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Travelling graphs for the forced mean curvature motion in an arbitrary space dimension
Ondes progressives pour le mouvement par courbure moyenne forcé en toute dimension d'espace

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Abstract: We construct travelling wave graphs of the form $z = -ct + \phi(x)$, $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$, $N \geq 2$, solutions to the N-dimensional forced mean curvature motion $V_n = -c_0 + \kappa$ $(c \geq c_0)$ with prescribed asymptotics. For any 1-homogeneous function ϕ_∞, viscosity solution to the eikonal equation $|D\phi_\infty| = \sqrt{(c/c_0)^2 - 1}$, we exhibit a smooth concave solution to the forced mean curvature motion whose asymptotics is driven by ϕ_∞. We also describe ϕ_∞ in terms of a probability measure on $\mathbb{S}^{N-2}$.

Résumé: Nous construisons des ondes progressives sous la forme de graphes $z = -ct + \phi(x)$, $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$, $N \geq 2$, solutions du mouvement par courbure moyenne forcé $V_n = -c_0 + \kappa$ $(c \geq c_0)$ en dimension N d'espace et avec un comportement asymptotique prescrit. Pour toute solution de viscosité ϕ_∞, 1-homogène en espace, de l'équation eikonale $|D\phi_\infty| = \sqrt{(c/c_0)^2 - 1}$, nous mettons en évidence une solution régulière et concave du mouvement par courbure moyenne forcé dont le comportement asymptotique est donné par ϕ_∞. Nous décrivons aussi ϕ_∞ en terme d'une mesure de probabilitÃ© sur la sphÃ¨re $\mathbb{S}^{N-2}$.

Keywords: forced mean curvature movement; eikonal equation; Hamilton-Jacobi equations; viscosity solution; reaction diffusion equations; travelling fronts;

1 Introduction

1.1 Setting of the problem

The question investigated here is the description of the travelling wave graph solutions to the forced mean curvature motion in any dimension $N \geq 2$, that is written under the general form

$$V_n = -c_0 + \kappa \tag{1}$$

where V_n is the normal velocity of the graph, κ its local mean curvature and c_0 a given strictly positive constant to be defined later. A graph satisfying (1) can be given by the equation $z = u(t,x)$ where $u : (t,x) \in \mathbb{R}^+ \times \mathbb{R}^{N-1} \mapsto u(t,x) \in \mathbb{R}$ is a solution to the parabolic equation

$$\frac{u_t}{\sqrt{1+|Du|^2}} = -c_0 + \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right), \quad t > 0\,,\, x \in \mathbb{R}^{N-1} \tag{2}$$

Indeed, at any time $t > 0$ fixed, the outer normal to the subgraph $\{(x,z) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid z \leq u(t,x)\}$ is given by

$$\vec{n} = \frac{1}{\sqrt{1+|Du|^2}} \begin{pmatrix} -D_x u \\ 1 \end{pmatrix}$$

its normal velocity V_n by $(0, \partial_t u)^T \cdot \vec{n}$ while its mean curvature by $\kappa = -\operatorname{div}_{(x,z)} \vec{n}$, see [9].

A travelling wave to (2) is a solution of the form $u(t,x) = -ct + \phi(x)$ where $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$ is the profile of the wave and $c \geq c_0$ is some given constant standing for its speed. Thus ϕ satisfies the following elliptic equation

$$-\operatorname{div}\left(\frac{D\phi}{\sqrt{1+|D\phi|^2}}\right) + c_0 - \frac{c}{\sqrt{1+|D\phi|^2}} = 0\,, \quad x \in \mathbb{R}^{N-1} \tag{3}$$

1.2 Connection with reaction diffusion equations

This work should provide us a better understanding of the multidimensional solutions to the non linear scalar reaction diffusion equation

$$\partial_t v = \Delta v + f(v)\,, \quad t > 0\,,\, (x,z) \in \mathbb{R}^{N-1} \times \mathbb{R} \tag{4}$$

where $v : (t,x,z) \in [0,+\infty) \times \mathbb{R}^{N-1} \times \mathbb{R} \mapsto v(t,x,z) \in \mathbb{R}$ and, especially the case of travelling waves in dimension N. In the case of a "bistable" nonlinearity f, that is to say when f is a continuously differentiable function on $\mathbb{R}$ satisfying

`i.` $f(0) = f(1) = 0$

`ii.` $f'(0) < 0$ and $f'(1) < 0$

`iii.` there exists $\theta \in (0,1)$ such that $f(v) < 0$ for $v \in (0,\theta)$, $f(v) > 0$ for $v \in (\theta,1)$

`iv.` $\displaystyle\int_0^1 f(v)\,\mathrm{d}v > 0,$

it is well-known [14] that there exists a one-dimensional travelling front $v(t,z) = \phi_0(z+c_0t)$ solution to (4) with $N = 1$. The speed c_0 is unique and strictly positive by [`iv`] while the profile ϕ_0 is unique up to translations. This result defines the constant $c_0 > 0$ that appears in equation (1).

In the case $N = 2$, multidimensional solutions to (4) are well understood. Paper [10] proves the existence of conical travelling waves solutions to (4), and paper [11] classifies all possible bounded non constant travelling waves solutions under rather weak conditions at infinity. In particular, it is proved in [11] that $c \geq c_0$ and, up to a shift in $x \in \mathbb{R}$, either u is a planar front $\phi_0(\pm x\cos\alpha + z\sin\alpha)$ with $\alpha = \arcsin(c_0/c) \in (0, \frac{\pi}{2}]$ or u is the unique conical front found in [10].

In higher dimensions, less is known. In [11], Hamel, Monneau and Roquejoffre proved the existence of conical travelling waves with cylindrical symmetry whose level sets are Lipschitz graphs moving away logarithmically from straight cones. Some special, non cylindrically symmetric pyramidal-shaped solutions (Taniguchi, [16]) are also known in the particular case $N = 3$.

Thus, in order to get a better understanding of the mechanisms at work, we further the idea of bridging reaction-diffusion equations with geometric motions. In particular, travelling wave graph solutions to the forced mean curvature motion go back to Fife [8]. He proved (in a formal fashion) that reaction-diffusion travelling fronts propagate with normal velocity

$$V_n = -c_0 + \frac{\kappa}{t} + O\left(\frac{1}{t^2}\right), \quad t >> 1.$$

For a mathematically rigorous treatment of these ideas, we refer for instance to de Mottoni, Schatzman [6] - small times, smooth solutions context - and Barles, Soner, Souganidis [1] - arbitrary large times, viscosity solutions context.

Related results must me mentioned in the case of a balanced bistable non-linearity f: assumption `iv.` is replaced by $\int_0^1 f(v)\,\mathrm{d}v = 0$ and (4) is called the balanced Allen-Cahn equation. In this case, $c_0 = 0$ and the forced mean curvature equation is replaced by the mean curvature equation. Chen, Guo, Hamel, Ninomiya and Roquejoffre [3] proved that there exist cylindrically symmetric traveling waves with paraboloid like interfaces solutions to (4) in dimension $N \geq 3$. Precisely, they proved that those solutions' level sets are asymptotically given by the equation $z = \frac{c}{2(N-1)}|x|^2$. On the other hand, Clutterbuck, Schnürer and Schulze [4] proved that there exists a unique rotationally symmetric, strictly convex, translating graph $u(t,x) = -ct + \phi(r)$ to the mean curvature motion (3) with $c_0 = 0$ and whose asymptotics is given by

$$\phi(r) = \frac{c}{2(N-1)}r^2 - \ln r + C + O\left(\frac{1}{r}\right)$$

Further works have also been done in the non radial case for the mean curvature equation. For instance, Xuan Hien Nguyen [15] built non radial and non convex translating graphs solution to (3) with $c_0 = 0$.

1.3 Main results

Our theorem 1.1 below states that, given a 1-homogeneous solution ϕ_∞ to the eikonal equation derived from (3) (i.e. the equation obtained by removing the curvature term) there exists a smooth solution ϕ to the forced mean curvature motion equation (3) whose asymptotic behaviour is prescribed by ϕ_∞. Here is the precise result.

Theorem 1.1 (Existence of solutions with prescribed asymptotics in dim. N) *Let $N \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal equation*

$$|D\phi_\infty(x)| = \cot\alpha \ , \quad x \in \mathbb{R}^{N-1} . \tag{5}$$

Then there exists a smooth concave solution $\phi \in C^\infty(\mathbb{R}^{N-1})$ to (3) such that

$$\phi(x) = \phi_\infty(x) + o(|x|) \quad \text{as } |x| \to +\infty. \tag{6}$$

This is the most possible general result. However, due to the possible complexity of a solution to the eikonal equation (5), it is useful to specialise our result to the particular case of a solution with a finite number of facets.

Theorem 1.2 (Solutions with finite number of facets in dimension N) *Let $N \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Choose ϕ^* a viscosity solution to the eikonal equation (5) given for any $x \in \mathbb{R}^{N-1}$ by*

$$\phi^*(x) = \inf_{\nu \in A} \left(-(\cot\alpha)\ x \cdot \nu + \gamma_\nu\right) \tag{7}$$

where A is a finite subset of cardinal $k \in \mathbb{N}^*$ of the sphere $\mathbb{S}^{N-2}$ and γ_ν are given real numbers. Then there exists a unique smooth concave solution $\phi \in C^\infty(\mathbb{R}^{N-1})$ to (3) such that

$$\begin{cases} -\dfrac{2\ln k}{c_0 \sin\alpha} \le \phi - \phi^* \le 0 \,, \quad x \in \mathbb{R}^{N-1} \\[2ex] \displaystyle\lim_{l \to +\infty} \sup_{dist(x, E_\infty) \ge l} |\phi(x) - \phi^*(x)| = 0 \end{cases} \tag{8}$$

where E_∞ is the set of edges defined as

$$E_\infty = \{x \in \mathbb{R}^{N-1} \,|\, \phi_\infty \text{ is not } C^1 \text{ at } x\}$$

with the 1-homogeneous function

$$\phi_\infty(x) = \inf_{\nu \in A} \left(-(\cot\alpha)\ x \cdot \nu\right)$$

In space dimension $N = 3$, we obtain a more precise result by considering solutions having a finite number of gradient jumps. Those solutions are still more complex than the infimum of a finite number of affine forms. Here is the precise result.

Theorem 1.3 (Solutions with finite number of gradient jumps and $N = 3$) *Let $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal equation (5) in dimension $N = 3$ with a finite number of singularities on $\mathbb{S}^1$. Then, there exist*

i. *a 2π-periodic continuous function $\psi_\infty : \theta \in [0, 2\pi] \mapsto \psi_\infty(\theta) \in [-\cot\alpha, \cot\alpha]$ and a finite number $k \in \mathbb{N}\setminus\{0\}$ of angles $\theta_1 < \cdots < \theta_k$ in $[0, 2\pi)$ such that*

$$\phi_\infty(r\cos\theta, r\sin\theta) = r\psi_\infty(\theta)\,, \quad (r,\theta) \in \mathbb{R}^+ \times [0, 2\pi)$$

Moreover, for any $i \in \{1, \ldots, k\}$,

a. *Either $\forall\theta \in [\theta_i, \theta_{i+1}]$, $\psi_\infty(\theta) = -(\cot\alpha)$ and we set $\sigma_i = 1$*

b. *Or*

$$\begin{cases} \forall\theta \in \left[\theta_i, \frac{\theta_i+\theta_{i+1}}{2}\right], & \psi_\infty(\theta) = -(\cot\alpha)\cos(\theta - \theta_i) \\ \forall\theta \in \left[\frac{\theta_i+\theta_{i+1}}{2}, \theta_{i+1}\right], & \psi_\infty(\theta) = -(\cot\alpha)\cos(\theta - \theta_{i+1}) \end{cases} \quad \text{and we set } \sigma_i = 0$$

By convention, $\theta_{k+1} = 2\pi + \theta_1$ and $\sigma_{k+1} = \sigma_1$. If $k \geq 2$, then $\sigma_i\sigma_{i+1} = 0$ for any $i \in \{1, ..., k\}$.

ii. *a smooth concave function $\phi \in C^\infty(\mathbb{R}^2)$ solution to equation (3) such that when $|x|$ goes to infinity*

$$\phi(x) = \phi_*(x) + O(1)$$

where

$$\phi_*(x) = -\frac{2}{c_0\sin\alpha}\ln\left(\int_{\mathbb{S}^1} e^{\frac{c_0\cos\alpha}{2}x\cdot\nu}\,\mathrm{d}\mu(\nu)\right) \tag{9}$$

and μ is the non negative measure on $\mathbb{S}^1$ with finite mass determined by ψ_∞ as follows: We set $\mu = \sum_{i=1}^k \mu_i$ where for any fixed $\lambda_0 > 0$, we set

a. *If $\sigma_i = 1$, then $\mu_i = \mathbb{1}_{(\theta_i,\theta_{i+1})}\,\mathrm{d}\theta + \lambda_0(\delta_{\theta_i} + \delta_{\theta_{i+1}})$*
(with the exception for $k = 1$: $\mu_1 = \mathbb{1}_{(\theta_1,\theta_1+2\pi)}\,\mathrm{d}\theta$).

b. *If $\sigma_i = 0$, then $\mu_i = \lambda_0(\delta_{\theta_i} + \delta_{\theta_{i+1}})$*

We plan to use our travelling graphs for the forced mean curvature motion exhibited in theorems 1.1 to 1.3 in order to construct multi-dimensional travelling fronts to the reaction diffusion equation (4); we plan to do it in a forthcoming paper.

That equation (5) prescribes the asymptotic behaviour of (3) has nothing surprising: let $\varepsilon > 0$ and denote by ϕ_ε the scaled function

$$\phi_\varepsilon(x) = \varepsilon\phi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^{N-1}$$

Since ϕ is a solution to (3), ϕ_ε satisfies

$$-\varepsilon\,\mathrm{div}\left(\frac{D\phi_\varepsilon}{\sqrt{1+|D\phi_\varepsilon|^2}}\right) + c_0 - \frac{c}{\sqrt{1+|D\phi_\varepsilon|^2}} = 0\,, \quad x \in \mathbb{R}^{N-1}$$

Let ε go to zero. If adequate estimates for ϕ_ε are known, (a subsequence of) $(\phi_\varepsilon)_{\varepsilon>0}$ converges to a function ϕ_∞ satisfying (5).

The proof of Theorem 1.1 is done by a sub and super solutions argument. We first construct a family of smooth sub-solutions to (3), which will give us some better insight in the equation. This step is quite general, and works in any space dimension. Then, we will construct a Lipschitz super-solution whose rescaled asymptotics is prescribed by ϕ_∞ and this will give us a smooth solution whose asymptotic behaviour is not well precise. To get a better asymptotics of the super-solution prescribed by the sub-solution, this will require a more delicate matching procedure which will limit us, for the moment, to any space dimension N with a finite number of facets (theorem 1.2) or to the space dimension $N = 3$ and a finite number of gradient jumps (theorem 1.3).

The rest of this paper is organised as follows. In section 2, we build and characterise all 1-homogeneous solutions to the eikonal equation (5). In section 3, we detail Perron's method in our context, and explain why it will yield a smooth concave solution. Sub-solutions are built in section 4, and super-solutions in section 5. Finally, section 6 sums up previous constructions to prove theorems 1.1 and 1.2. Section 7 presents a more precise approach in dimension $N = 3$ and details the proof of Theorem 1.3. An appendix is devoted to the Laplace's method that we use in our estimates.

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2 Eikonal equation

In this section, we classify the continuous viscosity solutions to the eikonal equation in any dimension $N \geq 2$:

$$|D\phi_\infty(x)| = \cot\alpha\,, \quad x \in \mathbb{R}^{N-1} \tag{10}$$

where $\alpha \in (0, \frac{\pi}{2}]$ is some given angle. In a first subsection, we are interested in the general case. In a second one, we reduce our study to 1-homogeneous functions and give a better description of those solutions in order to use them in both sections 4 and 5.

2.1 Characterisation of solutions to (10) in any dimension N

For any unit vector $\nu \in \mathbb{S}^{N-2}$ and $\gamma \in (-\infty, +\infty]$, let us define the affine map

$$\phi_{\nu,\gamma}(x) = -(\cot\alpha)\ \nu \cdot x + \gamma \in (-\infty, +\infty]\,, \quad x \in \mathbb{R}^{N-1}$$

Proposition 2.1 (A Liouville theorem for the eikonal equation)
Let $\phi_\infty \in C(\mathbb{R}^{N-1})$. Then ϕ_∞ is a viscosity solution to the eikonal equation (10) if and only if there exists a lower semi-continuous map $\gamma : \mathbb{S}^{N-2} \to (-\infty, +\infty]$ such that

$$\phi_\infty(x) = \inf_{\nu \in \mathbb{S}^{N-2}} \phi_{\nu,\gamma(\nu)}(x) \tag{11}$$

Moreover ϕ_∞ is 1-homogeneous if and only if for all $\nu \in \mathbb{S}^{N-2}$, $\gamma(\nu) \in \{0, +\infty\}$.

This result is most certainly known. Because we not only need the result but also an insight of the construction, we give a complete proof.

Proof of Proposition 2.1.

We first show the direct implication. Let $\phi_\infty \in C(\mathbb{R}^{N-1})$ be a viscosity solution to (10). We shall prove that ϕ_∞ is $(\cot\alpha)$-Lipschitz and concave before giving its characterisation as an infimum of affine maps.

Step 1: ϕ_∞ is locally Lipschitz

Let us consider the ball $B(a, R)$ centered in $a \in \mathbb{R}^{N-1}$ with radius $R > 2$ and define

$$C := \sup_{|x-y|\le 1,\ (x,y)\in B(a,R)^2} |\phi_\infty(x) - \phi_\infty(y)|$$

Because ϕ_∞ is continuous, $0 \le C < +\infty$. Denote $\bar{C} = \max(C, \cot\alpha) \in (0, +\infty)$. Then we claim that for any $(x, y) \in B(a, R-1)^2$ such that $|x - y| \le 1$, we have

$$|\phi_\infty(x) - \phi_\infty(y)| \le \bar{C}|x - y| \tag{12}$$

which asserts that ϕ_∞ is locally Lipschitz. Indeed, for any point $x_0 \in \overline{B(a, R-1)}$, any constant $\bar{\bar{C}} > \bar{C}$ and any $\lambda \ge 0$, we consider the function ψ_λ defined as

$$\psi_\lambda(x) := \lambda + \phi_\infty(x_0) + \bar{\bar{C}}|x - x_0|\,, \quad x \in \mathbb{R}^{N-1}$$

and we set

$$\lambda_* = \inf\left\{\lambda \in \mathbb{R}^+ \,|\, \forall\mu \ge \lambda\,,\ \forall x \in \overline{B(x_0, 1)}\,, \quad \psi_\mu(x) \ge \phi_\infty(x)\right\}$$

We shall prove by contradiction that $\lambda_* = 0$. If not, because $\psi_0 \ge \phi_\infty$ on $\{x_0\} \cup \partial B(x_0, 1)$, there exists a contact point z_0 between ψ_{λ_*} and ϕ_∞ which satisfies $z_0 \in B(x_0, 1)\setminus\{x_0\}$. Then ψ_{λ_*} is a test function for the viscosity subsolution ϕ_∞ at that point. Because $|\nabla\psi_{\lambda_*}(z_0)| = \bar{\bar{C}} > \cot\alpha$, we get a contradiction with the viscosity subsolution inequality. Therefore $\lambda_* = 0$ and $\psi_0 \ge \phi_\infty$ on $B(x_0, 1)$. Because this is true for any $\bar{\bar{C}} > \bar{C}$, we deduce that this is still true for $\bar{\bar{C}} = \bar{C}$ which implies (12).

Step 2: ϕ_∞ is $(\cot\alpha)$-Lipschitz

We now define

$$L = \limsup_{n\to+\infty} L_n \quad \text{with} \quad L_n := \sup\left\{\frac{\phi_\infty(y) - \phi_\infty(x)}{|y - x|}, \quad x \in \overline{B(a, R-2)}, \quad |y - x| \le \frac{1}{n}\right\}$$

Notice that for any $n \in \mathbb{N}^*$, $0 < L_n \le \bar{C}$. Moreover, there exists a sequence $(x_n, y_n)_{n\in\mathbb{N}^*}$ such that

$$\lim_{n\to+\infty} \frac{\phi_\infty(y_n) - \phi_\infty(x_n)}{|y_n - x_n|} = L \ \text{ and } |y_n - x_n| \le \frac{1}{n} \ \text{ with } x_n \in \overline{B(a, R-2)}$$

Define for any $x \in \mathbb{R}^{N-1}$

$$\varepsilon_n = |y_n - x_n|\,, \quad \phi_n(x) = \frac{\phi_\infty(x_n + \varepsilon_n x) - \phi_\infty(x_n)}{\varepsilon_n} \quad \text{and} \quad \nu_n = \frac{y_n - x_n}{\varepsilon_n} \in \mathbb{S}^{N-2}$$

Thus $(\phi_n(\nu_n))_{n\in\mathbb{N}^*}$ converges to L as n goes to infinity and for any $x \in \overline{B(0,1)}$, $|\phi_n(x)| \leq L_n|x|$. Because $\mathrm{Lip}(\phi_n; B(0,n)) \leq \bar{C}$, we see that up to a subsequence, $(\phi_n)_{n\in\mathbb{N}^*}$ converges locally uniformly on $\mathbb{R}^{N-1}$ to ϕ_0 a viscosity solution to (10). Moreover, $(\nu_n)_{n\in\mathbb{N}^*}$ converges to $\nu_0 \in \mathbb{S}^{N-2}$ with

$$\phi_0(\nu_0) = L \text{ and for any } x \in \mathbb{R}^{N-1}\,, \quad \phi_0(x) \leq L|x| =: \psi(x)$$

Because ψ touches ϕ_0 from above at ν_0, we conclude from the viscosity inequality for subsolutions that

$$L \leq \cot\alpha$$

Now for any $\epsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that $L_n \leq L + \varepsilon$ for any $n \geq n_\varepsilon$. In particular, for any $(x,y) \in \overline{B(a, R-2)}$ we can split the segment

$$[x,y] = \bigcup_{i=0,\dots,K-1} [x_i, x_{i+1}] \text{ with } x_0 = x\,,\ x_K = y \text{ and } |x_{i+1} - x_i| = \frac{|y-x|}{K} \leq \frac{1}{n}$$

This implies that

$$|\phi_\infty(x) - \phi_\infty(y)| \leq (L+\varepsilon)|x-y|$$

which is true for any $\varepsilon > 0$. This implies that ϕ is L-Lipschitz on $\overline{B(a, R-2)}$ with $L \leq \cot\alpha$.

Step 3: ϕ_∞ is concave

Because ϕ_∞ is a Lipschitz stationary viscosity solution to the evolution equation

$$u_t + H(Du) = 0\,, \quad x \in \mathbb{R}^{N-1} \text{ where } H(p) = \frac{1}{2}(p^2 - \cot^2\alpha)\,, \quad p \in \mathbb{R}^{N-1}$$

we can apply Lemma 4 page 131 in [7], and get that ϕ_∞ satisfies for any $t > 0$

$$\phi_\infty(x+x') - 2\phi_\infty(x) + \phi_\infty(x-x') \leq C_0 \frac{|x'|^2}{t}, \quad \text{for all } (x,x') \in \mathbb{R}^{2(N-1)}$$

and we can check that we have $C_0 = 1$. Letting t go to infinity shows that ϕ_∞ is concave in $\mathbb{R}^{N-1}$.

Step 4: Tangent cone

Since ϕ_∞ is Lipschitz continuous, it is differentiable almost everywhere by Rademacher's theorem. Let $D \subset \mathbb{R}^{N-1}$ be the set of differentiability of ϕ_∞ and fix $x_0 \in D$. Since ϕ_∞ is concave, for any $x \in \mathbb{R}^{N-1}$, we have

$$\phi_\infty(x) \leq \phi_\infty(x_0) + D\phi_\infty(x_0)\cdot(x - x_0)$$

Passing to the infimum on D, we get for any $x \in \mathbb{R}^{N-1}$,

$$\phi_\infty(x) \leq \psi(x) := \inf_{x_0\in D} \phi_\infty(x_0) + D\phi_\infty(x_0)\cdot(x - x_0)$$

Thus, ψ and ϕ_∞ are $(\cot\alpha)$-Lipschitz functions that coincide on D which is a dense set on $\mathbb{R}^{N-1}$. Therefore, they are in fact equal on $\mathbb{R}^{N-1}$. Using equation (10), we finally have

$$\phi_\infty(x) = \inf_{x_0\in D} -(\cot\alpha)\,\nu(x_0)\cdot x + g(x_0)$$

where for any $x_0 \in D$, $\nu(x_0) = -D\phi_\infty(x_0)/\cot\alpha \in \mathbb{S}^{N-2}$ and $g(x_0) = \phi_\infty(x_0) - x_0 \cdot D\phi_\infty(x_0) \in \mathbb{R}$. Defining γ as

$$\begin{array}{rccl} \gamma: & \mathbb{S}^{N-2} & \to & (-\infty, +\infty] \\ & \nu & \mapsto & \begin{cases} \inf_{x_0 \in A} g(x_0) \text{ if } A := \{x_0 \in D \,|\, \nu(x_0) = \nu\} \neq \emptyset \\ +\infty \text{ otherwise} \end{cases} \end{array} \tag{13}$$

we get the desired characterisation (11). Since ϕ_∞ is continuous, we also deduce from (11) that γ is lower semi-continuous.

Step 5: The 1-homogeneous case

We assume that ϕ_∞ is a 1-homogeneous continuous viscosity solution to (10). Then for any $x_0 \in \mathbb{R}^{N-1}$, there exists $p \in \mathbb{R}^{N-1}$ with $|p| = \cot\alpha$ such that by (11)

$$\forall x \in \mathbb{R}^{N-1}, \quad \phi_\infty(x) \leq \phi_\infty(x_0) + p \cdot (x - x_0)$$

On the one hand, considering $x = 0$, we get

$$p \cdot x_0 \leq \phi_\infty(x_0)$$

because ϕ_∞ is 1-homogeneous. On the other hand considering λx instead of x and taking the limit $\lambda \to +\infty$, we get

$$\psi(x) := p \cdot x \geq \phi_\infty(x) \quad \text{with equality at} \quad x = x_0\,.$$

Therefore if we call $\mathcal{L}_{\phi_\infty}$ the set of linear functions ψ satisfying $\psi \geq \phi_\infty$ such that $|\nabla\psi| = \cot\alpha$, we have

$$\phi_\infty = \inf_{\psi \in \mathcal{L}_{\phi_\infty}} \psi$$

because this is true at any point $x_0 \in \mathbb{R}^{N-1}$.

Step 6: Conclusion

Conversely, if a function ϕ_∞ is given by (11), then it is straightforward to check that ϕ_∞ is a viscosity solution to (10). ■

Remark 2.2 *In dimension $N = 2$, the previous proposition simply reads:*
If $N = 2$ and $\alpha \in (0, \frac{\pi}{2}]$, ϕ_∞ is a viscosity solution to (10) if and only if ϕ_∞ is affine or if there exists $(x_0, y_0) \in \mathbb{R}^2$ such that

$$\phi_\infty(x) = -(\cot\alpha)\,|x - x_0| + y_0\,, \quad x \in \mathbb{R} \tag{14}$$

Moreover, ϕ_∞ is 1-homogeneous if and only if $y_0 = 0$.

The proof of this proposition can also be done directly from definitions of viscosity solutions and we omit the details. Notice however the link with [10]: two-dimensional reaction diffusion waves are either planar fronts or the unique (up to translations) conical front whose level sets are asymptotics to the graph of ϕ_∞ just described.

2.2 The 1-homogeneous case

As stressed is theorem 1.1, we only build solutions to the forced mean curvature motion equation (3) whose asymptotics is prescribed by a 1-homogeneous solution to the eikonal equation (10). Therefore, it is worth emphasising this particular case.

Notice however that there exist viscosity solutions to the eikonal equation (10) defined in $\mathbb{R}^{N-1}$ that are not homogeneous of order 1. For instance, consider solutions given by (14) with $x \in \mathbb{R}^{N-1}$ and $y_0 \neq 0$. We can also consider any translation of a 1-homogeneous solution. Another example is for instance given in dimension $N = 3$ by a function $\phi_\infty = \inf_{i=1\dots4} \phi_i$ where $(\phi_i)_{i\in\{1\dots4\}}$ are four planar solutions defined for $x = (x_1, x_2) \in \mathbb{R}^2$ by

$$\phi_1(x) = -(\cot\alpha)\ x_1 + 2 \qquad \phi_2(x) = (\cot\alpha)\ x_1 + 2$$
$$\phi_3(x) = -(\cot\alpha)\ x_2 \qquad \phi_4(x) = (\cot\alpha)\ x_2$$

It is straightforward to check that ϕ_∞ satisfies $|D\phi_\infty| = \cot\alpha$ in the viscosity sense and that it is not homogeneous of order 1 since there exists $\lambda > 0$ such that $\phi_\infty(\lambda, 0) \neq \lambda\phi_\infty(1, 0)$.

In any case, a solution ϕ_∞ to the eikonal equation (10) is concave (see the proof of proposition 2.1, step 3). Therefore the function $g : \lambda \in \mathbb{R}^{+*} \mapsto g(\lambda) = \phi_\infty(\lambda x)/(\lambda|x|) \in \mathbb{R}$ is decreasing in $\lambda > 0$. Since ϕ_∞ is $(\cot\alpha)$-Lipschitz, g is bounded from below and for any $x \in \mathbb{S}^{N-2}$, the limit

$$\lim_{\lambda\to+\infty} \frac{\phi_\infty(\lambda x)}{|\lambda|}$$

exists and ϕ_∞ is asymptotically homogeneous. Thus we have a fairly general understanding of what is going on by restricting ourselves to homogeneous solutions to equation (10).

Proposition 2.3 ***(A countable characterisation of homogeneous solutions)*** *Let $\phi_\infty \in C(\mathbb{R}^{N-1})$. Then ϕ_∞ is a 1-homogeneous viscosity solution to the eikonal equation (10) if and only if there exists a sequence $(\nu_i)_{i\in\mathbb{N}}$ of $\mathbb{S}^{N-2}$ such that*

$$\phi_\infty(x) = \inf_{i\in\mathbb{N}} -(\cot\alpha)\ \nu_i \cdot x \tag{15}$$

Proof of Proposition 2.3.
Let $\phi_\infty \in C(\mathbb{R}^{N-1})$ be a 1-homogeneous viscosity solution to (10). According to proposition 2.1, there exists a lower semi continuous function γ defined from $\mathbb{S}^{N-2}$ to $\{0, +\infty\}$ such that

$$\phi_\infty(x) = \inf_{\nu\in\mathbb{S}^{N-2}} \phi_{\nu,\gamma(\nu)}(x)\,, \quad x \in \mathbb{R}^{N-1}$$

Then $K = \{\nu \in \mathbb{S}^{N-2} \,|\, \gamma(\nu) = 0\}$ is a compact set of $\mathbb{S}^{N-2}$. We claim (see lemma 2.4 and corollary 2.5 below) that there exists a sequence $(\nu_i)_{i\in\mathbb{N}}$ of $\mathbb{S}^{N-2}$ such that

$$K = \overline{\bigcup_{i\in\mathbb{N}} \{\nu_i\}}$$

Thus, $\phi_\infty(x)$ can be described as the infimum over $\nu \in K$ of the linear functions $-(\cot\alpha)\ x\cdot\nu$. Since $\cup_{i\in\mathbb{N}}\{\nu_i\}$ is dense in K, $\phi_\infty(x)$ can also be written as the infimum over $i \in \mathbb{N}$ of the linear functions $-(\cot\alpha)\ x \cdot \nu_i$. This ends the proof of proposition 2.3 since the converse implication is straightforward. ■

Lemma 2.4 ***(Decomposition of a compact set of $\mathbb{S}^{N-2}$ in cubes)***
For any compact set K of $\mathbb{S}^{N-2}$, there exists a countable family $(Q_i)_{i\in\mathbb{N}}$ of closed cubes of $\mathbb{R}^{N-1}$ such that

$$\left\{\begin{array}{l} \forall n \in \mathbb{N}\,, \quad K \subset \displaystyle\bigcup_{i\geq n} Q_i \\ \forall i \in \mathbb{N}\,, \quad Q_i \cap K \neq \emptyset \\ \limsup\limits_{i\to+\infty} diam(Q_i) = 0 \end{array}\right. \tag{16}$$

Proof of Lemma 2.4.
We built this decomposition in cubes by induction. Let $C_0 = [-1,1]^{N-1}$ be the first cube of width 2. Thus $K \subset C_0$. Since $C_0 \cap K$ is not empty, we divide C_0 in 2^{N-1} smaller cubes of width $2^0 = 1$. We call $C_{1,i}$ for $i = 1 \dots n_1$ those whose intersection with K is not empty. Then, $1 \leq n_1 \leq 2^{N-1}$ and

$$K \quad \subset \quad \bigcup_{i=1}^{n_1} C_{1,i}$$

In the same way, for $i = 1, ..., n_1$, we divide each cube $C_{1,i}$ in 2^{N-1} smaller cubes of width 2^{-1} and keep only those whose intersection with K is not empty. We call them $C_{2,k}$ for $k = 1 \dots n_2$ and $1 \leq n_2 \leq 2^{N-1} n_1$. Then, one can easily verify that $K \subset \cup_{k=1\dots n_2} C_{2,k}$.

Assume the cubes $C_{j,i}$ are built for $j \in \mathbb{N}$, $i = 1 \dots n_j$ and $1 \leq n_j \leq 2^{j(N-1)}$ such that

$$\left\{\begin{array}{l} K \subset \bigcup_{i=1}^{n_j} C_{j,i} \\ \forall i = 1 \dots n_j\,, \quad C_{j,i} \cap K \neq \emptyset \\ \mathrm{diam}(C_{j,i}) = 2^{-j+1} \end{array}\right.$$

Then we construct the cubes $C_{j+1,i}$ as follows. We divide each cube $C_{j,i}$ into 2^{N-1} smaller cubes of width 2^{-j} and keep only those whose intersection with K is not empty. We call them $C_{j+1,i}$ for $i = 1 \dots n_{j+1}$ and $1 \leq n_{j+1} \leq 2^{N-1} n_j \leq 2^{(j+1)(N-1)}$. By construction, it is easy to verify that

$$\left\{\begin{array}{l} K \subset \bigcup_{i=1}^{n_{j+1}} C_{j+1,i} \\ \forall i = 1 \dots n_{j+1}\,, \quad C_{j+1,i} \cap K \neq \emptyset \\ \mathrm{diam}(C_{j+1,i}) = 2^{-j} \end{array}\right.$$

The induction is then proved. We thus construct a countable family of cubes that we recall $(Q_j)_{j\in\mathbb{N}}$ for convenience with the desired assumptions (16). This ends the proof of lemma 2.4. ■

Corollary 2.5 ***(Representation of a compact set of $\mathbb{S}^{N-2}$)***
For any compact set K of $\mathbb{S}^{N-2}$, there exists a sequence $(\nu_j)_{j\in\mathbb{N}}$ of $\mathbb{S}^{N-2}$ such that

$$K = \overline{\bigcup_{j\in\mathbb{N}} \{\nu_j\}}$$

Proof of Corollary 2.5.
For K a compact set of $\mathbb{S}^{N-2}$, we define $(Q_j)_{j\in\mathbb{N}}$ a family of cubes as proposed in lemma 2.4. For any $j \in \mathbb{N}$, we choose $\nu_j \in K \cap Q_j$. Then, it is straightforward to check that $\overline{\cup_{j\in\mathbb{N}}\{\nu_j\}} \subset K$. Regarding the converse inclusion, we fix $x_0 \in K$ and $\varepsilon > 0$. By (16), there exists $n_\varepsilon \in \mathbb{N}$ such that the width of cube Q_i is smaller than ε provided $i \geq n_\varepsilon$. Since $K \subset \cup_{i\geq n_\varepsilon} Q_i$, there exists $i_\varepsilon \geq n_\varepsilon$ such that

$$x_0 \in Q_{i_\varepsilon} \text{ and } |x_0 - \nu_{i_\varepsilon}| \leq \varepsilon\sqrt{N-1}$$

This shows the density of $\cup_{j\in\mathbb{N}}\{\nu_j\}$ in K and ends the proof of corollary 2.5. ■

3 Perron's method and comparison principle

In this section, we are concerned with the forced mean curvature motion equation

$$-\operatorname{div}\left(\frac{D\phi}{\sqrt{1+|D\phi|^2}}\right) + c_0 - \frac{c}{\sqrt{1+|D\phi|^2}} = 0\,, \quad x \in \mathbb{R}^{N-1} \tag{17}$$

with the condition at infinity

$$\phi(x) = \phi_\infty(x) + o(|x|)\,, \quad x \in \mathbb{R}^{N-1} \tag{18}$$

where ϕ_∞ is a homogeneous viscosity solution to $|D\phi_\infty| = \cot\alpha$ found in section 2 with $\alpha = \arcsin(c_0/c) \in (0, \frac{\pi}{2}]$. We choose to solve (17) using Perron's method with sub and super-solutions (see [5] or [9]). Let us first recall the existence process and clarify the regularity of the solution in the following

Proposition 3.1 (Existence of a solution to (17) **in dimension** N**)**
Let $N \in \mathbb{N} \setminus \{0,1\}$, $(c_0, c) \in \mathbb{R}^2$ such that $c \geq c_0 > 0$. Assume that ϕ_* is a viscosity sub-solution and ϕ^* a viscosity super-solution to (17) *such that $\phi_* \leq \phi^*$ on $\mathbb{R}^{N-1}$.*
Then,
i) *there exists a function $\phi \in [\phi_*, \phi^*]$ viscosity solution to* (17).
ii) *Moreover, if ϕ^* is concave, and satisfies the following technical condition:*

$$\textit{there exists } p \in \mathbb{R}^{N-1} \textit{ such that } \limsup_{|x|\to+\infty} \frac{\phi^*(x) - p\cdot x}{|x|} < 0, \tag{19}$$

then ϕ can be chosen concave and smooth.

Proof of Proposition 3.1.
We build the solution ϕ using Perron's method directly in the framework of viscosity solutions to (17), that is to say ϕ is chosen as the maximal sub-solution to (17) (see the user's guide to viscosity solutions [5]).
Step 1: Concavity
We apply a result due to Imbert (see [12]) that we first recall. Denote F the following Hamiltonian

$$F(p,M) = -\frac{\operatorname{tr}M}{\sqrt{1+|p|^2}} + \frac{\operatorname{tr}(M\cdot(p\otimes p))}{(1+|p|^2)^{3/2}} + c_0 - \frac{c}{\sqrt{1+|p|^2}}\,, \quad (p,M) \in \mathbb{R}^{N-1}\times\mathbb{R}^{(N-1)\times(N-1)}_{\text{sym}}$$

where $\mathbb{R}^{(N-1)\times(N-1)}_{\text{sym}}$ is the set of $(N-1)$-square symmetric matrices.

Proposition 3.2 (Imbert's proposition 5 in [12])
Let u be a lower semi-continuous and epi-pointed function. If u is a supersolution to

$$F(Du(x), D^2u(x)) = 0\,, \quad x \in \mathbb{R}^{N-1}$$

then, so is its convex envelope.

In our context, $-\phi^*$ is epi-pointed because of the technical condition (19). Thus, the maximal sub-solution ϕ to (17) is concave (otherwise Imbert's result stated in proposition 3.2 contradicts the maximal property of ϕ). Then, ϕ is a concave viscosity solution to (17).

Step 2: Regularity

Once concavity is at hand, a Lipschitz bound is automatically available from the equation (17) itself:

$$|D\phi(x)| \leq \cot\alpha\,, \quad x \in \mathbb{R}^{N-1}$$

where $\alpha \in (0, \frac{\pi}{2}]$ is such that $c_0 = c\sin\alpha$. Then F becomes uniformly elliptic, thus allowing for $C^{1,1}$ estimates (see Theorem 4 in [12]). A bootstrap argument then shows that the solution is C^∞. This concludes the proof of proposition 3.1. ■

Remark 3.3 *Notice that the condition (19) is hidden in the statement of Proposition 9 in [12]. Thus, the proof of Proposition 5 in [12] uses Proposition 9.*

It now remains to find sub and super-solutions to (17).

4 Sub-solution

In this section we build smooth sub-solutions to the forced mean curvature equation (17) as global solutions to a viscous eikonal equation and we do believe that they are really close to the desired solutions.

4.1 Sub-solutions as solutions to a viscous eikonal equation

We have the following

Lemma 4.1 (Sufficient condition for a sub-solution to (17))
Fix $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Let ϕ_* be a concave smooth solution to

$$-\Delta\phi_* = \frac{c_0\sin\alpha}{2}\left(\cot^2\alpha - |D\phi_*|^2\right)\,, \quad x \in \mathbb{R}^{N-1} \tag{20}$$

such that

$$|D\phi_*(x)| \leq \cot\alpha\,, \quad x \in \mathbb{R}^{N-1} \tag{21}$$

Then ϕ_* is a smooth sub-solution to equation (17).

Proof of Lemma 4.1.
Let ϕ_* be any concave function verifying (20) and (21). Since ϕ_* is smooth and concave, we have

$$\begin{aligned} N[\phi_*] :=& -\operatorname{div}\left(\frac{D\phi_*}{\sqrt{1+|D\phi_*|^2}}\right) + c_0 - \frac{c}{\sqrt{1+|D\phi_*|^2}} \\ =& -\frac{\Delta\phi_*}{\sqrt{1+|D\phi_*|^2}} + \frac{D^2\phi_*(D\phi_*, D\phi_*)}{(1+|D\phi_*|^2)^{\frac{3}{2}}} + c_0 - \frac{c}{\sqrt{1+|D\phi_*|^2}} \\ \leq& \frac{1}{\sqrt{1+|D\phi|^2}}\left(-\Delta\phi_* + c_0\sqrt{1+|D\phi_*|^2} - c\right) \end{aligned}$$

From (21) and $c_0 = c\sin\alpha$, we deduce that

$$\begin{aligned} \cot^2\alpha - |D\phi_*|^2 &= \left(\frac{c}{c_0}\right)^2 - \left(\sqrt{1+|D\phi_*|^2}\right)^2 \\ &\leq \frac{2c}{c_0}\left(\frac{c}{c_0} - \sqrt{1+|D\phi_*|^2}\right) \end{aligned}$$

Using equation (20) satisfied by ϕ_*, we get

$$N[\phi_*] \leq \frac{1}{\sqrt{1+|D\phi_*|^2}}\left(\frac{c}{c_0} - \sqrt{1+|D\phi_*|^2}\right)\left(\frac{2c}{c_0}\frac{c_0^2}{2c} - c_0\right) = 0.$$

Thus, ϕ_* is a sub-solution to (17). ■

As it is well-known, equation (20) is readily transformed into a linear one by the Hopf-Cole transform

$$\tilde{\phi}_*(x) = \exp\left(-\frac{c_0\sin\alpha}{2}\phi_*\left(\frac{2x}{c_0\cos\alpha}\right)\right), \quad x \in \mathbb{R}^{N-1}, \quad \alpha \neq \frac{\pi}{2}$$

where $\tilde{\phi}_*$ is a positive solution to

$$-\Delta\tilde{\phi}_*(x) + \tilde{\phi}_*(x) = 0\,, \quad x \in \mathbb{R}^{N-1} \tag{22}$$

From [2], a positive solution $\tilde{\phi}_*$ to (22) has the form

$$\tilde{\phi}_*(x) = \int_{\mathbb{S}^{N-2}} e^{\nu\cdot x}\,\mathrm{d}\mu(\nu)\,, \quad x \in \mathbb{R}^{N-1},$$

where μ is a non negative measure on $\mathbb{S}^{N-2}$ with finite mass.

Now, for any non negative measure μ on the sphere $\mathbb{S}^{N-2}$, let us define

$$\phi_*(x) = -\frac{2}{c_0\sin\alpha}\ln\left(\int_{\mathbb{S}^{N-2}} e^{\frac{c_0\cos\alpha}{2}x\cdot\nu}\,\mathrm{d}\mu(\nu)\right), \quad x \in \mathbb{R}^{N-1}, \quad \alpha \in \left(0, \frac{\pi}{2}\right] \tag{23}$$

By construction, ϕ_* is a smooth solution to (20). Let us now prove that ϕ_* is a sub-solution to equation (17), with all the requirements.

Lemma 4.2 (Inequalities for the derivatives of ϕ_*)
Let μ be a non negative measure on $\mathbb{S}^{N-2}$ with finite mass, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Define ϕ_* as in (23)*. Then ϕ_* is a smooth concave solution to (20) and its gradient is uniformly bounded, that is to say, for any $(x, \xi) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$,*

$$|D\phi_*(x)| \leq \cot\alpha\,, \quad D^2\phi_*(x)(\xi,\xi) \leq 0 \tag{24}$$

Proof of Lemma 4.2.
Let μ and ϕ_* be so defined. We have

$$D\phi_*(x) = -(\cot\alpha)\,\frac{F_\nu(x)}{F_1(x)}\,, \quad x \in \mathbb{R}^{N-1}$$

where for any continuous (scalar or vector) function f defined on $\mathbb{S}^{N-2}$

$$F_f(x) = \int_{\mathbb{S}^{N-2}} e^{\frac{c_0\cos\alpha}{2}x\cdot\nu} f(\nu)\,\mathrm{d}\mu(\nu)$$

Remark that if we define for some fixed $x \in \mathbb{R}^{N-1}$, $\int_{\mathbb{S}^{N-2}} f(\nu)\,\mathrm{d}\overline{\mu}_x(\nu) = \frac{F_f(x)}{F_1(x)}$, then $\overline{\mu}_x$ is a probability measure on $\mathbb{S}^{N-2}$. We can then apply Jensen's inequality to the convex function $y \mapsto |y|^2$. This gives

$$\left|\frac{F_f(x)}{F_1(x)}\right|^2 = \left|\int_{\mathbb{S}^{N-2}} f(\nu)\,\mathrm{d}\overline{\mu}_x\right|^2 \leq \int_{\mathbb{S}^{N-2}} |f(\nu)|^2\,\mathrm{d}\overline{\mu}_x = \frac{F_{|f|^2}(x)}{F_1(x)} \tag{25}$$

for any continuous function f defined on $\mathbb{S}^{N-2}$. Applying this inequality to $f(\nu) = \nu$, we get the desired bound on the gradient of ϕ_*: $|D\phi_*(x)| \leq \cot\alpha$.

Regarding the concavity property of ϕ_*, we use the same type of arguments. Indeed, for any $\xi \in \mathbb{R}^{N-1}$ and $x \in \mathbb{R}^{N-1}$, we have

$$D^2\phi_*(x)(\xi,\xi) = -\frac{c_0\cos^2\alpha}{2\sin\alpha}\left(\frac{F_{f^2}(x)}{F_1(x)} - \left(\frac{F_f(x)}{F_1(x)}\right)^2\right)$$

where f is the continuous function defined on $\mathbb{S}^{N-2}$ by $f(\nu) = \nu\cdot\xi$. Applying again Jensen's inequality (25), we conclude that $D^2\phi_*(x)(\xi,\xi) \leq 0$ for any $\xi \in \mathbb{R}^{N-1}$ and $x \in \mathbb{R}^{N-1}$ which shows that ϕ_* is concave. ■

Finally, we proved the following proposition:

Proposition 4.3 ***(Existence of a sub-solution to*** (17)***)***
Fix $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Let μ be a non negative measure on $\mathbb{S}^{N-2}$ with finite mass. Define ϕ_* as in (23)*. Then, ϕ_* is a smooth concave sub-solution to* (17)*.*

Remark 4.4 *The way we choose the measure μ is decisive in the asymptotic behaviour of the sub-solution ϕ_* built as in proposition 4.3. Indeed, if we want the subsolution (and hence the solution) to the mean curvature equation* (17) *to follow asymptotically some given solution ϕ_∞ to the eikonal equation* (10)*, we will have to choose the measure μ carefully. In that procedure, information collected in section 2 will help.*

Of course, it will be also very interesting to assess whether each sub-solution built with a general probability measure gives rise to a solution to the mean curvature equation (17)*.*

5 Super-solution

A natural super-solution to the forced mean curvature equation (17) is a viscosity solution ϕ_∞ to the eikonal equation (10). Indeed, ϕ_∞ satisfies (in the distributional and viscosity sense)

$$-\operatorname{div}\left(\frac{D\phi_\infty}{\sqrt{1+|D\phi_\infty|^2}}\right)+c_0-\frac{c}{\sqrt{1+|D\phi_\infty|^2}}\geq 0\,,\quad x\in\mathbb{R}^{N-1}$$

However, this super-solution does not satisfy the right comparison with the previous sub-solution ϕ_*. For instance, if $N=3$, ϕ_∞ is the radially symmetric viscosity solution to the eikonal equation (10) and ϕ_* the sub-solution associated with the Lebesgue measure $\mu=\mathrm{d}\theta$ on $[0,2\pi]$ as in (23), then we can compute the asymptotic behaviour of both functions ϕ_∞ and ϕ_* for $x\in\mathbb{R}^2$ with $|x|$ large enough using Laplace's method (see appendix 8). We then observe that the sub-solution ϕ_* is above the super-solution ϕ_∞ in this area. This contradicts the crucial assumption $\phi_*\leq\phi^*$ on $\mathbb{R}^2$ in the Perron's method (see proposition 3.1).

5.1 Super-solutions as infimum of hyperplanes

Since we do believe that the sub-solution is close to the viscosity solution to the forced mean curvature equation (17) at infinity, we prefer to change the super-solution. In the general case of dimension N, we use the countable characterisation of the solution ϕ_∞ to the eikonal equation (10) that we want to approach (see proposition 2.3).

Proposition 5.1 ***(Existence of a super-solution to*** (17)***)***
Fix $\alpha\in(0,\frac{\pi}{2}]$, $c_0>0$ and $c=c_0/\sin\alpha$. Choose ϕ_∞ a 1-homogeneous solution to the eikonal equation (10)*. Define $(\nu_i)_{i\in\mathbb{N}}$ the sequence of $\mathbb{S}^{N-2}$ given by its countable characterisation in proposition 2.3.*

For any sequence $(\lambda_i)_{i\in\mathbb{N}}$ such that $\lambda_i>0$ and $\sum_{i\in\mathbb{N}}\lambda_i<+\infty$, we set

$$\phi_i(x)=-(\cot\alpha)\;x\cdot\nu_i-\frac{2}{c_0\sin\alpha}\ln\lambda_i\,,\quad i\in\mathbb{N}\,,\quad x\in\mathbb{R}^{N-1}$$

and

$$\phi^*(x)=\inf_{i\in\mathbb{N}}\phi_i(x)\,,\quad x\in\mathbb{R}^{N-1}$$

Then, ϕ^* is a concave continuous super-solution to (17)*.*

Proof of Proposition 5.1.
Since ϕ_i are exact solutions to the forced mean curvature equation (17), it is clear that ϕ^* is a super-solution to that equation. As the infimum of affine functions, it is concave and continuous. ■

Remark 5.2 *This construction is very easy. However, it is not clear whether the technical condition* (19) *is satisfied or not. It is even clear that when the set of $\{\nu_i\}_{i\in\mathbb{N}}$ is finite of cardinal less or equal to $N-1$, this condition is NOT verified. We will see later (see Step 4 of the proof of theorem 1.1) how to modify the sub- and super-solutions in order to satisfy condition* (19) *and then pass to the limit to recover the general case.*

Remark 5.3 *In the case when the set $\{\nu_i\}_{i\in\mathbb{N}}$ is infinite, the convergence of $\sum \lambda_i$ forces $(\lambda_i)_{i\in\mathbb{N}}$ to go to zero and the sequence $(-\ln\lambda_i)_{i\in\mathbb{N}}$ grows as i goes to infinity.*

6 General existence results

Now equipped with sub and super-solutions as well as a Perron's method, we are able to prove existence results. The general case in dimension $N \geq 2$ is the easiest one since the asymptotics is less precise. Let us explain our ideas in details depending on the degree of precision we want to obtain in our construction.

Let $N \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. It is worth noticing that some of our constructions do not work for $\alpha = \pi/2$. However, this case is obvious and leads to planar fronts. Therefore, we restrict ourselves to $\alpha \in (0, \frac{\pi}{2})$.

6.1 Proof of Theorem 1.1

Step 1: Sub and super-solutions
Choose ϕ_∞ a 1-homogeneous continuous viscosity solution to the eikonal equation (10) in $\mathbb{R}^{N-1}$. By proposition 2.3, there exists a sequence $(\nu_i)_{i\in\mathbb{N}}$ of $\mathbb{S}^{N-2}$ such that

$$\phi_\infty(x) = \inf_{i\in\mathbb{N}} -(\cot\alpha)\nu_i \cdot x$$

Let μ be the probability measure on $\mathbb{S}^{N-2}$ be defined as

$$\mu = \sum_{i\in\mathbb{N}} \lambda_i \delta_{\nu_i}$$

where $(\lambda_i)_{i\in\mathbb{N}}$ are chosen so that $\lambda_i > 0$ and $\sum_{i=0}^{+\infty} \lambda_i = 1$.

Build the sub-solution ϕ_* as in (23) with the above measure μ. Then by proposition 4.3, ϕ_* is a smooth concave sub-solution to the mean curvature motion equation (17). Build a concave continuous super-solution ϕ^* by proposition 5.1 as the infimum of hyperplanes where the $(\lambda_i)_{i\in\mathbb{N}}$ and $(\nu_i)_{i\in\mathbb{N}}$ are defined by the choice of μ. For any $x \in \mathbb{R}^{N-1}$,

$$\phi_*(x) = -\frac{2}{c_0\sin\alpha}\ln\left(\sum_{i=0}^{+\infty} \lambda_i\, e^{\frac{c_0\cos\alpha}{2}\nu_i\cdot x}\right) \leq -\frac{2}{c_0\sin\alpha}\ln\left(\lambda_i\, e^{\frac{c_0\cos\alpha}{2}\nu_i\cdot x}\right)$$

Since the last inequality holds for any $i \in \mathbb{N}$, we have

$$\phi_*(x) \leq \inf_{i\in\mathbb{N}}\left(-(\cot\alpha)\nu_i\cdot x - \frac{2}{c_0\sin\alpha}\ln\lambda_i\right) = \inf_{i\in\mathbb{N}} \phi_i(x) = \phi^*(x)$$

and the super-solution ϕ^* is above the sub-solution ϕ_*.
Step 2: Asymptotics of sub and super-solutions
Let us now precise their asymptotics: we claim that as $|x|$ goes to infinity

$$\phi_*(x) = \phi_\infty(x) + o(|x|) \text{ and } \phi^*(x) = \phi_\infty(x) + o(|x|) \tag{26}$$

To prove such a claim, the idea is to compare $\phi_\infty(x)$ with the limits as ε goes to zero of $\varepsilon\phi_*(x/\varepsilon)$ and $\varepsilon\phi^*(x/\varepsilon)$. In particular, we will prove the sequence of three inequalities: for any $x \in \mathbb{R}^{N-1}$

$$\phi_\infty(x) \le \lim_{\varepsilon\to 0} \varepsilon\phi_*(x/\varepsilon) \le \lim_{\varepsilon\to 0} \varepsilon\phi^*(x/\varepsilon) \le \phi_\infty(x) \tag{27}$$

which proves the desired claim (26).

The first step of the present proof leads easily to the second inequality in (27) since $\phi_* \le \phi^*$ on $\mathbb{R}^{N-1}$. As far as the first inequality is concerned, we have for any $i \in \mathbb{N}$ and $x \in \mathbb{R}^{N-1}$,

$$|x \cdot \nu_i| = |x| \cos(\theta_x - \theta_i) \le |x| \cos\delta_x$$

where δ_x is the angular distance between $x/|x|$ and $K := \overline{\cup_{i\in\mathbb{N}}\{\nu_i\}}$. Thus,

$$\begin{aligned}\phi_*(x) &\ge -\frac{2}{c_0 \sin\alpha} \ln\left(e^{\frac{c_0 \sin\alpha}{2}|x|\ \cos\delta_x} \mu(\mathbb{S}^{N-2})\right)\\ &= -(\cot\alpha)|x|\ \cos\delta_x = \inf_{\nu\in K} -(\cot\alpha)\ x\cdot\nu = \phi_\infty(x)\end{aligned}$$

where $\mu(\mathbb{S}^{N-2}) = \sum_i \lambda_i = 1$. Thus $\phi_\infty \le \phi_*$ on $\mathbb{R}^{N-1}$ and the homogeneity of ϕ_∞ gives the first inequality of (27).

Regarding the last inequality in (27), we know that for any $x \in \mathbb{R}^{N-1}$,

$$\phi^*(x) \le \phi_i(x) = -(\cot\alpha)\ x\cdot\nu_i - \frac{2}{c_0 \sin\alpha} \ln\lambda_i$$

Since $\lim_{\varepsilon\to 0} \varepsilon\phi_i(x/\varepsilon) = -(\cot\alpha)\ x\cdot\nu_i$, it is clear that

$$\lim_{\varepsilon\to 0} \varepsilon\phi^*\left(\frac{x}{\varepsilon}\right) \le \inf_{i\in\mathbb{N}} -(\cot\alpha)\ x\cdot\nu_i = \phi_\infty(x)$$

This ends the proof of the three inequalities (27) and hence of (26).

Step 3: Existence of a solution

By proposition 3.1, there exists a function $\phi \in [\phi_*, \phi^*]$ viscosity solution to (17) and by the previous step, ϕ verifies the right asymptotics

$$\phi(x) = \phi_\infty(x) + o(|x|)$$

However, in the statement of theorem 1.1, we claim that there exists a smooth concave solution to (17) and the above construction does not provide such information. By proposition 3.1, the regularity and concavity of the solution are at hand if the super-solution ϕ^* satisfies the technical assumption (19). If it does not, we will first modify the sub and the super-solutions in order to satisfy (19), then get a concave solution, and in a last step pass to the limit to find a solution (still concave) between ϕ^* and ϕ_*.

Step 4: Regularity and concavity

Let us consider for any $\varepsilon > 0$

$$\phi_*^\varepsilon(x) = -\frac{2}{c_0 \sin\alpha} \ln\left(\int_{\mathbb{S}^{N-2}} e^{\frac{c_0 \cos\alpha}{2} x\cdot\nu}\, \mathrm{d}\mu_\varepsilon(\nu)\right), \quad x \in \mathbb{R}^{N-1} \tag{28}$$

with $\mu_\varepsilon = \mu + \varepsilon\mu_1$ where

$$\mu_1 = \sum_{\pm} \sum_{j=1}^{N-1} \delta_{\pm e_j}$$

denoting $(e_i)_{i\in\{1,\dots,N-1\}}$ as the canonical orthonormal basis of $\mathbb{R}^{N-1}$. In the same way, we define

$$\phi^{\varepsilon *}(x) = \inf_{i\in\mathbb{N},\ j=1\dots N-1,\ \pm} \left(-(\cot\alpha)\ x\cdot\nu_i - \frac{2}{c_0\sin\alpha}\ln\lambda_i, -(\cot\alpha)\ x\cdot(\pm e_j) - \frac{2}{c_0\sin\alpha}\ln\varepsilon \right)$$

Then, ϕ_*^ε is a sub-solution, $\phi^{\varepsilon *}$ is a super-solution and $\phi_*^\varepsilon \le \phi^{\varepsilon *}$. It satisfies (19) for any $\varepsilon > 0$ and for $p = 0$. By proposition 3.1, there exists a concave smooth solution ϕ^ε satisfying equation (17), with ϕ^ε being $(\cot\alpha)$ - Lipschitz such that

$$\phi_*^\varepsilon(x) \le \phi^\varepsilon(x) \le \phi^{\varepsilon *}(x)\,, \quad x \in \mathbb{R}^{N-1}$$

Finally, we take the limit as ε goes to zero. The sub-solutions ϕ_*^ε go to ϕ_*. The super-solutions $\phi^{\varepsilon *}$ converge to ϕ^*. This follows from the expression of super-solutions as an infimum of hyperplanes, those associated to the ε weights going to $+\infty$. Moreover by Ascoli's theorem, $(\phi^\varepsilon)_{\varepsilon>0}$ converges (up to a subsequence) to some concave and $(\cot\alpha)$ - Lipschitz function ϕ^0 solution to (17) and satisfying

$$\phi_* \le \phi^0 \le \phi^*$$

Again a bootstrap argument shows that ϕ^0 is smooth. Therefore, ϕ^0 is the intended solution to the mean curvature equation (17). ■

6.2 Proof of Theorem 1.2

Step 1: Existence of a solution
Choose ϕ^* the viscosity solution to the eikonal equation (10) given by

$$\phi^*(x) = \inf_{\nu\in A}\left(-(\cot\alpha)\ x\cdot\nu + \gamma_\nu\right), \quad x \in \mathbb{R}^{N-1} \tag{29}$$

where $A = \{\nu_1, \dots, \nu_k\}$ is a finite subset of the sphere $\mathbb{S}^{N-2}$, $k \in \mathbb{N}^*$ and γ_ν are any given real numbers. We build a sub-solution ϕ_* as in proposition 4.3

$$\phi_*(x) = -\frac{2}{c_0\sin\alpha}\ln\left(\sum_{i=1}^{k}\lambda_i\ e^{\frac{c_0\cos\alpha}{2}x\cdot\nu_i}\right), \quad x \in \mathbb{R}^{N-1} \tag{30}$$

where λ_i is determined by the relation $\gamma_{\nu_i} = -\frac{2}{c_0\sin\alpha}\ln\lambda_i$ for $i = 1\dots k$. Let us notice that in the particular case when A is finite, the super-solution built in proposition 5.1 coincides with the solution ϕ^* to the eikonal equation. As in section 6.1, $\phi_* \le \phi^*$ and the assumptions of proposition 3.1 i) are satisfied. Thus, there exists a function $\phi \in [\phi_*, \phi^*]$ viscosity solution to (17). Dealing as in section 6.1 step 4, we can even find a smooth concave solution still denoted $\phi \in [\phi_*, \phi^*]$. It now remains to study $\phi_* - \phi^*$ to get a precise

asymptotics of the solution ϕ.

Step 2: Asymptotics (first line of (8))
Setting

$$\phi_i(x) = -(\cot\alpha)\ x\cdot\nu_i - \frac{2}{c_0\sin\alpha}\ln\lambda_i$$

we have

$$\begin{aligned}\phi_*(x) &= -\frac{2}{c_0\sin\alpha}\ln\left(\sum_{i=1}^k e^{-\frac{c_0\sin\alpha}{2}\phi_i(x)}\right) \geq -\frac{2}{c_0\sin\alpha}\ln\left(ke^{-\frac{c_0\sin\alpha}{2}\left(\min\limits_{i=1,\dots,k}\phi_i(x)\right)}\right)\\ &= -\frac{2}{c_0\sin\alpha}\ln\left(ke^{-\frac{c_0\sin\alpha}{2}\phi^*(x)}\right) = \phi^*(x) - \frac{2\ln k}{c_0\sin\alpha}\end{aligned}$$

This implies in particular that

$$-\frac{2\ln k}{c_0\sin\alpha} \leq \phi_* - \phi^* \leq 0 \tag{31}$$

which shows the first line of (8).

Step 3: Asymptotics (second line of (8))
We now notice that the set E_∞ of edges (where ϕ_∞ is not C^1) is characterized by

$$E_\infty = \left\{x\in\mathbb{R}^{N-1},\quad \max_{\nu\in A}\ x\cdot\nu = x\cdot\nu_{i_0} = x\cdot\nu_{i_1},\ \text{with } \nu_{i_0}\neq\nu_{i_1} \text{ and } (\nu_{i_0},\nu_{i_1})\in A^2\right\}$$

For each index $i_0\in\{1,...,k\}$, let us denote the convex set

$$K_{i_0} = \left\{x\in\mathbb{R}^{N-1},\quad x\cdot\nu_{i_0} = \max_{\nu\in A}\ x\cdot\nu\right\}$$

Then

$$\partial K_{i_0} \subset \bigcup_{j\neq i_0}(\nu_{i_0}-\nu_j)^\perp$$

For $x\in\mathrm{Int}(K_{i_0})$, let $x_{i_1}\in\partial K_{i_0}\subset E_\infty$ such that

$$\mathrm{dist}(x,E_\infty) = |x-x_{i_1}| \quad\text{with}\quad x_{i_1}\in(\nu_{i_0}-\nu_{i_1})^\perp.$$

For $j\neq i_0$, we define the orthogonal projection of x on $(\nu_{i_0}-\nu_j)^\perp$ as

$$x_j = \mathrm{Proj}_{|(\nu_{i_0}-\nu_j)^\perp}(x)$$

In particular $|x-x_j|\geq|x-x_{i_1}|$. Moreover

$$\begin{aligned}x\cdot\nu_j &= (x-x_j)\cdot\nu_j + x_j\cdot\nu_j = (x-x_j)\cdot\nu_j + x_j\cdot\nu_{i_0}\\ &= (x-x_j)\cdot(\nu_j-\nu_{i_0}) + x\cdot\nu_{i_0} = x\cdot\nu_{i_0} - |\nu_j-\nu_{i_0}||x-x_j|\\ &\leq x\cdot\nu_{i_0} - \delta\ \mathrm{dist}(x,E_\infty)\end{aligned}$$

with

$$\delta = \min_{\nu \neq \nu',\ \nu,\nu' \in A} |\nu - \nu'| > 0$$

Therefore

$$\begin{aligned}\phi_*(x) &= -\frac{2}{c_0 \sin\alpha} \ln\left(\sum_{i=1}^{k} \lambda_i e^{\frac{c_0 \cos\alpha}{2}\ x\cdot\nu_i}\right)\\ &\geq -\frac{2}{c_0 \sin\alpha} \ln\left(\sum_{i=1}^{k} \lambda_i e^{\frac{c_0 \cos\alpha}{2}\ \left(x\cdot\nu_{i_0} - \delta\ \mathrm{dist}(x,E_\infty)\right)}\right)\end{aligned}$$

and then for $x \in K_{i_0}$, we have

$$\phi^*(x) \geq \phi_*(x) \geq \phi^*(x) - \frac{2}{c_0 \sin\alpha} \ln\left(1 + \sum_{i \neq i_0} \frac{\lambda_i}{\lambda_{i_0}} e^{-\frac{c_0 \cos\alpha}{2}\ \delta\ \mathrm{dist}(x,E_\infty)}\right)$$

This shows that

$$\lim_{l \to +\infty} \sup_{\mathrm{dist}(x,E_\infty) \geq l} |\phi_*(x) - \phi^*(x)| = 0$$

which implies the second line of (8).

Step 4: Uniqueness

To end the proof of theorem 1.2, it only remains to prove uniqueness of the above smooth solution ϕ to the mean curvature equation (3) with the prescribed asymptotics given by ϕ^*. Let $\overline{\phi}$ and $\underline{\phi}$ be two solutions to (3) with the asymptotics (8). Let

$$\varepsilon := \inf\left\{\varepsilon' > 0 \,|\, \forall x \in \mathbb{R}^{N-1},\ \overline{\phi}(x) + \varepsilon' \geq \underline{\phi}(x)\right\}$$

then for any $x \in \mathbb{R}^{N-1}$,

$$\overline{\phi}(x) + \varepsilon \geq \underline{\phi}(x)$$

and there exists a sequence of points $(x_n)_n$ such that

$$\overline{\phi}(x_n) + \varepsilon - \underline{\phi}(x_n) \to 0 \ \text{ as } n \text{ goes to infinity}$$

Let us define for any $x \in \mathbb{R}^{N-1}$

$$\begin{cases} \overline{\phi}_n(x) = \overline{\phi}(x + x_n) - \overline{\phi}(x_n), \\ \underline{\phi}_n(x) = \underline{\phi}(x + x_n) - \overline{\phi}(x_n) \end{cases}$$

Then, up to the extraction of a subsequence, we have as n goes to infinity

$$\overline{\phi}_n \to \overline{\phi}_\infty \quad \text{and} \quad \underline{\phi}_n \to \underline{\phi}_\infty$$

with a uniform convergence on any compact sets of $\mathbb{R}^{N-1}$. Moreover $\overline{\phi}_\infty$ and $\underline{\phi}_\infty$ solve equation (3) and satisfy

$$\overline{\phi}_\infty + \varepsilon \geq \underline{\phi}_\infty \quad \text{with equality at} \quad x = 0$$

From the strong maximum principle, we deduce that for any $x \in \mathbb{R}^{N-1}$,

$$\overline{\phi}_\infty(x) + \varepsilon = \underline{\phi}_\infty(x) \tag{32}$$

Let us now assume that $\varepsilon > 0$. Because we have

$$E_\infty \subset \bigcup_{\nu \neq \nu',\ (\nu,\nu') \in A^2} (\nu - \nu')^\perp =: \hat{E}_\infty$$

we deduce that there exists $C > 0$ such that for any $R \geq 1$ and any $x \in \mathbb{R}^{N-1}$, we have

$$\sup_{y \in \overline{B_R(x)}} \mathrm{dist}(y, E_\infty) \geq \sup_{y \in \overline{B_R(x)}} \mathrm{dist}(y, \hat{E}_\infty) = R \sup_{y \in \overline{B_1(x/R)}} \mathrm{dist}(y, \hat{E}_\infty) \geq CR \tag{33}$$

with

$$C = \inf_{z \in \mathbb{R}^{N-1}} \left(\sup_{y \in \overline{B_1(z)}} \mathrm{dist}(y, \hat{E}_\infty) \right)$$

We easily check by contradiction that $C > 0$. Therefore by (8), let us choose R large enough such that

$$\sup_{\mathrm{dist}(y, E_\infty) \geq CR} |\phi(y) - \phi^*(y)| \leq \frac{\varepsilon}{4} \quad \text{for} \quad \phi = \overline{\phi}, \underline{\phi}$$

Then using (33), we get for some $y_n \in \overline{B_R(x_n)}$ with $\mathrm{dist}(y_n, E_\infty) \geq CR$,

$$\inf_{y \in \overline{B_R(x_n)}} |\overline{\phi}(y) - \underline{\phi}(y)| \leq |\overline{\phi}(y_n) - \underline{\phi}(y_n)| \leq |\overline{\phi}(y_n) - \phi^*(y_n)| + |\phi^*(y_n) - \underline{\phi}(y_n)| \leq \frac{\varepsilon}{2}$$

This implies that

$$\inf_{y \in \overline{B_R(0)}} |\overline{\phi}_\infty(y) - \underline{\phi}_\infty(y)| \leq \frac{\varepsilon}{2}$$

which is in contradiction with (32). Therefore $\varepsilon = 0$ and we get $\overline{\phi} \geq \underline{\phi}$. By symmetry, we also get $\underline{\phi} \geq \overline{\phi}$, which implies $\overline{\phi} = \underline{\phi}$ and shows the uniqueness of the solution. This ends the proof of the theorem 1.2. ■

7 Proof of further results in dimension $N = 3$

In this section, the space dimension is $N = 3$ and we denote any $x \in \mathbb{R}^2$ with its polar coordinates $(r, \theta_x) \in \mathbb{R}^+ \times [0, 2\pi)$ such that $x = r(\cos\theta_x, \sin\theta_x)$.

7.1 Classification in dimension $N = 3$ of solutions to the eikonal equation with a finite number of singularities

This subsection gives alternative statement and proof of proposition 2.3 in dimension $N = 3$, in the special case of a finite number of singularities (i.e. gradient jumps).

Proposition 7.1 (Classification with a finite number of singularities, $N = 3$)
Let $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal equation (10) *in dimension $N = 3$ with a finite number of singularities on $\mathbb{S}^1$. Then the* `i.` *of Theorem 1.3 holds.*

Proof of Proposition 7.1.
From Proposition 2.1, we know that there exists a (non empty) compact set $K = \gamma^{-1}(\{0\}) \subset \mathbb{S}^1$, such that

$$\phi_\infty(x) = \inf_{\nu \in K} \left(-(\cot\alpha)\ \nu \cdot x\right) \tag{34}$$

Thus, for any $\theta \in [0, 2\pi)$, $\psi_\infty(\theta) = \phi_\infty(\cos\theta, \sin\theta)$ defines a continuous function with values in $[-\cot\alpha, \cot\alpha]$. Firstly $\psi_{\infty|K} = -\cot\alpha$. Moreover for any maximal interval (a, b) contained in $\mathbb{S}^1 \backslash K$, we necessarily have

$$\psi_\infty(\theta) = \begin{cases} -(\cot\alpha)\ \cos(\theta - a) & \text{if} \quad \theta \in \left[a, \frac{a+b}{2}\right], \\ \\ -(\cot\alpha)\ \cos(\theta - b) & \text{if} \quad \theta \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Therefore ϕ_∞ has a singularity (gradient jump) at $\theta = \frac{a+b}{2}$. If ψ_∞ only has a finite number of singularities, then we get the characterization of ψ_∞ given in the `i.` of Theorem 1.3. This ends the proof of proposition 7.1.■

Remark 7.2 *Notice that without assuming that ϕ_∞ has a finite number of singularities on $\mathbb{S}^1$, the set K could be a Cantor set in (34).*

Remark 7.3 *Notice that the particular function $\phi_\infty(x) = -(\cot\alpha)|x|$ is the analogue (at the level of the eikonal equation) of the level sets of cylindrically symmetric solutions to reaction diffusion equation, constructed in [11] by Hamel, Monneau and Roquejoffre. Similarly, the particular case where the graph of ϕ_∞ is a pyramid is also the analogue of solutions contructed by Taniguchi in [16].*

7.2 Explicit construction of super-solutions in dimension $N = 3$

In the particular case $N = 3$, we construct super-solutions by hand and try to be more precise than in section 5, above all when ψ_∞ is constant and equal to $-(\cot\alpha)$ on some interval I. In that case, we construct our super-solution by hand. We explain our ideas on different elementary pieces that we bring together in the proof of theorem 1.3 to build a global super-solution ϕ^*. Those different elementary pieces are: **a cone**, **an edge** or **an arc** .

7.2.1 The cone case

Lemma 7.4 ***(Radially symmetric solutions)***
Let ϕ_∞ be the viscosity solution to eikonal equation (10) *whose graph is the straight cone i.e. $\phi_\infty(x) = -(\cot\alpha)|x|$ for $x \in \mathbb{R}^2$. Then, there exists a unique radially symmetric*

solution ϕ_c (unique up to an additive constant) to the forced mean curvature equation (17)*, satisfying*

$$\phi_c'(0) = 0 \text{ and } \phi_c(x) = \phi_\infty(x) + o(|x|)$$

Moreover ϕ_c is concave and $|D\phi_c| \leq \cot\alpha$. In the case $\alpha = \pi/2$, ϕ_c is zero (up to an additive constant). Otherwise, as $|x|$ goes to infinity, its asymptotics is more precisely given (up to a constant $C \in \mathbb{R}$) by

$$\phi_c(x) = -(\cot\alpha)|x| + \frac{1}{c_0 \sin\alpha}\ln|x| + C + \frac{2 - 3\sin^2\alpha}{c_0^2 \sin(2\alpha)|x|} + O\left(\frac{1}{|x|^2}\right), \quad \alpha \neq \frac{\pi}{2} \tag{35}$$

Moreover, let ϕ_* be the sub-solution defined by (23) *with $\mu = \frac{d\theta}{2\pi}$ and $N = 3$. Fix ϕ_c such that $C = C_0 := \frac{\ln(\pi c_0 \cos\alpha)}{c_0 \sin\alpha}$, then for any $x \in \mathbb{R}^2$, $\phi_c(x) \geq \phi_*(x)$ and as $|x|$ goes to infinity*

$$\phi_c(x) = \phi_*(x) + O\left(\frac{1}{\sqrt{|x|}}\right). \tag{36}$$

Proof of Lemma 7.4.
This result is proved using quite classical methods. The proof is sketched for the reader's convenience. With a slight misuse of notation, we denote in the case of radially symmetric solutions $\phi_c(x)$ by $\phi_c(|x|) = \phi_c(r)$ with $r = |x| \geq 0$. Then, equation (17) reads

$$-\frac{\phi_c'}{r} - \frac{\phi_c''}{1 + \phi_c'^2} + c_0\sqrt{1 + \phi_c'^2} - c = 0\,, \quad r > 0$$

Thus, ϕ_c satisfies an ODE involving only its first two derivatives and it can only be defined up to constants. Setting $v = \phi_c'$, we get

$$v' = (1 + v^2)\left(c_0\sqrt{1 + v^2} - c - \frac{v}{r}\right) := (1 + v^2)g(v, r)\,, \quad r > 0 \tag{37}$$

The proof of lemma 7.4 now reduces to the study of this ODE (existence, uniqueness and asymptotics).
Step 1: Existence
Since for any $r > 0$, $g(0, r) \leq 0$ and $g(v_0(r), r) = 0$ where

$$v_0(r) = -\frac{c^2 - c_0^2}{\frac{c}{r} + c_0\sqrt{\frac{1}{r^2} + c^2 - c_0^2}} \leq 0\,,$$

$v = 0$ is a super-solution and $v = v_0$ is a negative decreasing sub-solution to the ODE (37). Thus, for every $r_1 > 0$, there exist $r_2 > r_1$ and a solution $v \in C^\infty((r_1, r_2), \mathbb{R})$ to the ODE $v' = (1 + v^2)g(v, r)$ satisfying $v_0 \leq v \leq 0$ for any $r \in (r_1, r_2)$. Moreover we have $g_v'(v, r) \leq 0$ for $v \in [v_0, 0]$, and then we conclude that

$$v' \leq 0 \quad \text{for} \quad r \in (r_1, r_2) \tag{38}$$

Step 2: Qualitative properties
Since for any $r > 0$, $v_0(r) \in (-\cot\alpha, 0)$, the bounds of $v(r)$ by $v_0(r)$ and zero force v to

exist globally for $r > 0$. Moreover, as $\lim_{r\to 0} v_0(r) = 0$, v satisfies the same limit and we can extend v to 0 by continuity as $v(0) = 0$. This proves that v is a global smooth solution to (37) with initial condition $v(0) = 0$. Thus it is easy to check that any primitive function ϕ_c to v is a smooth radially symmetric solution to (17) satisfying $\phi_c'(0) = 0$. From (38), we conclude that ϕ_c is concave.

Step 3: Asymptotics

Since v is strictly decreasing on $\mathbb{R}^+$ and bounded from below, it converges to a finite limit $-\cot\alpha \le l < 0$ as r goes to infinity. Since v is uniformly bounded in $[-\cot\alpha, 0]$, l must satisfy $g(l, +\infty) = 0$ which leads to $l = -\cot\alpha$.

Linearising equation (37) around $-\cot\alpha$, we set $w = v + \cot\alpha$. As w is uniformly bounded on $\mathbb{R}^+$ and goes to zero at infinity, equation (37) reads

$$w'(r) = -c\,(\cot\alpha)w + \tilde{g}(w, r)\,, \quad r > 0$$

where $\tilde{g}(w, r) = O(w^2) + O(1/r)$ as r goes to infinity. By the Duhamel's formula, w follows exponentially fast the behaviour of the slowest term of $\tilde{g}$. Thus $w \sim C/r$ as r goes to infinity and a straight calculation gives $C = 1/(c_0 \sin\alpha)$. Repeating this method up to order 2, one gets

$$v(r) = -\cot\alpha + \frac{1}{c_0(\sin\alpha)r} + \frac{3\sin^2\alpha - 2}{c_0^2 \sin(2\alpha) r^2} + O\left(\frac{1}{r^3}\right)$$

This gives the desired asymptotics for ϕ_c up to constants.

Step 4: Uniqueness

Let ϕ_c^1 and ϕ_c^2 be two smooth radially symmetric solutions to (17). From step 3, we know that they satisfy the same asymptotic expansion as r goes to infinity and we assume the constants C are the same. Since $\phi_c^1 - \phi_c^2$ solves an elliptic equation with smooth cœfficients and no zero order term, the classical maximum principle applies. Hence $\phi_c^1 - \phi_c^2 = 0$ because $\lim_{r\to\infty}(\phi_c^1 - \phi_c^2)(r) = 0$. This proves the uniqueness of ϕ_c up to constants.

Step 5: Comparison with ϕ_*

Using Lemma 8.1, we can check (36) with the suitable value of the constant $C = C_0$ (see for instance the computation (44) with $N_0(x) \simeq 1/\sqrt{\pi}$). Finally, using the comparison principle (as in Step 4), we deduce that $\phi_* \le \phi_c$. This ends the proof of lemma 7.4. ■

7.2.2 The edge case

Lemma 7.5 ***(Edge super-solution)***
Assume ϕ_* is given by (23) *where the measure μ is the sum of two Dirac masses*

$$\mu = \mu_{\{\theta_1,\theta_2\}} = \lambda_1 \delta_{\theta_1} + \lambda_2 \delta_{\theta_2}$$

with $\lambda_i > 0$, $\theta_i \in [0, 2\pi)$ for $i = 1, 2$ such that $\theta_1 < \theta_2$ and δ_{θ_i} the Dirac mass in θ_i. In the case $\alpha \neq \pi/2$, define ϕ_e for any $x \in \mathbb{R}^2$ by

$$\phi_e(x) = \min(p_1(x), p_2(x)) \text{ with } p_i(x) = -(\cot\alpha)\; x \cdot \nu_i - \frac{2}{c_0 \sin\alpha} \ln\lambda_i \tag{39}$$

where $\nu_i = (\cos\theta_i, \sin\theta_i)$. Then, ϕ_e is a Lipschitz and piecewise smooth global super-solution to (17) *verifying $\phi_* \le \phi_e$ on $\mathbb{R}^2$. Moreover, as $|x|$ goes to infinity,*

$$\phi_e(x) = \phi_*(x) + O(1)\,, \quad x \in \mathbb{R}^2 \tag{40}$$

Proof of Lemma 7.5.
Notice first that $\phi_e = \phi^*$ with ϕ^* defined as a special case of Proposition 5.1. This shows that ϕ_e is a concave (Lipschitz) supersolution. Finally (40) follows from (31). This ends the proof of the lemma. ■

7.2.3 The arc case

Here we wish to describe a super-solution to (17) which, from above, looks like an arc, i.e. is made up of two non parallel straight lines connected by a circle.

Lemma 7.6 ***(Arc super-solution)***
Assume ϕ_* is given by (23) *where the measure μ is the sum of two Dirac masses and a Lebesgue measure*

$$\mu = \mu_{[\theta_1,\theta_2]} = \lambda\delta_{\theta_1} + \lambda\delta_{\theta_2} + \mathbb{1}_{(\theta_1,\theta_2)}\,\mathrm{d}\theta,$$

where $\lambda > 0$, $\theta_i \in [0, 2\pi)$ for $i = 1, 2$ and $\theta_1 < \theta_2$.

Define ϕ_e as in the edge case (39) *with $\lambda_1 = \lambda_2 = \lambda$. Define ϕ_c as in the cone case (lemma 7.4) where the constant $C \in \mathbb{R}$ in* (35) *is chosen such that $\phi_c(0) = \phi_e(0) = -\frac{2}{c_0 \sin\alpha}\ln\lambda$.*

Finally, define ϕ_a on $\mathbb{R}^2$ by

$$\forall x \in \mathbb{R}^2\,, \quad \phi_a(x) = \begin{cases} -\dfrac{2}{c_0\sin\alpha}\ln\lambda & \text{if } x = 0 \\ \min(\phi_c(x), \phi_e(x)) & \text{if } \theta_x \in (\theta_1, \theta_2) \\ \phi_e(x) & \text{otherwise,} \end{cases} \tag{41}$$

Then ϕ_a is a Lipschitz continuous global super-solution to (17)*. Moreover, as $|x|$ goes to infinity*

$$\phi_a(x) = \phi_*(x) + O(1)$$

The shape of ϕ_a is sketched on Figures 1,2, 3.

Proof of Lemma 7.6
Step 1: ϕ_a is a global continuous super-solution
By definition and lemmas 7.4 and 7.5, ϕ_a is a super-solution to (17) where it is locally the minimum of supersolutions, i.e. everywhere except on the two half lines $\theta_x = \theta_i$ for $i = 1, 2$. However, we have $\phi_c(0) = \phi_e(0)$ and $D\phi_e(x) = -(\cot\alpha)\nu_i$ while $\phi_c'(r) \in (-\cot\alpha, 0]$ for any $x \in \mathbb{R}^2$ with $\theta_x = \theta_i$, $i = 1$ or 2. Thus, $\phi_e(x) \le \phi_c(x)$ on a neighborhood $\mathcal{N}$ (not containing the origin) of the two half lines $\theta_x = \theta_i$ for $i = 1, 2$. This implies $\phi_a = \phi_e$ on $\mathcal{N}$ and then ϕ_a is at least a supersolution on $\mathbb{R}^2 \setminus \{0\}$. Moreover, ϕ_a is a supersolution on the whole $\mathbb{R}^2$. Indeed, $\phi_a = \phi_e$ for $\theta_x \notin [\theta_1, \theta_2]$, then ϕ_a has a gradient jump along the edge $\theta = (\theta_1 + \theta_2)/2 + \pi$ up to the origin. And this gradient jump implies that there is no C^2 test function touching ϕ_a from below at $x = 0$.

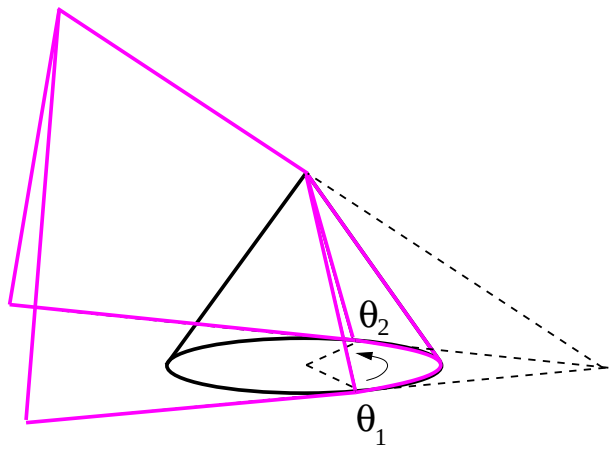

Figure 1: Sketch of ϕ_a for $\theta_2 - \theta_1 < \pi$

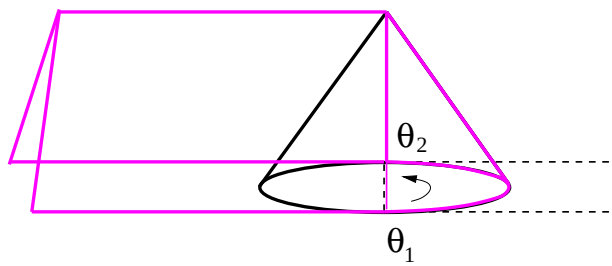

Figure 2: Sketch of ϕ_a for $\theta_2 - \theta_1 = \pi$

Step 2: Relative positions of ϕ_e and ϕ_c
Let us now study the relative positions of both graphs of ϕ_e and ϕ_c. Since $\phi_c(x) \in (-|x|\cot\alpha + \phi_c(0), \phi_c(0)]$, we have

$$\begin{aligned} p_1(x) = \phi_c(x) &\Longleftrightarrow -r(\cot\alpha)\cos(\theta_x - \theta_1) + \phi_c(0) = \phi_c(r) \\ &\Longleftrightarrow \theta_x = \theta_1 \pm \arccos\left(\frac{\phi_c(r) - \phi_c(0)}{-r\cot\alpha}\right) := \theta_1 \pm \bar{\theta}(r) \end{aligned}$$

where $\bar{\theta}(r) \in (0, \pi/2)$ for $r > 0$ (see Figure 4). Notice that from the concavity of ϕ_c, we deduce that the set

$$\{x, \quad p_1(x) \leq \phi_c(x)\} = \left\{x, \quad \theta_x \in [\theta_1 - \overline{\theta}(r), \theta_1 + \overline{\theta}(r)]\right\}$$

is a convex set. Therefore we deduce that

$$\phi_e(x) \leq \phi_c(x) \Leftrightarrow \theta_x \in [\theta_1 - \bar{\theta}(r), \theta_1 + \bar{\theta}(r)] \cup [\theta_2 - \bar{\theta}(r), \theta_2 + \bar{\theta}(r)]$$

and $\phi_a(x) = \phi_c(x)$ if and only if $\theta_x \in I_r = [\theta_1 + \bar{\theta}(r), \theta_2 - \bar{\theta}(r)]$. Since we choose ϕ_c such that $\phi_c'(0) = 0$, we get

$$\lim_{r\to 0} \bar{\theta}(r) = +\frac{\pi}{2}$$

This forces both curves $\theta_x = \theta_1 + \bar{\theta}(r)$ and $\theta_x = \theta_2 - \bar{\theta}(r)$ to intersect at some point $(x, z) \neq (0, \phi_c(0))$ as soon as $\theta_2 - \theta_1 < \pi$. In that case, it is worth noticing that the above interval I_r is empty for sufficiently small r (see Figures 5,6).

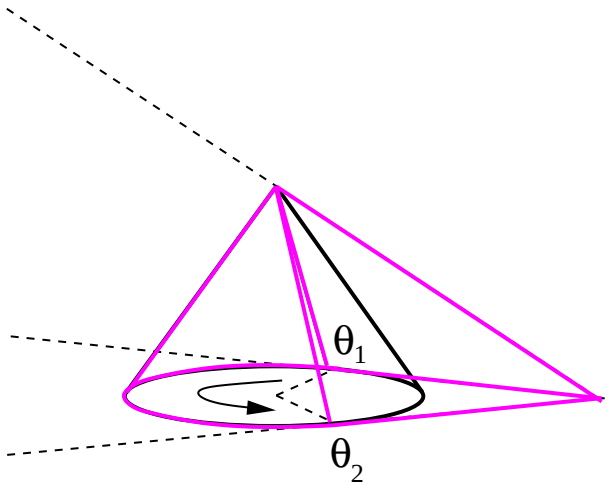

Figure 3: Sketch of ϕ_a for $\theta_2 - \theta_1 > \pi$

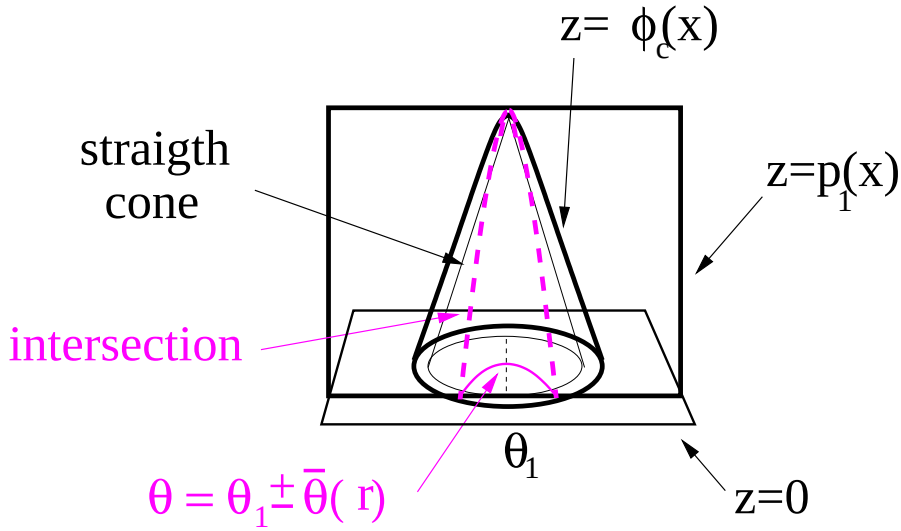

Figure 4: Intersection of $z = -(\cot\alpha)\ v_1 \cdot x$ with $z = \phi_c(x)$

On the other hand, using the asymptotics (35) of ϕ_c found in lemma 7.4, we get as r goes to infinity

$$\cos\bar{\theta}(r) = 1 - \frac{1}{c_0\cos\alpha}\frac{\ln r}{r} + \frac{\phi_c(0) - C}{r\cot\alpha} + O\left(\frac{1}{r^2}\right), \quad \alpha \neq \frac{\pi}{2} \tag{42}$$

where C is the constant given by (35) and fixed by the choice $\phi_c(0) = \phi_e(0)$.
Step 3: $\phi_* \leq \phi_a + constant$
Let ϕ_* be the sub-solution given by (23) where the measure μ is $\mu_{[\theta_1,\theta_2]}$, i.e. with $b = c_0\cos\alpha$:

$$\phi_*(x) = -\frac{2}{c_0\sin\alpha}\ln\left(\lambda e^{\frac{br}{2}\cos(\theta_x-\theta_1)} + \lambda e^{\frac{br}{2}\cos(\theta_x-\theta_2)} + \int_{\theta_1}^{\theta_2} e^{\frac{br}{2}\cos(\theta_x-\theta)}\ d\theta\right)$$

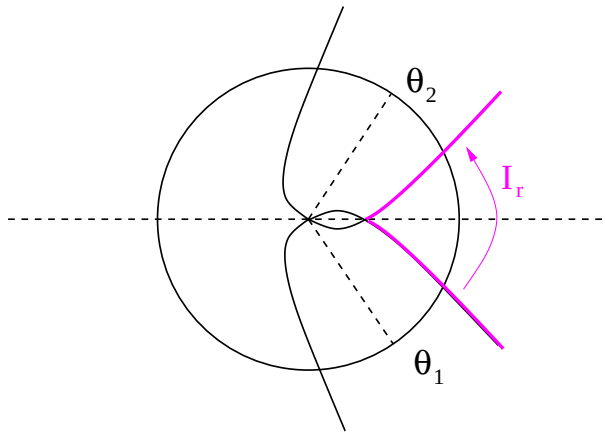

Figure 5: The set I_r if $\theta_2 - \theta_1 < \pi$

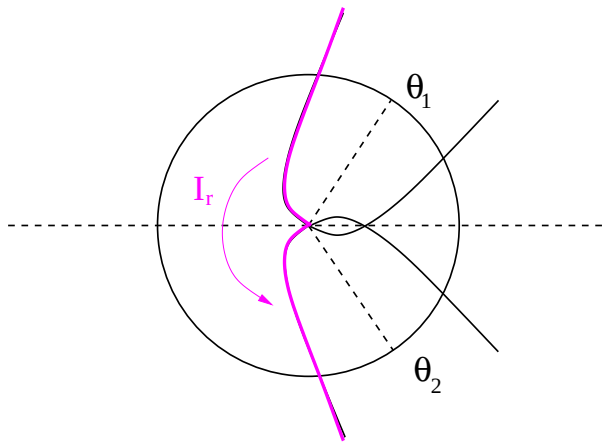

Figure 6: The set I_r if $\theta_2 - \theta_1 > \pi$

Each term in the Logarithm being non negative, we have

$$\phi_* \leq \phi_e \quad \text{on} \quad \mathbb{R}^2. \tag{43}$$

To prove that ϕ_a is above the sub-solution ϕ_* up to an additive constant, it remains to compare ϕ_* and ϕ_c when $\theta_x \in I_r = [\theta_1 + \bar{\theta}(r), \theta_2 - \bar{\theta}(r)]$ and r sufficiently large.

According to lemma 8.1, one gets that for any $x \in \mathbb{R}^2$ for $r = |x|$ sufficiently large and uniformly in $\theta_x \in [\theta_1, \theta_2]$

$$\phi_*(x) = -(\cot\alpha)r + \frac{\ln r}{c_0 \sin\alpha} - \frac{2}{c_0 \sin\alpha}\ln(\Phi(x)) \tag{44}$$

$$\text{with} \quad \Phi(x) := \frac{2\pi N_0(x)}{\sqrt{b}} + \lambda\sqrt{r}e^{\frac{br}{2}(\cos(\theta_x-\theta_1)-1)} + \lambda\sqrt{r}e^{\frac{br}{2}(\cos(\theta_x-\theta_2)-1)} + O\left(\frac{1}{\sqrt{r}}\right)$$

where

$$N_0(x) = \int_{\sqrt{r}g(\theta_1-\theta_x)}^{\sqrt{r}g(\theta_2-\theta_x)} e^{-\frac{u^2}{4}}\frac{\mathrm{d}u}{2\pi}$$

as defined in lemma 8.1. Since g is odd and $\theta_x \in [\theta_1, \theta_2]$, we see that

$$N_0(x) \geq \int_0^{\sqrt{r} g\left(\frac{\theta_2-\theta_1}{2}\right)} e^{-\frac{u^2}{4}} \frac{\mathrm{d}u}{2\pi} = \frac{1}{2\sqrt{\pi}} + o_r(1)$$

We deduce in particular that for r large enough and uniformly in $\theta_x \in [\theta_1, \theta_2]$:

$$\Phi(x) \geq \frac{\sqrt{\pi}}{2\sqrt{b}}$$

Therefore, from the asymptotics (35) of ϕ_c, we deduce that there exist $r_1 > 0$, $C_1 > 0$ such that

$$\forall r \geq 0\,, \quad \forall \theta_x \in [\theta_1, \theta_2]\,, \quad r \geq r_1 \Rightarrow \phi_*(x) \leq \phi_c(x) + C_1 \tag{45}$$

Now from (43) and (45), we deduce that (up to increasing the constant C_1),

$$\forall r \geq 0\,, \quad \forall \theta_x \in [\theta_1, \theta_2]\,, \quad r \geq r_1 \Rightarrow \phi_*(x) \leq \phi_a(x) + C_1$$

Step 4: $\phi_* \geq \phi_a - constant$

Case 1: $\theta_x \in I_r$
We start with the asymptotics (44). Using (42), we see that there exist $r_2 > 0$, $C_2 > 0$ such that

$$\forall i = 1, 2\,, \quad \forall r \geq 0\,, \quad \forall \theta_x \in I_r\,, \quad r \geq r_2 \Rightarrow \sqrt{r} e^{\frac{br}{2}(\cos(\theta_x - \theta_i) - 1)} \leq \sqrt{r} e^{\frac{br}{2}(\cos(\overline{\theta}(r)) - 1)} \leq C_2$$

Using also the fact that $N_0(x) \leq 1/(2\sqrt{\pi})$, we deduce that Φ is bounded for r large enough and then (up to increasing r_2 and C_2)

$$\forall r \geq 0\,, \quad \forall \theta_x \in I_r\,, \quad r \geq r_2 \Rightarrow \phi_*(x) \geq \phi_c(x) - C_2 \tag{46}$$

Case 2: $\theta_x \in [\theta_1, \theta_2] \backslash I_r$
Let us assume that $\theta_x \in [\theta_1, \theta_1 + \overline{\theta}(r))$ (the symmetric case is similar). Then there exist $r_3 > 0$, $C_3 > 0$ such that

$$\forall r \geq 0\,, \quad \forall \theta_x \in [\theta_1, \theta_1 + \overline{\theta}(r))\,, \quad r \geq r_3 \Rightarrow \sqrt{r} e^{\frac{br}{2}(\cos(\theta_x - \theta_1) - 1)} \geq \sqrt{r} e^{\frac{br}{2}(\cos(\overline{\theta}(r)) - 1)} \geq C_3 > 0$$

Therefore,(up to increasing the constants r_3 and C_3) for any $r \geq 0$ and any $\theta_x \in [\theta_1, \theta_1 + \overline{\theta}(r))$,

$$r \geq r_3 \Rightarrow \Phi(x) \leq C_3 \left(\lambda \sqrt{r} e^{\frac{br}{2}(\cos(\theta_x - \theta_1) - 1)} + \lambda \sqrt{r} e^{\frac{br}{2}(\cos(\theta_x - \theta_2) - 1)} \right)$$

and then

$$\begin{aligned} \phi_*(x) &\geq -\frac{2}{c_0 \sin\alpha} \ln \left\{ \lambda e^{\frac{br}{2}\cos(\theta_x - \theta_1)} + \lambda e^{\frac{br}{2}\cos(\theta_x - \theta_2)} \right\} - \frac{2 \ln C_3}{c_0 \sin\alpha} \\ &\geq -\frac{2}{c_0 \sin\alpha} \ln \left(2 e^{\frac{c_0 \sin\alpha}{2} \phi_e(x)} \right) - \frac{2 \ln C_3}{c_0 \sin\alpha} \\ &= \phi_e(x) - \frac{2 \ln(2C_3)}{c_0 \sin\alpha} \end{aligned} \tag{47}$$

Case 3: $\theta_x \in \mathbb{S}^1 \backslash [\theta_1, \theta_2]$
Notice that the set $\mathbb{S}^1 \backslash [\theta_1, \theta_2]$ is not empty because $\theta_2 - \theta_1 < 2\pi$ (as a consequence of $\theta_1, \theta_2 \in [0, 2\pi)$). In that case, we define

$$\theta'_m = \theta_m - \pi \quad \text{with} \quad \theta_m = \frac{\theta_1 + \theta_2}{2}.$$

Then, it satisfies $\theta_2 - 2\pi < \theta'_m < \theta_1$. Let us assume that (the other case is similar):

$$\theta_x \in [\theta'_m, \theta_1). \tag{48}$$

We also define

$$\theta'_x = \theta_x + \pi$$

Then we have

$$\int_{\theta_1}^{\theta_2} e^{\frac{br}{2}\cos(\theta_x - \theta)}\, d\theta = \int_{[\theta_1,\theta_2]\cap[\theta_x,\theta'_x]} (...)\, d\theta + \int_{[\theta_1,\theta_2]\backslash[\theta_x,\theta'_x]} (...)\, d\theta$$

We have

$$\int_{[\theta_1,\theta_2]\cap[\theta_x,\theta'_x]} e^{\frac{br}{2}\cos(\theta_x - \theta)}\, d\theta \leq \pi\, e^{\frac{br}{2}\cos(\theta_x - \theta_1)}$$

Using (48), we also see that

$$\int_{[\theta_1,\theta_2]\backslash[\theta_x,\theta'_x]} e^{\frac{br}{2}\cos(\theta_x - \theta)}\, d\theta \leq \pi\, e^{\frac{br}{2}\cos(\theta_x - \theta_1)}$$

Therefore, we conclude that in this third case,

$$\begin{aligned} \phi_*(x) &\geq -\frac{2}{c_0 \sin\alpha} \ln\left\{(\lambda + 2\pi)e^{\frac{br}{2}\cos(\theta_x - \theta_1)} + (\lambda + 2\pi)e^{\frac{br}{2}\cos(\theta_x - \theta_2)}\right\} \\ &\geq \phi_e(x) - \frac{2\ln\left(\frac{2(\lambda+2\pi)}{\lambda}\right)}{c_0 \sin\alpha} \end{aligned} \tag{49}$$

Conclusion
Putting (46), (47) and (49) together, we get that there exists a constant $C > 0$ such that for r large enough and uniformly in $\theta_x \in \mathbb{S}^1$.

$$\phi_*(x) \geq \phi_a(x) - C$$

The functions ϕ_* and ϕ_a being continuous, the result still holds for any $r \geq 0$ (up to increasing the constant C). This concludes the proof of lemma 7.6. ■

7.3 Proof of Theorem 1.3

Step 1: Existence of a solution
Let $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$. The case $\alpha = \pi/2$ is obvious and we omit it. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal equation (10) in dimension

$N = 3$ with a finite number m of singularities. By proposition 7.1, the `i.` of Theorem 1.3 is already established, and we can consider the measure μ given in the `ii.` of Theorem 1.3. Then Proposition 4.3 implies that the function ϕ_* given by (9) is a smooth concave subsolution of (17).

If $k = 1$, ϕ_∞ has no gradient jump and the corresponding measure is $\mu = \mathrm{d}\theta$ or $\mu = \lambda_0(\delta_{\theta_1} + \delta_{\theta_1+2\pi})$. In the first case, we saw in lemma 7.4 that ϕ_c, to which a suitable constant is added, is a smooth solution to (17) with the right asymptotics at infinity. In the second one, $\phi = \phi_\infty$ is a suitable solution to (17).

We now turn to the case $k \geq 2$. For any $i \in \{1, \dots, k\}$, choose $\lambda_0 > 0$ a given positive constant. We have

$$\mu = \sum_j \mu_j \geq \tilde{\mu}_i$$

with

$$\tilde{\mu}_i = 2\lambda_0(\delta_{\theta_i} + \delta_{\theta_{i+1}}) + \sigma_i \ \mathbb{1}_{(\theta_i, \theta_{i+1})} \, \mathrm{d}\theta \tag{50}$$

Let $\tilde{\phi}_{i*}$ be the subsolution defined in (23) with the measure $\tilde{\mu}_i$. If $\tilde{\mu}_i$ corresponds to an arc ($\sigma_i = 1$), denote $\tilde{\phi}_i^*$ the global supersolution defined in lemma 7.6 with $\lambda = 2\lambda_0$. Notice that there is a constant $C > 0$ (that can be chosen independently of the index i) such that

$$\tilde{\phi}_i^* - C \leq \tilde{\phi}_{i*} \leq \tilde{\phi}_i^* + C \tag{51}$$

If $\tilde{\mu}_i$ corresponds to an edge ($\sigma_i = 0$), denote $\tilde{\phi}_i^*$ the global supersolution defined in lemma 7.5 with $\lambda_1 = \lambda_2 = 2\lambda_0$, which satisfies in particular (51). Finally, define on $\mathbb{R}^2$ the function $\tilde{\phi}^*$ as the infimum over $i \in \{1, \dots, k\}$ of $\tilde{\phi}_i^*$. Notice that, by construction, we have

$$\tilde{\phi}^*(x) = \tilde{\phi}_i^*(x) \quad \text{if} \quad \theta_x \in [\theta_i, \theta_{i+1}] \tag{52}$$

We also have in particular

$$\phi_* \leq \tilde{\phi}_{i*} \leq \tilde{\phi}_i^* + C \leq \tilde{\phi}^* + C =: \phi^*$$

We claim that at infinity

$$\phi^*(x) = \phi_*(x) + O(1). \tag{53}$$

We shall first finish the proof of theorem 1.3 and come back to the proof of that claim in a second step. Thus ϕ^* is a global supersolution above the subsolution ϕ_*. Moreover, either there exists $\sigma_i = 1$ and then we have (see in particular Figures 1, 2, 3)

$$\text{there exists } p \in \mathbb{R}^{N-1} \text{ such that } \limsup_{|x| \to +\infty} \frac{\phi^*(x) - p \cdot x}{|x|} < 0. \tag{54}$$

Or $\sigma_i = 0$ for any i, and condition (54) is satisfied if $k \geq 3$. The special case $k = 2$ and $\sigma_1 = \sigma_2 = 0$ corresponds to an edge for which we already know the existence of a smooth

concave solution, by Theorem 1.2. In the other cases, condition (54) and Proposition 3.1 imply the existence of a smooth concave solution $\phi \in [\phi_*, \phi^*]$.

Step 2: Proof of (53) in the case $k \geq 2$

Let $x \in \mathbb{R}^2$, then there exists $i \in \{1, \dots, k\}$ such that $\theta_x \in [\theta_i, \theta_{i+1}]$.

We can write:

$$\mu = \tilde{\mu}_i + \bar{\mu}_i$$

where $\tilde{\mu}_i$ is defined by (50). So

$$\text{supp}(\bar{\mu}_i) \quad \subset \quad \mathbb{S}^1 \backslash (\theta_i, \theta_{i+1}) \quad \neq \emptyset \tag{55}$$

Then $k \geq 2$ implies that

$$l_x := \min\{\theta_x - \theta_i, \theta_{i+1} - \theta_x\} < \pi$$

and

$$\int_{\mathbb{S}^1} e^{\frac{br}{2}\cos(\theta_x - \theta)} \, d\bar{\mu}_i(\theta) \leq \bar{\mu}_i(\mathbb{S}^1) e^{\frac{br}{2}\cos l_x} \leq \frac{\bar{\mu}_i(\mathbb{S}^1)}{2\lambda_0} \int_{\mathbb{S}^1} e^{\frac{br}{2}\cos(\theta_x - \theta)} \, d\tilde{\mu}_i(\theta)$$

with

$$\int_{\mathbb{S}^1} e^{\frac{br}{2}\cos(\theta_x - \theta)} \, d\tilde{\mu}_i(\theta) = 2\lambda_0 e^{\frac{br}{2}\cos(\theta_x - \theta_i)} + 2\lambda_0 e^{\frac{br}{2}\cos(\theta_x - \theta_{i+1})} + \sigma_i \int_{\theta_i}^{\theta_{i+1}} e^{\frac{br}{2}\cos(\theta_x - \theta)} \, \mathrm{d}\theta$$

Therefore we have

$$\begin{aligned}
\phi_*(x) &\geq -\frac{2}{c_0 \sin\alpha} \ln\left\{\left(1 + \frac{\bar{\mu}_i(\mathbb{S}^1)}{2\lambda_0}\right) \int_{\mathbb{S}^1} e^{\frac{br}{2}\cos(\theta_x - \theta)} \, d\tilde{\mu}_i(\theta)\right\} \\
&= \tilde{\phi}_{i*}(x) - \frac{2}{c_0 \sin\alpha} \ln\left(1 + \frac{\bar{\mu}_i(\mathbb{S}^1)}{2\lambda_0}\right) \\
&\geq \tilde{\phi}_i^*(x) - C'
\end{aligned}$$

where we have used (51) in the last line. Using (52), we see that this implies

$$\phi_*(x) \geq \phi^*(x) - C'' \quad \text{for} \quad \theta_x \in [\theta_i, \theta_{i+1}] \tag{56}$$

Finally, this implies (53) and ends the proof of theorem 1.3. ■

8 Appendix: Laplace's method

For the reader's convenience, we reproduce here Laplace's method. It investigates asymptotics as r goes to infinity of integrals involving expressions of the form e^{-rJ}, J denoting some given function. Our interest is to find uniform estimates as $x = r(\cos\theta_x, \sin\theta_x)$ lies in a given angle sector $[\theta_1, \theta_2]$. The proof develops ideas that can be found for a simpler case in [7], chapter 4.5.2 page 204. Lemma 8.1 below is only used in Step 3 and 4 of the proof of Lemma 7.6.

Lemma 8.1 ***(Uniform asymptotics in a sector*** $[\theta_1, \theta_2]$***)***
Define for any $x = r(\cos\theta_x, \sin\theta_x) \in \mathbb{R}^2$ *with* $\theta_x \in [0, 2\pi)$

$$F(x) = \lambda_1\, e^{\frac{br}{2}\cos(\theta_1-\theta_x)} + \lambda_2\, e^{\frac{br}{2}\cos(\theta_2-\theta_x)} + \int_{\theta_1}^{\theta_2} e^{\frac{br}{2}\cos(\theta-\theta_x)} f(\theta)\frac{\mathrm{d}\theta}{2\pi}$$

where

i. $b = c_0\cos\alpha > 0$, $\lambda_i \in \mathbb{R}$, $\theta_i \in [0, 2\pi]$ *for* $i = 1, 2$ *and* $\theta_1 < \theta_2$

ii. $f \in C^1([0, 2\pi], \mathbb{C})$ *is* 2π*-periodic.*

As r *goes to infinity, we have the following asymptotics uniform in the angular sector* $\theta_x \in [\theta_1, \theta_2]$

$$F(x) = \lambda_1\, e^{\frac{br}{2}\cos(\theta_1-\theta_x)} + \lambda_2\, e^{\frac{br}{2}\cos(\theta_2-\theta_x)} + e^{\frac{br}{2}}\left(\frac{f(\theta_x)}{\sqrt{br}}N_0(x) + \frac{R(x)}{r}\right)$$

where

$$N_0(x) = \int_{\sqrt{r}g(\theta_1-\theta_x)}^{\sqrt{r}g(\theta_2-\theta_x)} e^{-\frac{u^2}{4}}\frac{\mathrm{d}u}{2\pi} \quad \in [0, 1/\sqrt{\pi}]$$

and

$$g(\theta) = \begin{cases} sign(\theta)\sqrt{2b(1-\cos\theta)} & for \quad \theta \in [-\pi, \pi] \\ sign(\theta)2\sqrt{b} & for \quad \theta \in \mathbb{R}\backslash[-\pi, \pi] \end{cases} \tag{57}$$

Moreover, there exists a constant $C > 0$ *such that for any* $x \in \mathbb{R}^2$*, if* $r > 1$ *and* $\theta_x \in [\theta_1, \theta_2]$ *then* $|R(x)| \leq C$*.*

Proof of Lemma 8.1.
It is straightforward to check that g defined by (57) is an odd C^3-diffeomorphism from $[-\pi, \pi]$ to $[-2\sqrt{b}, 2\sqrt{b}]$ satisfying $g(0) = 0$, $g'(0) = \sqrt{b}$ and $g"(0) = 0$. We have also chosen to extend g to the real line by continuity. However, when we speak about g^{-1}, it has to be understood as the inverse of g on $[-\pi, \pi]$.

Afterwards, for any $x \in \mathbb{R}^2$, we define

$$I(x) := \int_{\theta_1}^{\theta_2} e^{\frac{br}{2}\cos(\theta-\theta_x)} f(\theta)\frac{\mathrm{d}\theta}{2\pi} = \int_{\theta_1-\theta_x}^{\theta_2-\theta_x} e^{\frac{br}{2}\cos\theta} f(\theta+\theta_x)\frac{\mathrm{d}\theta}{2\pi}$$

Assume $\theta_x \in [\theta_1, \theta_2]$. In order to get a bound on I uniform in the angle θ_x, we fix some $\delta > 0$ and set

$$\theta_* = \begin{cases} \theta_1 - \theta_x & \text{if } \theta_1 - \theta_x \geq -\pi + \delta \\ -\pi + \delta & \text{otherwise} \end{cases} \qquad \theta^* = \begin{cases} \theta_2 - \theta_x & \text{if } \theta_2 - \theta_x \leq \pi - \delta \\ \pi - \delta & \text{otherwise} \end{cases}$$

We then cut the integral I into three parts, integrating between $\theta_1 - \theta_x$ and θ_*, between θ_* and θ^* and finally between θ^* and $\theta_2 - \theta_x$. We call those three integrals I_1, I_2 and I_3 respectively.

Regarding I_1 and I_3, $\cos\theta$ can be bounded in both cases by $\cos(\pi-\delta)$ and f by its L^∞ norm on the compact set $[0,2\pi]$. Thus, there exists a constant $C>0$ such that for any $x\in\mathbb{R}^2$ with $\theta_x\in[\theta_1,\theta_2]$,

$$I_1+I_3\leq Ce^{\frac{br}{2}\cos(\pi-\delta)}$$

For sufficiently small $\delta>0$, the right hand term decreases exponentially fast and the contribution of I_1 and I_3 in I is exponentially small as r goes to infinity uniformly in θ_x.

Using the change of variables $u=\sqrt{r}g(\theta)$, we rewrite I_2 as

$$I_2(x)=\frac{e^{\frac{br}{2}}}{\sqrt{br}}\int_{\sqrt{r}g(\theta_*)}^{\sqrt{r}g(\theta^*)}e^{-\frac{u^2}{4}}\,h\left(\frac{u}{\sqrt{r}}\right)\frac{\mathrm{d}u}{2\pi}$$

where $h(t)=f(\theta_x+g^{-1}(t))/\sqrt{1-(t^2/(4b))}$. Since $h(t)=h(0)+\int_0^t h'(s)\,\mathrm{d}s$, we have

$$I_2(x)=e^{\frac{br}{2}}\left(\frac{f(\theta_x)}{\sqrt{br}}N_0^*(x)+\frac{R(x)}{r}\right)$$

where N_0^* is defined as in lemma 8.1 with θ_* or θ^* when needed, but it only changes the desired asymptotics with an exponentially small correction as above. The remainder term R is defined as

$$R(x)=r^{\frac{1}{2}}\int_{\sqrt{r}g(\theta_*)}^{\sqrt{r}g(\theta^*)}e^{-\frac{u^2}{4}}\int_0^{\frac{u}{\sqrt{r}}}\frac{h'(s)}{\sqrt{b}}\,\mathrm{d}s\,\frac{\mathrm{d}u}{2\pi}$$

Since h is smooth, h' is uniformly bounded on $[g(\theta_*),g(\theta^*)]$ and the bound only depends on δ. A straight calculation then shows that there exists $C>0$ such that

$$R(x)\leq C\int_{\sqrt{r}g(\theta_*)}^{\sqrt{r}g(\theta^*)}|u|e^{-\frac{u^2}{4}}\,\mathrm{d}u\leq C\int_{-\infty}^{+\infty}|u|e^{-\frac{u^2}{4}}\,\mathrm{d}u$$

Putting finally I_1, I_2 and I_3 together, we get the desired asymptotics. ■

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