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Lower compactness estimates for scalar balance laws

Fabio Ancona*, Olivier Glass†, Khai T. Nguyen*

Abstract

In this paper, we study the compactness in L^1_{loc} of the semigroup $(S_t)_{t \geq 0}$ of entropy weak solutions to strictly convex scalar conservation laws in one space dimension. The compactness of S_t for each $t > 0$ was established by P. D. Lax [9]. Upper estimates for the Kolmogorov's ε -entropy of the image through S_t of bounded sets in $L^1 \cap L^\infty$ were given by C. De Lellis and F. Golse [4]. Here, we provide lower estimates on this ε -entropy of the same order as the one established in [4], thus showing that such an ε -entropy is of size $\approx (1/\varepsilon)$. Moreover, we extend these estimates of compactness to the case of convex balance laws.

1 Introduction

Consider a scalar conservation law in one space dimension

$$u_t + f(u)_x = 0, \quad (1)$$

where $u = u(t, x)$ is the state variable, and $f \in C^2(\mathbb{R}; \mathbb{R})$ is a (uniformly) strictly convex function:

$$f''(u) \geq c > 0 \quad \forall u \in \mathbb{R}. \quad (2)$$

Without loss of generality, we will suppose

$$f'(0) = 0, \quad (3)$$

since one may always reduce the general case to this one by performing the space-variable and flux transformations $x \rightarrow x + tf'(0)$ and $f(u) \rightarrow f(u) - uf'(0)$. We recall that problems of this type do not possess classical solutions since discontinuities arise in finite time even if the initial data are smooth. Hence, it is natural to consider weak solutions in the sense of distributions that, for sake of uniqueness, satisfy an entropy criterion for admissibility [3]:

$$u(t, x-) \geq u(t, x+) \quad \text{for a.e. } t > 0, \quad \forall x \in \mathbb{R}, \quad (4)$$

where $u(t, x\pm)$ denote the one-sided limits of $u(t, \cdot)$ at x . The equation (1) generates an L^1 -contractive semigroup of solutions $(S_t)_{t \geq 0}$ that associates, to every given initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the unique entropy admissible weak solution $S_t u_0 \doteq u(t, \cdot)$ of the corresponding Cauchy problem (cfr. [3, 8]). This yields the existence of a continuous semigroup $(S_t)_{t \geq 0}$ acting on the whole space $L^1(\mathbb{R})$. Such a semigroup S_t was shown by Lax [9] to be compact as a mapping from $L^1(\mathbb{R})$ to $L^1_{loc}(\mathbb{R})$, for every $t > 0$. De Lellis and Golse [4], following a suggestion by Lax [9], used the Kolmogorov's ε -entropy concept, which is recalled below, to provide a quantitative version of this compactness effect.

Definition 1. *Let (X, d) be a metric space and K a totally bounded subset of X . For $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε . Then the ε -entropy of K is defined as*

$$H_\varepsilon(K \mid X) \doteq \log_2 N_\varepsilon(K).$$

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Throughout the paper, we will call an ε -cover, a cover of K by subsets of X having diameter no larger than 2ε .

De Lellis and Golse obtained an upper bound for the ε -entropy of the set of solutions to (1) at any given time $t > 0$, as $\varepsilon \rightarrow 0^+$; that is to say, they showed how strong is the compactifying effect. Precisely, they established the following result.

Theorem 1 ([4]). *Consider $f \in C^2(\mathbb{R}; \mathbb{R})$ satisfying (2) and (3). Given any $L, m, M > 0$, define the set of bounded, compactly supported initial data*

$$\mathcal{C}_{[L,m,M]} \doteq \left\{ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \mid \text{Supp}(u_0) \subset [-L, L], \|u_0\|_{L^1} \leq m, \|u_0\|_{L^\infty} \leq M \right\}. \quad (5)$$

Then for $\varepsilon > 0$ sufficiently small, one has

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid L^1(\mathbb{R})) \leq \frac{4}{\varepsilon} \left(\frac{4L(T)^2}{cT} + 4L(T)\sqrt{\frac{2m}{cT}} \right) \quad \forall T > 0, \quad (6)$$

with

$$L(T) \doteq L + 2 \sup_{|z| \leq M} |f''(z)| \sqrt{2mT/c}. \quad (7)$$

The aim of this paper is to show that the ε -entropy estimates provided by Theorem 1 turn out to be optimal, since we shall establish a lower bound on such an ε -entropy which is of the same order as De Lellis and Golse's upper bounds. Hence, we deduce that $H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid L^1(\mathbb{R}))$ is exactly of size $\approx \varepsilon^{-1}$. Precisely, we prove the following.

Theorem 2. *Under the assumptions and in the same setting of Theorem 1, for any $T > 0$, and for $\varepsilon > 0$ sufficiently small, one has*

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \cdot \frac{L^2}{48 \cdot \ln(2) \cdot |f''(0)| T}. \quad (8)$$

As suggested in [9], the knowledge of the ε -entropy magnitude of the solution set of (1) may play an important role to provide estimates on the accuracy and resolution of numerical methods for (1).

The main steps of the proof of the lower bound (8) consist in:

1. Introducing a suitable class of piecewise affine functions and showing that any element of such a class can be obtained, at any given time t , as the value $u(t, \cdot)$ of an entropy admissible weak solution of (1), with initial data in $\mathcal{C}_{[L,m,M]}$.
2. Providing an optimal estimate of the maximum number of functions in such a class that can be contained in a subset of $S_T(\mathcal{C}_{[L,m,M]})$ having diameter no larger than 2ε . This estimate is established with a similar combinatorial argument as the one used in [1].

Remark 1. *Since by (2), (7), we have $L(T)\sqrt{\frac{2m}{cT}} \leq \frac{L(T)^2}{2cT}$, one derives from (6) the estimate*

$$H_\varepsilon(S_T(\mathcal{C}_{[L,m,M]}) \mid L^1(\mathbb{R})) \leq \frac{1}{\varepsilon} \cdot \frac{24L(T)^2}{cT}.$$

Therefore, the size $\frac{1}{\varepsilon} \cdot \frac{L^2}{|f''(0)|T}$ of the lower bound (8) turns out to be the same as the one of the upper bound on the ε -entropy of $S_T(\mathcal{C}_{[L,m,M]})$ provided by Theorem 1, upon replacing L with $L(T)$, and $|f''(0)|$ with c .

Next, we address the more general case of convex balance laws. Namely, given $f \in C^2(\mathbb{R}; \mathbb{R})$ satisfying (2), (3) and $g \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$, we will analyze the compactifying effect of the balance law

$$u_t + f(u)_x = g(t, x, u). \quad (9)$$

As for (1) we will consider weak solutions of (9) that satisfy the entropy admissibility condition (4). We will make the following assumptions on the source term g :

$$g(t, x, 0) = 0 \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (10)$$

$$D_x g(t, x, u) \geq 0 \quad \forall (t, x, u) \in \mathbb{R}^+ \times \mathbb{R}^2, \quad (11)$$

$$\exists \omega \in L^1_{loc}(\mathbb{R}^+) \text{ s.t. } |D_u g(t, x, u)| \leq \omega(t) \quad \text{for a.e. } t \in \mathbb{R}^+, \forall (x, u) \in \mathbb{R}^2. \quad (12)$$

In particular, (10), (12) together imply

$$|g(t, x, u)| \leq \omega(t) \cdot |u| \quad \text{for a.e. } t \in \mathbb{R}^+, \forall (x, u) \in \mathbb{R}^2. \quad (13)$$

Under assumptions (12) or (13), for each $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists a unique entropy admissible solution $u(t, x)$ of (9) with initial condition $u(0, \cdot) = u_0$, see [3, 6, 8].

Remark 2. Condition (10) in particular implies the fact that the source term g , if not zero, does depend on u , since otherwise one would have $g = g(t, x) = 0$, for all t, x . Because of (10), all solutions $u(t, \cdot)$ to (9) with initial data of compact support remain compactly supported for all times $t > 0$. Instead, condition (11) guarantees that, at any given time $T > 0$, all constant states $w(x) \equiv w \in \mathbb{R}$ can be obtained as the value $u(T, \cdot)$ of an admissible solution to (9) with suitable initial data. In fact, thanks to (11), the backward generalized characteristics emanating from time $T > 0$ can never cross at a time $0 < t < T$. Hence, using the method of the generalized characteristics [2, 3], one can always reconstruct a solution on the time interval $[0, T]$, starting from the terminal value $u(T, \cdot) \equiv w$.

We shall denote by E_t the evolution operator that associates, to every initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the entropy admissible solution $E_t u_0 \doteq u(t, \cdot)$ of the corresponding Cauchy problem for (9). We establish the following.

Theorem 3. Suppose that $f \in C^2(\mathbb{R}; \mathbb{R})$ and $g \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfy (2), (3), (10), (11) and (12). Then, in the same setting of Theorem 1, for any $T > 0$ and for $\varepsilon > 0$ sufficiently small, one has

$$H_\varepsilon(E_T(\mathcal{C}_{[L, m, M]}) \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \cdot \frac{L^2 \cdot \exp(-\|\omega\|_{L^1(0, T)})}{48 \cdot \ln(2) \cdot |f''(0)| T}. \quad (14)$$

Since balance laws are not considered in [4], following the same lines of the proof in [4] we also establish the same type of upper bound for $H_\varepsilon(E_T(\mathcal{C}_{[L, m, M]}) \mid L^1(\mathbb{R}))$ as the one given in Theorem 1.

Let us introduce the following notations. Given $t > 0$, $M > 0$ and $a, b \in \mathbb{R}$ with $a < b$, we set

$$\Delta_{a, b, t}(M) \doteq \left\{ (s, x) \mid s \in [0, t], \quad a - (t - s) \cdot \|f'\|_{L^\infty(-M, M)} \leq x \leq b + (t - s) \cdot \|f'\|_{L^\infty(-M, M)} \right\}, \quad (15)$$

and

$$k_{a, b, t}(M) \doteq \max \{ |D_x g(s, x, u)|; (s, x) \in \Delta_{a, b, t}(M), u \in [-M, M] \}. \quad (16)$$

We obtain the following result.

Theorem 4. In the same setting of Theorem 1, assume that $f \in C^2(\mathbb{R}; \mathbb{R})$ and $g \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfy (2), (3), (10), (12). Then, for $\varepsilon > 0$ sufficiently small, one has

$$H_\varepsilon(E_T(\mathcal{C}_{[L, m, M]}) \mid L^1(\mathbb{R})) \leq \frac{1}{\varepsilon} \cdot \frac{8 L(T)^2 \cdot \left(1 + 2(1 + cT^2 K_{L, T}) \exp(\|\omega\|_{L^1(0, T)}) \right)}{cT} \quad \forall T > 0,$$

where

$$L(T) \doteq L + 2\|f''\|_{L^\infty(-M_T, M_T)} \sqrt{\frac{2mT}{c}} \left[1 + T\sqrt{cK_{L, T}} \right] \cdot \exp(\|\omega\|_{L^1(0, T)}),$$

$$K_{L, T} \doteq k_{-L_T, L_T, T}(M_T),$$

with

$$L_T \doteq L + \|f''\|_{L^\infty(-M_T, M_T)} \cdot M_T T,$$

and

$$M_T \doteq \exp(\|\omega\|_{L^1(0, T)}) \cdot M.$$

Remark 3. As observed in Remark 2, the condition (11) guarantees that the equation (9) be controllable to constant states at any time $T > 0$. However, it remains an open question whether such a condition is really necessary in order to show that the class of piecewise affine functions introduced in the proof of Theorem 2 be reachable by solutions of (9) at any given time $T > 0$. On the other hand, Theorem 4 implies that the solution set of (9) is compact in L^1_{loc} without requiring that the assumption (11) be satisfied. Thus, it remains as well open the problem of determining whether such an assumption is necessary to establish the lower bound (14) on the ε -entropy of the solution set of (9).

Remark 4. Theorems 3 and 4 remain true if the source term has the form $g = g(t, u)$, and satisfies only the condition (12), together with $g(\cdot, 0) \in L^1_{loc}$. Clearly, in this case, the solution $u(t, \cdot)$ of (9) will not be in general compactly supported, but instead the difference between $u(t, \cdot)$ and the solution of (9) with zero initial data has always compact support. So, it will be convenient to compute the ε -entropy of the translated set $E_T(\mathcal{C}_{[L, m, M]}) - E_T 0$, which obviously coincides with the one of $E_T(\mathcal{C}_{[L, m, M]})$. In this way we will see in Subsection 4.3 that one can establish, for $\varepsilon > 0$ sufficiently small, the estimate

$$H_\varepsilon(E_T(\mathcal{C}_{[L, m, M]}) \mid L^1(\mathbb{R})) \leq \frac{1}{\varepsilon} \cdot \frac{8L(T)^2 \cdot (1 + 2\exp(\|\omega\|_{L^1(0, T)}))}{cT} \quad \forall T > 0, \quad (17)$$

where

$$L(T) \doteq L + 2\|f''\|_{L^\infty(-M_T^g, M_T^g)} \sqrt{\frac{2mT}{c}} \cdot \exp(\|\omega\|_{L^1(0, T)}),$$

with

$$M_T^g \doteq \exp(\|\omega\|_{L^1(0, T)})(M + \|g(\cdot, 0)\|_{L^1(0, T)}).$$

Moreover, for any $T > 0$, and for $\varepsilon > 0$ sufficiently small, we derive the estimate

$$H_\varepsilon(E_T(\mathcal{C}_{[L, m, M]}) \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \cdot \frac{L^2 \cdot \exp(-\|\omega\|_{L^1(0, T)})}{24 \cdot \ln(2) \cdot \|f''\|_{L^\infty(-G_T, G_T)} T}. \quad (18)$$

where

$$G_T \doteq 1 + \|g(\cdot, 0)\|_{L^1(0, T)} \exp(\|\omega\|_{L^1(0, T)}). \quad (19)$$

As a final remark, we observe that it would be interesting to provide upper and lower quantitative compactness estimates for the solution set of genuinely nonlinear 2×2 systems of conservation laws (whose L^1_{loc} compactness follows from the estimates provided in [5], as observed in [10]), while it remains a completely open problem whether such a compactness property continues to hold (and possibly derive similar quantitative estimates) for general systems of N conservation laws with genuinely nonlinear characteristic fields.

The paper is organized as follows. In Section 2 we provide a tight lower bound for the ε -entropy of the solution set of a convex conservation law, establishing Theorem 2. In Section 3 we extend the results of Section 2 to the case of convex balance laws, proving Theorem 3. Finally, in Section 4 we derive an upper bound for the ε -entropy of the solution set of a convex balance law, proving Theorem 4; also, we prove Remark 4.

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2 Lower compactness estimates for conservation laws

2.1 Proof of Theorem 2

For arbitrary positive constants L , M , m and b , let us consider the set

$$\mathcal{A}_{[L, m, M, b]} \doteq \left\{ u_T \in \text{BV}(\mathbb{R}) \mid \text{Supp}(u_T) \subset [-L, L], \|u_T\|_{L^1} \leq m, \|u_T\|_{L^\infty} \leq M, Du_T \leq b \right\},$$

where the last inequality has to be understood in the sense of measures, i.e. the Radon measure Du_T satisfies $Du_T(J) \leq b \cdot |J|$ for every Borel set $J \subset \mathbb{R}$, $|J|$ being the Lebesgue measure of J . We obtain a proof of Theorem 2 as a consequence of the following two propositions that shall be established below.

Proposition 1. *Suppose that $f \in C^2(\mathbb{R}; \mathbb{R})$ satisfies (2), (3). Given any $L, m, M, T > 0$, for*

$$h \leq \min \left(M, \frac{m}{L}, \frac{L}{8T|f''(0)|} \right), \quad (20)$$

sufficiently small, one has

$$\mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]} \subset S_T(\mathcal{C}_{[L, m, M]}), \quad (21)$$

with

$$L_T \doteq L - 2T|f''(0)| \cdot h. \quad (22)$$

Proposition 2. *Given $L, m, M, b > 0$, for any $\varepsilon > 0$ satisfying*

$$\varepsilon \leq \frac{\min(m, LM)}{6}, \quad (23)$$

one has

$$H_\varepsilon(\mathcal{A}_{[L, m, M, b]} \mid L^1(\mathbb{R})) \geq \frac{1}{\varepsilon} \cdot \frac{2bL^2}{27 \ln(2)}. \quad (24)$$

Notice that the lower bound (24) is independent on m and M , which appear only in the constraint (23). Moreover, because of (20), the constant L_T given by (22) satisfies $L_T \geq (3/4)L$. Hence, applying (24), with $L = L_T$, $b = (2T|f''(0)|)^{-1}$, and relying on (21), we recover the estimate (8), which proves Theorem 2.

2.2 Proof of Proposition 1

1. We will show that any element u_T of the set on the left-hand side of (21) can be obtained as the value at time T of a weak admissible solution to (1), that is backward construct starting from u_T by reversing the direction of time. Namely, given

$$u_T \in \mathcal{A}_{[L_T, Lh, h, (2T|f''(0)|)^{-1}]}, \quad (25)$$

set

$$w_0(x) \doteq u_T(-x) \quad \forall x \in \mathbb{R}, \quad (26)$$

and consider the entropy weak solution $w(t, x) \doteq S_t w_0$ of (1) with initial data w_0 . By well-known properties of solutions to scalar conservation laws, and because of (20), (25), w verifies the L^1 and L^∞ bounds (cfr. [3, Theorem 6.2.3, Theorem 6.2.6]):

$$\begin{aligned} \|w(t, \cdot)\|_{L^1} &\leq \|w_0\|_{L^1} \leq Lh \leq m, \\ \|w(t, \cdot)\|_{L^\infty} &\leq \|w_0\|_{L^\infty} \leq h \leq M, \end{aligned} \quad \forall t > 0. \quad (27)$$

Moreover, recalling that the solution propagates along generalized characteristics (cfr. [3, Chapters X & XI]), and due to (3), (25), (27), we derive the bound on the support of w :

$$\text{Supp}(w(t, \cdot)) \subset [-l(t), l(t)] \quad \forall t > 0, \quad (28)$$

where

$$l(t) \doteq L_T + t \cdot \|f'\|_{L^\infty(-h, h)} \leq L_T + th \cdot \|f''\|_{L^\infty(-h, h)}. \quad (29)$$

Since, by the continuity assumption on f'' , we may assume that

$$|f''(h)| \leq 2|f''(0)|, \quad (30)$$

for $|h|$ sufficiently small, we deduce from (22), (29), that

$$l(t) \leq L \quad \forall t \in [0, T].$$

Together with (28), this yields the uniform bound on the support of w :

$$\text{Supp}(w(t, \cdot)) \subset [-L, L] \quad \forall t \in [0, T]. \quad (31)$$

Next, observe that the function

$$u(t, x) \doteq w(T - t, -x), \quad (t, x) \in [0, T] \times \mathbb{R},$$

is a weak solution of (1) in the sense of distribution, which, by (26), clearly satisfies

$$u(T, \cdot) = u_T.$$

On the other hand, if we show that $w(t, x)$ is Lipschitz continuous with respect to x , for all $x \in \mathbb{R}$, $t \in]0, T[$, it would follow that $u(t, x)$ automatically satisfies the admissibility condition (4) as an equality, and thus provides an entropy weak solution of (1) which attains the value u_T at time T , and has initial data $u(0, \cdot) = w(T, -\cdot) \in \mathcal{C}_{[L, m, M]}$ by virtue of (27), (31).

2. Therefore, to complete the proof of the proposition we only need to show that

$$w(t, \cdot) \in \text{Lip}(\mathbb{R}) \quad \forall t \in]0, T[. \quad (32)$$

Toward this end, observe that since w is an entropy admissible weak solution of (1), with initial data $w_0 \in \text{BV}(\mathbb{R})$ by virtue of (25)-(26), one has $w(t, \cdot) \in \text{BV}(\mathbb{R})$ for all $t > 0$ (cfr. [3, Theorem 6.2.6]). It will not be restrictive to assume that $w(t, \cdot)$ is right continuous (possibly redefining $w(t, \cdot)$ on the countable set of jump points). Moreover, by the strict convexity assumption (2) on f , w satisfies the one-sided Oleinik inequality [11]:

$$\frac{w(t, y) - w(t, x)}{y - x} \leq \frac{1}{ct}, \quad (33)$$

for all $t > 0$, and all $x, y \in \mathbb{R}$ with $x < y$. We will show that the initial lower bound

$$w_0(y-) - w_0(x+) = Dw_0(]x, y[) \geq -\frac{y - x}{2T|f''(0)|}, \quad (34)$$

induced by (25)-(26) propagates over the time interval $]0, T[$, so that w satisfies also a lower semi-Lipschitz estimate which, together with (33), yields (32). Namely, we will prove that, for any $t \in]0, T[$, there exists some constant $C_t > 0$ such that, for all $x, y \in \mathbb{R}$, with $x < y$, there holds:

$$\frac{w(t, y) - w(t, x)}{y - x} \geq -C_t. \quad (35)$$

Observe that, if $w(t, y) - w(t, x-) \geq 0$, relying on (4) we immediately deduce (35). Hence, we will consider only the case where

$$w(t, y) - w(t, x-) < 0. \quad (36)$$

We shall rely on Dafermos' theory of generalized characteristics (in the sense of Filippov) for entropy solutions of conservation laws (cfr. [3, Chapters X & XI]). Let $\xi^x(\cdot)$ and $\xi^y(\cdot)$ be, respectively, the minimal and maximal backward generalized characteristics emanating from (t, x) and (t, y) . Since these characteristics are shock-free, they are straight lines of slope $f'(w(t, x-))$ and $f'(w(t, y))$, respectively, and satisfy

$$\begin{aligned} w(t, x-) &= w(s, \xi^x(s)) \leq w_0(\xi^x(0)+) \\ w(t, y) &= w(s, \xi^y(s)) \geq w_0(\xi^y(0)-) \end{aligned} \quad \forall 0 < s < t, \quad (37)$$

(see [3, Theorems 11.1.1 & 11.1.3].) Moreover, $\xi^x(\cdot)$ and $\xi^y(\cdot)$ do not cross at any time $s \in]0, t[$ ([3, Theorem 10.3.2 & Corollary 11.1.2]), so that one has $\xi^y(0) \geq \xi^x(0)$. Notice that, if $\xi^y(0) = \xi^x(0)$, it follows that

$$f'(w(t, y)) - f'(w(t, x-)) = \frac{y - x}{t} > 0,$$

which, by the strict convexity of f , implies $w(t, y) - w(t, x-) > 0$ in contrast with (36). Therefore, we may assume that $\xi^y(0) > \xi^x(0)$. Due to (34), (37) we then have

$$\begin{aligned} w(t, y) - w(t, x-) &\geq w_0(\xi^y(0)-) - w_0(\xi^x(0)+) \\ &\geq -\frac{\xi^y(0) - \xi^x(0)}{2T|f''(0)|} \\ &= -\frac{y - x - t(f'(w(t, y)) - f'(w(t, x-)))}{2T|f''(0)|}. \end{aligned}$$

Thanks to (27), (30) and (36), this yields

$$w(t, y) - w(t, x-) \geq -\frac{y - x}{2T|f''(0)|} + \frac{t}{T}[w(t, y) - w(t, x-)]. \quad (38)$$

In turn, (38) together with (4) yields (35) with

$$C_t \doteq \frac{1}{2(T-t)|f''(0)|}.$$

It follows from (33) and (35) that, for any fixed $t \in]0, T[$, there holds

$$|w(t, y) - w(t, x)| \leq \max\{(at)^{-1}, C_t\} \cdot |y - x| \quad \forall x < y,$$

establishing (32), and thus concluding the proof of Proposition 1. \square

2.3 Proof of Proposition 2

1. Following a similar strategy as the one pursued in [1], we will establish a lower bound on the covering number $N_\varepsilon(\mathcal{A}_{[L, m, M, b]})$, by first introducing a two-parameters class of piecewise affine functions in $\mathcal{A}_{[L, m, M, b]}$, and next providing an estimate of the maximum number of such functions contained in a subset of $\mathcal{A}_{[L, m, M, b]}$ having diameter no larger than 2ε . Namely, given any integer $n \geq 2$, and a constant $h > 0$, for every n -tuple $\iota = (\iota_i)_{i=0, \dots, n-1} \in \{-1, 1\}^n$ consider the function $\mathcal{F}_\iota : \mathbb{R} \rightarrow [-h, h]$, with support contained in $[-L, L]$, defined by (see Figure 1):

$$\mathcal{F}_\iota(x) = \begin{cases} \frac{hn}{2L} \left(x + L - k \frac{2L}{n} \right) & \text{if } \iota_k = 1, \\ \frac{hn}{2L} \left(x + L - (k+1) \frac{2L}{n} \right) & \text{if } \iota_k = -1, \end{cases} \quad \forall x \in \left[-L + k \frac{2L}{n}, -L + (k+1) \frac{2L}{n} \right],$$

$k \in \{0, \dots, n-1\}.$

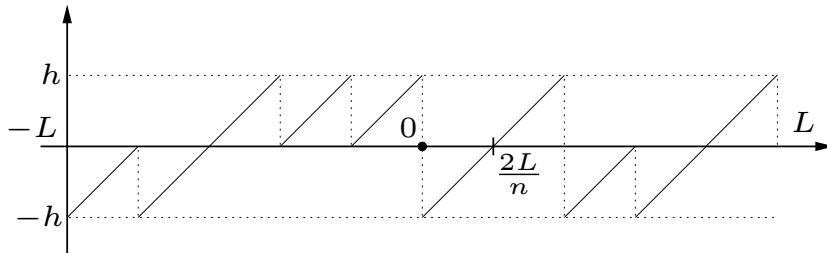


Figure 1: The function \mathcal{F}_ι for $n = 10$ and $\iota = (-1, -1, 1, 1, 1, -1, 1, -1, -1, 1)$

Notice that every \mathcal{F}_ι , $\iota \in \{-1, 1\}^n$, belongs to $\mathcal{A}_{[L, m, M, b]}$ provided that

$$h \leq M, \quad h \leq \frac{m}{L}, \quad nh \leq 2Lb. \quad (39)$$

Moreover, given any $\iota, \bar{\iota} \in \{-1, 1\}^n$, one has

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} = \frac{2hL}{n} d(\iota, \bar{\iota}),$$

where

$$d(\iota, \bar{\iota}) \doteq \text{Card} \{k \in \{1, \dots, n\} \mid \iota_k \neq \bar{\iota}_k\}.$$

It follows that

$$\|\mathcal{F}_\iota - \mathcal{F}_{\bar{\iota}}\|_{L^1} \leq \varepsilon \iff d(\iota, \bar{\iota}) \leq \frac{n\varepsilon}{2hL}.$$

Notice that, given any fixed $\bar{\iota} \in \{-1, 1\}^n$, the set $\mathcal{I}_{\bar{\iota}}(\varepsilon)$ of n -tuples $\iota \in \{-1, 1\}^n$ such that $d(\iota, \bar{\iota}) \leq \frac{n\varepsilon}{2hL}$ depends on $\bar{\iota}$, but the number of elements of $\mathcal{I}_{\bar{\iota}}(\varepsilon)$ is independent of the choice of $\bar{\iota}$. Denote $\mathcal{C}(\varepsilon)$ such a number. By standard combinatorial properties, counting the n -tuples that differ for a given number of entries, we compute

$$\mathcal{C}(\varepsilon) = \sum_{\ell=0}^{\lfloor \frac{n\varepsilon}{2hL} \rfloor} \binom{n}{\ell}, \quad (40)$$

where $\lfloor \alpha \rfloor = \max\{z \in \mathbb{Z} \mid z \leq \alpha\}$ denotes the integer part of α . In order to provide an estimate of $\mathcal{C}(\varepsilon)$, we rewrite the right-hand side of (40) using the fact that, if X_1, \dots, X_n are independent random variables with Bernoulli distribution $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$, then for any $k \leq n$ one has

$$\mathbb{P}(X_1 + \dots + X_n \leq k) = \frac{1}{2^n} \sum_{\ell=0}^k \binom{n}{\ell}. \quad (41)$$

We may estimate the left-hand side of (41) setting $S_n = X_1 + \dots + X_n$, and using Hoeffding's inequality ([7, Theorem 2]) that, for any fixed $\mu > 0$, gives

$$\mathbb{P}(S_n - \mathbb{E}(S_n) \leq -\mu) \leq \exp\left(-\frac{2\mu^2}{n}\right), \quad (42)$$

where $\mathbb{E}(S_n)$ denotes the expectation of S_n . Since, by the above assumptions on X_1, \dots, X_n , we have $\mathbb{E}(S_n) = \frac{n}{2}$, taking $\mu = \frac{n}{2} - \lfloor \frac{n\varepsilon}{2hL} \rfloor$, $k = \lfloor \frac{n\varepsilon}{2hL} \rfloor$, and assuming

$$\frac{n\varepsilon}{2hL} \leq \frac{n}{2}, \quad (43)$$

we deduce from (40)-(42) that

$$\begin{aligned} \frac{1}{2^n} \mathcal{C}(\varepsilon) &\leq \exp\left(-2\frac{\left(\frac{n}{2} - \lfloor \frac{n\varepsilon}{2hL} \rfloor\right)^2}{n}\right) \\ &\leq \exp\left(-\frac{n}{2}\left(1 - \frac{\varepsilon}{hL}\right)^2\right). \end{aligned} \quad (44)$$

2. To obtain a large lower bound on the covering number of $\mathcal{A}_{[L, M, m, b]}$, let us maximize the map

$$\Psi(h, n) \doteq \frac{n}{2} \left(1 - \frac{\varepsilon}{hL}\right)^2,$$

with the parameters h, n , subject to (39) and (43). If we first optimize $\Psi(h, n)$ with respect to h (letting n be sufficiently large so that the first two constraints in (39) be satisfied) we find that the maximum is attained for

$$h_n \doteq \frac{2bL}{n}. \quad (45)$$

Next, optimizing $\Psi(h_n, n)$ for n satisfying (43), we deduce that the maximum is attained for

$$\bar{n} \doteq \frac{2bL^2}{3\varepsilon}. \quad (46)$$

One can check that

$$\frac{\bar{n}\varepsilon}{2h_{\bar{n}}L} = \frac{\bar{n}}{6} < \frac{\bar{n}}{2},$$

so that, with $h_{\bar{n}}, \bar{n}$ defined by (45), (46), conditions (39), (43) are both verified provided that ε satisfies (23). Hence, we deduce from (44) that

$$\frac{\mathcal{C}(\varepsilon)}{2^n} \leq \exp(-\Psi(h_{\bar{n}}, \bar{n})) = \exp\left(-\frac{1}{\varepsilon} \cdot \frac{4bL^2}{27}\right).$$

Now observe that any ε -cover of $\mathcal{A}_{[L, M, m, b]}$, in particular, contains the set

$$\mathcal{F} \doteq \{\mathcal{F}_\iota : \mathbb{R} \rightarrow [-h_{\bar{n}}, h_{\bar{n}}]; \iota \in \{-1, 1\}^n\},$$

and that each element of this cover contains at most $\mathcal{C}(2\varepsilon)$ functions of \mathcal{F} . Since the cardinality of \mathcal{F} is 2^n , it follows that the number of sets in an ε -cover of $\mathcal{A}_{[L, M, m, b]}$ is at least

$$N_\varepsilon(\mathcal{A}_{[L, M, m, b]}) \geq \frac{2^n}{\mathcal{C}(2\varepsilon)} \geq \exp\left(\frac{1}{\varepsilon} \cdot \frac{2bL^2}{27}\right),$$

which yields (24), thus completing the proof of Proposition 2. \square

3 Lower compactness estimates for balance laws

3.1 Proof of Theorem 3

In order to establish Theorem 3, we will make use of a local Oleinik type estimate for balance laws (9), that extends (33) taking into account the contribution of the source term. An inequality of this kind was established in [12, Theorem 1.2]. Here, we provide a slightly more accurate estimate, determining how the constant C appearing in [12, Theorem 1.2] depends on the time t and on the set of points x, y for which the inequality holds. For source terms of the form $g = g(u)$, a global Oleinik type estimate was obtained in [6, Section 4]. Recall the notation $k_{a,b,t}(M)$ from (16).

Lemma 1. *Suppose that $f \in C^2(\mathbb{R}; \mathbb{R})$ and $g \in C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfy (2), (10) and (12). Given $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, let $u : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding entropy admissible solution of (9) with initial condition $u(0, \cdot) = u_0$, and set*

$$M_t \doteq \|u_0\|_{L^\infty} \cdot \exp(\|\omega\|_{L^1(0,t)}) \quad t \geq 0. \quad (47)$$

Then, for all $t > 0$, and for any given $a, b \in \mathbb{R}$, $a < b$, there holds

$$\frac{u(t, y) - u(t, x)}{y - x} \leq \frac{(1 + ct^2 k_{a,b,t}(M_t)) \cdot \exp(\|\omega\|_{L^1(0,t)})}{ct} \quad \forall x, y \in [a, b], \quad x < y. \quad (48)$$

Remark 5. *From the proof of the lemma it will be clear that, when the source term has the form $g = g(t, u)$, and satisfies (12), but not necessarily (10), one obtains the global Oleinik estimate*

$$\frac{u(t, y) - u(t, x)}{y - x} \leq \frac{\exp(\|\omega\|_{L^1(0,t)})}{ct} \quad \forall x, y \in \mathbb{R}, \quad x < y, \quad (49)$$

which is a bit more accurate than the one provided in [6] (showing that the constant C appearing in [6, Section 4] is precisely $1/c$).

Relying on Lemma 1, we will establish the analogous result of Proposition 1 in the case of balance laws, which together with Proposition 2 yields the conclusion of Theorem 3.

Proposition 3. *Under the assumptions of Lemma 1, assume that f, g satisfy also (3) and (11), respectively. Then, given any $L, m, M, T > 0$, for*

$$h \leq \min\left(M, \frac{m}{2L}, \frac{L}{8T|f''(0)|}\right) \exp(-\|\omega\|_{L^1(0,T)}), \quad (50)$$

sufficiently small, one has

$$\mathcal{A}_{[\tilde{L}_T, Lh, h, (2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)}))^{-1}]} \subset E_T(\mathcal{C}_{[L, m, M]}), \quad (51)$$

with

$$\tilde{L}_T \doteq L - 2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)}) \cdot h. \quad (52)$$

Note that, thanks to (50), the constant \tilde{L}_T given in (52) satisfies $\tilde{L}_T \geq (3/4)L$. Hence, relying on (51), and applying (24), with $L = \tilde{L}_T$, $b = (2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)}))^{-1}$, we recover the estimate (14), which proves Theorem 3.

Remark 6. For balance laws (9) with source term not satisfying condition (11), a result as the one provided by Proposition 3 is in general false. If we consider for example the balance law

$$u_t + (u^2/2)_x = -x, \quad (53)$$

with initial data $u_0 \equiv 0$, by a direct computation along the characteristics we find that a shock discontinuity is generated in the corresponding solution at time $T = \pi/2$. It follows that the zero constant cannot be attained as the value of a solution to (53) at any time $T > \pi/2$. A way to see this is that, should a solution reach 0 at time $T > \pi/2$, it would not contain a shock for $t \in [0, T]$ as a consequence of [3, Theorem 11.9.4], and hence it would be reversible in time, which yields a contradiction with the shock generation mentioned above. Another way to see this is that by tracing characteristics backwards on such a solution, one would find a centered rarefaction wave at positive time.

But on the contrary, one clearly has $0 \in \mathcal{A}_{[L', m', M', b']}$ for any $L', m', M', b > 0$. Hence, if we let E_t denote the evolution operator associated to (53), the inclusion $\mathcal{A}_{[L', m', M', b]} \subset E_T(\mathcal{C}_{[L, m, M]})$ will be never satisfied for $T > \pi/2$, no matter how we choose the constants L', m', M', b .

3.2 Proof of Lemma 1

1. It will be sufficient to prove (48) when the initial data $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, since one can then recover (48) for general data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ relying on the L^1 -continuity of the evolution operator E_t , $t > 0$, and on the lower semicontinuity of the positive variation. Therefore, we may assume that $u(t, \cdot) \doteq E_t u_0 \in BV(\mathbb{R})$ for all $t > 0$, and thus we can use again Dafermos' theory of generalized characteristics (we refer to [3, Section 11.9] for the general case of balance laws). Moreover, it is not restrictive to suppose that $u(t, \cdot)$ is right continuous, and to establish (48) only at points $x, y \in [a, b]$ where $u(t, \cdot)$ is continuous (since one then derives (48) at the points of discontinuity taking the right limits of $u(t, \cdot)$).

Observe that, if $\xi(\cdot)$ denotes the maximal backward generalized characteristic emanating from a point (t, x) , then by [3, Theorem 11.9.1] there is some C^1 function $v(\cdot)$ that, together with $\xi(\cdot)$, satisfies on $]0, t[$ the characteristic equation

$$\begin{cases} \dot{\xi}(s) = f'(v(s)), \\ \dot{v}(s) = g(s, \xi(s), v(s)), \end{cases} \quad (54)$$

with

$$\xi(t) = x, \quad v(t) = u(t, x). \quad (55)$$

Furthermore, there holds $u(s, \xi(s) \pm) = v(s)$ for all $s \in]0, t[$, and

$$u_0(\xi(0)-) \leq v(0) \leq u_0(\xi(0)+). \quad (56)$$

Therefore, since (13), (54) imply

$$\frac{d}{ds} |v(s)| \leq \omega(s) \cdot |v(s)|, \quad (57)$$

applying Gronwall's lemma, and using (55), (56), we deduce

$$|u(t, x)| \leq \max\{|u_0(\xi(0)-)|, |u_0(\xi(0)+)|\} \cdot \exp(\|\omega\|_{L^1(0,t)}).$$

In turn, this yields

$$\|u(s, \cdot)\|_{L^\infty} \leq M_t \quad \forall s \in [0, t], \quad t \geq 0, \quad (58)$$

with M_t defined by (47). Relying on (58), and because of (54), we deduce that the set $\Delta_{a,b,t}(M_t)$ defined in (15) is a backward domain of determinacy relative to the interval $[a, b]$ and to the time t , since it contains all backward generalized characteristics emanating from points (t, x) , $x \in [a, b]$.

2. Fix $t > 0$, $a, b \in \mathbb{R}$, $a < b$, and consider $x < y$ two points of continuity of $u(t, \cdot)$ inside $[a, b]$. Let $\xi^x(\cdot)$ and $\xi^y(\cdot)$ be the (unique) backward generalized characteristics (cfr. [3, Theorems 11.9.5]) emanating from (t, x) and (t, y) , respectively. By [3, Theorems 11.9.1 & 11.9.3] there will be some C^1 functions $v^x(\cdot), v^y(\cdot)$, so that $(\xi^x(\cdot), v^x(\cdot))$ and $(\xi^y(\cdot), v^y(\cdot))$ satisfy on $]0, t[$ the characteristic equations (54) with

$$\xi^x(t) = x, \quad v^x(t) = u(t, x) \quad \text{and} \quad \xi^y(t) = y, \quad v^y(t) = u(t, y). \quad (59)$$

Observe that, if $u(t, x) \geq u(t, y)$, the inequality (48) is certainly satisfied since its right-hand side is always positive. Therefore, by virtue of (59) we will consider only the case $v^x(t) < v^y(t)$. Then, set

$$\bar{\sigma} \doteq \inf \{ \sigma \in]0, t[\mid v^x(s) < v^y(s) \quad \forall s \in [\sigma, t] \}, \quad (60)$$

and observe that, by the strict convexity assumption (2) on f , and because of (54), one has

$$y - x \geq \xi^y(s) - \xi^x(s) \quad \forall s \in [\bar{\sigma}, t]. \quad (61)$$

Moreover, since $\xi^x(\cdot)$ and $\xi^y(\cdot)$ do not cross at any time $s \in]0, t[$ (cfr. [3, Section 11.9]), it follows that

$$\xi^y(s) \geq \xi^x(s) \quad \forall s \in [0, t]. \quad (62)$$

Then, relying on (12), (54), (58), (61), and recalling (16), we deduce that for all $s \in]\bar{\sigma}, t[$, there holds

$$\begin{aligned} \dot{v}^y(s) - \dot{v}^x(s) &= g(s, \xi^y(s), v^y(s)) - g(s, \xi^y(s), v^x(s)) + g(s, \xi^y(s), v^x(s)) - g(s, \xi^x(s), v^x(s)) \\ &\leq \omega(s)(v^y(s) - v^x(s)) + k_{a,b,t}(M_t)(\xi^y(s) - \xi^x(s)) \\ &\leq \omega(s)(v^y(s) - v^x(s)) + k_{a,b,t}(M_t)(y - x). \end{aligned} \quad (63)$$

Hence, using Gronwall's lemma, from (63) we derive

$$\begin{aligned} v^y(s) - v^x(s) &\geq (v^y(t) - v^x(t)) \exp\left(-\int_s^t \omega(\tau) d\tau\right) - (t-s)k_{a,b,t}(M_t)(y-x) \\ &\geq (v^y(t) - v^x(t)) \exp\left(-\int_0^t \omega(\tau) d\tau\right) - tk_{a,b,t}(M_t)(y-x) \end{aligned} \quad \forall s \in [\bar{\sigma}, t]. \quad (64)$$

Two cases now may occur. If $\bar{\sigma} > 0$, by the definition (60) and because of the continuity of $v^x(\cdot), v^y(\cdot)$ it follows that $v^x(\bar{\sigma}) = v^y(\bar{\sigma})$, which together with (64), and recalling (59), yields

$$u(t, y) - u(t, x) \leq tk_{a,b,t}(M_t) \exp\left(\int_0^t \omega(\tau) d\tau\right)(y-x). \quad (65)$$

Instead, if $\bar{\sigma} = 0$, using (64), and relying on (2), (62), we deduce

$$\begin{aligned} 0 &\leq \xi^y(0) - \xi^x(0) \\ &= y - x - \int_0^t (f'(v^y(s)) - f'(v^x(s))) ds \\ &\leq y - x - c \int_0^t (v^y(s) - v^x(s)) ds \\ &\leq (1 + ct^2 k_{a,b,t}(M_t))(y-x) - ct(v^y(t) - v^x(t)) \exp\left(-\int_0^t \omega(\tau) d\tau\right), \end{aligned}$$

which, by virtue of (59), yields

$$u(t, y) - u(t, x) \leq \frac{(1 + ct^2 k_{a,b,t}(M_t))(y - x)}{ct} \exp\left(\int_0^t \omega(\tau) d\tau\right). \quad (66)$$

Hence, from (65), (66) we recover the inequality (48) concluding the proof of the lemma.

Notice that the assumption (10) was used, in conjunction with (12), only to establish the a-priori bound (58) on the L^∞ norm of the solution, which in turn was needed to define a bound on $D_x g$ over a domain of determinacy of the solution. Therefore, as observed in Remark 5, the conclusion of the lemma continues to hold (with $k_{a,b,t}(M_t) = 0$ in (48)) in the case the source term $g = g(t, u)$ satisfies only the assumption (12). \square

3.3 Proof of Proposition 3

1. In the same spirit of the proof of Proposition 1, we will show that any element u_T of the set on the left-hand side of (51) can be obtained as the value at time T of a weak admissible solution to (9) constructed by reversing the direction of time, starting from u_T . Namely, given

$$u_T \in \mathcal{A}_{[\tilde{L}_T, Lh, h, (2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)}))^{-1}]}, \quad (67)$$

setting

$$\tilde{g}(t, x, u) \doteq -g(T - t, -x, u),$$

we consider the entropy weak solution $w(t, x) \doteq \tilde{E}_t w_0$ of

$$w_t + f(w)_x = \tilde{g}(t, x, w),$$

with initial data

$$w_0(x) \doteq u_T(-x) \quad \forall x \in \mathbb{R}. \quad (68)$$

Observe that, by the estimate (58) established in the proof of Lemma 1, and because of (50), (67), (68), there holds

$$\begin{aligned} \|w(t, \cdot)\|_{L^\infty} &\leq \exp(\|\omega\|_{L^1(0,T)}) \|u_T\|_{L^\infty} \\ &\leq \exp(\|\omega\|_{L^1(0,T)}) h \leq M. \end{aligned} \quad \forall t \in [0, T]. \quad (69)$$

Moreover, as discussed in the proof of Lemma 1, the solution propagates along generalized characteristics that satisfy the equation (54), and hence, by virtue of (67), (69), we derive the bound on the support of $w(t, \cdot)$:

$$\text{Supp}(w(t, \cdot)) \subset [-l(t), l(t)] \quad \forall t \in [0, T], \quad (70)$$

with

$$l(t) \doteq \tilde{L}_T + t \|f'\|_{L^\infty(-\rho_h, \rho_h)}, \quad \rho_h \doteq \exp(\|\omega\|_{L^1(0,T)}) h. \quad (71)$$

On the other hand, observing that by the continuity of f'' we may assume that

$$|f''(\rho_h)| \leq 2|f''(0)|, \quad (72)$$

for $|h|$ sufficiently small, relying on (3) we derive

$$\begin{aligned} l(t) &\leq \tilde{L}_T + t \rho_h \|f''\|_{L^\infty(-\rho_h, \rho_h)} \\ &\leq \tilde{L}_T + 2T |f''(0)| \rho_h. \end{aligned} \quad (73)$$

Hence, recalling (52), we deduce from (70), (71) and (73) the uniform bound on the support of w :

$$\text{Supp}(w(t, \cdot)) \subset [L, L] \quad \forall t \in [0, T], \quad (74)$$

which, in turn, together with (50), (69), yields

$$\|w(t, \cdot)\|_{L^1} \leq 2L \exp(\|\omega\|_{L^1(0,T)}) h \leq m \quad \forall t \in [0, T]. \quad (75)$$

Next, observe that the function

$$u(t, x) \doteq w(T - t, -x), \quad \forall (t, x) \in [0, T] \times \mathbb{R},$$

is a weak distributional solution of (9), which, by (68), satisfies

$$u(T, \cdot) = u_T,$$

and thanks to (69), (74), (75), verifies $u(0, \cdot) \in \mathcal{C}_{[L, m, M]}$. Moreover, note that Lemma 1 guarantees that $w(t, x)$ verifies the corresponding upper one-side inequality (48). Therefore, to conclude the proof of Proposition 3 it will be sufficient to show that w satisfies also a lower semi-Lipschitz estimate, since then one deduces that $u(t, \cdot) = w(T - t, -\cdot)$ is locally Lipschitz and hence automatically satisfies the entropy condition (4).

2. We will show that the initial lower bound

$$\frac{w_0(y-) - w_0(x+)}{y - x} \geq -\frac{1}{2T|f''(0)| \exp(\|\omega\|_{L^1(0, T)})} \quad \forall x, y \in \mathbb{R}, \quad x < y, \quad (76)$$

induced by (67), (68), propagates over the time interval $]0, T[$, i.e. we will prove that, for any fixed $t \in]0, T[$, there exists some constant $\tilde{C}_t > 0$ such that there holds:

$$\frac{w(t, y) - w(t, x)}{y - x} \geq -\tilde{C}_t \quad \forall x, y \in \mathbb{R}, \quad x < y. \quad (77)$$

As in the proof of Lemma 1, it is not restrictive to establish (77) only at points $x, y \in \mathbb{R}$ where $w(t, \cdot)$ is continuous. Observe that (77) is automatically verified when $w(t, y) - w(t, x) \geq 0$. Hence, we only need to consider the case where

$$w(t, x) > w(t, y). \quad (78)$$

Letting $\xi^x(\cdot)$, $\xi^y(\cdot)$ denote the backward generalized characteristics emanating from (t, x) and (t, y) , respectively, there will be some C^1 functions $v^x(\cdot)$ and $v^y(\cdot)$ so that $(\xi^x(\cdot), v^x(\cdot))$ and $(\xi^y(\cdot), v^y(\cdot))$ satisfy on $]0, t[$ the characteristic equation (54), with \tilde{g} in place of g , and

$$\xi^x(t) = x, \quad v^x(t) = w(t, x) \quad \text{and} \quad \xi^y(t) = y, \quad v^y(t) = w(t, y). \quad (79)$$

Furthermore, there holds

$$v^x(0) \leq w_0(\xi^x(0)+), \quad v^y(0) \geq w_0(\xi^y(0)-). \quad (80)$$

We will now prove that the inequality (78) between $v^x(t)$ and $v^y(t)$ is preserved on the whole interval $]0, t[$, i.e. there holds

$$v^x(s) > v^y(s) \quad \forall s \in]0, t[. \quad (81)$$

Toward a proof of this claim, set

$$\bar{\sigma} \doteq \inf \{ \sigma \in]0, t[\mid v^x(s) > v^y(s) \quad \forall s \in [\sigma, t] \}, \quad (82)$$

and observe that, by the strict convexity of f , and because of (54), one has

$$\xi^x(s) < \xi^y(s) \quad \forall s \in [\bar{\sigma}, t]. \quad (83)$$

Moreover, the assumption (11) on $D_x g$ implies $\tilde{D}_x \tilde{g} \geq 0$. Then, relying on (12), (54), (83), and by the definition (82), we find

$$\begin{aligned} \dot{v}^x(s) - \dot{v}^y(s) &= \tilde{g}(s, \xi^x(s), v^x(s)) - \tilde{g}(s, \xi^x(s), v^y(s)) + \tilde{g}(s, \xi^x(s), v^y(s)) - \tilde{g}(s, \xi^y(s), v^y(s)) \\ &\leq \omega(s)(v^x(s) - v^y(s)), \end{aligned} \quad (84)$$

for all $s \in]\bar{\sigma}, t[$. Hence, using Gronwall's lemma, from (84) we derive

$$\begin{aligned} v^x(s) - v^y(s) &\geq (v^x(t) - v^y(t)) \exp\left(-\int_s^t \omega(\tau) d\tau\right) \\ &\geq (v^x(t) - v^y(t)) \exp\left(-\int_0^t \omega(\tau) d\tau\right) > 0 \end{aligned} \quad \forall s \in [\bar{\sigma}, t], \quad (85)$$

which, in particular, implies $v^x(\bar{\sigma}) > v^y(\bar{\sigma})$. Thus, by the definition of $\bar{\sigma}$ in (82), and by the continuity of $v^x(\cdot), v^y(\cdot)$, it follows that $\bar{\sigma} = 0$.

3. Since $\bar{\sigma} = 0$, by (83) and (85) we deduce that

$$\xi^x(0) < \xi^y(0), \quad v^x(0) > v^y(0). \quad (86)$$

Relying on (69), (72), (76), (80), (81), (86), and using (85) with $\bar{\sigma} = 0$, we find

$$\begin{aligned} \Delta_0 &\doteq w_0(\xi^x(0)+) - w_0(\xi^y(0)-) \leq \frac{\xi^y(0) - \xi^x(0)}{2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)})} \\ &= \frac{y - x + \int_0^t (f'(v^x(s)) - f'(v^y(s))) ds}{2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)})} \\ &\leq \frac{y - x}{2T|f''(0)| \exp(\|\omega\|_{L^1(0,T)})} + \frac{t}{T} \cdot \Delta_0 \end{aligned}$$

which yields

$$w_0(\xi^x(0)+) - w_0(\xi^y(0)-) \leq \frac{y - x}{2(T-t)|f''(0)| \exp(\|\omega\|_{L^1(0,T)})}. \quad (87)$$

Hence, applying again (85) with $\bar{\sigma} = 0$, thanks to (80), (79) and (87) we obtain

$$\begin{aligned} w(t, y) - w(t, x) &\geq \exp(\|\omega\|_{L^1(0,T)}) (w_0(\xi^y(0)-) - w_0(\xi^x(0)+)) \\ &\geq -\frac{y - x}{2(T-t)|f''(0)|}, \end{aligned}$$

which proves (77) with

$$\tilde{C}_t \doteq \frac{1}{2(T-t)|f''(0)|},$$

thus completing the proof of the proposition. \square

4 Upper compactness estimates for balance laws

4.1 Proof of Theorem 4

Following the arguments of De Lellis and Golse in [4], we shall establish an upper estimate on the ε -entropy of $E_T(\mathcal{C}_{[L,m,M]})$ relying on the upper bound on the ε -entropy of a class of nondecreasing functions provided by:

Lemma 2. ([4, Lemma 3.1]) *Given any, $L, V > 0$, setting*

$$\mathcal{I}_{[L,V]} \doteq \{v : [0, L] \rightarrow [0, V] \mid v \text{ is nondecreasing}\},$$

for $0 < \varepsilon < \frac{LV}{6}$, there holds

$$H_\varepsilon(\mathcal{I}_{[L,V]} \mid L^1([0, L])) \leq 4 \left\lceil \frac{LV}{\varepsilon} \right\rceil.$$

In order to derive an a-priori bound on the support of solutions to balance laws in terms of the L^1 norm of their initial data, we will use the next technical lemma whose proof is provided below.

Lemma 3. *Given $v \in \text{BV}(\mathbb{R})$, compactly supported and satisfying*

$$Dv \leq B \quad \text{in the sense of measures,} \quad (88)$$

for some constant $B > 0$, there holds

$$\|v\|_{L^\infty} \leq \sqrt{2B\|v\|_{L^1}}. \quad (89)$$

Now, supposing Lemma 3 proven, one establishes Theorem 4 as follows.

1. Given any $u_0 \in \mathcal{C}_{[L,m,M]}$, we shall first derive an a-priori bound on the L^1 norm of the corresponding entropy weak solution $u(t, \cdot) \doteq E_t u_0$ of (9), relying on the Oleinik type estimates provided by Lemma 1. To this end, observe that by standard arguments for conservation laws (e.g. cfr. proof of [3, Theorem 11.8.2]), and thanks to (13), we find

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^1} \leq \|g(t, \cdot, u(t, \cdot))\|_{L^1} \leq \omega(t) \cdot \|u(t, \cdot)\|_{L^1}.$$

Hence, applying Gronwall's lemma, and by the definition (5), we deduce

$$\|u(t, \cdot)\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|u_0\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,T)}) \cdot m \quad \forall t \in [0, T]. \quad (90)$$

Moreover, as discussed in the proof of Proposition 3, because of (5) and (58), we have

$$\|u(t, \cdot)\|_{L^\infty} \leq M_T \doteq \exp(\|\omega\|_{L^1(0,T)}) \cdot M \quad \forall t \in [0, T], \quad (91)$$

and with (3), (56) and (57), that

$$\text{Supp}(u(t, \cdot)) \subset [-l(t), l(t)] \quad \forall t \in [0, T], \quad (92)$$

with

$$l(t) \leq L + \|f''\|_{L^\infty(-M_T, M_T)} \int_0^t \|u(t, \cdot)\|_{L^\infty} dt. \quad (93)$$

On the other hand, observe that (91)-(93) and the estimate (48) imply $u(t, \cdot) \in \text{BV}(\mathbb{R})$, and

$$Du(t) \leq \frac{(1 + ct^2 K_{L,T}) \cdot \exp(\|\omega\|_{L^1(0,T)})}{ct} \quad \forall t \in]0, T], \quad (94)$$

with

$$K_{L,T} \doteq k_{-L_T, L_T, T}(M_T), \quad L_T \doteq L + \|f''\|_{L^\infty(-M_T, M_T)} \cdot M_T T,$$

($k_{-L_T, L_T, T}(M_T)$ being defined in (16)). Hence, applying Lemma 3 and relying on (90), (94), we derive

$$\|u(t, \cdot)\|_{L^\infty} \leq \sqrt{\frac{2m}{c}} \cdot \exp(\|\omega\|_{L^1(0,T)}) \sqrt{\frac{(1 + ct^2 K_{L,T})}{t}} \quad \forall t \in]0, T], \quad (95)$$

which, together with (93), yields

$$\begin{aligned} l(t) &\leq L + \|f''\|_{L^\infty(-M_T, M_T)} \sqrt{\frac{2m}{c}} \cdot \exp(\|\omega\|_{L^1(0,T)}) \int_0^t \sqrt{\frac{(1 + ct^2 K_{L,T})}{t}} dt \\ &\leq L(T) \doteq L + 2\|f''\|_{L^\infty(-M_T, M_T)} \sqrt{\frac{2mT}{c}} \left[1 + T\sqrt{cK_{L,T}} \right] \cdot \exp(\|\omega\|_{L^1(0,T)}). \end{aligned} \quad (96)$$

Then, thanks to (2) and (95)-(96), we deduce

$$\|u(T, \cdot)\|_{L^\infty} \leq \frac{L(T)}{2cT}. \quad (97)$$

2. In connection with $u_T \doteq E_T u_0$, consider now the function

$$u_\#(x) \doteq \frac{(1 + cT^2 K_{L,T}) \exp(\|\omega\|_{L^1(0,T)})}{cT} x + \frac{L(T)}{2cT} - u_T(x - L(T)), \quad x \in [0, 2L(T)].$$

By virtue of (94), $u_\#$ is nondecreasing and, thanks to (97), satisfies

$$0 \leq u_\#(x) \leq \frac{\left(1 + 2(1 + cT^2 K_{L,T}) \exp(\|\omega\|_{L^1(0,T)})\right) L(T)}{cT} \quad \forall x \in [0, 2L(T)].$$

Hence, one has

$$u_\# \in \mathcal{I} \left[2L(T), \left(1 + 2(1 + cT^2 K_{L,T}) \exp(\|\omega\|_{L^1(0,T)})\right) \frac{L(T)}{cT} \right].$$

Since $u_\#$ is obtained from the restriction of u_T to $[-L(T), L(T)]$ by a change of sign, a translation by a fixed function, and a shift of a fixed constant, it follows that the ε -entropy relative to $L^1(0, 2L(T))$ of the set gathering all functions $u_\#$ obtained from $E_T u_0$, $u_0 \in \mathcal{C}_{[L,m,M]}$, in this way, equals the one of $E_T(\mathcal{C}_{[L,m,M]})$ relative to $L^1(-L(T), L(T))$. Therefore, the conclusion of Theorem 4 follows by an application of Lemma 2, and observing that, because of (92) and (96), one clearly has

$$H_\varepsilon(E_T(\mathcal{C}_{[L,m,M]}) | L^1(\mathbb{R})) = H_\varepsilon(E_T(\mathcal{C}_{[L,m,M]}) | L^1(-L(T), L(T))).$$

□

4.2 Proof of Lemma 3

We shall first establish the conclusion of the Lemma 3 for compactly supported functions v , that belong to $C_0^\infty(\mathbb{R})$, and thus by (88) satisfy $v'(x) \leq B$ for all $x \in \mathbb{R}$. Assume that $\text{Supp}(v) \subset [-L, L]$, and consider a point $\bar{x} \in [-L, L]$ such that $|v(\bar{x})| = \|v\|_\infty$. We discuss two cases according to the sign of $v(\bar{x})$.

If $v(\bar{x}) > 0$, defining $y \doteq \min\{x \in [-L, \bar{x}] \mid v > 0 \text{ on }]x, \bar{x}]\}$, one has $v(y) = 0$, and

$$v^2(\bar{x}) = 2 \int_y^{\bar{x}} v(x) v'(x) dx \leq 2B \int_y^{\bar{x}} v(x) dx \leq 2B \|v\|_{L^1(\mathbb{R})}.$$

If $v(\bar{x}) < 0$, defining $y \doteq \max\{x \in [\bar{x}, L] \mid v < 0 \text{ on } [\bar{x}, x[\}$, one has

$$v^2(\bar{x}) = -2 \int_{\bar{x}}^y v(x) v'(x) dx \leq -2B \int_{\bar{x}}^y v(x) dx \leq 2B \|v\|_{L^1(\mathbb{R})}.$$

Hence, in both cases we get the estimate (89) when v is smooth.

For general $v \in \text{BV}(\mathbb{R})$ satisfying the assumptions of the lemma, consider $\rho \in C_0^\infty(\mathbb{R})$, with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho = 1$, define the mollifier $\rho_\nu(x) \doteq \frac{1}{\nu} \rho\left(\frac{x}{\nu}\right)$, for $\nu > 0$, and then introduce a smooth approximation of v setting

$$v_\nu \doteq \rho_\nu * v.$$

Observe that, by standard properties of convolutions, and applying the integration-by-parts formula for BV functions, relying on (88) one gets

$$v'_\nu - B = \rho'_\nu * v - \rho_\nu * B = \rho_\nu * (Dv - B) \leq 0.$$

Hence, by the above conclusion we can apply (89) to v_ν , finding

$$\|v_\nu\|_{L^\infty} \leq \sqrt{2B \|v_\nu\|_{L^1}}. \quad (98)$$

Since $v_\nu \rightarrow v$ in $L^1(\mathbb{R})$ as $\nu \rightarrow 0^+$, and because there holds $\|v_\nu\|_\infty \rightarrow \|v\|_\infty$ as $\nu \rightarrow 0^+$, we then recover from (98) the estimate (89) for v , thus completing the proof of the lemma. □

4.3 Proof of Remark 4

Let us discuss the case when $g = g(t, u)$ satisfies (12), but does not necessarily satisfy (10). As observed in Remark 5, in this case we can still rely on the Oleinik type estimate (49). On the other hand, the solution of (9) will be in general not compactly supported, since the solution with zero initial data will be in general different from zero. Then, to establish the same type of estimates of Theorem 3 and Theorem 4,

it will be more appropriate to consider the ε -entropy of the set of functions $E_T(\mathcal{C}_{[L,m,M]}) - E_T 0$ (which are compactly supported by the finite speed of propagation of solutions along characteristics) rather than the ε -entropy of $E_T(\mathcal{C}_{[L,m,M]})$. Clearly, the two sets have the same ε -entropy, being obtained one from the other by a translation.

1. Upper estimate. Observe now that, if $t \mapsto v^0(t)$ denotes the solution of the Cauchy problem $\dot{v} = g(t, v)$, $v(0) = 0$, then the function defined by setting $v(t, x) \doteq v^0(t)$, for all $x \in \mathbb{R}$, results to be the (admissible) solution of (9) with initial data $u_0 \equiv 0$. Hence, one has

$$E_t 0 = v^0(t), \quad \forall t > 0. \quad (99)$$

Thus, setting

$$L_1(t) \doteq \inf \{x \in \text{Supp}(E_t u_0 - E_t 0) \mid u_0 \in \mathcal{C}_{[L,m,M]}\}, \quad L_2(t) \doteq \sup \{x \in \text{Supp}(E_t u_0 - E_t 0) \mid u_0 \in \mathcal{C}_{[L,m,M]}\}, \quad (100)$$

for every $u_0 \in \mathcal{C}_{[L,m,M]}$ there holds

$$E_t u_0(x) = v^0(t) \quad \forall t > 0, \forall x < L_1(t) \text{ or } x > L_2(t). \quad (101)$$

Relying on (12), (101), one then deduces as in the proof of Theorem 4 that

$$\frac{d}{dt} \|E_t u_0 - E_t 0\|_{L^1} \leq \|g(t, E_t u_0) - g(t, E_t 0)\|_{L^1} \leq \omega(t) \cdot \|E_t u_0 - E_t 0\|_{L^1},$$

which, in turn, yields the estimate on the L^1 norm

$$\|E_t u_0 - E_t 0\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|u_0\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,T)}) \cdot m \quad \forall t \in [0, T]. \quad (102)$$

With similar arguments, one can derive as in the proof of Lemma 1 the a-priori bound on the L^∞ norm

$$\|E_t u_0 - E_t 0\|_{L^\infty} \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|u_0\|_{L^\infty} \leq M_T \doteq \exp(\|\omega\|_{L^1(0,T)}) \cdot M \quad \forall t \in [0, T]. \quad (103)$$

On the other hand, applying Lemma 3, and thanks to (49) and (102), we derive

$$\|E_t u_0 - E_t 0\|_{L^\infty} \leq \sqrt{\frac{2m}{ct}} \cdot \exp(\|\omega\|_{L^1(0,T)}) \quad \forall t \in]0, T]. \quad (104)$$

Given any $u_1, u_2 \in \mathcal{C}_{[L,m,M]}$, we introduce

$$l_1(u_1, t) \doteq \inf \text{Supp}(E_t u_1 - E_t 0) \quad \text{and} \quad l_2(u_2, t) \doteq \sup \text{Supp}(E_t u_2 - E_t 0).$$

We consider the maximal backward characteristic emanating from $(t, l_1(u_1, t))$ associated to $E_{(\cdot)} u_1$, denoted $\xi_1(\cdot)$, and the minimal backward characteristic emanating from $(t, l_2(u_2, t))$ associated to $E_{(\cdot)} u_2$, denoted $\xi_2(\cdot)$. Then there are some C^1 functions $v_1(\cdot)$ and $v_2(\cdot)$ so that $(\xi_1(\cdot), v_1(\cdot))$ and $(\xi_2(\cdot), v_2(\cdot))$ satisfy on $]0, t[$ the characteristic equation (54), with $g(s, v(s))$ in place of $g(s, \xi(s), v(s))$, and with $\xi_1(t) = l_1(u_1, t)$, $v_1(t) = E_t u_1(l_1(u_1, t))$, and $\xi_2(t) = l_2(u_2, t)$, $v_2(t) = E_t u_2(l_2(u_2, t))$. Observe that by the properties of the characteristics one clearly has $-L \leq \xi_i(0) \leq L$, $i = 1, 2$, and

$$|v_i(s)| = |E_s(u_i)(\xi_i(s))| \leq \|E_s(u_i)\|_{L^\infty} \quad \forall s \in]0, t[, \quad i = 1, 2.$$

Moreover, using that $g(\cdot, 0) \in L^1_{loc}$, and applying Gronwall's inequality we find

$$|v^0(s)| \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|g(\cdot, 0)\|_{L^1(0,t)} \quad \forall s \in [0, t]. \quad (105)$$

Hence, relying on (99), (103), (105), we derive the estimates

$$\max(|v_1(s)|, |v_2(s)|) \leq M_T^g \doteq \exp(\|\omega\|_{L^1(0,T)}) (M + \|g(\cdot, 0)\|_{L^1(0,T)}) \quad \forall s \in [0, t],$$

and using (3),

$$|l_2(u_2, t) - l_1(u_1, t)| \leq 2L + \|f''\|_{L^\infty(-M_T^g, M_T^g)} \int_0^t |v_2(s) - v_1(s)| ds. \quad (106)$$

On the other hand, thanks to (104), we get

$$\begin{aligned} |v_2(s) - v_1(s)| &\leq \|E_s(u_2) - E_s(0)\|_{L^\infty} + \|E_s(u_1) - E_s(0)\|_{L^\infty} \\ &\leq 2\sqrt{\frac{2m}{cs}} \cdot \exp(\|\omega\|_{L^1(0,T)}) \quad \forall s \in]0, t]. \end{aligned} \quad (107)$$

Recalling (100), by the arbitrariness of $u_1, u_2 \in \mathcal{C}_{[L,m,M]}$, we deduce from (106), (107) the estimate

$$L_2(t) - L_1(t) \leq 2L(T) \doteq 2L + 4\|f''\|_{L^\infty(-M_T^g, M_T^g)} \sqrt{\frac{2mT}{c}} \cdot \exp(\|\omega\|_{L^1(0,T)}) \quad \forall t \in [0, T],$$

which in turn, together with (2), (104), yields

$$\|E_T u_0 - E_T 0\|_{L^\infty} \leq \frac{L(T)}{2cT}. \quad (108)$$

Then, for any given $u_0 \in \mathcal{C}_{[L,m,M]}$, setting $u_T \doteq E_T u_0 - E_T 0$, we consider the function

$$u_\#(x) \doteq \frac{\exp(\|\omega\|_{L^1(0,T)})}{cT} x + \frac{L(T)}{2cT} - u_T(x + L_1(T)), \quad x \in [0, 2L(T)],$$

that results to be an element of $\mathcal{I}_{[2L(T), \frac{L(T)}{cT}(1+2\exp(\|\omega\|_{L^1(0,T)}))]}$, by virtue of (49) and (108). With the same arguments of the proof of Theorem 4, applying Lemma 2, we thus obtain the estimate (17).

2. Lower estimate. For what concerns the lower bound on the ε -entropy of the set of solutions of (9), with $g = g(t, u)$ satisfying (12) together with $g(\cdot, 0) \in L^1_{loc}$, following the same lines of the proof of Proposition 3 one can show that, for

$$h \leq \left(1, M, \frac{m}{L}, \frac{L}{4T\|f''\|_{L^\infty(-G_T, G_T)}}\right) \exp(-\|\omega\|_{L^1(0,T)}), \quad (109)$$

sufficiently small, there holds

$$\mathcal{A}_T \doteq \mathcal{A}_{[\tilde{L}_T, Lh, h, (T\|f''\|_{L^\infty(-G_T, G_T)} \exp(\|\omega\|_{L^1(0,T)}))^{-1}]} \subseteq \mathcal{T}_{\tau_T}(E_T(\mathcal{C}_{[L,m,M]})) - E_T 0, \quad (110)$$

with

$$\tilde{L}_T \doteq L - T\|f''\|_{L^\infty(-G_T, G_T)} \cdot \exp(\|\omega\|_{L^1(0,T)})h, \quad (111)$$

where G_T is defined by (19), and \mathcal{T}_{τ_T} denotes the shift operator

$$u(x) \mapsto \mathcal{T}_{\tau_T}(u)(x) \doteq u(x - \tau_T),$$

associated to some constant τ_T to be defined later (see (116)). In fact, given any $u_T \in \mathcal{A}_T$, consider the entropy weak solution $w(t, x) \doteq \tilde{E}_t w_0$ of

$$w_t + f(w)_x = -g(T - t, w),$$

with initial data

$$w_0(x) \doteq u_T(-x) + v^0(T) \quad \forall x \in \mathbb{R}, \quad (112)$$

and observe that the function $u(t, x) \doteq w(T - t, -x)$ is a weak distributional solution of (9), which, by (99), (112), satisfies

$$u(T, \cdot) = u_T + E_T 0. \quad (113)$$

Moreover, note that because of (99), (112) one has $\tilde{E}_t v^0(T) = v^0(T-t)$, and $\tilde{E}_t w_0 = v^0(T-t)$ for all x with $|x|$ large enough. Hence, with the same arguments as above, by virtue of (12), (109), (112), and because $u_T \in \mathcal{A}_T$, one deduces that

$$\|w(t, \cdot) - \tilde{E}_t v^0(T)\|_{L^\infty} \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|u_T\|_{L^\infty} \leq \exp(\|\omega\|_{L^1(0,T)}) \cdot h \leq M \quad \forall t \in [0, T], \quad (114)$$

$$\|w(t, \cdot) - \tilde{E}_t v^0(T)\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,t)}) \cdot \|u_T\|_{L^1} \leq \exp(\|\omega\|_{L^1(0,T)}) \cdot Lh \leq m \quad \forall t \in [0, T]. \quad (115)$$

Recalling the definition (19) of G_T , from (105), (109) and (114) we derive

$$\|w(s, \cdot)\|_{L^\infty} \leq G_T, \quad \forall s \in [0, T].$$

Thus, observing that $\tilde{E}_T v_0(T) = v_0(0) = 0$, setting

$$\tilde{L}_1(T) \doteq -\sup\{x \in \text{Supp}(\tilde{E}_T w_0) \mid w_0(\cdot) \in \mathcal{A}_T + v^0(T)\},$$

$$\tilde{L}_2(T) \doteq -\inf\{x \in \text{Supp}(\tilde{E}_T w_0) \mid w_0(\cdot) \in \mathcal{A}_T + v^0(T)\},$$

and relying on (109) and (111), we deduce that

$$\text{Supp}(w(T, \cdot)) \subset [\tilde{L}_1(T), \tilde{L}_2(T)], \quad \tilde{L}_2(T) - \tilde{L}_1(T) \leq 2\tilde{L}_T + 2T\|f''\|_{L^\infty(-G_T, G_T)} \cdot \exp(\|\omega\|_{L^1(0,T)})h \leq 2L.$$

Then, setting

$$\tau_T \doteq \tilde{L}_1(T) + L, \quad (116)$$

we find that $\text{Supp}(\mathcal{T}_{-\tau_T}(w(T, \cdot))) = \text{Supp}(\mathcal{T}_{-\tau_T}(\tilde{E}_T w_0(\cdot))) \subset [-L, L]$, which, together with the estimates (114)-(115), yields

$$\mathcal{T}_{-\tau_T}(u(0, \cdot)) = \mathcal{T}_{-\tau_T}(w(T, \cdot)) \in \mathcal{C}_{[L, m, M]}.$$

Therefore, since $w(t, x)$ verifies the upper one-side inequality (49), if we establish the lower bound (77), for some constant $\tilde{C}_t > 0$, and for all $t \in]0, T[$, it would follow that $u(t, \cdot) \in \text{Lip}_{loc}(\mathbb{R})$ for all $t \in (0, T)$, and hence, observing that $\mathcal{T}_{-\tau_T}(u(t, \cdot)) = E_t(\mathcal{T}_{-\tau_T}(u(0, \cdot)))$, we deduce by (113) that

$$\mathcal{T}_{-\tau_T}(u_T) \in E_T(\mathcal{C}_{[L, m, M]}) - E_T 0.$$

In turn, this relation clearly implies $u_T \in \mathcal{T}_{\tau_T}(E_T(\mathcal{C}_{[L, m, M]})) - E_T 0$, proving (110).

Concerning (77), as observed in the proof of Proposition 3 it will be sufficient to derive such an estimate for all points $x, y \in \mathbb{R}$, $x < y$, where $w(t, \cdot)$ is continuous, and such that $w(t, x) > w(t, y)$. To this end, note that, by the definition of \mathcal{A}_T in (110), $u_T \in \mathcal{A}_T$ and (112) imply

$$\frac{w_0(y-) - w_0(x+)}{y - x} \geq -\frac{1}{T\|f''\|_{L^\infty[-G_T, G_T]} \exp(\|\omega\|_{L^1(0,T)})} \quad \forall x, y \in \mathbb{R}, \quad x < y.$$

Then, with the same notations and with the same procedure of the proof of Proposition 3, relying on (12), (54), (114), we obtain

$$\begin{aligned} \Delta_0 &\doteq w_0(\xi^x(0)+) - w_0(\xi^y(0)-) \leq \frac{\xi^y(0) - \xi^x(0)}{T\|f''\|_{L^\infty[-G_T, G_T]} \exp(\|\omega\|_{L^1(0,T)})} \\ &= \frac{y - x + \int_0^t (f'(v^x(s)) - f'(v^y(s))) ds}{T\|f''\|_{L^\infty[-G_T, G_T]} \exp(\|\omega\|_{L^1(0,T)})} \\ &\leq \frac{y - x}{T\|f''\|_{L^\infty[-G_T, G_T]} \exp(\|\omega\|_{L^1(0,T)})} + \frac{t}{T} \cdot \Delta_0. \end{aligned}$$

In turn, this yields

$$\begin{aligned} w(t, y) - w(t, x) &\geq \exp(\|\omega\|_{L^1(0,T)}) (w_0(\xi^y(0)-) - w_0(\xi^x(0)+)) \\ &\geq -\frac{y-x}{(T-t)\|f''\|_{L^\infty_{[-G_T, G_T]}}}, \end{aligned}$$

thus establishing (77), with $\tilde{C}_t \doteq ((T-t)\|f''\|_{L^\infty_{[-G_T, G_T]}})^{-1}$. This completes the proof of (110). Since by (109), (111) we have $\tilde{L}_T \geq (3/4)L$, relying on (110), observing that

$$H_\varepsilon(E_T(\mathcal{C}_{[L,m,M]} | L^1(\mathbb{R}))) = H_\varepsilon(\mathcal{T}_{\tau_T}(E_T(\mathcal{C}_{[L,m,M]})) - E_T 0 | L^1(\mathbb{R})),$$

and applying (24), with $b = (T\|f''\|_{L^\infty_{[-G_T, G_T]}} \exp(\|\omega\|_{L^1(0,T)}))^{-1}$, $L = \tilde{L}_T$, we derive the estimate (18).

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