbf{u} - \mathbf{u}^h\| + h\|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}$ , for any  $\alpha > 0$  sufficiently small, we deduce

$$(28) \quad \begin{aligned} & \int_{\Gamma_C} \gamma(\pi_0^H \lambda - \lambda)(\lambda^H - R_h(\mathbf{u}^h)) d\Gamma \\ & \lesssim \alpha \|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C}^2 + \alpha h^2 \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}^2 + \alpha^{-1} h H \|\lambda\|_{1/2,\Gamma_C}^2. \end{aligned}$$

Then, by using the inequalities (22), (26), (27), (28) and Proposition 3.4 the proof of the theorem follows.  $\square$

### 3.2.2 Second contact condition: $M^{H-} = M_1^{H-}$

Now, we focus on the case of nonpositive continuous piecewise affine multipliers where  $M^{H-}$  is given by (10).

**Theorem 3.6.** *Let  $(\mathbf{u}, \lambda)$  be the solution to Problem (6). Assume that  $\mathbf{u}_r \in (H^2(\Omega))^2$ . Let  $\gamma_0$  be small enough and let  $(\mathbf{u}^h, \lambda^H)$  be the solution to the discrete problem (12) where  $M^{H-} = M_1^{H-}$ . Then, we have for any  $\eta > 0$*

$$\left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\| \lesssim h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega} + (H^{\frac{1-\eta}{2}} + h^{1/2}) \|\lambda\|_{1/2,\Gamma_C} + H^{\frac{1-\eta}{2}} \|\mathbf{u}\|_{3/2-\eta,\Omega}.$$

**Proof.** We choose  $\mu = 0$  in (24) which implies

$$\inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma \leq - \int_{\Gamma_C} \lambda^H \llbracket u_n \rrbracket d\Gamma \leq 0.$$

In the infimum (23) we choose  $\mu^H = 0$ . So

$$(29) \quad \begin{aligned} & \inf_{\mu^H \in M_1^{H-}} \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ & \leq - \int_{\Gamma_C} \lambda (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ & = - \int_{\Gamma_C} \lambda r^H (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ & \quad - \int_{\Gamma_C} \lambda (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) - r^H (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ & \leq - \int_{\Gamma_C} \lambda (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) - r^H (\llbracket u_n^h \rrbracket + \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ & = \int_{\Gamma_C} \lambda (r^H \llbracket u_n^h \rrbracket - \llbracket u_n^h \rrbracket) d\Gamma + \int_{\Gamma_C} \lambda (r^H (\gamma(\lambda^H - R_h(\mathbf{u}^h))) - \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma, \end{aligned}$$

where  $r^H : L^1(\Gamma_C) \mapsto W_1^H$  is a quasi-interpolation operator which preserves the nonpositivity defined for any function  $v$  in  $L^1(\Gamma_C)$  by

$$r^H v = \sum_{\mathbf{x} \in N^H} \alpha_{\mathbf{x}}(v) \psi_{\mathbf{x}},$$

where  $N^H$  represents the set of nodes  $\mathbf{x}_0, \dots, \mathbf{x}_N$  in  $\overline{\Gamma_C}$ ,  $\psi_{\mathbf{x}}$  is the scalar basis function of  $W_1^H$  (defined on  $\overline{\Gamma_C}$ ) at node  $\mathbf{x}$  satisfying  $\psi_{\mathbf{x}}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$  for all  $\mathbf{x}' \in N^H$  and

$$\alpha_{\mathbf{x}}(v) = \left( \int_{\Gamma_C} v \psi_{\mathbf{x}} d\Gamma \right) \left( \int_{\Gamma_C} \psi_{\mathbf{x}} d\Gamma \right)^{-1}.$$

The approximation properties of  $r^H$  are proved in [25]. We simply recall hereafter the two main results without proof. The first result is concerned with  $L^2$ -stability property of  $r^H$ .

**Lemma 3.7.** *For any  $v \in L^2(\Gamma_C)$  and any  $E \in T^H$  we have*

$$\|r^H v\|_{0,E} \lesssim \|v\|_{0,\gamma_E},$$

where  $\gamma_E = \cup_{\{F \in T^H: \bar{F} \cap \bar{E} \neq \emptyset\}} \bar{F}$ .

Note that the proof of this lemma given in [25] uses the assumption that the mesh  $T^H$  is quasi-uniform (the quasi uniformity is needed in [25] since inverse inequalities were used). A straightforward calculation shows that the quasi-uniformity assumption is not necessary to obtain the  $L^2$ -stability. The second result is concerned with the  $L^2$ -approximation properties of  $r^H$ .

**Lemma 3.8.** *For any  $v \in H^\eta(\Gamma_C)$ ,  $0 \leq \eta \leq 1$ , and any  $E \in T^H$  we have*

$$(30) \quad \|v - r^H v\|_{0,E} \lesssim H^\eta \|v\|_{\eta,\gamma_E},$$

where  $\gamma_E = \cup_{\{F \in E^H: \bar{F} \cap \bar{E} \neq \emptyset\}} \bar{F}$ .

Consequently, the first integral term in (29) is estimated using (30) as follows

$$\begin{aligned} \int_{\Gamma_C} \lambda(r^H \llbracket u_n^h \rrbracket - \llbracket u_n^h \rrbracket) d\Gamma &\leq \int_{\Gamma_C} \lambda(r^H(\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket) - (\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket)) d\Gamma + \int_{\Gamma_C} \lambda(r^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket) d\Gamma, \\ &\lesssim \|\lambda\|_{0,\Gamma_C} H^{1/2} \|\mathbf{u} - \mathbf{u}^h\| + \|\lambda\|_{0,\Gamma_C} H^{1-\eta} \|\llbracket u_n \rrbracket\|_{1-\eta,\Gamma_C}, \\ &\lesssim \|\lambda\|_{1/2,\Gamma_C} H^{1/2} \|\mathbf{u} - \mathbf{u}^h\| + \|\lambda\|_{1/2,\Gamma_C} H^{1-\eta} \|\llbracket u_n \rrbracket\|_{1-\eta,\Gamma_C}, \\ &\lesssim H^{1/2} \|\lambda\|_{1/2,\Gamma_C} \|\mathbf{u} - \mathbf{u}^h\| + H^{\frac{1-\eta}{2}} \|\lambda\|_{1/2,\Gamma_C} H^{\frac{1-\eta}{2}} \|\llbracket u_n \rrbracket\|_{1-\eta,\Gamma_C}. \end{aligned}$$

Therefore, for any  $\alpha > 0$  we write

$$(31) \quad \begin{aligned} &\int_{\Gamma_C} \lambda(r^H \llbracket u_n^h \rrbracket - \llbracket u_n^h \rrbracket) d\Gamma \\ &\lesssim \alpha \|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha H^{1-\eta} \|\mathbf{u}\|_{3/2-\eta,\Omega}^2 + \alpha^{-1} (H^{1-\eta} + H) \|\lambda\|_{1/2,\Gamma_C}^2. \end{aligned}$$

Now, we consider the second integral term in (29):

$$\begin{aligned} &\int_{\Gamma_C} \lambda(r^H(\gamma(\lambda^H - R_h(\mathbf{u}^h))) - \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ &\leq \|\lambda\|_{0,\Gamma_C} \|r^H(\gamma(\lambda^H - R_h(\mathbf{u}^h))) - \gamma(\lambda^H - R_h(\mathbf{u}^h))\|_{0,\Gamma_C} \\ &\lesssim \|\lambda\|_{0,\Gamma_C} \|\gamma(\lambda^H - R_h(\mathbf{u}^h))\|_{0,\Gamma_C} \\ &\lesssim \gamma_0^{1/2} h^{1/2} \|\lambda\|_{0,\Gamma_C} \left\| \gamma^{1/2} \left( (\lambda^H - \lambda) + \sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u}) + R_h(\Pi^h \mathbf{u} - \mathbf{u}^h) \right) \right\|_{0,\Gamma_C} \\ &\lesssim \gamma_0^{1/2} h^{1/2} \|\lambda\|_{1/2,\Gamma_C} \left( \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C} + h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega} + \gamma_0^{1/2} \|\mathbf{u} - \mathbf{u}^h\| \right). \end{aligned}$$

As a consequence, for any  $\alpha > 0$  we have

$$(32) \quad \begin{aligned} &\int_{\Gamma_C} \lambda(r^H(\gamma(\lambda^H - R_h(\mathbf{u}^h))) - \gamma(\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ &\lesssim \alpha (\|\mathbf{u} - \mathbf{u}^h\|^2 + \|\gamma^{1/2}(\lambda^H - \lambda)\|_{0,\Gamma_C}^2) + \alpha h^2 \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega}^2 + \alpha^{-1} h \|\lambda\|_{1/2,\Gamma_C}^2. \end{aligned}$$

The proof of the theorem then follows by using the inequalities (22), (29), (31), (32) and Proposition 3.4.  $\square$



### 3.2.3 Third contact condition: $M^{H-} = M_{1,*}^{H-}$

This choice corresponds to “weakly nonpositive” continuous piecewise affine multipliers where  $M^{H-}$  is given by (11).

**Theorem 3.9.** *Let  $(\mathbf{u}, \lambda)$  be the solution to Problem (6). Assume that  $\mathbf{u}_r \in (H^2(\Omega))^2$ . Let  $\gamma_0$  be small enough and let  $(\mathbf{u}^h, \lambda^H)$  be the solution to the discrete problem (12) where  $M^{H-} = M_{1,*}^{H-}$ . Then, for any  $\eta > 0$  we have*

$$\left\| \left( \mathbf{u} - \mathbf{u}^h, \lambda - \lambda^H \right) \right\| \lesssim h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2,\Omega} + (h^{1/2} + H^{3/2-\eta}) \|\lambda\|_{1/2,\Gamma_C} + h^{-1/2} H^{1-\eta} \|\mathbf{u}\|_{3/2-\eta,\Omega}.$$

**Proof.** By setting  $\mu = 0$  in (24) we obtain

$$\begin{aligned} \inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma &\leq - \int_{\Gamma_C} \lambda^H \llbracket u_n \rrbracket d\Gamma, \\ &= \int_{\Gamma_C} \lambda^H (I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket) d\Gamma - \int_{\Gamma_C} \lambda^H I^H \llbracket u_n \rrbracket d\Gamma, \\ &\leq \int_{\Gamma_C} \lambda^H (I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket) d\Gamma, \\ &= \int_{\Gamma_C} (\lambda^H - \lambda) (I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} \lambda (I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket) d\Gamma, \\ &\leq \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0,\Gamma_C} \|\gamma^{-1/2} (I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket)\|_{0,\Gamma_C}, \\ &\quad + \|\lambda\|_{0,\Gamma_C} \|I^H \llbracket u_n \rrbracket - \llbracket u_n \rrbracket\|_{0,\Gamma_C}, \\ &\lesssim H^{1-\eta} h^{-1/2} \|\mathbf{u}\|_{3/2-\eta,\Omega} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0,\Gamma_C} + \|\lambda\|_{1/2,\Gamma_C} H^{1-\eta} \|\mathbf{u}\|_{3/2-\eta,\Omega}, \end{aligned}$$

where  $I^H$  is the Lagrange interpolation operator onto  $W_1^H$ . The operator  $I^H$  is defined for any  $v \in \mathcal{C}(\Gamma_C)$  and satisfies the following error estimates for any  $1/2 < r \leq 2$ :

$$\|v - I^H v\|_{0,\Gamma_C} \lesssim H^r \|v\|_{r,\Gamma_C}.$$

Therefore, for any  $\alpha > 0$  we have

$$(33) \quad \begin{aligned} &\inf_{\mu \in M^-} \int_{\Gamma_C} (\mu - \lambda^H) \llbracket u_n \rrbracket d\Gamma \\ &\lesssim \alpha h^{-1} H^{2(1-\eta)} \|\mathbf{u}\|_{3/2-\eta,\Omega}^2 + \alpha^{-1} (\|\gamma^{1/2} (\lambda^H - \lambda)\|_{0,\Gamma_C}^2 + h \|\lambda\|_{1/2,\Gamma_C}^2) \end{aligned}$$

In the infimum (23) we choose  $\mu^H = \pi_1^H \lambda$  where  $\pi_1^H$  denotes the  $L^2(\Gamma_C)$ -projection onto  $W_1^H$ . The operator  $\pi_1^H$  is defined for any  $v \in L^2(\Gamma_C)$  by

$$\pi_1^H v \in W_1^H, \quad \int_{\Gamma_C} (v - \pi_1^H v) \mu d\Gamma = 0, \quad \forall \mu \in W_1^H,$$

and satisfies, for any  $0 \leq r \leq 2$ , the following error estimates

$$(34) \quad H^{-1/2} \|v - \pi_1^H v\|_{-1/2,\Gamma_C} + \|v - \pi_1^H v\|_{0,\Gamma_C} \leq C H^r \|v\|_{r,\Gamma_C}.$$

Clearly  $\pi_1^H \lambda \in M_{1,*}^{H-}$ , so that

$$(35) \quad \begin{aligned} &\inf_{\mu^H \in M_{1,*}^{H-}} \int_{\Gamma_C} (\mu^H - \lambda) (\llbracket u_n^h \rrbracket + \gamma (\lambda^H - R_h(\mathbf{u}^h))) d\Gamma \\ &\leq \int_{\Gamma_C} (\pi_1^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma + \int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (\lambda^H - R_h(\mathbf{u}^h)) d\Gamma. \end{aligned}$$

The first integral term in (35) is estimated using (34) as follows

$$\begin{aligned}
\int_{\Gamma_C} (\pi_1^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma &= \int_{\Gamma_C} (\pi_1^H \lambda - \lambda) (\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} (\pi_1^H \lambda - \lambda) \llbracket u_n \rrbracket d\Gamma, \\
&= \int_{\Gamma_C} (\pi_1^H \lambda - \lambda) (\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket) d\Gamma + \int_{\Gamma_C} (\pi_1^H \lambda - \lambda) (\llbracket u_n \rrbracket - \pi_1^H \llbracket u_n \rrbracket) d\Gamma, \\
&\leq \|\pi_1^H \lambda - \lambda\|_{-1/2, \Gamma_C} \|\llbracket u_n^h \rrbracket - \llbracket u_n \rrbracket\|_{1/2, \Gamma_C} + \|\pi_1^H \lambda - \lambda\|_{0, \Gamma_C} \|\llbracket u_n \rrbracket - \pi_1^H \llbracket u_n \rrbracket\|_{0, \Gamma_C}, \\
&\lesssim H \|\lambda\|_{1/2, \Gamma_C} \|\mathbf{u} - \mathbf{u}^h\| + H^{1/2} \|\lambda\|_{1/2, \Gamma_C} H^{1-\eta} \|\mathbf{u}\|_{3/2-\eta, \Omega}.
\end{aligned}$$

Therefore, for any  $\alpha > 0$ , we have

$$\begin{aligned}
&\int_{\Gamma_C} (\pi_1^H \lambda - \lambda) \llbracket u_n^h \rrbracket d\Gamma \\
(36) \quad &\lesssim \alpha \left( \|\mathbf{u} - \mathbf{u}^h\|^2 + H^{3/2-\eta} \|\mathbf{u}\|_{3/2-\eta, \Omega}^2 \right) + \alpha^{-1} (H^2 + H^{3/2-\eta}) \|\lambda\|_{1/2, \Gamma_C}^2.
\end{aligned}$$

For the second integral term in (35) by using the bounds given in (34), (13), (21) we get

$$\begin{aligned}
\int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (\lambda^H - R_h(\mathbf{u}^h)) d\Gamma &= \int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (\lambda^H - \lambda) d\Gamma \\
&+ \int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u})) d\Gamma \\
&+ \int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (R_h(\Pi^h \mathbf{u}) - R_h(\mathbf{u}^h)) d\Gamma \\
&\lesssim \gamma_0^{1/2} h^{1/2} \|\pi_1^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0, \Gamma_C} \\
&+ \gamma_0^{1/2} h^{1/2} \|\pi_1^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} (\sigma_n(\mathbf{u}) - R_h(\Pi^h \mathbf{u}))\|_{0, \Gamma_C} \\
&+ \gamma_0^{1/2} h^{1/2} \|\pi_1^H \lambda - \lambda\|_{0, \Gamma_C} \|\gamma^{1/2} R_h(\Pi^h \mathbf{u} - \mathbf{u}^h)\|_{0, \Gamma_C} \\
&\lesssim \gamma_0^{1/2} h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0, \Gamma_C} \\
&+ \gamma_0^{1/2} h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} h \|\mathbf{u} - \chi \mathbf{u}_s\|_{2, \Omega} \\
&+ \gamma_0 h^{1/2} H^{1/2} \|\lambda\|_{1/2, \Gamma_C} \|\mathbf{u}^h - \Pi^h \mathbf{u}\|.
\end{aligned}$$

Since  $\|\mathbf{u}^h - \Pi^h \mathbf{u}\| \leq \|\mathbf{u} - \mathbf{u}^h\| + Ch \|\mathbf{u} - \chi \mathbf{u}_s\|_{2, \Omega}$ , for any small  $\alpha > 0$  we get

$$\begin{aligned}
&\int_{\Gamma_C} \gamma (\pi_1^H \lambda - \lambda) (\lambda^H - R_h(\mathbf{u}^h)) d\Gamma \\
(37) \quad &\lesssim \alpha \|\mathbf{u} - \mathbf{u}^h\|^2 + \alpha \|\gamma^{1/2} (\lambda^H - \lambda)\|_{0, \Gamma_C}^2 + \alpha h^2 \|\mathbf{u} - \chi \mathbf{u}_s\|_{2, \Omega}^2 + \alpha^{-1} h H \|\lambda\|_{1/2, \Gamma_C}^2.
\end{aligned}$$

Finally, the theorem is established by combining Proposition 3.4 and the inequalities (33), (35), (36), (37) and (22).  $\square$

## 4 Numerical experiments

The numerical tests are performed on a non-cracked square defined by

$$\bar{\Omega} = [0, 1] \times [-0.5, 0.5],$$

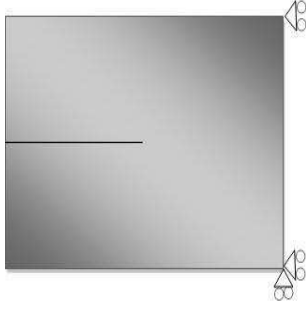


Figure 5: Cracked specimen.

and the considered crack is the line segment  $\Gamma_C = ]0, 0.5[ \times \{0\}$  (see Fig. 5). Three degrees of freedom are blocked in order to eliminate the rigid body motions (Fig. 5). In order to have both a contact zone and a non contact zone between the crack lips, we impose the following body force vector field

$$\mathbf{f}(x, y) = \begin{pmatrix} 0 \\ 3.5x(1-x)y \cos(2\pi x) \end{pmatrix}.$$

Neumann boundary conditions are prescribed as follows:

$$\begin{aligned} \mathbf{g}(0, y) = \mathbf{g}(1, y) &= \begin{pmatrix} 0 \\ 4 \cdot 10^{-2} \sin(2\pi y) \end{pmatrix} & -0.5 \leq y \leq 0.5, \\ \mathbf{g}(x, -0.5) = \mathbf{g}(x, 0.5) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 \leq x \leq 1. \end{aligned}$$

An example of a non structured mesh used is presented in Fig. 6. The Numerical tests are performed with GETFEM++, the C++ finite-element library developed by our team (see [37]).

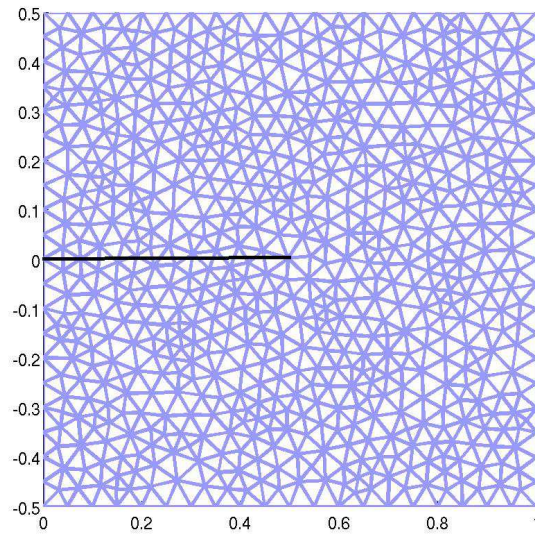


Figure 6: Non-structured mesh.

## 4.1 Numerical solving

The algebraic formulation of Problem (12) is given as follows

$$(38) \quad \begin{cases} \text{Find } U \in \mathbb{R}^N \text{ and } L \in \overline{M}^{H^-} \text{ such that} \\ (K - K_\gamma)U - (B - C_\gamma)^T L = F, \\ (\overline{L} - L)^T ((B - C_\gamma)U + D_\gamma L) \geq 0, \quad \forall \overline{L} \in \overline{M}^{H^-}, \end{cases}$$

where  $U$  is the vector of degrees of freedom (d.o.f.) for  $\mathbf{u}^h$ ,  $L$  is the vector of d.o.f. for the multiplier  $\lambda^H$ ,  $\overline{M}^{H^-}$  is the set of vectors  $L$  such that the corresponding multiplier lies in  $M^{H^-}$ ,  $K$  is the classical stiffness matrix coming from the term  $a(\mathbf{u}^h, \mathbf{v}^h)$ ,  $F$  is the right-hand side corresponding to the Neumann boundary condition and the volume forces, and  $B, K_\gamma, C_\gamma, D_\gamma$  are the matrices corresponding to the terms  $b(\lambda^H, \mathbf{v}^h)$ ,  $\int_{\Gamma_C} \gamma R_h(\mathbf{u}^h) R_h(\mathbf{v}^h) d\Gamma$ ,  $\int_{\Gamma_C} \gamma \lambda^H R_h(\mathbf{v}^h) d\Gamma$ ,  $\int_{\Gamma_C} \gamma \lambda^H \mu^H d\Gamma$ , respectively.

The inequality in (38) can be expressed as an equivalent projection

$$(39) \quad L = P_{\overline{M}^{H^-}}(L - r((B - C_\gamma)U + D_\gamma L)),$$

where  $r$  is a positive augmentation parameter. This last step transforms the contact condition into a nonlinear equation and we have to solve the following system:

$$(40) \quad \begin{cases} \text{Find } U \in \mathbb{R}^N \text{ and } L \in \overline{M}^{H^-} \text{ such that} \\ (K - K_\gamma)U - (B - C_\gamma)^T L - F = 0, \\ -\frac{1}{r} \left[ L - P_{\overline{M}^{H^-}}(L - r((B - C_\gamma)U + D_\gamma L)) \right] = 0. \end{cases}$$

This allows us to use the semi-smooth Newton method (introduced for contact and friction problems in [2]) to solve Problem (40). The term ‘semi-smooth’ comes from the fact that projections are only piecewise differentiable. Practically, it is one of the most robust algorithms to solve contact problems with or without friction. In order to write a Newton step, one has to compute the derivative of the projection (39). An analytical expression can only be obtained when the projection itself is simple to express. This is the case for instance when the set  $M^{H^-}$  is chosen to be the set of multipliers having non-positive values on each finite-element node of the contact boundary (such as  $M_0^{H^-}$  or  $M_1^{H^-}$ ). In this case, the projection can be expressed component-wise (see [28]).

In order to keep the independence between the mesh and the crack, the approximation space  $W^H$  for the multiplier is chosen to be the trace on  $\Gamma_C$  of a Lagrange finite-element method defined on the same mesh as  $\mathbf{V}^h$  (in that sense  $H = h$ ) and its degree will be specified in the following. Let us denote  $X^h$  the space corresponding to the Lagrange finite-element method. The choice of a basis of the trace space  $W^H = X^h|_{\Gamma_C}$  is not completely straightforward. Indeed, the traces on  $\Gamma_C$  of the shape functions of  $X^h$  may be linearly dependent. A way to overcome this difficulty is to eliminate the redundant functions. Our approach in the presented numerical experiments is as follows. In a first time, we eliminate locally dependent columns of the mass matrix  $\int_{\Gamma_C} \psi_i \psi_j d\Gamma$ , where  $\psi_i$  is the finite-element shape functions of  $X^h$ , with a block-wise Gram-Schmidt algorithm. In a second time, we detect the potential remaining kernel of the mass matrix with a Lanczos algorithm.

## 4.2 Numerical tests

In this section, we present numerical tests of the stabilized and non stabilized unilateral contact problem for the following, differently enriched, finite-element methods:  $P_2/P_1, P_2/P_0, P_1 + /P_1,$

$P_1/P_1$ ,  $P_1/P_0$ . The notation  $P_i/P_j$  (resp.  $P_1+/P_1$ ) means that the displacement is approximated with a  $P_i$  extended finite-element method (resp. a  $P_1$  extended finite-element method with an additional cubic bubble function) and the multiplier with a continuous  $P_j$  finite-element method for  $j > 0$  (resp. continuous  $P_1$  finite-element method).

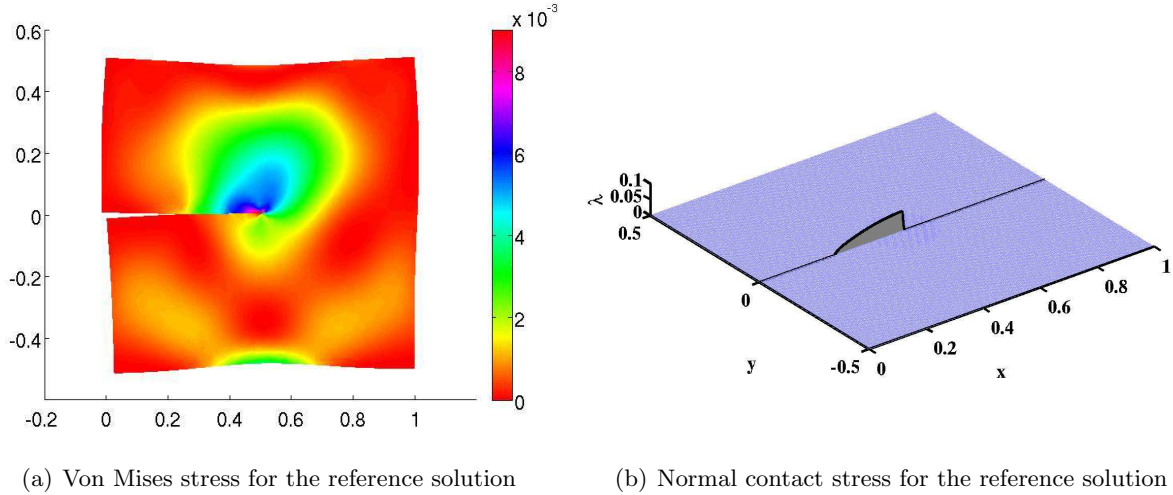
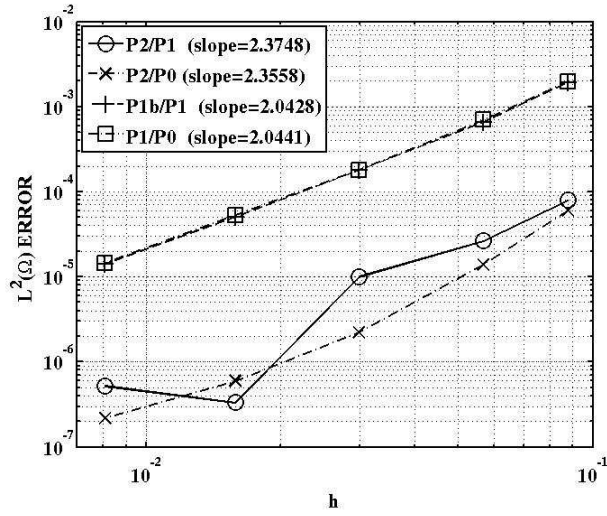


Figure 7: Von Mises stress and normal contact stress for the reference solution

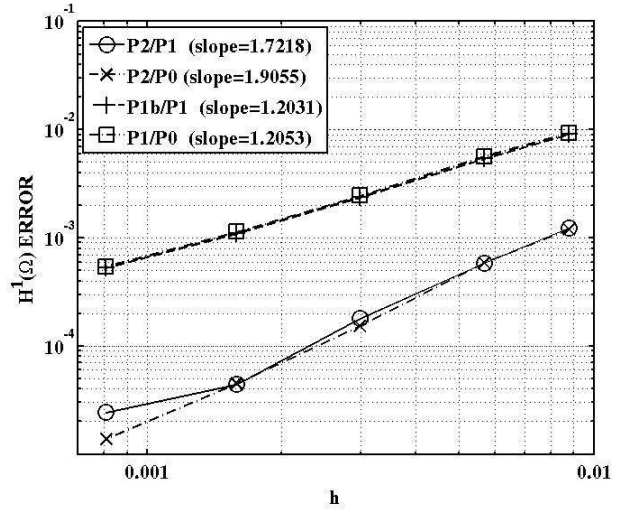
The numerical tests are performed on non-structured meshes with  $h = 0.088, 0.057, 0.03, 0.016, 0.008$  respectively. The reference solution is obtained with a structured  $P_2/P_1$  method and  $h = 0.0027$ . The von Mises stress of the reference solution is presented in Fig. 7(a). Its distribution shows that the von Mises stress is not singular at the crack lips. The normal contact stress of the reference solution is presented in Fig. 7(b). The normal contact stress is not singular at the crack lips which confirms the theoretical result presented in Lemma A.1.

**Without stabilization:** The curves in the non stabilized case are given in Fig. 8(a) for the error in the  $L^2(\Omega)$ -norm on the displacement, in Fig. 8(b) for the error in the  $H^1(\Omega)$ -norm on the displacement and in Fig. 8(c) for the error in the  $L^2(\Gamma_C)$ -norm on the contact stress. The  $P_1/P_1$  method is not plotted because it does not work without stabilization. The  $P_2/P_1$  and  $P_1/P_0$  versions generally work without stabilization even though a uniform inf-sup condition cannot be proven. Fig. 8(a) shows that the rate of convergence in the error  $L^2(\Omega)$ -norm is of order 2.4 for the  $P_2/P_j$  methods and of order 2 for the  $P_1/P_j$  methods. This rate of convergence is close to optimality because the singularity due to the transition between contact and non contact is expected to be in  $H^{5/2-\eta}(\Omega)$  for any  $\eta > 0$ . Theoretically, this limits the convergence rate to  $3/2 - \eta$  in the  $H^1(\Omega)$ -norm. Fig. 8(b) shows that the rate of convergence in energy norm is optimal for all pairs of elements considered. Fig. 8(c) shows that, except the  $P_1/P_0$  method, the rate of convergence in the  $L^2(\Gamma_C)$ -norm is optimal but there are very large oscillations. For the  $P_1/P_0$  method the rate of convergence in the  $L^2(\Gamma_C)$ -norm is not optimal (of order 0.42). It seems that the presence of some spurious modes affects this rate of convergence.

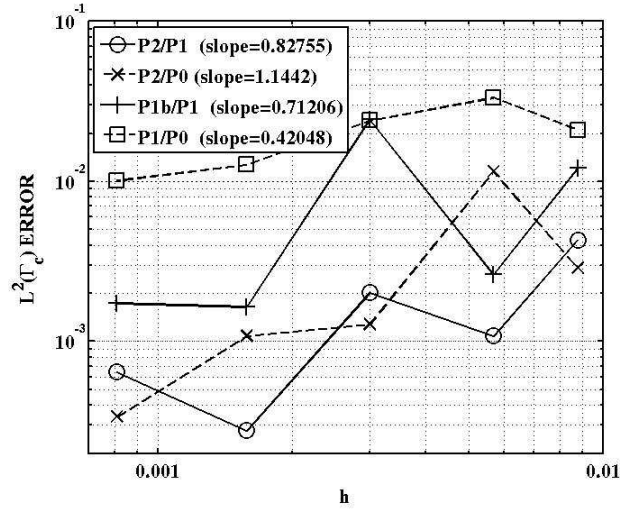
**Stabilized method:** The curves in the stabilized case are given in Fig. 9(a) for the error in the  $L^2(\Omega)$ -norm on the displacement, in Fig. 9(b) for the error in the  $H^1(\Omega)$ -norm on the displacement and in Fig. 9(c) for the error in the  $L^2(\Gamma_C)$ -norm of the contact stress. Similarly to the non stabilized method, Fig. 9(b) shows that we have an optimal rate of convergence, with a slight difference, for the error in the  $H^1(\Omega)$ -norm on the displacement. Concerning the error in the  $L^2(\Omega)$ -norm the rate of convergence is affected by the stabilization for the quadratic



(a) Error in  $L^2(\Omega)$ -norm of the displacement

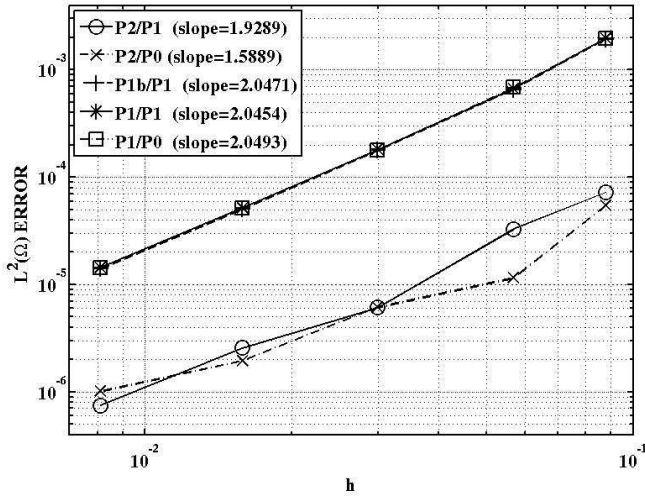


(b) Error in  $H^1(\Omega)$ -norm of the displacement

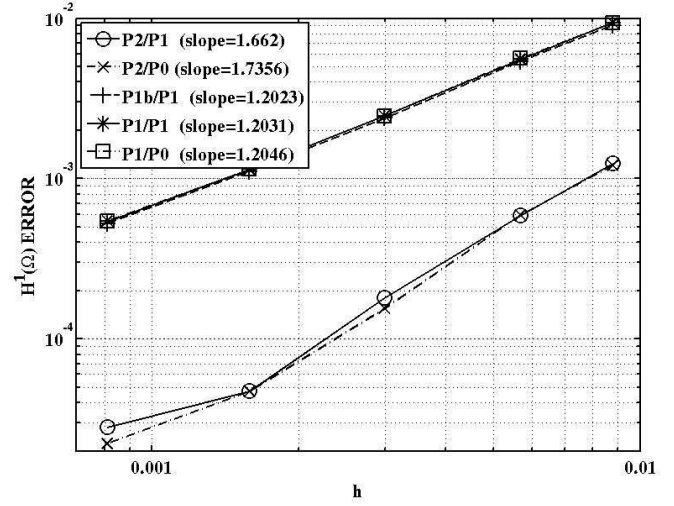


(c) Error in  $L^2(\Gamma_C)$ -norm of the contact stress

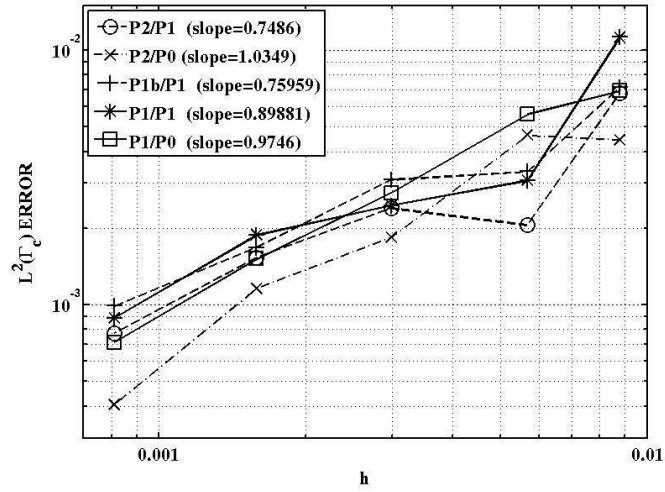
Figure 8: Convergence curves in the non stabilized case



(a) Error in  $L^2(\Omega)$ -norm of the displacement



(b) Error in  $H^1(\Omega)$ -norm of the displacement



(c) Error in  $L^2(\Gamma_C)$ -norm of the contact stress

Figure 9: Convergence curves in the stabilized case



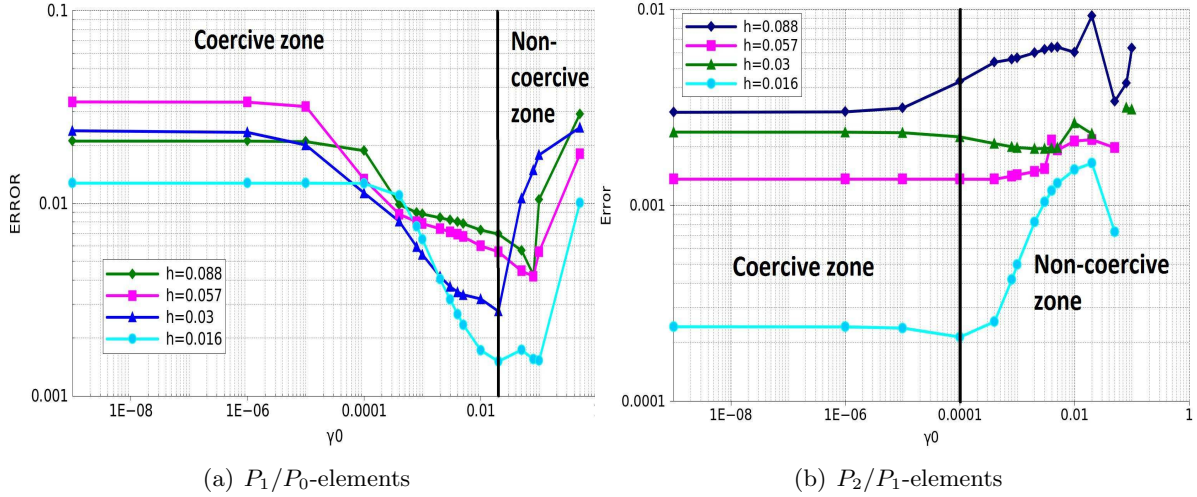


Figure 10: Influence of the stabilization parameter in  $L^2(\Gamma_C)$ -norm of the contact stress

elements  $P_2/P_1$  and  $P_2/P_0$ . For the error in the  $L^2(\Gamma_C)$ -norm of the contact stress, Fig. 9(c) shows that the Barbosa-Hughes stabilization eliminates the spurious modes for the  $P_1/P_1$  and  $P_1/P_0$  methods. For the remaining pairs of elements, the stabilization also allows to reduce the oscillations in the convergence of the contact stress.

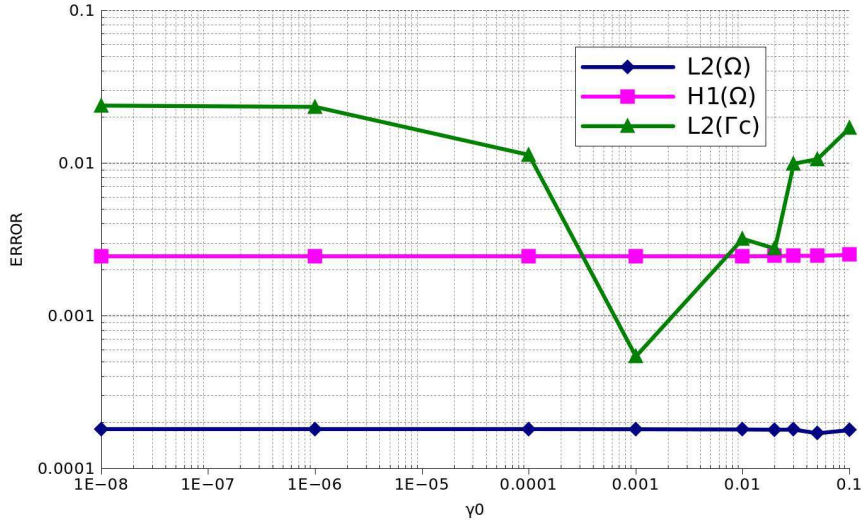


Figure 11: Influence of the stabilization parameter for  $P_1/P_0$  method

The stabilization parameter is chosen in such a way that it is as large as possible but keeps the coercivity of the stiffness matrix. To check the coercivity, we calculate the smallest eigenvalue of the stiffness matrix. For the  $L^2(\Gamma_C)$ -norm on the contact stress, the value of the stabilization parameter can be divided into two zones. A coercive area where the error decreases when increasing the stabilization parameter  $\gamma_0$  and a non-coercive zone where the error evolves randomly according to the stabilization parameter (see Fig. 10(a) and 10(b)). Fig. 11 shows that the stabilization parameter has no influence on the error in  $L^2(\Omega)$  and  $H^1(\Omega)$ -norms of the



displacement.

## Conclusion

Concerning the three contact conditions we considered theoretically, the given a priori error estimates are obviously sub-optimal. This limitation of the mathematical analysis is not specific to the approximation of contact problems in the framework of XFEM. It is in fact particularly true for the approximation of the contact condition with Lagrange multiplier. This is probably due to technical reasons. The approximation with Lagrange multiplier is made necessary here to apply the Barbosa-Hughes stabilization technique (see [26]).

In the numerical tests we considered, the stabilized methods have indeed an optimal rate of convergence. More surprisingly, the unstabilized methods have also an optimal rate of convergence concerning the displacement (except the  $P_1/P_1$  method whose linear system was not invertible). This may lead to the conclusion that no locking phenomenon were present in the numerical situation we studied despite the non-satisfaction of the discrete inf-sup condition. The fact that such a locking situation may exist or not in the studied framework (contact problem on crack lips for a linear elastic body) is an open question.

## Appendix A : Singularity of the contact stress

**Lemma A.1.** *Assume that we have a finite number of transition points between the contact and the non contact zones on the crack lips, then the contact stress  $\sigma_n$  is in  $H^{1/2}(\Gamma_C)$ .*

**Proof.** Let  $\mathbf{m}$  be a transition point which delimits two zones of nonzero length, a non contact zone ( $u_n < 0$ ) and a zone where the contact is effective ( $u_n = 0$ ). Moussaoui et al. [34] show that the asymptotic displacement near this transition point is no more singular than  $r^{3/2} \sin\left(\frac{3}{2}\theta\right)$  where  $(r, \theta)$  are the polar coordinate relative to  $\mathbf{m}$  and the crack. Consequently, the normal contact stress is not singular near the transition points between the contact and the non contact zones.

Now, we compute the singular part of the stress in the vicinity of the crack-tip. We can restrict the study to the case of a contact occurring on a neighborhood of the crack-tip, since  $\lambda = 0$  if there is no contact at the crack-tip.

Using the div-rot lemma, we rewrite the stress components in terms of an Airy function  $\phi$  as follows:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

In two-dimensional isotropic elasticity, the Hooke's law is given by:

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy}, \\ \sigma_{yy} &= (\lambda + 2\mu)\varepsilon_{yy} + \lambda\varepsilon_{xx}, \\ \sigma_{xy} &= \mu(\varepsilon_{xy} + \varepsilon_{yx}) = 2\mu\varepsilon_{xy}. \end{aligned}$$

So

$$\begin{aligned}\varepsilon_{xy} &= \varepsilon_{yx} = -\frac{1}{2\mu} \frac{\partial^2 \phi}{\partial x \partial y}, \\ \varepsilon_{xx} &= \frac{1}{4\mu(\lambda + \mu)} \left( (\lambda + 2\mu) \frac{\partial^2 \phi}{\partial y^2} - \lambda \frac{\partial^2 \phi}{\partial x^2} \right), \\ \varepsilon_{yy} &= -\frac{1}{4\mu(\lambda + \mu)} \left( \lambda \frac{\partial^2 \phi}{\partial y^2} - (\lambda + 2\mu) \frac{\partial^2 \phi}{\partial x^2} \right).\end{aligned}$$

The compatibility relations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0,$$

lead to the bi-harmonic equation:

$$\frac{\lambda + 2\mu}{4\mu(\lambda + \mu)} \left[ \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \right] = 0 \iff \Delta^2 \phi = 0,$$

whose general solution in polar coordinates is a linear combination of the following elementary functions:

$$r^{s+1} \cos(s-1)\theta, \quad r^{s+1} \cos(s+1)\theta, \quad r^{s+1} \sin(s-1)\theta, \quad r^{s+1} \sin(s+1)\theta.$$

Let  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$  and  $\sigma_{r\theta}$  be the polar stress components. By using  $\mathbf{e}_r = (\cos \theta, \sin \theta)$ ,  $\mathbf{e}_\theta = (-\sin \theta, \cos \theta)$  and the fact that  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k})$  is direct and  $\nabla \phi \wedge \mathbf{k}$  is independent of  $x$ ,  $y$ , we obtain

$$\sigma_{rr} = \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r}.$$

Besides, we have

$$\begin{aligned}\sigma_{xx} &= (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y}, \\ \sigma_{yy} &= (\lambda + 2\mu) \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_x}{\partial x}, \\ \sigma_{xy} &= \mu(\varepsilon_{xy} + \varepsilon_{yx}) = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right),\end{aligned}$$

and  $\nabla \mathbf{u} = \left( \frac{\partial u_r}{\partial r} \mathbf{e}_r + \frac{\partial u_\theta}{\partial r} \mathbf{e}_\theta \right) \otimes \mathbf{e}_r + \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} \mathbf{e}_r + \frac{1}{r} u_r \mathbf{e}_\theta - \frac{1}{r} u_\theta \mathbf{e}_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \mathbf{e}_\theta \right) \otimes \mathbf{e}_\theta$  where  $u_r$  and  $u_\theta$  are the radial and angular components of the displacement. So in polar coordinates, it becomes

$$\begin{aligned}\sigma_{rr} &= (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + \frac{\lambda}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right), \\ \sigma_{\theta\theta} &= \frac{(\lambda + 2\mu)}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right) + \lambda \frac{\partial u_r}{\partial r}, \\ \sigma_{r\theta} &= \mu \left( \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{1}{r} u_\theta \right).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} &= (\lambda + 2\mu) \frac{\partial u_r}{\partial r} + \frac{\lambda}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right), \\ \frac{\partial^2 \phi}{\partial r^2} &= \frac{(\lambda + 2\mu)}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right) + \lambda \frac{\partial u_r}{\partial r}, \\ \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta \partial r} &= \mu \left( \frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{1}{r} u_\theta \right).\end{aligned}$$

In [18], Grisvard gives the corresponding displacement in polar coordinates with  $\rho = 1 + \frac{2\mu}{\lambda + \mu}$ :

$$\begin{aligned}u_r &= r^s (a \sin(s+1)\theta + b \cos(s+1)\theta + c(\rho - s) \sin(s-1)\theta - d(\rho - s) \cos(s-1)\theta), \\ u_\theta &= r^s (a \cos(s+1)\theta - b \sin(s+1)\theta - c(\rho + s) \cos(s-1)\theta - d(\rho + s) \sin(s-1)\theta),\end{aligned}\tag{41}$$

where  $a, b, c, d$  are generic constants. The  $P_1$  finite-element method will not optimally approximate the terms of this form which are not in  $H^2(\Omega)$ . So we have to determine the terms for which the real part of  $s$  is such that  $0 < \text{Re}(s) < 1$ .

The boundary conditions for the effective contact without friction on the crack with  $\theta = \pi$  can be expressed as:

$$\begin{aligned}u_\theta(r, \pi) - u_\theta(r, -\pi) &= 0, \\ \sigma_{\theta\theta}(r, \pi) - \sigma_{\theta\theta}(r, -\pi) &= 0, \\ \sigma_{r\theta}(r, \pi) = \sigma_{r\theta}(r, -\pi) &= 0.\end{aligned}$$

The first equality expresses the contact condition: the jump of the normal displacement is equal to zero because we are not in the opening mode, the second equation represents the action-reaction law and the last equality expresses the absence of friction.

By using (41), these conditions read as:

$$\begin{aligned}u_\theta(r, \pi) - u_\theta(r, -\pi) &= 2r^s (-b \sin(s+1)\pi - d(\rho + s) \sin(s-1)\pi) \\ &= 2r^s (b \sin(s\pi) + d(\rho + s) \sin(s\pi)), \\ \sigma_{r\theta}(r, \pi) &= \mu r^{s-1} (2as \cos(s+1)\pi - 2bs \sin(s+1)\pi - 2cs^2 \cos(s-1)\pi \\ &\quad - 2ds^2 \sin(s-1)\pi) \\ &= 2\mu r^{s-1} (-as \cos(s\pi) + bs \sin(s\pi) + cs^2 \cos(s\pi) + ds^2 \sin(s\pi)), \\ \sigma_{r\theta}(r, -\pi) &= 2\mu r^{s-1} (-as \cos(s\pi) - bs \sin(s\pi) + cs^2 \cos(s\pi) - ds^2 \sin(s\pi)), \\ \sigma_{\theta\theta}(r, \pi) - \sigma_{\theta\theta}(r, -\pi) &= r^{s-1} (\lambda(2as \sin(s+1)\pi + 2c(\rho - s)s \sin(s-1)\pi) \\ &\quad + (\lambda + 2\mu)(-2as \sin(s+1)\pi + 2cs(\rho + s - 2) \sin(s-1)\pi)) \\ &= r^{s-1} (4\mu as \sin(s\pi) - 4cs\mu(s+1) \sin(s\pi)).\end{aligned}$$

The determinant of the corresponding linear system can be written as:

$$\begin{aligned}D &= 32\mu^3 s^3 r^{4s-3} \sin^2(s\pi) \begin{vmatrix} 0 & 1 & -\cos(s\pi) & -\cos(s\pi) \\ 1 & 0 & \sin(s\pi) & -\sin(s\pi) \\ 0 & -s-1 & s \cos(s\pi) & s \cos(s\pi) \\ \rho+s & 0 & s \sin(s\pi) & -s \sin(s\pi) \end{vmatrix} \\ &= 64\mu^3 s^3 r^{4s-3} \rho \sin^3(\pi s) \cos(s\pi).\end{aligned}$$

So  $D = 0$  reduces to  $\sin^3(\pi s) \cos(s\pi) = 0$  and the only solution satisfying  $0 < Re(s) < 1$  is  $s = \frac{1}{2}$ .

For  $s = \frac{1}{2}$ , we obtain:

$$a = \frac{3c}{2}, b = 0, d = 0$$

which means that only one singular mode is present. For this singular mode we have also:  $\sigma_{\theta\theta}(r, \pi) = \sigma_{\theta\theta}(r, -\pi) = 0$ . This property corresponds to the classical Neumann boundary condition on the crack lips. The consequence is there is no supplementary singular mode to the classical shear mode and the normal stress component is not singular on the crack lips.  $\square$

## Appendix B : Proof of Lemma 3.3

In order to prove Lemma 3.3, we distinguish three cases: totally enriched triangles, partial enriched triangles and the triangle containing the crack tip.

First, for a totally enriched triangle (Fig. 4(a)) we have  $\Pi^h \mathbf{u}_r|_{T \cap \Omega_i} = I^h \tilde{\mathbf{u}}_r^i|_{T \cap \Omega_i}$  (see Lemma 3.2). Then,  $E_T^i \mathbf{u}_r = I^h \tilde{\mathbf{u}}_r^i$  and we have

$$\begin{aligned} \|\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r\|_{0,T} &= \|\tilde{\mathbf{u}}_r^i - I^h \tilde{\mathbf{u}}_r^i\|_{0,T}, \\ &\lesssim h^2 \|\tilde{\mathbf{u}}_r^i\|_{2,T}, \\ \|\tilde{\mathbf{u}}_r^i - E_T^i \mathbf{u}_r\|_{1,T} &\lesssim h \|\tilde{\mathbf{u}}_r^i\|_{2,T}. \end{aligned}$$

Second, for a partially enriched triangle and considering the particular case shown in Fig. 4(b) we have:

$$\begin{aligned} \Pi^h \mathbf{u}_r|_{T \cap \Omega_1} &= \mathbf{u}_r^1(\mathbf{x}_1) \varphi_1 + \mathbf{u}_r^2(\mathbf{x}_2) \varphi_2 + \mathbf{u}_r^1(\mathbf{x}_3) \varphi_3, \\ &= \Pi^h \tilde{\mathbf{u}}_r^1 + (\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)) \varphi_2, \\ \Pi^h \mathbf{u}_r|_{T \cap \Omega_2} &= \mathbf{u}_r^1(\mathbf{x}_1) \varphi_1 + \mathbf{u}_r^2(\mathbf{x}_2) \varphi_2 + \tilde{\mathbf{u}}_r^2(\mathbf{x}_3) \varphi_3, \\ &= \Pi^h \tilde{\mathbf{u}}_r^2 + (\tilde{\mathbf{u}}_r^2(\mathbf{x}_1) - \mathbf{u}_r^1(\mathbf{x}_1)) \varphi_1. \end{aligned}$$

In this case  $E_T^1 \mathbf{u}_r = \Pi^h \tilde{\mathbf{u}}_r^1 + (\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)) \varphi_2$  and  $E_T^2 \mathbf{u}_r = \Pi^h \tilde{\mathbf{u}}_r^2 + (\tilde{\mathbf{u}}_r^2(\mathbf{x}_1) - \mathbf{u}_r^1(\mathbf{x}_1)) \varphi_1$ . Then we have:

$$\begin{aligned} \|\tilde{\mathbf{u}}_r^1 - E_T^1 \mathbf{u}_r\|_{0,T} &= \|\tilde{\mathbf{u}}_r^1 - \Pi^h \tilde{\mathbf{u}}_r^1 - (\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)) \varphi_2\|_{0,T}, \\ &\lesssim \|\tilde{\mathbf{u}}_r^1 - \Pi^h \tilde{\mathbf{u}}_r^1\|_{0,T} + |\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)| \|\varphi_2\|_{0,T}, \\ \|\tilde{\mathbf{u}}_r^1 - E_T^1 \mathbf{u}_r\|_{1,T} &\lesssim \|\tilde{\mathbf{u}}_r^1 - \Pi^h \tilde{\mathbf{u}}_r^1\|_{1,T} + |\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)|. \end{aligned}$$

Furthermore, we have from [35]:

$$|\tilde{\mathbf{u}}_r^1(\mathbf{x}_2) - \mathbf{u}_r^2(\mathbf{x}_2)| \lesssim h_T |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)},$$

and since  $\|\varphi_2\|_{0,T} \lesssim h$  we can conclude that:

$$\begin{aligned} \|\tilde{\mathbf{u}}_r^1 - E_T^1 \mathbf{u}_r\|_{0,T} &\lesssim h_T^2 \left( \|\tilde{\mathbf{u}}_r^1\|_{2,T} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right), \\ \|\tilde{\mathbf{u}}_r^1 - E_T^1 \mathbf{u}_r\|_{1,T} &\lesssim h_T \left( \|\tilde{\mathbf{u}}_r^1\|_{2,T} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right). \end{aligned}$$

In the same way we have:

$$\begin{aligned}\|\tilde{\mathbf{u}}_r^2 - E_T^2 \mathbf{u}_r\|_{0,T} &\lesssim h_T^2 \left( \|\tilde{\mathbf{u}}_r^2\|_{2,T^+} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right), \\ \|\tilde{\mathbf{u}}_r^2 - E_T^2 \mathbf{u}_r\|_{1,T} &\lesssim h_T \left( \|\tilde{\mathbf{u}}_r^2\|_{2,T^+} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right).\end{aligned}$$

A similar reasoning can be applied to the other situations of partially enriched triangles to obtain the same result.

Finally, for the triangle containing the crack tip, and in the particular case described in Fig. 4(c) we have:

$$\begin{aligned}\Pi^h \mathbf{u}_r|_{T \cap \Omega_1} &= \mathbf{u}_r^1(\mathbf{x}_1)\varphi_1 + \mathbf{u}_r^2(\mathbf{x}_2)\varphi_2 + \mathbf{u}_r^2(\mathbf{x}_3)\varphi_3, \\ &= \Pi^h \tilde{\mathbf{u}}_r^1 + (\mathbf{u}_r^2(\mathbf{x}_2) - \tilde{\mathbf{u}}_r^1(\mathbf{x}_2))\varphi_2 + (\mathbf{u}_r^2(\mathbf{x}_3) - \tilde{\mathbf{u}}_r^1(\mathbf{x}_3))\varphi_3, \\ \Pi^h \mathbf{u}_r|_{T \cap \Omega_2} &= \mathbf{u}_r^1(\mathbf{x}_1)\varphi_1 + \mathbf{u}_r^2(\mathbf{x}_2)\varphi_2 + \mathbf{u}_r^2(\mathbf{x}_3)\varphi_3, \\ &= \Pi^h \tilde{\mathbf{u}}_r^2 + (\mathbf{u}_r^1(\mathbf{x}_1) - \tilde{\mathbf{u}}_r^2(\mathbf{x}_1))\varphi_1.\end{aligned}$$

Thus, we have  $E_T^1 \mathbf{u}_r = \Pi^h \tilde{\mathbf{u}}_r^1 + (\mathbf{u}_r^2(\mathbf{x}_2) - \tilde{\mathbf{u}}_r^1(\mathbf{x}_2))\varphi_2 + (\mathbf{u}_r^2(\mathbf{x}_3) - \tilde{\mathbf{u}}_r^1(\mathbf{x}_3))\varphi_3$  and  $E_T^2 \mathbf{u}_r = \Pi^h \tilde{\mathbf{u}}_r^2 + (\mathbf{u}_r^1(\mathbf{x}_1) - \tilde{\mathbf{u}}_r^2(\mathbf{x}_1))\varphi_1$ . Note that we have (see [35]):

$$|\mathbf{u}_r^i(\mathbf{x}_j) - \tilde{\mathbf{u}}_r^l(\mathbf{x}_j)| \lesssim h_T |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)},$$

with  $j \in \{1, 2, 3\}$ ,  $i \in \{1, 2\}$ ,  $l = 3 - i$  and  $\mathbf{x}_j$  a node belonging to a partially enriched triangle or triangle containing the crack tip. Then, by the same way in the case of partially enriched triangle we have the following result for  $i \in \{1, 2\}$ :

$$\begin{aligned}\|\tilde{\mathbf{u}}_r^1 - E_T^i \mathbf{u}_r\|_{0,T} &\lesssim h_T^2 \left( \|\tilde{\mathbf{u}}_r^i\|_{2,T^+} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right), \\ \|\tilde{\mathbf{u}}_r^2 - E_T^i \mathbf{u}_r\|_{1,T} &\lesssim h_T \left( \|\tilde{\mathbf{u}}_r^i\|_{2,T^+} + |\tilde{\mathbf{u}}_r^1 - \tilde{\mathbf{u}}_r^2|_{2,B(\mathbf{x}^*, h_T)} \right).\end{aligned}$$

This concludes the proof, since a similar reasoning can be applied to the other situations of a triangle containing the crack tip.  $\square$

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