A geometric derivation of the linear Boltzmann equation for a particle interacting with a Gaussian random field

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A GEOMETRIC DERIVATION OF THE LINEAR BOLTZMANN EQUATION FOR A PARTICLE INTERACTING WITH A GAUSSIAN RANDOM FIELD

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Abstract. The linear Boltzmann equation is derived from a microscopic quantum model of a particle interacting, in the weak coupling limit, with a translation invariant and centered Gaussian random field. The result holds in dimension $d \geq 3$ for initial states given as a pure families of states, in the sense of semiclassical measures and an assumption about renewing the stochastic potential is needed in order to force the asymptotic Markovian property, unlike in the works of Erdös and Yau. This derivation uses an isomorphism between the Hilbert space of square integrable function w.r.t. the random parameter and the bosonic Fock space over $L^2(\mathbb{R}^d)$. The solutions for the microscopic model are approximately coherent states with a parameter in the phase space, providing a geometric viewpoint.

1. Introduction

The derivation of the linear Boltzmann equation has been studied for both classical and quantum microscopic models.

In the classical case the article [15] provided a derivation of the linear Boltzmann equation for Green functions in the case of a Lorentz gas. Later the article [32] presented a review of different classical microscopic models and of kinetic equations obtained as limits of these models, with emphasis on the approximated Markovian behaviour of the microscopic dynamics (some quantum models are also studied). The article [6] gives a derivation of the linear Boltzmann equation for the density of particles in the case of the Lorentz model.

In the quantum case, the weak coupling limit of a Fermi gas in a translation invariant Gaussian potential (and other random potentials) is studied in [24]. Their proofs make use of combinatorics and graphs technics. We are here interested in the bosonic case. In this setting results of Erdös and Yau, and more recently of Poupaud and Vasseur exist.

The article of Erdös and Yau [13]: differs from our work in several points.

- They need initial data in the WKB form $h^{d/2} f(hx) \exp(\frac{iS(hx)}{h})$ for f and S in the Schwartz class, whereas we can use any bounded pure (in the sense of semiclassical measures) family of states in $\mathcal{L}_1^+(L^2(\mathbb{R}^d))$.
- We need a renewal of the stochastics to get the Markovian behaviour, whereas they don't need this additional assumption.
- Our result holds in dimension $d \geq 3$ whereas they can handle $d \geq 2$.

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- They need a radial symmetry on V defined by $\hat{V} = \sqrt{\hat{G}}$ and we don't need any symmetry.
- Our viewpoint is more geometric and their viewpoint is more combinatorial.

The article of Poupaud and Vasseur [28]: provides another derivation of the linear Boltzmann equation. Their assumptions on the random potential are different from ours. Indeed the potentials they consider are necessarily almost everywhere bounded, and this is not the case for a Gaussian potential. Thus there is no implication between our and their results. Note nonetheless that they needed an assumption of renewal of their random potential.

Setting and strategy. We are interested in the derivation of the linear Boltzmann equation

$$\partial_t \mu_t(x,\xi) + 2\xi.\partial_x \mu_t(x,\xi) = \int \sigma(\xi,\xi')\delta(|\xi|^2 - |\xi'|^2)(\mu_t(x,\xi') - \mu_t(x,\xi))\,\mathrm{d}\xi'$$

for a particle interacting with a translation invariant centered Gaussian random field described by the Schrödinger equation

$$\begin{cases} ih\partial_t u = -\Delta_x u + \mathcal{V}_\omega^h(x)\, u \\ u_{t=0} = \psi_0^h \in L^2(\mathbb{R}^d;\mathbb{C}) \end{cases} \tag{1.1}$$

where the potential $\mathcal{V}_\omega^h(x)$ depends on a random parameter ω in a probability space $(\Omega_{\mathbb{P}}, \mathbb{P})$. In the weak coupling limit the dependence of the random potential with respect to h is

$$\mathcal{V}_\omega^h = \sqrt{h}\mathcal{V}_\omega$$

where h represent the ratio between the microscopic and macroscopic scale. We will consider the limit $h \to 0$. Note that the expression are the same for the case of the weak density limit as we are considering a Gaussian random field.

We use the isomorphism between the Gaussian space $L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$ associated with $L^2(\mathbb{R}^d;\mathbb{R})$ and the symmetric Fock space $\Gamma L^2(\mathbb{R}^d)$ associated with $L^2(\mathbb{R}^d;\mathbb{C})$ (denoted by $L^2(\mathbb{R}^d)$) given by

$$\frac{1}{\sqrt{n!}} : \Phi_G(f_1)\cdots\Phi_G(f_n) : \ \leftrightarrow f_1 \vee \cdots \vee f_n$$

for f_j in $L^2(\mathbb{R}^d;\mathbb{R})$. (The general Gaussian random fields Φ_G and the Wick powers are defined in Sections 4.2 and 4.3.) The field operator $\sqrt{2}\Phi(f)$ then corresponds to the multiplication by $\Phi_G(f)$ and with some assumptions we can write $\mathcal{V}_\omega(x) = \Phi_G(V(x-\cdot))$ for some function V. We will give more details on this correspondence in Section 4. A useful reference on this subject is [31]. We can thus express the Hamiltonian as in the Fock space as

$$-\Delta_x + \sqrt{2h}\Phi\left(V\left(x-\cdot\right)\right).$$

We can then approximate this equation by an explicitly solvable one whose solutions are coherent states. The geometric idea behind the computations is then that as the initial state is the coherent state whose parameter in the phase space is in zero (the empty state) we can use this geometric information in the phase space at

least for short times. The computations done with this solution allow us to recover the dual linear Boltzmann equation for short times for the observables.
Some remarks.

Quantum field theory and geometry in phase space: Quantum field theory allows us to see how geometry in phase space is involved. We use the viewpoint of [1] but in a case that is not in the framework chosen by the authors. Indeed we are not in a situation of mean field limit and the introduction of a parameter ε is an artifact that allows us to keep track of the importance of the different terms.

No graphs: Unlike the work of Erdös and Yau we avoided the use of graphs to try to keep a more geometric viewpoint. Yet we cannot reach times of order 1 as they do.

Trotter-Kato: As we don't get the approximate Markovian behaviour in a satisfying way we need to introduce a renewal of the random potential. The works [4, 3] deal in a more sophisticated manner with interactions defined piecewise in time. Other Ansätze may give a better approximation of the solution of the initial problem and give the Markovian behaviour of the evolution.

Positivity and *a priori* estimates: One of the important tools in our derivation of the linear Boltzmann equation is the use of *a priori* estimates to show that we don't lose too much mass in the measures during our approximations. The mass conservation and positivity properties of the linear Boltzmann equation then allow us to conclude.

Dimension and dispersion: Our result holds in dimension $d \geq 3$ as dispersion inequalities for the free Schrödinger group provide the time integrability needed for some expressions. It may be possible to reach the limit case of dimension $d = 2$.

Outline of the article. In Section 2 we describe the quantum model, state the main result and give the structure of the proof. We then recall some facts about the linear Boltzmann equation in Section 3. We specify the link between the Gaussian random field and the symmetric Fock space in Section 4 and thus obtain a new expression for the dynamics, and a candidate for an approximated dynamics. We solve these approximated dynamics in Section 5. We use this explicit solution to compute the measure of an observable for short times in Section 6. We control the error done using this approximation in Section 7. And finally we glue together the different pieces to achieve the proof in Section 8.

For the convenience of the reader we recall in the Appendices A, B and C some results about stochastics, semiclassical measures and semigroups. The Appendix D is devoted to small results we need about approximate identities.

2. Model and result

In this text the Hilbert spaces are always separable. The integer $d \geq 1$ denotes the dimension of the space $\mathbb{R}^d_x$. Our result requires $d \geq 3$.

2.1. Rescaled quantum random field.

Let $G : \mathbb{R}^d \to \mathbb{R}$ positive definite, such that $\hat{G} = |\hat{V}|^2$ with $\hat{V} \in \mathcal{S}\left(\mathbb{R}^d; \mathbb{R}\right)$ and $\mathcal{V}^h_\omega(x)$, $(\omega, x) \in \Omega_{\mathbb{P}} \times \mathbb{R}^d$, the translation invariant centered Gaussian random field with mean zero and covariance $hG(x - x')$.

We consider the Schrödinger equation

$$\begin{cases} ih\partial_t u_{t,\omega}(x) = H^h_\omega u_{t,\omega}(x) \\ \qquad u_{0,\omega}(x) = \psi_0(x) \in L^2_x \end{cases} \tag{2.1}$$

with the Hamiltonian

$$H^h_\omega = -\Delta_x + \mathcal{V}^h_\omega(x)\ . \tag{2.2}$$

Let us fix a time T and an integer N. Let $t_k = k\Delta t$ with $\Delta t = \frac{T}{N}$. The dynamics is defined piecewise in the intervals $[t_{k-1}, t_k]$ by the Hamiltonians

$$H_{h,\omega_k} = -\Delta_x + \mathcal{V}_{h,\omega_k}(x)$$

with independent random fields $\mathcal{V}_{h,\omega_k}$, $\omega_k \in \Omega_k$. Thus we get, for an initial data $\psi_0 \in L^2_x$,

$$G_{N,\frac{\Delta t}{h},\bar\omega_N} = e^{-i\frac{\Delta t}{h}H_{h,\omega_N}} e^{-i\frac{\Delta t}{h}H_{h,\omega_{N-1}}} \cdots e^{-i\frac{\Delta t}{h}H_{h,\omega_1}}\ , \tag{2.3}$$

$$\psi_T(x,\bar\omega_N) = G_{N,\frac{\Delta t}{h},\bar\omega_N}\psi_0\,,$$

with $\bar\omega_k := (\omega_1,\dots,\omega_k) \in \bar\Omega_k = \Omega_1\times\cdots\times\Omega_k$, $\bar{\mathbb{P}}_k = \mathbb{P}_1\times\cdots\times\mathbb{P}_k$ and $L^2(\bar\Omega^k,\bar{\mathbb{P}}^k) \simeq \bigotimes_{j=1}^k L^2(\Omega_j,\mathbb{P}_j)$.

Definition 2.1. Let ρ be a normal state on $\mathcal{L}(L^2_x)$, *i.e.* $\rho \in \mathcal{L}^+_1(L^2_x)$ and $\operatorname{Tr}\rho = 1$. We define

$$\rho^h_t = \int_{\Omega_{\mathbb{P}}} e^{-i\frac{t}{h}H_{h,\omega}}\,\rho\, e^{i\frac{t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}(\omega)\,, \tag{2.4}$$

$$\rho^h_{N,\Delta t}(\bar\omega_N) = G_{N,\frac{\Delta t}{h},\bar\omega_N}\,\rho\, G^{-1}_{N,\frac{\Delta t}{h},\bar\omega_N}\ , \tag{2.5}$$

$$\rho^h_{N,\Delta t} = \int_{\bar\Omega_N} \rho^h_{N,\Delta t}(\bar\omega_N)\,\mathrm{d}\bar{\mathbb{P}}_N(\bar\omega_N)\,. \tag{2.6}$$

2.2. **The main result.** Let b be a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$. The measure of the observable $b^W(hx,D_x)$ in a normal state ρ is given by

$$\operatorname{Tr}\left[b^W(hx,D_x)\,\rho\right]$$

where the Weyl quantization is defined by

$$b^W(hx,D_x)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_\xi}\int_{\mathbb{R}^d_{x'}} e^{i(x-x')\cdot\xi} b\Big(h\frac{x+x'}{2},\xi\Big)\,u(x')\,\mathrm{d}x'\,\mathrm{d}\xi\,.$$

One can refer for example to [27] about the properties of the Weyl quantization.

Consider the dynamic given by Equations (2.1) and (2.2) with renewal as in Equation (2.3), $\Delta t = h^\alpha$, $N = N^h = T/h^\alpha$, $\alpha \in]\frac34, 1[$.

We say that a family of states (ρ^h), $h \in]0,h_0]$ is pure if there is a measure μ_0 on $\mathbb{R}^{2d}_{x,\xi}$ such that

$$\forall b \in \mathcal{C}^\infty_0\big(\mathbb{R}^{2d}_{x,\xi}\big),\ \lim_{h\to0^+}\operatorname{Tr}\left[\rho^h\, b^W(hx,D_x)\right] = \int_{\mathbb{R}^{2d}_{x,\xi}} b\ \mathrm{d}\mu_0\,.$$

We refer the reader to Appendix B and [8, 16, 17, 26, 1, 2] for general information about semiclassical measures.

Theorem 2.2. *Assume that* $(\rho^h)_{h\in]0,h_0]}$ *is pure with* $\mu_0(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)=1$. *Then*

$$\forall b\in\mathcal{C}_0^\infty\left(\mathbb{R}^{2d}_{x,\xi}\right),\ \lim_{h\to 0}\mathrm{Tr}\left[\rho^h_{N,\Delta t}\,b^W(hx,D_x)\right]=\int b\ \mathrm{d}\mu_T$$

where $\mu_T=\mu_{t=T}$ *with, for* $t\in(0,T)$,

(2.7)
$$\begin{cases}\partial_t\mu_t(x,\xi)+2\xi.\partial_x\mu_t(x,\xi)=\displaystyle\int\sigma(\xi,\xi')\,\delta\big(|\xi|^2-|\xi'|^2\big)(\mu_t(x,\xi')-\mu_t(x,\xi))\,\mathrm{d}\xi'\\ \mu_{t=0}=\mu_0\end{cases},$$

and $\sigma(\xi,\xi')=2\pi|\hat V(\xi-\xi')|^2$.

Remark 2.3. The meaning of measures solving the linear Boltzmann Equation (2.7) is specified in Section 3.

The result says that the family $(\rho^h_{N,\Delta t})$ remains pure for every $T\,(=N\Delta t)$ as soon as (ρ^h) is pure.

The justification of the choice of the scaling in the Weyl quantization is the following. Physically the parameter h is the quotient of the microscopic scale over the macroscopic scale, either in time or in position. Thus if we consider an observable $b(X,\Xi)$ varying on a macroscopic scale, the corresponding observable on the microscopic scale will be $b(hx,\xi)$.

The scaling of the random field according to the covariance $hG(x-x')$, is done on a mesoscopic scale imposed by the kinetic regime (see Section 4.1).

Sketch of the Proof. Let μ_T in $\mathcal{M}(\rho^h_{N,\Delta t},h\in]0,h_0[)$ (the set of semiclassical measures defined after extraction of subsequences, see Appendix B). We denote by $\mathcal{B}(t)$ the flow associated with the Boltzmann equation (2.7) and $\mathcal{B}^T(t)$ the flow associated with the dual equation, see Section 3 for more details about $\mathcal{B}(t)$ and $\mathcal{B}^T(t)$. For any non-negative b in $\mathcal{C}_0^\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ we shall prove

(1) $\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}b\mathrm{d}\mu_T\geq\liminf_{h\to0}\mathrm{Tr}[\rho^h_{N,\Delta t}\,b^W(hx,D_x)]$ by the definition of μ_T,
(2) $\liminf_{h\to0}\mathrm{Tr}[\rho^h_{N,\Delta t}\,b^W(hx,D_x)]\ \geq\ \int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}(\mathcal{B}^T(T)b)\,\mathrm{d}\mu_0$ (see the remark below),
(3) $\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}(\mathcal{B}^T(T)b)\,\mathrm{d}\mu_0=\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}b\ \mathrm{d}(\mathcal{B}(T)\mu_0)$ by the definition of $\mathcal{B}(T)$.

Taking this for granted, it implies the lower bound

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}b\ \mathrm{d}\mu_T\geq\int_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}b\ \mathrm{d}(\mathcal{B}(T)\mu_0)\ .$$

Since this inequality holds for any non-negative b in $\mathcal{C}_0^\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ which is dense in $\mathcal{C}^0_\infty(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$ with dual $\mathcal{M}_b(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)$, we get

$$\mu_T|_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}\geq\mathcal{B}(T)\mu_0|_{\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi}\ .$$

But we also have $\mathcal{B}(T)\mu_0(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi)=1$ from the properties of the linear Boltzmann equation and $\mu_T(\mathbb{R}^d_x\times\mathbb{R}^d_\xi)\leq1$ from the properties of semiclassical measures. So, necessarily,

$$\mu_T\big(\mathbb{R}^d_x\times\mathbb{R}^{d*}_\xi\big)=1$$
$$\mu_T\big(\mathbb{R}^d_x\times\{0\}_\xi\big)=0$$

and $\mu_T=\mathcal{B}(T)\mu_0$. Thus we have the result. □

Remark 2.4. The step

$$\liminf_{h\to 0} \operatorname{Tr}\left[\rho^h_{N,\Delta t}\, b^W(hx, D_x)\right] \geq \int_{\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi} \left(\mathcal{B}^T(T)b\right) \mathrm{d}\mu_0$$

will be the technical part which requires various estimates for the quantum dynamics of the whole system particle-random field.

To prove this result we consider first the case without renewal of the stochastics, *i.e.* $N = 1$ for short times in Sections 5, 6, 7 and then glue together the estimates obtained this way N times for N "big" in Section 8. To simplify the problem of finding estimates for short times we approximate the equation by a simpler one which is solved and studied in Section 5. In Section 6, using the solution to the approximated equation, we carry out explicit computations which give rise to the different terms of the dual linear Boltzmann equation. Then we control the error between the solutions of the approximated equation and the exact equation in Section 7. All these computations are done within the framework of quantum field theory. This allows us

- to use conveniently the geometric content of coherent states,
- to keep track of the different orders of importance of the different terms by using the Wick quantization with a parameter ε.

For the reader's convenience, we recall the correspondence between the stochastic and Fock space viewpoints in Section 4.

Remark 2.5. Our initial data $(\rho^h)_{h\in]0,h_0]}$ are assumed to belong to $\mathcal{L}^+_1 L^2_x$ with $\operatorname{Tr}\rho^h = 1$. We will thus make estimates for states ρ in $\mathcal{L}^+_1 L^2_x$, with $\operatorname{Tr}\rho = 1$ with constants independent of ρ.

3. The linear Boltzmann equation

Information on the linear Boltzmann equation can be found in [10, 11, 30].
In this part suppose that $\sigma \in \mathcal{C}^\infty(\mathbb{R}^d_\xi \times \mathbb{R}^d_{\xi'}; \mathbb{R})$ and $\sigma \geq 0$.

3.1. The formal linear Boltzmann equation.

Since all the objects we use are diagonal in $|\xi|$, the following definitions will be convenient.
Notation: Let $0 < r < r' < +\infty$ we define the Sobolev spaces

$$H^n\left[r, r'\right] = H^n\left(\mathbb{R}^d_x \times A_\xi\left[r, r'\right]; \mathbb{R}\right)$$

where $A_\xi[r, r']$ is the annulus $\{\xi,\, |\xi| \in]r, r'[\}$ in the variable ξ. When there is no ambiguity we write A_ξ for $A_\xi[r, r']$. We also define $L^2[r, r'] = H^0[r, r']$.

Definition 3.1. The *collision operator* Q is defined for $b \in L^2[r, r']$ by

$$Qb = Q_+ b - Q_- b\,, \tag{3.1}$$

with

$$Q_+ b\,(x,\xi) = \int_{\mathbb{R}^d_{\xi'}} b(x,\xi')\,\sigma(\xi,\xi')\,\delta\big(\,|\xi|^2 - |\xi'|^2\,\big)\,\mathrm{d}\xi'\,,$$

$$Q_- b\,(x,\xi) = b(x,\xi) \int_{\mathbb{R}^d_{\xi'}} \sigma(\xi,\xi')\,\delta\big(\,|\xi|^2 - |\xi'|^2\,\big)\,\mathrm{d}\xi'\,.$$

Remark 3.2. For a given ξ these integrals only involve the values of $\sigma(\xi, |\xi|\,\omega)$ and $b(x, |\xi|\,\omega)$ for $\omega \in \mathbb{S}^{d-1}$.

Definition 3.3. The *linear Boltzmann equation* is formally the equation

$$\begin{cases} \partial_t f = \left\{ f, |\xi|^2 \right\} + Qf \\ f_{t=0} = f_0 \end{cases}$$

and its *dual equation* is

$$\begin{cases} \partial_t b = -\left\{ b, |\xi|^2 \right\} + Qb = 2\xi.\partial_x b + Qb \\ b_{t=0} = b_0 \end{cases} .$$

We will see in the next sections that the dual linear Boltzmann equation is solved by a group $(\mathcal{B}^T(t))_{t\in\mathbb{R}}$ of operators on $\mathcal{C}^0_\infty(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi; \mathbb{R})$ and this defines by duality a group $(\mathcal{B}(t))_{t\in\mathbb{R}}$ of operators on $\mathcal{M}_b(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi; \mathbb{R})$.

3.2. **Properties.** We recall here the main properties of the dual linear Boltzmann equation. (The arguments are the same as for the linear Boltzmann equation.)

Some standard notations and results about semigroups are given in Appendix C. We begin by solving the dual linear Boltzmann equation in $L^2[r, r']$ in the sense of semigroups. We observe that $2\xi.\partial_x$ generates a strongly continuous contraction semigroup on $L^2[r, r']$. Since the operator Q is bounded on $L^2[r, r']$ we get that $2\xi.\partial_x + Q$ generates a semigroup $(\mathcal{B}^T(t))_{t\geq 0}$ bounded by $\exp(t \|Q\|_{\mathcal{L}(L^2[r,r'])})$ and the associated domain is $D(2\xi.\partial_x)$.

Proposition 3.4. *Let $0 < r < r' < +\infty$. The operator Q on $H^n[r, r']$ is well defined and bounded, with*

$$\|Q\|_{\mathcal{L}(H^n[r,r'])} \leq C_d \sup_{|\alpha|\leq n} \|\partial^\alpha \sigma\|_{\infty, A^2_\xi[r,r']} .$$

The group of (space-)translation $(e^{2t\xi.\partial_x})_t$ preserves $H^n[r, r']$.

Proposition 3.5. *Let $0 < r < r' < +\infty$. The strongly continuous group $(\mathcal{B}^T(t))_{t\geq 0}$ of infinitesimal generator $2\xi.\partial_x + Q$ preserves*

1. *the Sobolev spaces $H^n[r, r']$, for $n \in \mathbb{N}$,*
2. *the set of functions with compact support,*
3. *the set of infinitely differentiable functions with compact support in $\mathbb{R}^d_x \times A_\xi[r, r']$, $\mathcal{C}^\infty_0(\mathbb{R}^d_x \times A_\xi[r, r']; \mathbb{R})$,*
4. *the set of non-negative functions, for $t \geq 0$.*

Proof. For (1) we use Proposition 3.4.

For (2) we can use the Trotter approximation

$$\mathcal{B}^T(t) = \lim_{n\to\infty} \left(e^{2\frac{t}{n}\xi.\partial_x} e^{\frac{t}{n}Q} \right)^n ,$$

the fact that Q is "local" in $(x, |\xi|)$, and that the speed of propagation of the (space-) translations is finite when $\xi \in A_\xi [r, r']$.

For (3) we use (1), (2) and

$$\mathcal{C}^\infty_0 \left(\mathbb{R}^d_x \times A_\xi [r, r'] ; \mathbb{R} \right) = \bigcap_{n=0}^{\infty} H^n [r, r'] \bigcap \{ f, \text{ Supp } f \text{ compact} \} .$$

For (4) we use both the Trotter approximation

$$\mathcal{B}^T(t) = \lim_{n\to\infty} \left(e^{2\frac{t}{n}\xi.\partial_x} e^{\frac{t}{n}Q_+} e^{-\frac{t}{n}Q_-} \right)^n$$

and the fact that $e^{2\frac{t}{n}\xi.\partial_x}$ preserves the non-negative functions as a translation, $e^{\frac{t}{n}Q_+}$ preserves the non-negative functions for $t \geq 0$ because Q_+ does, $e^{-\frac{t}{n}Q_-}$ preserves the non-negative functions as a multiplication operator by a positive function. □

Since $\mathcal{C}_0^\infty\left(\mathbb{R}_x^d \times A_\xi; \mathbb{R}\right) \subset D\left(2\xi.\partial_x\right)$ we can give the following result.

Proposition 3.6. *For every $b_0 \in \mathcal{C}_0^\infty\left(\mathbb{R}_x^d \times A_\xi; \mathbb{R}\right)$ there exists a unique function $b_t = \mathcal{B}^T(t)\, b_0 \in \mathcal{C}^1\left(\mathbb{R}^+; L^2[r, r']\right) \cap \mathcal{C}^0\left(\mathbb{R}^+; D\left(2\xi.\partial_x\right)\right)$ such that for every $t \in \mathbb{R}$,*

$$\begin{cases} \partial_t b_t = 2\xi.\partial_x b_t + Q b_t \\ b_{t=0} = b_0 \end{cases}.$$

Moreover $\forall t \in \mathbb{R}$, $b_t \in \mathcal{C}_0^\infty\left(\mathbb{R}_x^d \times A_\xi; \mathbb{R}\right)$. If b_0 is non-negative, then $\forall t \geq 0$, b_t is non-negative.

3.3. **The linear Boltzmann equation. Notation:** For X a locally compact, Hausdorff space we denote by $\mathcal{M}_b(X; \mathbb{R})$ the set of Radon measures and by $\mathcal{C}_\infty^0(X; \mathbb{R})$ the set of functions f on X such that for all $\varepsilon > 0$ there exists a compact $K_{f,\varepsilon}$ such that $|f(x)| < \varepsilon$ outside of $K_{f,\varepsilon}$ (*i.e.* the set of functions that vanish at infinity).

For a topological space X, locally compact and Hausdorff,

$$\mathcal{M}_b(X; \mathbb{R}) = \left(\mathcal{C}_\infty^0(X; \mathbb{R})\right)'.$$

Proposition 3.7. *The semigroup $(\mathcal{B}^T(t))_{t\geq 0}$ defined on $\mathcal{C}_0^0(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$ can be extended to a strongly continuous group on $(\mathcal{C}_\infty^0(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R}), \|\cdot\|_\infty)$ and thus defines by duality a (weak* continuous) group $\mathcal{B}(t)$ on $\mathcal{M}_b(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$.*

Proof. Using a partition of the unity, we can extend $\mathcal{B}^T(t)$ to $\mathcal{C}^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$. As $\mathcal{B}^T(t)$ is positive, we have $\mathcal{B}^T(t)(\|b\|_\infty \pm b) \geq 0$ for all b in $\mathcal{C}_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$ and so

$$\left\|\mathcal{B}^T(t)\, b\right\|_\infty \leq \|b\|_\infty.$$

We can thus extend continuously $\mathcal{B}^T(t)$ from $\mathcal{C}_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$ to $\mathcal{C}_\infty^0(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}; \mathbb{R})$. □

Definition 3.8. The *linear Boltzmann group* $(\mathcal{B}(t))$ is defined on $\mathcal{M}_b(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*})$ by duality, let $\mu \in \mathcal{M}_b(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*})$, then, for any $t \in \mathbb{R}$,

$$\forall b \in \mathcal{C}_\infty^0\left(\mathbb{R}_x^d \times \mathbb{R}_\xi^{d*}\right), \qquad \langle \mathcal{B}(t)\mu, b\rangle = \left\langle \mu, \mathcal{B}^T(t) b\right\rangle.$$

3.4. **A Trotter-type approximation.** In this part we will prove a result in the spirit of the approximation of Trotter

$$e^{A+B} = \lim_{N\to\infty}\left(e^{A/N} e^{B/N}\right)^N.$$

Notation: For $n \in \mathbb{N}$ and $b \in \mathcal{C}_0^\infty(\mathbb{R}_{x,\xi}^{2d})$, set

$$\mathcal{N}_n(b) := \sup_{|\alpha|\leq n} \left\|\partial^\alpha b\right\|_\infty. \tag{3.2}$$

Proposition 3.9. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}_{x,\xi}^{2d})$, $T > 0$ and $n \in \mathbb{N}$. There are constants $C_{n,Q}$ and $C_{T,b}$ such that for all $N \in \mathbb{N}^*$*

$$\mathcal{N}_n\left(e^{T(2\xi.\partial_x + Q)} b - \left(e^{\frac{T}{N}Q} e^{\frac{T}{N}2\xi.\partial_x}\right)^N b\right) \leq e^{T(2n + C_{n,Q})} C_{T,b} \frac{T^2}{N}.$$

Definition 3.10. Let $Q_t \in \mathcal{L}\left(L^2\left[r,r'\right]\right)$ be the operator defined by $Q_t = e^{t2\xi.\partial_x} Q e^{-t2\xi.\partial_x}$, *i.e.* $Q_t = Q_{+,t} - Q_-$ with

$$Q_{+,t} b\,(x,\xi) = \int_{\mathbb{R}^d_{\xi'}} \sigma(\xi,\xi')\,\delta\big(\,|\xi|^2 - |\xi'|^2\,\big)\,b(x-2t(\xi'-\xi),\xi')\,\mathrm{d}\xi'$$

We also define $Q_{-,t} = Q_-$ to have consistent notations in the sequel.

Let $G_Q(t,t_0)$ be the dynamical system associated with the family (Q_t) in $\mathcal{C}(\mathbb{R};\mathcal{L}(L^2[r,r']))$ given by

$$\begin{cases} \partial_t b_t = Q_t\, b_t \\ b_{t=t_0} = b_0 \in L^2_{x,\xi} \end{cases}, \qquad b_t = G_Q\,(t,t_0)\,b_0\,.$$

Note the relation

$$\mathcal{B}^T(t) = G_Q(t,0) e^{2t\xi.\partial_x} = e^{2t\xi.\partial_x} G_Q(0,-t)\,.$$

Lemma 3.11. *For any $n \in \mathbb{N}$, $s \geq 0$ and $b \in \mathcal{C}^\infty_0\left(\mathbb{R}^d_x \times A_\xi\left[r,r'\right]\right)$, there exist constants C_1, and C_2 depending on d, r and r' such that*

(1) $\mathcal{N}_n(Qb) \leq C_1\,\mathcal{N}_n(b)$,
(2) $\mathcal{N}_n((Q-Q_t)b) \leq C_2\,t(1+2\,|t|)^n\,\mathcal{N}_{n+1}(b)$,
(3) $\mathcal{N}_n(e^{2t\xi.\partial_x} b) \leq (1+2\,|t|)^n\,\mathcal{N}_n(b)$.

Proof. The first point is clear from the integral expression of Qb.

For the second point differentiate and estimate the integral formula for $b\,(x-2t\xi,\xi) - b\,(x,\xi)$, with $|\alpha| \leq n$,

$$\begin{aligned} \left|\partial^\alpha\big(b(x-2t\xi,\xi) - b(x,\xi)\big)\right| &\leq \int_0^t \left|\partial^\alpha\big(2\xi.\partial_x b(x-2s\xi,\xi)\big)\right| \mathrm{d}s \\ &\leq 2\,|\xi|\,t\,(1+2t)^n\,\mathcal{N}_{n+1}(b)\,. \end{aligned}$$

The last point results from $\big(e^{2t\xi.\partial_x} b\big)(x,\xi) = b(x+2t\xi,\xi)$. □

Definition 3.12. For $b \in \mathcal{C}^\infty_0\left(\mathbb{R}^d_x \times A_\xi\left[r,r'\right]\right)$, let

$$\mathcal{N}_n(Q) = \sup_{b\neq 0} \frac{\mathcal{N}_n(Qb)}{\mathcal{N}_n b} \text{ and } \mathcal{N}_{n+1,n}(s,Q-Q_s) = \sup_{b\neq 0} \frac{\mathcal{N}_n\big((Q-Q_s)b\big)}{s\,(1+2\,|s|)^n\,\mathcal{N}_{n+1} b}\,.$$

Lemma 3.13. *Let b, $\tilde{b} \in \mathcal{C}^\infty_0\left(\mathbb{R}^d_x \times A_\xi\left[r,r'\right]\right)$, then for all $t \geq 0$,*

$$e^{tQ}\tilde{b} - G_Q\,(t,0)\,b = e^{tQ}\left(\tilde{b}-b\right) + \int_0^t e^{(t-s)Q}\,(Q-Q_s)\,G_Q\,(s,0)\,b\,\mathrm{d}s$$

and we have the estimate

$$\begin{aligned} &\mathcal{N}_n\big(e^{tQ}\tilde{b} - G_Q(t,0)b\big) \\ &\quad\leq e^{t\mathcal{N}_n Q}\,\mathcal{N}_n(\tilde{b}-b) \\ &\qquad + t^2(1+2t)^n e^{t\mathcal{N}_n Q} \sup_{s\in[0,t]} \big\{\mathcal{N}_{n+1,n}(s,Q-Q_s)\,\mathcal{N}_{n+1}\big(G_Q(s,0)\big)\big\}\,\mathcal{N}_{n+1}(b)\,. \end{aligned}$$

Proof. The equality is clear once we have computed that both sides satisfy the equation

$$\partial_t \Delta_t = Q\Delta_t + (Q-Q_t)G_Q(t,0)b\,.$$

The inequality then follows from Lemma 3.11. □

Proof of Proposition 3.9. We fix N and forget the N's in the notations concerning $\tilde{b}$. We set $b_t = \mathcal{B}^T(t)\, b$ and define $\tilde{b}_t$ piecewise on $[0,T]$ by setting $t_k = \frac{kT}{N}$, $\tilde{b}_{t_k} = \left(e^{\frac{T}{N}Q} e^{\frac{T}{N} 2\xi.\partial_x}\right)^k b_0$ and, for $t \in [t_k, t_{k+1}[$, $\tilde{b}_t = e^{(t-t_k)Q} e^{(t-t_k)2\xi.\partial_x} \tilde{b}_{t_k}$. Let $\delta_k = \mathcal{N}_n\big(b_{t_k} - \tilde{b}_{t_k}\big)$; we get

$$e^{\frac{T}{N}Q} e^{\frac{T}{N}2\xi.\partial_x} \tilde{b}_{t_k} - e^{\frac{T}{N}(2\xi.\partial_x + Q)} b_{t_k} = e^{\frac{T}{N}Q} e^{\frac{T}{N}2\xi.\partial_x} \tilde{b}_{t_k} - G_Q\big(\tfrac{T}{N}, 0\big) e^{\frac{T}{N}2\xi.\partial_x} b_{t_k}$$

and we can then use Lemma 3.13 to obtain

$$\begin{aligned}
\delta_{k+1} &\leq e^{\frac{T}{N}\mathcal{N}_n Q} \left(1 + 2\tfrac{T}{N}\right)^n \delta_k + \left(\tfrac{T}{N}\right)^2 \left(1 + 2\tfrac{T}{N}\right)^n e^{\frac{T}{N}\mathcal{N}_n Q} \\
&\qquad \sup_{s \in [t_k, t_{k+1}]} \mathcal{N}_{n+1,n}\big(s - t_k, Q - Q_{s-t_k}\big) \\
&\qquad \sup_{s \in [t_k, t_{k+1}]} \mathcal{N}_{n+1}\big(G_Q(s - t_k, 0)\, e^{\frac{T}{N}2\xi.\partial_x} b_{t_k}\big) \\
&\leq e^{\frac{T}{N}\mathcal{N}_n Q} e^{2\frac{nT}{N}} \delta_k + \left(\tfrac{T}{N}\right)^2 e^{\frac{T}{N}\mathcal{N}_n Q} C_{N,T}
\end{aligned}$$

where we defined

$$\begin{aligned}
C_{N,T,b} = \left(1 + 2\tfrac{T}{N}\right)^n &\sup_{s \in [0, T/N]} \mathcal{N}_{n+1,n}(s, Q - Q_s) \\
&\sup_{k \in \{0, \ldots, N-1\}} \sup_{s \in [0, T/N]} \mathcal{N}_{n+1}\big(G_Q(s, 0)\, e^{-\frac{T}{N}Q} b_{t_{k+1}}\big)\,.
\end{aligned}$$

Then we get the recursive formula

$$\delta_{k+1} \leq e^{\frac{T}{N}(2n + \mathcal{N}_n Q)}\, \delta_k + C_{N,T,b} \left(\tfrac{T}{N}\right)^2 e^{\frac{T}{N}\mathcal{N}_n Q}$$

so that

$$\delta_N \leq e^{T\,(2n + \mathcal{N}_n Q)}\, C_{N,T,b} \tfrac{T^2}{N}\,.$$

The only thing remaining is to observe that $C_{N,T,b} \leq C_{T,b}$, with

$$C_{T,b} := (1 + 2T)^n \sup_{s \in [0,T]} \mathcal{N}_{n+1,n}(s, Q - Q_s) \sup_{s_j \in [0,T]} \mathcal{N}_{n+1}\big(G_Q(s_1, 0)\, e^{-s_2 Q} b_{s_3}\big)$$

and for a fixed T this quantity $C_{T,b}$ is finite, so that we get the result. □

4. From stochastics to the Fock space

The relation between Gaussian random processes and the Fock space is treated in [31, 22], we recall some facts without proofs which clarify this relation.

4.1. **Classical kinetic regime.** In microscopic variable, consider a particle moving among obstacles with a velocity $v \propto 1$ and a distance of interaction $R \propto 1$. During a time T the particle sweeps a volume of order vTR^{d-1}. In the kinetic regime it is assumed that during a long microscopic time $T = t/h$ with $t \propto 1$ the macroscopic time, the average particle encounters a number $\propto 1$ obstacle.

We denote by ρ the density of obstacles and thus obtain $\rho = 1/vTR^{d-1} \propto h$. To get this density of obstacles we need the distance between two nearest obstacles to be of order $h^{-1/d}$.

Thus we consider a Schrödinger equation of the form

$$i\partial_T \psi = -\Delta_x \psi + \mathcal{V}^h_\omega(x)\, \psi$$

that is

$$ih\partial_t \psi = -\Delta_x \psi + \mathcal{V}^h_\omega(x)\, \psi\,.$$

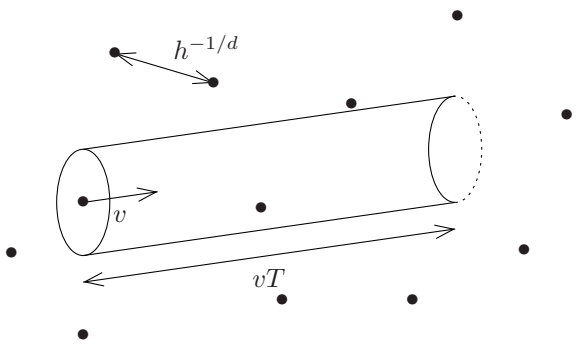

FIGURE 4.1. Kinetic regime.

A translation invariant Gaussian random field of covariance $G(x-x')$, $\hat{G} = |\hat{V}|^2$ is $V * W_\omega$ where W_ω is the spatial white noise (see Appendix A) and V describes the interaction potential. In the kinetic regime the obstacles are spread at the mesoscopic scale $h^{1/d}$. Only the white noise W_ω^h is rescaled (and not V) according to

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d;\mathbb{R})\,, \qquad \int \varphi(h^{1/d}x)\, W_\omega^h(x)\,\mathrm{d}x = \int \varphi(x)\, W_\omega(x)\,\mathrm{d}x\,,$$

i.e. $W_\omega^h(x) = hW_\omega(h^{1/d}x)$. Thus we get $\mathcal{V}_\omega^h = hV * W_\omega(h^{1/d}\cdot)$ and $G^h = hG$. See Appendix A for more details.

4.2. **General Gaussian random fields.** We introduce a different viewpoint on Gaussian random fields.

Definition 4.1. Let $(\Omega_{\mathbb{P}}, \mathcal{G}, \mathbb{P})$ be a probability measure space. Let E be a (real) vector space. A *general random field* indexed by E is a map Φ from E to the random variables on $\Omega_{\mathbb{P}}$, so that (almost everywhere)

$$\begin{aligned} \Phi(v+w) &= \Phi(v) + \Phi(w)\,, && \forall v, w \in E\,,\\ \Phi(\alpha\, v) &= \alpha\, \Phi(v)\,, && \forall \alpha \in \mathbb{R},\ \forall v \in E\,. \end{aligned}$$

Definition 4.2. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. The *general centered Gaussian random field* indexed by $\mathcal{H}_{\mathbb{R}}$ is a random field Φ_G indexed by $\mathcal{H}_{\mathbb{R}}$ so that

(1) $\mathcal{G}$ is the smallest σ-algebra for which all the $\Phi_G(v)$, $v \in \mathcal{H}_{\mathbb{R}}$ are measurable,
(2) each $\Phi_G(v)$ is a centered Gaussian random variable,
(3) $\mathbb{E}\left[\Phi_G(v)\,\Phi_G(w)\right] = \langle v, w\rangle$ with $\langle\cdot,\cdot\rangle$ the inner product on $\mathcal{H}_{\mathbb{R}}$.

One can refer to [31] for the following two theorems.

Theorem 4.3. *Let Φ_G and Φ'_G be two general centered Gaussian random field indexed by $\mathcal{H}_{\mathbb{R}}$ on probability measure spaces $(\Omega_{\mathbb{P}}, \mathcal{G}, \mathbb{P})$ and $(\Omega'_{\mathbb{P}}, \mathcal{G}', \mathbb{P}')$ respectively. Then there exists an isomorphism between the two probability measure spaces so that for every $v \in \mathcal{H}_{\mathbb{R}}$, $\Phi_G(v)$ corresponds to $\Phi'_G(v)$ under the isomorphism.*

Theorem 4.4. *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. A general centered Gaussian random field indexed by $\mathcal{H}_{\mathbb{R}}$ exists and it is unique (in the sense of the preceding theorem).*

Proposition 4.5. *Let $G \in L^1(\mathbb{R}^d;\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R}^d;\mathbb{C})$ positive definite, we can choose $V \in L^2(\mathbb{R}^d;\mathbb{R})$ such that*

$$\hat{G} = |\hat{V}|^2\,.$$

Then the Gaussian random field of mean zero and covariance $\Sigma(x,y) = G(x-y)$ is also the random field obtained as $\Phi_G(\tau_x V)$ where Φ_G is the general Gaussian random field indexed by $L^2(\mathbb{R}^d;\mathbb{R})$.

Proof. From Bochner's theorem we deduce that $\hat{G} \in L^1(\mathbb{R}^d;\mathbb{C})$ has real positive values. Thus we can set $\hat{V} = \sqrt{\hat{G}}$. Then it suffices to prove that the covariance function $\Sigma(x,y)$ of $\Phi_G(\tau_x V)$ is $G(x-y)$.

$$\begin{aligned}\Sigma(x,y) &= \mathbb{E}\left[\Phi_G(\tau_x V)\,\Phi_G(\tau_y V)\right] \\ &= \langle \tau_{x-y} V, V\rangle_{L^2(\mathbb{R}^d_x;\mathbb{R})} \\ &= \frac{1}{(2\pi)^d}\left\langle e^{-i(x-y).\xi}\hat{V}, \hat{V}\right\rangle_{L^2(\mathbb{R}^d_\xi;\mathbb{C})} \\ &= \mathcal{F}^{-1}\left[|\hat{V}|^2\right](y-x)\,. \end{aligned}$$

□

Remark 4.6. If we replace G by $G^h = hG$, the field $\sqrt{h}\Phi_G(\tau_x V)$ gives the expected covariance function.

4.3. **Wick powers.**

Definition 4.7. Let f be a random variable with finite moments, for $n \in \mathbb{N}^*$, $:f^n: \in \mathbb{C}[X]$, the n-th *Wick power* of f is defined recursively by

$$:f^0: = 1\,, \quad \partial_X :f^n: = n:f^{n-1}: \quad \text{and} \quad \tilde{\mathbb{E}}[:f^n:] = 0\,,$$

where $\tilde{\mathbb{E}} : \mathbb{C}[X] \to \mathbb{C}$ is the linear map defined by $\tilde{\mathbb{E}}[X^n] = \mathbb{E}[f^n]$. We still denote by $:f^n:$ the random variable $:f^n:(f)$.

Remark 4.8. For the first terms we have

$$:f^0: = 1\,, \quad :f^1: = f - \mathbb{E}f\,, \quad \text{and} \quad :f^2: = f^2 - 2\mathbb{E}[f]\,f - \mathbb{E}[f^2] + 2\mathbb{E}[f]^2\,.$$

4.4. **The isomorphism with the Fock space.**

Definition 4.9. Let $\mathcal{H}_\mathbb{C}$ be a complex Hilbert space, $\vee$ the symmetric tensor product, the *symmetric Fock space* over $\mathcal{H}_\mathbb{C}$ is

$$\Gamma\mathcal{H}_\mathbb{C} = \bigoplus_{n=0}^{+\infty} \Gamma_n\mathcal{H}_\mathbb{C}$$

where $\Gamma_n\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{C}^{\vee n}$ is the Hilbert completion for the norm inherited from the scalar product over $\mathcal{H}_\mathbb{C}$ of the algebraic symmetric n-th power of $\mathcal{H}_\mathbb{C}$, and the sum is also the Hilbert completion of the algebraic sum. We denote by $\Gamma_F\mathcal{H}_\mathbb{C}$ the algebraic sum (but with a completed tensor product) we will eventually refer to this set as the *finite particle vectors*. We define the empty state $\Omega = (1,0,0,\dots) \in \Gamma\mathcal{H}_\mathbb{C}$. The creation $a^*(f)$ and annihilation $a(f)$ operators are defined on $\Gamma_F\mathcal{H}_\mathbb{C}$ by

- $a^*(f)(g^{\vee n}) := (n+1)^{\frac{1}{2}} f \vee g^{\vee n}$,
- $a(f)(g^{\vee n}) := n^{\frac{1}{2}}\,\langle f, g\rangle\, g^{\vee n-1}$,

for $f, g \in \mathcal{H}_\mathbb{C}$. The field operator $\Phi(f) = (a^*(f) + a(f))/\sqrt{2}$ is then essentially self-adjoint for $\Gamma_F\mathcal{H}_\mathbb{C}$ is a dense set of analytic vectors and we still denote by $\Phi(f)$ its closure.

One can refer to [31] for the following theorem.

Theorem 4.10. *Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and $\mathcal{H}_{\mathbb{C}}$ its complexification. Let Φ_G the general centered Gaussian random field indexed by $\mathcal{H}_{\mathbb{R}}$ over a probability space $(\Omega_{\mathbb{P}}, \mathcal{G}, \mathbb{P})$. The symmetric Fock space $\Gamma\mathcal{H}_{\mathbb{C}}$ is unitarily equivalent to $L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$ under a unitary $D : \Gamma\mathcal{H}_{\mathbb{C}} \to L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$ such that*

- $D\,\Omega = 1$,
- $D\,\Gamma_n\mathcal{H}_{\mathbb{C}} =$ *the closed subspace of* $L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$ *generated by*

$$\{ :\Phi_G(f_1)\cdots\Phi_G(f_n): \,,\, f_j \in \mathcal{H}_{\mathbb{R}}\} \,,$$

- $D\,\sqrt{2}\Phi(f)\,D^{-1} = \Phi_G(f)$ *for* $f \in \mathcal{H}_{\mathbb{R}}$, *with* $\Phi_G(f)$ *seen as a multiplication operator on* $L^2(\Omega_{\mathbb{P}}, \mathbb{P}; \mathbb{C})$,
- $D\,f_1 \vee \cdots \vee f_n = \frac{1}{\sqrt{n!}} :\Phi_G(f_1)\cdots\Phi_G(f_n):$ *for* $f_1, \ldots, f_n \in \mathcal{H}_{\mathbb{R}}$.

4.5. **The expression of the dynamic in the Fock space.** We will apply the results of Section 4.4 with $\mathcal{H}_{\mathbb{R}} = L^2(\mathbb{R}^d_y; \mathbb{R})$ and $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}^d_y; \mathbb{C}) = L^2_y$ and get an isomorphism

$$D : \Gamma L^2_y \to L^2(\Omega_{\mathbb{P}}, \mathbb{P})\,.$$

We set $\mathrm{Ad}\{A\}[B] = ABA^{-1}$ and for Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, $\mathrm{Tr}_{\mathcal{H}'}[A]$ denotes the partial trace of an operator A on $\mathcal{H} \otimes \mathcal{H}'$.

Note that with the stochastic presentation

$$\begin{aligned}
\rho^h_{N,\Delta t} &= \int_{\bar\Omega_N} \rho^h_{N,\Delta t}(\bar\omega_N)\,\mathrm{d}\mathbb{P}_N(\bar\omega_N) \\
&= \int_{\bar\Omega_N} \mathrm{Ad}\left\{G_{N,\frac{\Delta t}{h},\bar\omega_N}\right\}[\rho]\,\mathrm{d}\mathbb{P}_N(\bar\omega_N) \\
&= \int_{\Omega_N} \mathrm{Ad}\left\{e^{-i\frac{\Delta t}{h}H_{h,\omega_N}}\right\}\Big[\cdots\int_{\Omega_1}\mathrm{Ad}\left[e^{-i\frac{\Delta t}{h}H_{h,\omega_1}}\right]\{\rho\}\,\mathrm{d}\mathbb{P}(\omega_1)\cdots\Big]\,\mathrm{d}\mathbb{P}(\omega_N) \\
&= \int_{\Omega_N} \mathrm{Ad}\left\{e^{-i\frac{\Delta t}{h}H_{h,\omega_N}}\right\}\Big[\int_{\bar\Omega_{N-1}}\rho^h_{N-1,\Delta t}(\bar\omega_{N-1})\,\mathrm{d}\mathbb{P}_{N-1}(\bar\omega_{N-1})\Big]\,\mathrm{d}\mathbb{P}(\omega_N) \\
&= \int_{\Omega_N} \mathrm{Ad}\left\{e^{-i\frac{\Delta t}{h}H_{h,\omega_N}}\right\}\left[\rho^h_{N-1,\Delta t}\right]\mathrm{d}\mathbb{P}(\omega_N)\,.
\end{aligned}$$

The last integral can be expressed using a partial trace as

$$\begin{aligned}
&\int e^{-i\frac{\Delta t}{h}H_{h,\omega}}\rho e^{i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}\,(\omega) \\
&= \int e^{-i\frac{\Delta t}{h}H_{h,\omega}}\rho 1\,(\omega)\,1\,(\omega)\,e^{i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}\,(\omega) \\
&= \operatorname*{Tr}_{L^2(\Omega_{\mathbb{P}},\mathbb{P})}\left[\int^{\oplus} e^{-i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}\,(\omega)\,\rho\otimes|1\rangle\langle 1|\int^{\oplus} e^{i\frac{\Delta t}{h}H_{h,\omega'}}\,\mathrm{d}\mathbb{P}\,(\omega')\right].
\end{aligned}$$

The isomorphism

$$U_{G\leftarrow F} = \mathrm{Id}_{L^2_x}\otimes D : L^2_x\otimes\Gamma L^2_y \to L^2_x\otimes L^2\,(\Omega_{\mathbb{P}},\mathbb{P})$$

is such that

$$\begin{aligned}
U^{-1}_{G\leftarrow F}\int^{\oplus} e^{-i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}\,(\omega)\,U_{G\leftarrow F} &= e^{-i\frac{\Delta t}{h}H_h}\,, \\
U^{-1}_{G\leftarrow F}\rho\otimes|1\rangle\langle 1|\,U_{G\leftarrow F} &= \rho\otimes|\Omega\rangle\langle\Omega|
\end{aligned}$$

with $U_{G\leftarrow F}^{-1}\int^{\oplus} H_{h,\omega}\,\mathrm{d}\mathbb{P}(\omega)\,U_{G\leftarrow F}=H_h$. And thus

$$\operatorname*{Tr}_{L^2(\Omega_{\mathbb{P}},\mathbb{P})}\left[\int^{\oplus} e^{-i\frac{\Delta t}{h}H_{h,\omega}}\,\mathrm{d}\mathbb{P}\left(\omega\right)\rho\otimes\left|1\right\rangle\left\langle 1\right|\int^{\oplus} e^{i\frac{\Delta t}{h}H_{h,\omega'}}\,\mathrm{d}\mathbb{P}\left(\omega'\right)\right]$$
$$=\operatorname*{Tr}_{\Gamma L_y^2}\left[e^{-i\frac{\Delta t}{h}H_h}\rho\otimes\left|\Omega\right\rangle\left\langle\Omega\right|e^{i\frac{\Delta t}{h}H_h}\right].$$

The only thing left is to compute H_h, but $U_{G\leftarrow F}^{-1}\left(\mathcal{V}_\omega^h(x)\times\right)U_{G\leftarrow F}=\Phi_G(\tau_x V)$. We get that in the Fock space formalism

$$\rho_{N,\Delta t}^h=\left(\operatorname*{Tr}_{\Gamma L_y^2}\left[\mathrm{Ad}\left\{e^{-i\frac{\Delta t}{h}H_h}\right\}\left[\cdot\otimes\left|\Omega\right\rangle\left\langle\Omega\right|\right]\right]\right)^N\left[\rho\right]$$

with $H_h=U_{G\leftarrow F}^{-1}\,H_{h,\omega}\,U_{G\leftarrow F}=-\Delta_x+\sqrt{2h}\Phi(\tau_x V)$.

4.6. **Existence of the dynamic.** We will show that the dynamic of the system is well defined and to do so we will show that the Hamiltonian is essentially self-adjoint on a certain domain. We make use of Nelson's commutator theorem which can be found in [29]. Since we work with a fixed $h>0$ the value of h will be unimportant we take $h=1$ in this section to clarify our exposition.

Theorem 4.11. *Let N' be a self-adjoint operator with $N'\geq 1$. Let H be a symmetric operator with domain D' which is a core for N'. Suppose that:*

(1) *For some $C_1>0$ and all $u\in D'$,*
$$\left\|Hu\right\|\leq C_1\left\|N'u\right\|,$$
(2) *For some $C_2>0$ and all $u\in D'$,*
$$\left|\left\langle Hu,N'u\right\rangle-\left\langle N'u,Hu\right\rangle\right|\leq C_2\left\|N'^{1/2}u\right\|^2.$$

Then H is essentially self-adjoint on D' and its closure is essentially self-adjoint on any other core for N'.

Let $D':=\mathcal{C}_0^\infty(\mathbb{R}^d)\otimes^{\mathrm{alg}}\Gamma_F L_y^2$ be the domain of both

$$N'=\mathrm{Id}-\Delta_x+N,$$
$$H=\quad-\Delta_x+\sqrt{2}\Phi(\tau.V)$$

where

- $-\Delta_x$ denotes the operator $-\Delta_x\otimes\mathrm{Id}_{\Gamma L_y^2}$,
- N denotes the operator $\mathrm{Id}_{L_x^2}\otimes N$ with N the number operator on ΓL_y^2 and
- $\Phi_F(\tau.V)$ denotes the operator defined on $L^2(\mathbb{R}^d;\Gamma L_y^2)\simeq L^2(\mathbb{R}^d)\otimes\Gamma L_y^2$ by
$$L^2(\mathbb{R}^d;\Gamma L_y^2)\to L^2(\mathbb{R}^d;\Gamma L_y^2)$$
$$u\mapsto\Phi(\tau.V)u$$
with $[\Phi(\tau.V)u](x):=[\Phi(\tau_x V)][u(x)]$.

We still denote by N' the closure of the essentially self-adjoint operator N' defined on D'. Then D' is a core for this operator. We remark that $N'\geq I$ on D' and thus also on $D\left(N'\right)$ as D' is a core for N'.

Proposition 4.12. *Suppose that V belongs to $H^2(\mathbb{R}^d)$. Then the Hamiltonian H satisfies the hypotheses of Theorem 4.11.*

Proof. Let $u \in D'$, then

$$\begin{aligned}\|Hu\|_{L^2_x\otimes\Gamma L^2_y} &\leq \|-\Delta_x u\|_{L^2_x\otimes\Gamma L^2_y} + 2\|V\|_{L^2}\left\|\sqrt{N+1}u\right\|_{L^2_x\otimes\Gamma L^2_y}, \\ &\leq (1+2\|V\|_{L^2})\|N'u\|_{L^2_x\otimes\Gamma L^2_y}\end{aligned}$$

which is the first estimate. We also observe that in the sense of quadratic forms

$$\begin{aligned}[H,N'] &= \sqrt{2}\left[\Phi(\tau.V), -\Delta_x + N\right], \\ &= \sqrt{2}\Phi(\tau.\nabla V).\nabla_x + \sqrt{2}\Phi(\tau.\Delta V) + (a^*(\tau.V) - a(\tau.V))\end{aligned}$$

so that

$$\begin{aligned}&|\langle Hu, N'u\rangle - \langle N'u, Hu\rangle| \\ &\qquad\leq \left(\sqrt{2}\|\nabla V\|_{L^2} + \sqrt{2}\|\Delta V\|_{L^2} + 2\|V\|_{L^2}\right)\left\|N'^{1/2}u\right\|^2\end{aligned}$$

which is the second estimate. □

5. An approximated equation and its solution

5.1. **The scaling for field operators.** The ε parameter is an intermediate scale which allows to easily identify the graduation in Wick powers. It will in the end be adjusted with respect to h. Let $(D_\varepsilon f)(y) = \varepsilon^{-d/2} f\left(\frac{y}{\varepsilon}\right)$ and

$$H_{h,\varepsilon} = \mathrm{Ad}\left\{\mathrm{Id}_{L^2_x}\otimes\Gamma D_\varepsilon\right\}[H_h] = -\Delta_x\otimes I_{\Gamma_y} + \sqrt{2h}\Phi\left(\varepsilon^{-d/2}V\left(x-\frac{y}{\varepsilon}\right)\right).$$

We now introduce some definitions and notations that will be useful to deal with the Wick quantization and the scaled versions of our objects in the Fock space.

5.2. **The second quantization.** The method of second quantization is exposed in the books [5, 7], an introduction to quantum field theory and second quantization can be found in [14]. The series of articles [18, 19, 20, 21] uses this framework with a small parameter to handle classical or mean field limits by developing the Hepp method [23]. We will use the notation and framework of [1, 2] to handle the second quantization with a small parameter. For the convenience of the reader we expose briefly this framework. We also recall some formulae in Appendix E.1.

Most of our operators on the Fock space will arise as Wick quantizations of polynomials.

Definition 5.1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space (the scalar product is $\mathbb{C}$-antilinear with respect to the left variable). The symmetric tensor product is denoted by $\vee$. The *polynomials* with variable in $\mathcal{H}$ are the finite linear combinations of monomials $Q : \mathcal{H} \to \mathbb{C}$ of the form

$$Q(z) = \left\langle z^{\vee q}, \tilde{Q} z^{\vee p}\right\rangle$$

where $p, q \in \mathbb{N}$, $\tilde{Q} \in \mathcal{L}(\mathcal{H}^{\vee p}, \mathcal{H}^{\vee q})$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product on $\mathcal{H}^{\vee q}$. The set of such polynomials is denoted by $\mathcal{P}(\mathcal{H})$.

The symmetric *Fock space* associated to $\mathcal{H}$ is

$$\Gamma\mathcal{H} = \bigoplus_{n=0}^{+\infty}\Gamma_n\mathcal{H}$$

with $\Gamma_n\mathcal{H} = \mathcal{H}^{\vee n}$ the completed n-th symmetric power of $\mathcal{H}$ and the sum is completed, the set of *finite particle vectors* $\Gamma_F\mathcal{H}$ is defined as the Fock space but with an algebraic sum.

The *Wick quantization* of a polynomial is defined as the linear combination of the Wick quantizations of its monomials, and for a monomial Q we define Q^{Wick} : $\Gamma_F\mathcal{H} \to \Gamma_F\mathcal{H}$ as the linear operator such that

$$\begin{aligned} Q^{Wick}\big|_{\mathcal{H}^{\vee n}} &= 1_{[p,+\infty)}(n)\, \tfrac{\sqrt{n!(n-p+q)!}}{(n-p)!}\, \varepsilon^{\frac{p+q}{2}} \left(\tilde{Q} \bigvee \mathrm{Id}_{\mathcal{H}^{\vee n-p}}\right), \\ &\in \mathcal{L}\big(\mathcal{H}^{\vee n}, \mathcal{H}^{\vee n-p+q}\big)\,. \end{aligned}$$

The field operators $\Phi_\varepsilon(f)$ $(f \in \mathcal{H})$ are defined as the closure of the essentially self-adjoint operators $\left(\langle z, f\rangle + \langle f, z\rangle\right)^{Wick}/\sqrt{2}$.

The Weyl operators are then defined by $W(f) = \exp(i\Phi_\varepsilon(f))$. The empty state Ω is $(1, 0, 0, \dots)$ and the coherent states are defined as $E(f) = W\big(\frac{\sqrt{2}}{i\varepsilon} f\big)\,\Omega$.

Proposition 5.2. *For any $Q \in \mathcal{P}(\mathcal{H})$, Q^{Wick} is closable and the domain of its closure contains*

$$\{W(f)\,\phi,\ \phi \in \Gamma_F\mathcal{H},\ f \in \mathcal{H}\}\,.$$

Definition 5.3. For a self-adjoint operator A on $\mathcal{H}$, the self-adjoint operator $\mathrm{d}\Gamma_\varepsilon(A)$ is defined by

$$\begin{aligned} \mathrm{d}\Gamma_\varepsilon(A)\big|_{D(A)^{\vee n},\mathrm{alg}} &= \varepsilon n A \vee \mathrm{Id}_{\mathcal{H}}^{\vee n-1} \\ &= \varepsilon\left(A \otimes \mathrm{Id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{H}} + \cdots + \mathrm{Id}_{\mathcal{H}} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{H}} \otimes A\right) \end{aligned}$$

and for a unitary U on $\mathcal{H}$, the unitary operator $\Gamma(U)$ on $\Gamma\mathcal{H}$ is defined by

$$\Gamma(U)\big|_{\mathcal{H}^{\vee n}} = U^{\vee n} = U \otimes \cdots \otimes U$$

and thus $\Gamma(e^{itA}) = \exp\left(\frac{it}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon(A)\right)$.

5.3. **Space translation in the fields and Fourier transform.** We introduce a notation for an object $X = (X_1, \dots, X_d)$ with d components, like $\xi \in \mathbb{R}^d$, $D_x = (\partial_{x_1}, \dots, \partial_{x_d})$ or $\mathrm{d}\Gamma(D_y)$,

$$X^{\cdot 2} := X_1^2 + \cdots + X_d^2\,.$$

We would rather have a field operator with no dependence in x. Then we recall that $e^{-ix.D_y} = \tau_x$ and thus

$$\begin{aligned} &\Gamma\big(e^{i\varepsilon x.D_y}\big)\, H_{h,\varepsilon}\, \Gamma\big(e^{-i\varepsilon x.D_y}\big) \\ &\qquad = D_x^{\cdot 2} - 2D_x.\,\mathrm{d}\Gamma_\varepsilon(D_y) + \mathrm{d}\Gamma_\varepsilon(D_y)^{\cdot 2} + \sqrt{2}\Phi_\varepsilon\Big(\varepsilon^{-d/2}\sqrt{\tfrac{h}{\varepsilon}}V\left(-\tfrac{y}{\varepsilon}\right)\Big) \end{aligned}$$

where we use an ε-dependent operator $\mathrm{d}\Gamma_\varepsilon$. After a conjugation by the Fourier transform in both the particle and the field variables we get

$$\hat{H}_{h,\varepsilon} = \xi^{\cdot 2} - \mathrm{d}\Gamma_\varepsilon(2\xi.\eta) + \mathrm{d}\Gamma_\varepsilon(\eta)^{\cdot 2} + \sqrt{2}\Phi_\varepsilon(f_{h,\varepsilon})$$

with $f_{h,\varepsilon}(\eta) = \varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat{V}(-\varepsilon\eta)$, *i.e.* $\hat{H}_{h,\varepsilon} = Q_{h,\varepsilon}^{Wick}$ with

$$Q_{h,\varepsilon}(z) = \xi^{\cdot 2} + \left\langle z, \left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right) z\right\rangle + \langle z, \eta z\rangle^{\cdot 2} + 2\Re\,\langle z, f_{h,\varepsilon}\rangle\,.$$

When we neglect the quartic part $\langle z, \eta z\rangle^{\cdot 2}$ and thus get another polynomial

$$Q_{h,\varepsilon}^{app}(z) = \xi^{\cdot 2} + \left\langle z, \left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right) z\right\rangle + 2\Re\,\langle z, f_{h,\varepsilon}\rangle$$

we can solve explicitly the evolution associated with the Hamiltonian

$$\hat{H}^{app}_{h,\varepsilon} = Q^{app,Wick}_{h,\varepsilon} = \xi^{\cdot 2} + \mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right) + \sqrt{2}\Phi_\varepsilon\left(f_{h,\varepsilon}\right) .$$

Definition 5.4. Let $\rho \in \mathcal{L}_1\left(L^2_x\right)$, then we define

$$\begin{aligned}
\boldsymbol{\rho}_t &= \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}H_{h,\varepsilon}}\right\}\left[\rho\otimes \mathrm{proj}\,\Omega\right] , \qquad \boldsymbol{\rho}^{app}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}H^{app}_{h,\varepsilon}}\right\}\left[\rho\otimes \mathrm{proj}\,\Omega\right] ,\\
\hat{\boldsymbol{\rho}}_t &= \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}\hat{H}_{h,\varepsilon}}\right\}\left[\hat{\rho}\otimes \mathrm{proj}\,\Omega\right] , \qquad \hat{\boldsymbol{\rho}}^{app}_t = \mathrm{Ad}\left\{e^{-i\frac{t}{\varepsilon}\hat{H}^{app}_{h,\varepsilon}}\right\}\left[\hat{\rho}\otimes \mathrm{proj}\,\Omega\right] ,\\
\rho^{\varepsilon}_t &= \operatorname*{Tr}_{\Gamma L^2_x}\boldsymbol{\rho}_t , \qquad \rho^{\varepsilon,app}_t = \operatorname*{Tr}_{\Gamma L^2_x}\boldsymbol{\rho}^{app}_t .
\end{aligned}$$

This definition is consistent with the previous one given for ρ^h_t as $\rho^h_t = \rho^{\varepsilon}_{\frac{\varepsilon}{h}t}$ and the dilatation acts only in the Fock space part of $L^2_x\otimes\Gamma L^2_y$.

5.4. **The approximated equation and its solution.**

5.4.1. *Results.*

Definition 5.5. Let $\psi_0 \in L^2_x$. We define

$$\hat{\Psi}_{h,\varepsilon,t} = e^{-i\frac{t}{\varepsilon}\hat{H}_{h,\varepsilon}}\Omega\otimes\hat{\psi}_0 \qquad \text{and} \qquad \hat{\Psi}^{app}_{h,\varepsilon,t} = e^{-i\frac{t}{\varepsilon}\hat{H}^{app}_{h,\varepsilon}}\Omega\otimes\hat{\psi}_0 .$$

We will show three results in this section.

Proposition 5.6. *We have*

$$\hat{\Psi}^{app}_{h,\varepsilon,t} = e^{-i\frac{\omega_{h,\varepsilon,t}}{\varepsilon}}W\left(\tfrac{\sqrt{2}}{i\varepsilon}z_{h,\varepsilon,t}\right)\Omega\otimes\hat{\psi}_0$$

with $z_{h,\varepsilon,t} = -i\int_0^t e^{-i\frac{s}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2} - 2\xi.\varepsilon\eta\right)}f_{h,\varepsilon}\,\mathrm{d}s$ *and* $\omega_{h,\varepsilon,t} = t\xi^2 + \int_0^t \Re\left\langle z_s, f_{h,\varepsilon}\right\rangle \mathrm{d}s$.

We have an estimate on the size of z_t.

Proposition 5.7. *There exists a constant $C_{G,d}$ depending only on G and the dimension d such that*

$$\left\||\eta|^{\nu} z_{h,\varepsilon,t}\right\|_{L^2_\eta} \leq C_{G,d}\sqrt{\frac{ht}{\varepsilon}}\varepsilon^{1/2-\nu} .$$

We also have an estimate on the error on $\hat{\Psi}_t$ when considering $\hat{\Psi}^{app}_t$.

Proposition 5.8. *Let $T_0 > 0$. There exists a constant $C_{T_0,G,d}$ such that for $\frac{ht}{\varepsilon} \leq T_0$,*

$$\left\|\hat{\Psi}_{h,\varepsilon,t} - \hat{\Psi}^{app}_{h,\varepsilon,t}\right\| \leq C_{T_0,G,d}\left(\frac{ht}{\varepsilon}\right)^2 h^{-1} .$$

5.4.2. *A transformation.*

First we get rid of the quadratic part (*i.e.* $\mathrm{d}\Gamma_\varepsilon$).

Definition 5.9. Let

$$\tilde{\hat{\Psi}}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}}e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right)}\Psi_{h,\varepsilon,t} \text{ and } \tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}}e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2} - 2\xi.\eta\right)}\Psi^{app}_{h,\varepsilon,t} .$$

Proposition 5.10. *Then $\tilde{\hat{\Psi}}_t$ (resp. $\tilde{\hat{\Psi}}^{app}_t$) is solution of the equation*

$$i\varepsilon\partial_t\tilde{\Psi}_{h,\varepsilon,t} = \tilde{Q}^{Wick}_{h,\varepsilon}\tilde{\Psi}_{h,\varepsilon,t}$$

(resp. $i\varepsilon\partial_t\tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t} = \tilde{Q}^{app,Wick}_{h,\varepsilon}\tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t}$) with the initial condition $\tilde{\hat{\Psi}}_{h,\varepsilon,t=0} = \Omega$ (resp. $\tilde{\hat{\Psi}}^{app}_{h,\varepsilon,t=0} = \Omega$) and $\tilde{Q}_{h,\varepsilon,t}(z) = 2\Re\langle z, \tilde{f}_{h,\varepsilon,t}\rangle + \langle z,\eta z\rangle^{\cdot 2}$ (resp. $\tilde{Q}^{app}_{h,\varepsilon,t}(z) = 2\Re\langle z,\tilde{f}_{h,\varepsilon,t}\rangle$) with $\tilde{f}_{h,\varepsilon,t} = e^{i\frac{t}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2} - 2\xi.\varepsilon\eta\right)}f_{h,\varepsilon}$.

Proof. Indeed

$$\begin{aligned} i\varepsilon\partial_t\tilde{\hat{\Psi}}_t &= i\varepsilon\partial_t\left[e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}}e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2}-2\xi.\eta\right)}\hat{\Psi}_t\right] \\ &= e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}}e^{i\frac{t}{\varepsilon}\,\mathrm{d}\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2}-2\xi.\eta\right)}\left[2\Re\left\langle z,f\right\rangle+\left\langle z,\eta z\right\rangle^{\cdot 2}\right]^{Wick}\hat{\Psi}_t \\ &= \left[2\Re\left\langle z,e^{it\left(\varepsilon\eta^{\cdot 2}-2\xi.\eta\right)}f\right\rangle+\left\langle z,\eta z\right\rangle^{\cdot 2}\right]^{Wick}e^{i\frac{t}{\varepsilon}\xi^{\cdot 2}}e^{i\frac{t}{\varepsilon}d\Gamma_\varepsilon\left(\varepsilon\eta^{\cdot 2}-2\xi.\eta\right)}\hat{\Psi}_t \\ &= \tilde{Q}_t^{Wick}\tilde{\hat{\Psi}}_t\,. \end{aligned}$$

And we can proceed analogously with $\tilde{\Psi}_t^{app}$. □

5.4.3. *The classical movement associated with the approximated equation.* The classical movement is the solution to the equation

$$\begin{cases} i\partial_t\tilde{z}_{h,\varepsilon,t}=\partial_{\bar{z}}\tilde{Q}_{h,\varepsilon,t}\left(\tilde{z}_{h,\varepsilon,t}\right)=\tilde{f}_{h,\varepsilon,t} \\ \tilde{z}_{h,\varepsilon,t}=0 \end{cases} \tag{5.1}$$

i.e.

$$\tilde{z}_{h,\varepsilon,t}=-i\int_0^t\tilde{f}_{h,\varepsilon,s}\,\mathrm{d}s=-i\int_0^t e^{i\frac{s}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}f_{h,\varepsilon}\,\mathrm{d}s\,.$$

With this simpler dynamics the translation of Proposition 5.7 is the following.

Proposition 5.11. *There exists a constant $C_{G,d}$ depending only on G and the dimension d such that*

$$\left\||\eta|^\nu\,\tilde{z}_{h,\varepsilon,t}\right\|_{L^2_\eta}\leq C_{G,d}\sqrt{\frac{ht}{\varepsilon}}\varepsilon^{1/2-\nu}\,.$$

Proof. We compute $\left\||\eta|^\nu\,\tilde{z}_{h,\varepsilon,t}\right\|^2_{L^2_\eta}$

$$\left\||\eta|^\nu\,\tilde{z}_{h,\varepsilon,t}\right\|^2_{L^2_\eta}=\int_0^t\int_0^t\int_{\mathbb{R}^d_\eta}e^{i\frac{s-s'}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}\left|\eta\right|^{2\nu}\left|f_{h,\varepsilon}(\eta)\right|^2\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}s'$$

A change of variable $\eta'=\varepsilon\eta-\xi$ gives

$$\begin{aligned} &\int_{\mathbb{R}^d_\eta}e^{i\frac{s-s'}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}\left|\eta\right|^{2\nu}\left|f_{h,\varepsilon}(\eta)\right|^2\mathrm{d}\eta \\ &\qquad=\varepsilon^{-2\nu}\frac{h}{\varepsilon}e^{-i\frac{s-s'}{\varepsilon}\xi^{\cdot 2}}\int_{\mathbb{R}^d_\eta}e^{i\frac{s-s'}{\varepsilon}\eta^{\cdot 2}}\left|\eta+\xi\right|^{2\nu}\hat{G}(\eta+\xi)\,\mathrm{d}\eta \end{aligned}$$

as $f_{h,\varepsilon}(\eta)=\varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat{V}(-\varepsilon\eta)$ and $\frac{h}{\varepsilon}\,\hat{G}(\varepsilon\eta)\,\varepsilon^d=\left|f_{h,\varepsilon}(\eta)\right|^2$.

For $s\neq s'$

$$\begin{aligned} &\left|\int_{\mathbb{R}^d_\eta}e^{i\frac{s-s'}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}\left|\eta\right|^{2\nu}\left|f_{h,\varepsilon}(\eta)\right|^2\mathrm{d}\eta\right| \\ &\qquad=\left(\frac{\pi\varepsilon}{s'-s}\right)^{d/2}\varepsilon^{-2\nu}\frac{h}{\varepsilon}\left\|\mathcal{F}\left(\eta\mapsto\left|\eta+\xi\right|^{2\nu}\hat{G}(\eta+\xi)\right)\right\|_{L^1} \\ &\qquad=\left(\frac{\pi\varepsilon}{s'-s}\right)^{d/2}\varepsilon^{-2\nu}\frac{h}{\varepsilon}\left\|\mathcal{F}\left(\eta\mapsto\left|\eta\right|^{2\nu}\hat{G}(\eta)\right)\right\|_{L^1} \end{aligned}$$

is uniformly bounded by $C_G\,\varepsilon^{-2\nu}\frac{h}{\varepsilon}$. The squared norm $\|\,|\eta|^{\nu}\,\tilde{z}_t^{app}\|^2_{L^2_\eta}$ is then bounded by

$$\begin{aligned}\|\,|\eta|^{\nu}\,\tilde{z}_{h,\varepsilon,t}\|^2_{L^2_\eta} &\leq C_G\frac{h}{\varepsilon}\varepsilon^{-2\nu}\int_0^t\int_0^t \min\left\{\left(\frac{\pi\varepsilon}{s'-s}\right)^{d/2},1\right\}\mathrm{d}s\,\mathrm{d}s'\\ &\leq C_G\frac{h}{\varepsilon}\varepsilon^{-2\nu}\left[\pi^{d/2}\varepsilon^{d/2}\int_{|s-s'|\geq 2\delta,s,s'\in[0,t]}\frac{ds ds'}{(s'-s)^{d/2}}+2\sqrt{2}t\delta\right]\\ &\leq C_G\frac{h}{\varepsilon}\varepsilon^{-2\nu}\left[\pi^{d/2}\varepsilon^{d/2}2^{d/4}2\sqrt{2}t\frac{2}{d-2}\delta^{1-d/2}+2\sqrt{2}t\delta\right]\end{aligned}$$

which is optimal when $\delta=\varepsilon$. This achieves the proof. □

Remark 5.12. The same estimate holds for z_t with a similar proof.

5.4.4. *Resolution of the approximated solution and comparison with the exact solution.*

Proposition 5.13. *The solution to the equation*

$$i\varepsilon\partial_t\tilde{\Psi}^{app}_{h,\varepsilon,t}=\tilde{Q}^{app,Wick}_{h,\varepsilon}\tilde{\Psi}^{app}_{h,\varepsilon,t}$$

with initial data $\tilde{\Psi}^{app}_{h,\varepsilon,t=0}=\Omega$ *is*

$$\tilde{\Psi}^{app}_{h,\varepsilon,t}=e^{-i\frac{\tilde{\omega}_{h,\varepsilon,t}}{\varepsilon}}W\left(\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_{h,\varepsilon,t}\right)\Omega$$

with $\tilde{\omega}_{h,\varepsilon,t}=\int_0^t\Re\left\langle\tilde{z}_{h,\varepsilon,s},\tilde{f}_{h,\varepsilon,s}\right\rangle\mathrm{d}s$.

Proof. Indeed let us apply $i\varepsilon\partial_t$ to the term on the right hand side:

$$\begin{aligned}&i\varepsilon\partial_t e^{-i\frac{\tilde{\omega}_t}{\varepsilon}}W\Big(\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t\Big)\,\Omega\\ &=\left(\partial_t\tilde{\omega}-i\varepsilon\frac{i\varepsilon}{2}\Im\left\langle\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t,-\tfrac{\sqrt{2}}{\varepsilon}\tilde{f}_t\right\rangle+i\varepsilon i\Phi\Big(-\tfrac{\sqrt{2}}{\varepsilon}\tilde{f}_t\Big)\right)e^{-i\frac{\omega_t}{\varepsilon}}W\Big(\tfrac{\sqrt{2}}{i\varepsilon}\tilde{z}_t\Big)\,\Omega\\ &=\left(\partial_t\tilde{\omega}-\Im\left\langle\tfrac{1}{i}\tilde{z}_t,\tilde{f}_t\right\rangle+\sqrt{2}\Phi(\tilde{f}_t)\right)\tilde{\tilde{\Psi}}^{app}_t\end{aligned}$$

since $\frac{1}{t}\left\langle\varphi,\left[W(z+tu)-W(z)\right]\psi\right\rangle\xrightarrow[t\to 0]{}\left\langle\varphi,\left[-\frac{i\varepsilon}{2}\Im\langle z,u\rangle+i\Phi(u)\right]W(z)\psi\right\rangle$. □

We then compare $\tilde{\Psi}_t$ and $\tilde{\Psi}^{app}_t$.

Proposition 5.14. *Let* $\Delta\tilde{\tilde{\Psi}}_{h,\varepsilon,t}=\tilde{\tilde{\Psi}}_{h,\varepsilon,t}-\tilde{\tilde{\Psi}}^{app}_{h,\varepsilon,t}$ *and* $\Delta\tilde{Q}_t(z)=\langle z,\eta z\rangle^{\cdot 2}$, *then*

$$\Delta\tilde{\tilde{\Psi}}_{h,\varepsilon,t}=-\frac{i}{\varepsilon}\int_0^t e^{-i\frac{t-s}{\varepsilon}\tilde{Q}^{Wick}_{h,\varepsilon}}\Delta\tilde{Q}^{Wick}\tilde{\tilde{\Psi}}^{app}_{h,\varepsilon,s}\,\mathrm{d}s\,.$$

Proof. It suffices to remark that

$$i\varepsilon\partial_t\Delta\tilde{\tilde{\Psi}}_t=\tilde{Q}^{Wick}\Delta\tilde{\tilde{\Psi}}_t+\Delta\tilde{Q}^{Wick}\tilde{\tilde{\Psi}}^{app}_t$$

and that the integral expression satisfies the same differential equation. □

Proposition 5.15. *The difference* $\Delta\tilde{\tilde{\Psi}}_{h,\varepsilon,t}$ *can be controlled as*

$$\left\|\Delta\tilde{\tilde{\Psi}}_{h,\varepsilon,t}\right\|\leq\frac{1}{\varepsilon}\int_0^t\left\|\Delta\tilde{Q}^{Wick}\tilde{\tilde{\Psi}}^{app}_{h,\varepsilon,s}\right\|\mathrm{d}s=\frac{1}{\varepsilon}\int_0^t\left\|\Delta\tilde{Q}^{Wick}\,E(\tilde{z}_{h,\varepsilon,s})\right\|\mathrm{d}s\,.$$

Proposition 5.16. *Let $T_0 > 0$. There exists a constant $C_{T_0,G,d}$ such that for $\frac{ht}{\varepsilon} \leq T_0$,*

$$\left\| \Delta\tilde{Q}^{Wick}\, E\big(\tilde{z}^{app}_{h,\varepsilon,t}\big) \right\| \leq C_{T_0,G,d} \frac{ht}{\varepsilon}\,.$$

Proof. We make use of the relation valid for coherent states

$$\big\langle E(\tilde{z}_{h,\varepsilon,t}), \big(\Delta\tilde{Q}^{Wick}\big)^* \Delta\tilde{Q}^{Wick} E(\tilde{z}_{h,\varepsilon,t})\big\rangle = \overset{Wick}{\mathrm{Symb}}\Big(\big(\Delta\tilde{Q}^{Wick}\big)^* \Delta\tilde{Q}^{Wick}\Big)(\tilde{z}_{h,\varepsilon,t})\,.$$

Since

$$\begin{aligned}
&\overset{Wick}{\mathrm{Symb}}\left(\Big(\big(\langle z,\eta z\rangle^{.2}\big)^{Wick}\Big)^2\right) \\
&\quad = \big(\langle z,\eta z\rangle^{.2}\big)^2 + 4\varepsilon\left(\langle z,\eta z\rangle . \langle \eta z|\right)\left(|\eta z\rangle . \langle z,\eta z\rangle\right) + 2\varepsilon^2\big(\langle \eta z|^{.\otimes 2}\big)\big(|\eta z\rangle^{.\otimes 2}\big)\,,
\end{aligned}$$

using Proposition 5.11, we obtain that

$$\left\| \Delta\tilde{Q}^{Wick}\, E(\tilde{z}_{h,\varepsilon,t}) \right\|^2 \leq C_{T_0,G,d}\left(\left(\frac{ht}{\varepsilon}\right)^4 + 4\varepsilon\left(\frac{ht}{\varepsilon}\right)^2 \frac{ht}{\varepsilon^2} + 2\varepsilon^2\left(\frac{ht}{\varepsilon^2}\right)^2\right)$$

which gives the result for $\frac{ht}{\varepsilon} \leq T_0$. □

Proposition 5.17. *Let $T_0 > 0$. There exists a constant $C_{T_0,G,d}$ such that for $\frac{ht}{\varepsilon} \leq T_0$,*

$$\begin{aligned}
\left\| \Delta\tilde{\hat{\Psi}}_{h,\varepsilon,t} \right\| &\leq \frac{1}{\varepsilon}\int_0^t \left\| \Delta\tilde{Q}^{Wick}\, E(\tilde{z}_{h,\varepsilon,t}) \right\| \mathrm{d}s \\
&\leq C_{T_0,G,d}\, h^{-1}\left(\frac{ht}{\varepsilon}\right)^2 .
\end{aligned}$$

6. Measure of an observable at a mesoscopic scale for the approximated dynamics

6.1. **Result.** In this section we make the connection with the linear Boltzmann equation.

Let b be a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ and $\rho \in \mathcal{L}^+_1 L^2_x$, $\mathrm{Tr}\,\rho = 1$. The measure of the observable $b^W(hx, D_x)$ in the state ρ is denoted by

$$m(b,\rho) = \mathrm{Tr}\left[b^W(hx, D_x)\,\rho\right]\,.$$

Proposition 6.1. *Let b be a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ and $\rho \in \mathcal{L}^+_1 L^2_x$, $\mathrm{Tr}\,\rho \leq 1$ such that the kernel of $\hat{\rho} = \mathrm{Ad}\{\mathcal{F}_x\}[\rho]$ has a bounded support. Let $\alpha \in [0,1[$. Introduce the symbol $b_t = e^{tQ}e^{2t\xi.\partial_x} b$ where Q is the collision operator introduced in Equation 3.1 with here $\sigma(\xi,\xi') = 2\pi\hat{G}(\xi'-\xi) = 2\pi|\hat{V}(\xi-\xi')|^2$. When $h^\alpha \leq \frac{ht}{\varepsilon} \leq 1$, the inequality*

$$m(b, \rho^{\varepsilon,app}_t) \geq m\big(b_{\frac{ht}{\varepsilon}}, \rho\big) - \mathcal{E}_6$$

then holds with $\mathcal{E}_6 = C_{b,\mu}\frac{ht}{\varepsilon}\left(\frac{ht}{\varepsilon} + h + \left[h\left(\frac{ht}{\varepsilon}\right)^{-1}\right]^{d/2-1} + h^{\mu(d,\alpha)}\right)$ for some constant $C_{b,\mu} > 0$ and $\mu(d,\alpha) > 0$.

Remark 6.2. This result also holds with b a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{d*}_\xi;\mathbb{C})$. The proof is the same as for Proposition 6.1, with the symplectic Fourier transform $\mathcal{F}^\sigma$ replaced by the usual Fourier transform. The special case when $b(\xi) = b_1(|\xi|^2)$ is of particular interest and the symbol b_t in the previous statement does not depend on t.

Proposition 6.1 is a by-product of the following stronger result.

Proposition 6.3. *Let $b_s \in \mathcal{C}^1(\mathbb{R};\mathcal{D}(\mathbb{R}^{2d}_{x,\xi}))$ such that for some $R>1$, and for all s, $\mathrm{Supp}_\xi\, b_s \subset B_R - B_{R^{-1}}$. Let $\rho \in \mathcal{L}^+_1 L^2_x$, $\mathrm{Tr}\,\rho \le 1$ such that the kernel of $\hat\rho = \mathrm{Ad}\,\{\mathcal{F}_x\}[\rho]$ has a bounded support. Then*

$$m\big(b_{\frac{ht}{\varepsilon}}, \rho_t^{\varepsilon,app}\big) \\ \ge m(b,\rho) - \frac{i}{\varepsilon}\int_0^t m\Big(i\varepsilon\partial_s b_s - ih\left\{b_s, \xi^{\cdot 2}\right\} + ihQ_{-\frac{ht}{\varepsilon}} b_s, \rho_s^{\varepsilon,app}\Big)\,\mathrm{d}s - \mathcal{E}_6\,.$$

Remark 6.4. The conservation of the support in ξ is important and will be provided by the properties of the dual linear Boltzmann equation in the application of this proposition.

Proof that Proposition 6.3 implies Proposition 6.1. Since one can make mistakes between the notations of those two propositions we use notations with tildes, $\tilde b$ for Proposition 6.1 and without tildes for Proposition 6.3. Thus we want

$$\tilde b = b_{\frac{ht}{\varepsilon}}\,, \qquad \tilde b_{\frac{ht}{\varepsilon}} = b\,.$$

Denote by $\tilde G\,(t,t_0)$ the dynamical system associated with $(-2\xi.\partial_x - Q_{-t})_t$ given by

$$\begin{cases} \partial_t b_t = (-2\xi.\partial_x - Q_{-t})\, b_t \\ b_{t=t_0} = b_0 \end{cases}, \qquad b_t = \tilde G(t,t_0)\, b_0\,.$$

To have a vanishing term for b in the integral we require $b_{ht/\varepsilon} = \tilde G(\frac{ht}{\varepsilon},0)b$, so that with $\tilde b_{ht/\varepsilon} = \tilde G(0,-\frac{ht}{\varepsilon})\tilde b$, we will get the expected result. The only thing remaining to prove is $\tilde G(0,-t) = e^{tQ}e^{2t\xi.\partial_x}$. It is equivalent to show that

$$e^{2t\xi.\partial_x}\tilde G\,(t,0) = e^{-tQ}\,,$$

which is clear by derivation and using that $Q_t = e^{t2\xi.\partial_x}Qe^{-t2\xi.\partial_x}$. □

6.2. **Expression of the measure of an observable for the approximated equation.** We carry out an explicit computation using only the approximated equation. We recall (see Proposition 5.13) that the solution of the approximated equation with initial data $\Psi^{app}_{t=0} = \psi_0\otimes\Omega$ is (after translation and Fourier transform)

$$\hat\Psi^{app}_{h,\varepsilon,t} = \hat\psi_0\,(\xi)\, e^{-i\frac{\omega^\xi_{h,\varepsilon,t}}{\varepsilon}} W\Big(\tfrac{\sqrt{2}}{i\varepsilon} z^\xi_{h,\varepsilon,t}\Big)\,\Omega$$

with $z_{h,\varepsilon,t} = -i\int_0^t e^{-i\frac{s}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2} - 2\xi.\varepsilon\eta\right)} f_{h,\varepsilon}\,\mathrm{d}s$ and $\omega_t = t\xi^2 + \int_0^t \Re\,\langle z_{h,\varepsilon,s}, f_{h,\varepsilon}\rangle\,\mathrm{d}s$ and $f_{h,\varepsilon}(\eta) = \varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat V(-\varepsilon\eta)$.

Definition 6.5. Let $\sigma(X_1,X_2) = \xi_1.x_2 - x_1.\xi_2$ ($X_j = (x_j,\xi_j) \in \mathbb{R}^{2d}_{x,\xi}$) be the standard symplectic form on $\mathbb{R}^{2d}_{x,\xi}$.

Let $X' = (x',\xi') \in \mathbb{R}^{2d}_{x,\xi}$, the Weyl operators on L^2_x are defined by

$$\tau^h_{X'} = \Big(e^{-i\sigma(\cdot,X')}\Big)^W(hx,D_x) = e^{-i\sigma(\cdot,X')^W(hx,D_x)} = e^{i(\xi'\cdot hx - x'\cdot D_x)}\,,$$

their Fourier transform is denoted by $\hat\tau^h_P := \mathrm{Ad}\,\{\mathcal{F}_x\}\left[\tau^h_P\right]$.

The symplectic Fourier transform $\mathcal{F}^\sigma$ is defined on $L^2(\mathbb{R}^{2d}_{x,\xi};\mathbb{C})$ by

$$\mathcal{F}^\sigma b\,(X) = \int_{\mathbb{R}^{2d}} e^{-i\sigma(X,X')} b\,(X')\mathrm{đ}X'$$

with $\mathrm{đ}X = \mathrm{d}X/(2\pi)^d$.

Proposition 6.6. *Let b be a symbol in $\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ and $\rho \in \mathcal{L}_1^+ L^2_x$, $\operatorname{Tr}\rho \leq 1$, then*

$$m(b, \rho_t^{\varepsilon,app}) = \iiint \mathcal{F}^\sigma b(P)\, e^{-\frac{i[\omega]+[\varphi]_1-[\varphi,p_x]_2}{\varepsilon}}\, \hat\tau_P^h(\xi_2,\xi_1)\, \hat\rho(\xi_1,\xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{đ}P$$

with

$$[\omega] = \omega_t^{\xi_1} - \omega_t^{\xi_2} \tag{6.1}$$

$$[\varphi]_1 = [\varphi]_{1,1} + [\varphi]_{1,2} \tag{6.2}$$

$$[\varphi]_{1,j} = \frac{1}{2}|z_t^{\xi_j}|^2\,,\ j = 1,2 \tag{6.3}$$

$$[\varphi, p_x]_2 = \left\langle z_t^{\xi_2}, e^{ip_x\cdot\varepsilon\eta} z_t^{\xi_1}\right\rangle\,. \tag{6.4}$$

Remark 6.7. From $e^{i\varepsilon x.\lambda}\,\tau_P^h\, e^{-i\varepsilon x.\lambda} = e^{i\varepsilon\lambda.p_x}\,\tau_P^h$ and taking λ as the spectral parameter of $\mathrm{d}\Gamma_\varepsilon(D_y)$,

$$\Gamma(e^{i\varepsilon x.D_y})\,\tau_P^h\,\Gamma(e^{-i\varepsilon x.D_y}) = \Gamma(e^{ip_x\cdot\varepsilon D_y})\,\tau_P^h$$

and after conjugating with the Fourier transforms, we obtain

$$\mathrm{Ad}\left\{(\mathcal{F}_x\otimes\Gamma\mathcal{F}_y)\,\Gamma(e^{i\varepsilon x D_y})\right\}\left[\tau_P^h\right] = \Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat\tau_P^h\,.$$

Proof. As $b^W(hx, D_x) = \int \mathcal{F}^\sigma b(P)\,\tau_P^h\,\mathrm{đ}P$, we have for $\rho\in\mathcal{L}_1^+$

$$m(b,\rho) = \int \mathcal{F}^\sigma b(P)\,\mathrm{Tr}\big[\tau_P^h\,\rho\big]\,\mathrm{đ}P\,.$$

By translating and Fourier transforming we get the expression

$$m(b,\rho_t^{\varepsilon,app}) = \int \mathcal{F}^\sigma b(P)\,\mathrm{Tr}\big[\hat{\boldsymbol{\rho}}_t^{app}\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat\tau_P^h\big]\mathrm{đ}P\,.$$

We conclude with the Lemma 6.8 below. □

Lemma 6.8. *The kernel K_P of the operator $\mathrm{Tr}_{\Gamma L^2_\eta}[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})]$ on L^2_ξ is*

$$K_P(\xi_1,\xi_2) = e^{-i\frac{\omega_{h,\varepsilon,t}^{\xi_1}-\omega_{h,\varepsilon,t}^{\xi_2}}{\varepsilon}}\,\hat\rho(\xi_1,\xi_2)e^{-\frac{1}{2\varepsilon}\left(|z_{h,\varepsilon,t}^{\xi_1}|^2-|z_{h,\varepsilon,t}^{\xi_2}|^2+2\langle z_{h,\varepsilon,t}^{\xi_2},e^{ip_x\cdot\varepsilon\eta}z_{h,\varepsilon,t}^{\xi_1}\rangle\right)}\,.$$

Proof. Using $\hat\rho\otimes|\Omega\rangle\langle\Omega| = \int_{\xi_1}^{\oplus}\int_{\xi_2}^{\oplus}\hat\rho(\xi_1,\xi_2)\;|\Omega\rangle\langle\Omega|\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$ we get

$$\begin{aligned}&\operatorname*{Tr}_{\Gamma L^2_\eta}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\right]\\ &= \operatorname*{Tr}_{\Gamma L^2_\eta}\left[\int_{\mathbb{R}^d_{\xi_1}}^{\oplus}\int_{\mathbb{R}^d_{\xi_2}}^{\oplus}\left|E(z_t^{\xi_1})\right\rangle\left\langle E(z_t^{\xi_2})\right|e^{-i\frac{\omega_t^{\xi_1}}{\varepsilon}}e^{i\frac{\omega_t^{\xi_2}}{\varepsilon}}\hat\rho(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\Gamma(e^{ip_x\cdot\varepsilon\eta})\right]\end{aligned}$$

and by the rules of calculus on coherent states we obtain

$$\begin{aligned}K_P\,(\xi_1,\xi_2) &= e^{-i\frac{\omega_t^{\xi_1}-\omega_t^{\xi_2}}{\varepsilon}}\,\hat\rho\,(\xi_1,\xi_2)\left\langle E(z_t^{\xi_2})\right|\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\left|E(z_t^{\xi_1})\right\rangle\\ &= e^{-i\frac{\omega_t^{\xi_1}-\omega_t^{\xi_2}}{\varepsilon}}\,\hat\rho\,(\xi_1,\xi_2)\,e^{-\left|z_t^{\xi_1}\right|^2/2\varepsilon-\left|z_t^{\xi_2}\right|^2/2\varepsilon+\frac{1}{\varepsilon}\left\langle z_t^{\xi_2},e^{ip_x\cdot\varepsilon\eta}z_t^{\xi_1}\right\rangle}\end{aligned}$$

which is the result of the Lemma. □

Definition 6.9. For $j = \{,\}, -, +$ we define

$$m_j = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\, \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\, \Gamma(e^{ip_x\cdot\varepsilon\eta})\, \mathscr{A}_{j,P}\right] \text{đ}P$$

where the operators $\mathscr{A}_{j,P}$ are defined by their kernels, according to the notations of Equations (6.1), (6.2), (6.3) and (6.4),

$$ih\, \mathscr{A}_{\{,\},P}(\xi_1,\xi_2) = \hat{\tau}_P^h(\xi_2,\xi_1)\, \partial_t\left[\omega\right], \tag{6.5}$$

$$ih\, \mathscr{A}_{\{,\},j,P}(\xi_j) = \hat{\tau}_P^h(\xi_2,\xi_1)\, \partial_t \omega_t^{\xi_j}\ ,\ j=1,2\,, \tag{6.6}$$

$$ih\, \mathscr{A}_{-,P}(\xi_1,\xi_2) = i\partial_t\left[\varphi\right]_1\ \hat{\tau}_P^h(\xi_2,\xi_1)\,, \tag{6.7}$$

$$ih\, \mathscr{A}_{-,j,P}(\xi_j) = i\partial_t\left[\varphi\right]_{1,j}\ \hat{\tau}_P^h(\xi_2,\xi_1)\,,\ j=1,2\,, \tag{6.8}$$

$$ih\, \mathscr{A}_{+,P}(\xi_1,\xi_2) = i\partial_t\left[\varphi,p_x\right]_2\ \hat{\tau}_P^h(\xi_2,\xi_1) \tag{6.9}$$

and $\mathscr{A}_{\{,\}} = \mathscr{A}_{\{,\},1} - \mathscr{A}_{\{,\},2}$, $\mathscr{A}_- = \mathscr{A}_{-,1} + \mathscr{A}_{-,2}$.

The indexes $\{,\}$, $-$ and $+$ were chosen to recall the terms of the linear Boltzmann equation, $\{,\}$ corresponding to $\left\{\xi^2,\cdot\right\}$, $+$ to Q_+ and $-$ to Q_-.

Proposition 6.10. *Let $b_t \in \mathcal{C}^1(\mathbb{R};\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi}))$, then the equality*

$$i\varepsilon\partial_t m(b_t,\rho_t^{\varepsilon,app}) = m(i\varepsilon\partial_t b_t,\rho_t^{\varepsilon,app}) + ih\left(m_{\{,\}} - m_- + m_+\right)$$

holds.

Remark 6.11. Later we will put each of those terms m_j in the form

$$m_j = m(c_j,\rho_t^{\varepsilon,app}) + \Delta_j$$

where Δ_j denotes a small error term.

Proof of Proposition 6.10. Indeed

$$\begin{aligned} &i\varepsilon\partial_t m(b,\rho_t^{\varepsilon,app}) \\ &\quad= \iiint_{P,\xi_1,\xi_2} \left[\mathcal{F}^\sigma i\varepsilon\partial_t b(P) + \mathcal{F}^\sigma b(P)\left\{\partial_t\left[\omega\right] - i\partial_t\left[\varphi\right]_1 + i\partial_t\left[\varphi,p_x\right]_2\right\}\right] \\ &\qquad\qquad e^{-\frac{i[\omega]+[\varphi]_1-[\varphi,p_x]_2}{\varepsilon}}\hat{\tau}_P^h(\xi_2,\xi_1)\,\hat{\rho}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\text{đ}P \end{aligned}$$

and so it suffices to prove the following lemma. □

Lemma 6.12. *For $j = \{,\}$, $-$, $+$, the formula*

$$\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\mathscr{A}_{j,P}\right] = \iint \mathscr{A}_j(P,\xi_1,\xi_2)\, e^{-\frac{i[\omega]+[\varphi]_1-[\varphi,p_x]_2}{\varepsilon}}\hat{\rho}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

holds.

Proof. Indeed

$$\begin{aligned} &\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\mathscr{A}_{j,P}\right] \\ &= \iint \hat{\rho}(\xi_1,\xi_2)\left\langle E(z_t^{\xi_2})\right|\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\left|E(z_t^{\xi_1})\right\rangle e^{-i\frac{\omega_t^{\xi_1}-\omega_t^{\xi_2}}{\varepsilon}}\mathscr{A}_j(P,\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &= \iint \hat{\rho}(\xi_1,\xi_2)\, e^{-\frac{i[\omega]+[\varphi]_1+[\varphi,p_x]_2}{\varepsilon}}\mathscr{A}_j(P,\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,. \end{aligned}$$

□

6.3. **Two estimates.** We will need several times these estimates to get rid of the term $\Gamma(e^{ip_x\cdot\varepsilon\eta})$ and then control small errors on the operators $\mathcal{A}_P$.

Proposition 6.13. *Let $\mathscr{A}_P$ be a P-dependent family of operators in $\mathcal{L}(L^2_\xi)$. Then*

$$\langle P\rangle^{-k}\left|\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\mathscr{A}_P\right]\right|\leq\frac{ht}{\varepsilon}\sup_{P\in\mathbb{R}^{2d}}\langle P\rangle^{-k}\left\|\mathscr{A}_P\right\|_{\mathcal{L}(L^2_\xi)}$$

and

$$\left|\int_{\mathbb{R}^{2d}_P}\mathcal{F}^\sigma b(P)\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\mathscr{A}_P\right]\mathrm{d}\!\!^{-}P\right|$$
$$\leq\frac{ht}{\varepsilon}\left\|\langle\cdot\rangle^k\mathcal{F}^\sigma b\right\|_{L^1_P}\sup_P\langle P\rangle^{-k}\left\|\mathscr{A}_P\right\|_{\mathcal{L}\left(L^2_\xi\right)}.$$

This can be proved in two steps.

Remark 6.14. It suffices to prove this property with $\rho=|\psi\rangle\langle\psi|$ with a $\hat{\psi}$ with bounded support as any $\rho\in\mathcal{L}_1^+L^2_x$, $\mathrm{Tr}\,\rho=1$ can be decomposed as

$$\rho=\sum_{j\geq0}\lambda_j\,|\psi_j\rangle\langle\psi_j|$$

with positive λ_j's and $\sum_j\lambda_j=1$, and

$$\mathrm{Supp}\,\hat{\rho}\,(\xi,\xi')\subset B^2_M\Leftrightarrow\forall j,\ \mathrm{Supp}\,\hat{\psi}_j\subset B_M\,.$$

Lemma 6.15. *Let $\mathscr{A}_P$ be a P-dependent family of operators in $\mathcal{L}(L^2_\eta)$ and $\hat{\Psi}$ be a normed vector in $L^2_\xi\otimes\Gamma L^2_\eta$. Then*

$$\left|\mathrm{Tr}\left[\mathrm{proj}\,\hat{\Psi}\left(\Gamma\left(e^{ip_x\cdot\varepsilon\eta}\right)-\mathrm{Id}\right)\mathscr{A}_P\right]\right|\leq\left\|\left(\Gamma\left(e^{ip_x\cdot\varepsilon\eta}\right)-\mathrm{Id}\right)\hat{\Psi}\right\|\left\|\mathscr{A}_P\right\|_{\mathcal{L}\left(L^2_\xi\right)}.$$

Lemma 6.16. *There exists a constant $C_{G,d}$ which depends only on G and d such that*

$$\left\|\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\hat{\Psi}^{app}_{h,\varepsilon,t}\right\|\leq C_{G,d}\frac{ht}{\varepsilon}\,.$$

Proof. The calculus rules on coherent states give

$$\left\|\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\hat{\Psi}^{app}_{h,\varepsilon,t}\right\|^2=\sup_\xi\left\|E\left(e^{ip_x\cdot\varepsilon\eta}z^\xi_{h,\varepsilon,t}\right)-E\left(z^\xi_{h,\varepsilon,t}\right)\right\|^2$$
$$=\sup_\xi 2\left(1-\cos\left(\frac{1}{\varepsilon}\Im\left\langle e^{ip_x.\varepsilon\eta}z^\xi_{h,\varepsilon,t},z^\xi_{h,\varepsilon,t}\right\rangle\right)\right)$$
$$\leq C_{G,d}\left(\frac{ht}{\varepsilon}\right)^2,$$

where the last inequality is obtained using $|1-\cos t|\leq t^2/2$ and the estimates on $\|z_t\|$. □

Proposition 6.17. *Let $\mathscr{E}_P$ be a P-dependent family of operators in $\mathcal{L}(L^2_\xi)$ and $\hat{\boldsymbol{\rho}}$ be a state on $L^2_\xi\otimes\Gamma L^2_\eta$. Then for any integer k (with possibly infinite quantities)*

$$\left|\int_{\mathbb{R}^{2d}_P}\mathcal{F}^\sigma b(P)\,|\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}\,\mathscr{E}_P\right]|\mathrm{d}\!\!^{-}P\right|\leq\left\|\langle\cdot\rangle^k\mathcal{F}^\sigma b\right\|_{L^1_P}\sup_P\left\|\langle P\rangle^{-k}\,\mathscr{E}_P\right\|_{\mathcal{L}\left(L^2_\xi\right)}.$$

6.4. **The transport term $m_{\{,\}}$.** The result of this section is the following.

Proposition 6.18. *Let $\rho \in \mathcal{L}_1^+ L_x^2$, $\operatorname{Tr}\rho \leq 1$ and $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ such that $\operatorname{Supp}\hat{\rho}(\xi,\xi') \subset B_R^2$, and $\operatorname{Supp}_\xi b \subset B_R$ for some $R>0$ then*

$$m_{\{,\}} = m\big(-\{b,\xi^{\cdot 2}\},t\big) + \Delta_{\{,\}}$$

with $|\Delta_{\{,\}}| \leq C_{G,R,b}\,(\frac{ht}{\varepsilon} + h + (\frac{\varepsilon}{t})^{d/2})$.

It is a consequence of the following more accurate result.

Proposition 6.19. *Suppose the hypotheses of Proposition 6.18 are satisfied and keep the same notations. The term Δ_1 can be decomposed as*

$$\Delta_{\{,\}} = \sum_{j=1}^{3} \Delta_{\{,\},j}$$

with, for some integer k,

(1) $\big|\Delta_{\{,\},1}\big| \leq 2\frac{ht}{\varepsilon} \| \langle\cdot\rangle^k \mathcal{F}^\sigma b\|_{L^1_P} \mathcal{O}\left(1 + h + [h(\frac{ht}{\varepsilon})^{-1}]^{d/2-1}\right)$,
(2) $\big|\Delta_{\{,\},2}\big| \leq \left(\|\mathcal{F}^\sigma b\|_{L^1_P} + \| \langle\cdot\rangle^k \mathcal{F}^\sigma b\|_{L^1_P}\right) \mathcal{O}\left(h + (\frac{\varepsilon}{t})^{\frac{d}{2}-1}\right)$,
(3) $\big|\Delta_{\{,\},3}\big| \leq \frac{ht}{\varepsilon} \|\mathcal{F}^\sigma\{b,\xi^{\cdot 2}\}\|_{L^1_P}$.

Remark 6.20. The operator $\mathscr{A}_{\{,\},P}$ is actually

$$\mathscr{A}_{\{,\},P} = \frac{1}{ih}\left[\hat{\tau}_P^h\,,\,\partial_t\omega\times\right]\,.$$

Remark 6.21. We can introduce a cutoff function $\chi_R \in \mathcal{C}_0^\infty(\mathbb{R}^d_\xi)$ such that $\chi_R(B_R) = \{1\}$, $\chi_R(\mathbb{R}^d_\xi - B_{R+1}) = \{0\}$ and $\chi_R(\mathbb{R}^d_\xi) \subset [0,1]$.

This result will be proved by considering successively every error term. These error terms $\Delta_{\{,\},j}$, $j=1,2,3$ are given by the following approximation process (where we write shortly B^W for $B^W(-hD_\xi,\xi)$).

$$\begin{aligned}
m_{\{,\}} &= \int_P \mathcal{F}^\sigma b(P)\operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\frac{1}{ih}\left[\hat{\tau}_P^h\,,\partial_t\omega\times\right]\right]\mathrm{d}P \\
&= \int_P \mathcal{F}^\sigma b(P)\operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\frac{1}{ih}\left[\hat{\tau}_P^h\,,\chi_R\partial_t\omega\times\right]\right]\mathrm{d}P \\
&= \int_P \mathcal{F}^\sigma b(P)\operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\frac{1}{ih}\left[\hat{\tau}_P^h\,,\chi_R\partial_t\omega\times\right]\right]\mathrm{d}P + \Delta_{\{,\},1} \\
&= \operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\frac{1}{ih}\left[b^W\,,\chi_R\partial_t\omega\times\right]\right]\mathrm{d}P + \Delta_{\{,\},1} \\
&= -\operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\{b\,,\chi_R\xi^{\cdot 2}\}^W\right]\mathrm{d}P + \sum_{j=1}^{2}\Delta_{\{,\},j} \\
&= \int_P \mathcal{F}^\sigma\big(-\{b\,,\xi^{\cdot 2}\}\big)(P)\operatorname{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\hat{\tau}_P^h\right]\mathrm{d}P + \sum_{j=1}^{2}\Delta_{\{,\},j}
\end{aligned}$$

$$= \int_P \mathcal{F}^\sigma\big(-\{b\,,\xi^{\cdot 2}\}\big)\,(P)\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat{\tau}_P^h\right]\mathrm{đ}P + \sum_{j=1}^3 \Delta_{\{,\},j}$$

$$= m\big(-\{b\,,\xi^{\cdot 2}\},t\big) + \sum_{j=1}^3 \Delta_{\{,\},j}\,.$$

The quantities $\Delta_{\{,\},j}$ are defined by

$$\Delta_{\{,\},1} = \int_P \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\frac{1}{ih}\left[\hat{\tau}_P^h\,,\chi_R\partial_t\omega\times\right]\right]\mathrm{đ}P\,,$$

$$\Delta_{\{,\},2} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\frac{1}{ih}\left(\left[b\,,\chi_R\partial_t\omega\times\right] - \frac{h}{i}\left\{b\,,\chi_R\xi^{\cdot 2}\right\}^W\right)\right]\mathrm{đ}P\,,$$

$$\Delta_{\{,\},3} = \int_P \mathcal{F}^\sigma\big(-\{b\,,\xi^{\cdot 2}\}\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\mathrm{Id}-\Gamma(e^{ip_x\cdot\varepsilon\eta})\right)\hat{\tau}_P^h\right]\mathrm{đ}P\,.$$

We will use the structure of $\partial_t\omega$:

Proposition 6.22. *The time derivative of ω is given by*

$$\partial_t\omega_{h,\varepsilon,t} = \xi^{\cdot 2} - h\Im\int_0^{t/\varepsilon}\!\!\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,.$$

Proof. Differentiating ω with respect to t,

$$\begin{aligned}\partial_t\omega_{h,\varepsilon,t} &= \xi^{\cdot 2} + \Re\left\langle z^\xi_{h,\varepsilon,t}, f_{h,\varepsilon}\right\rangle \\ &= \xi^{\cdot 2} + \Re\int_{\mathbb{R}^d_\eta} i\int_0^t e^{-i\frac{s}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}\left|f_{h,\varepsilon}(\eta)\right|^2\mathrm{d}s e^{i\frac{t}{\varepsilon}\left(\varepsilon^2\eta^{\cdot 2}-2\xi.\varepsilon\eta\right)}\,\mathrm{d}\eta\end{aligned}$$

which is the result once we replace $f_{h,\varepsilon}$ by its expression in terms of $\hat{V}$, use $\hat{G}=|\hat{V}|^2$ and make a change of variable. □

Lemma 6.23. *We have, for some integer k,*

$$\left[\hat{\tau}_P^h,\chi_R\partial_t\omega_{h,\varepsilon,t}\times\right] = \frac{h}{i}\left\{e^{i\sigma(P,X)},\chi_R\xi^2\right\}^W(-hD_\xi,\xi) + h\mathcal{O}\Big(\langle P\rangle^k\,h + \left(\frac{\varepsilon}{t}\right)^{\frac{d}{2}-1}\Big)\,.$$

and in particular $\left\|\left[\hat{\tau}_P^h,\chi_R\partial_t\omega\times\right]\right\|_{\mathcal{L}\left(L^2_\xi\right)} \le \langle P\rangle^k\,\mathcal{O}(h)$.

Remark 6.24. We use in this proposition that $G\in L^1_x$.

Proof of Lemma 6.23. We split the commutator in three parts

$$\left[\hat{\tau}_P^h,\chi_R\partial_t\omega\times\right] = \left[\hat{\tau}_P^h,\chi_R(\xi)\xi^{\cdot 2} - hg_1(\xi) + hR_1\left(\frac{t}{\varepsilon},\xi\right)\times\right]$$

with

$$g_1(\xi) := \chi_R(\xi)\Im\lim_{M\to+\infty}\int_0^M\!\!\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$

$$R_1(u,\xi) := \chi_R(\xi)\Im\lim_{M\to+\infty}\int_u^M\!\!\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,.$$

The biggest part, in $\xi^{\cdot 2}$, gives the only relevant contribution

$$\left[\hat{\tau}_P^h,\chi_R\xi^{\cdot 2}\times\right] = \frac{h}{i}\left\{e^{i\sigma(P,X)},\chi_R\xi^{\cdot 2}\times\right\}^{Weyl} + \langle P\rangle^k\,\mathcal{O}_{h\to 0}\left(h^2\right)\,.$$

One of the other parts can be estimated without using the commutator structure

$$\begin{aligned}\left\|\left[\hat{\tau}_P^h, R_1\left(\tfrac{t}{\varepsilon},\xi\right)\times\right]\right\|_{\mathcal{L}(L^2_\xi)} &\leq 2\left\|\hat{\tau}_P^h\right\|_{\mathcal{L}(L^2_\xi)}\left\|R_1\left(\tfrac{t}{\varepsilon},\xi\right)\times\right\|_{L^\infty_\xi}\\ &\leq C\left(\frac{\varepsilon}{t}\right)^{\frac{d}{2}-1}\end{aligned}$$

since

$$\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta = e^{-is\xi^{\cdot 2}}\int_{\mathbb{R}^d_x} G(x)\,e^{-ix.\xi}\left(\frac{2\pi}{|s|}\right)^{d/2} e^{id\,\mathrm{sign}\,s\frac{\pi}{4}} e^{\frac{x^2}{2is}}\,\mathrm{d}x$$

and thus

$$\left|\int_{\mathbb{R}^d_\eta} e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\right| \leq \left(\frac{2\pi}{|s|}\right)^{d/2}\|G\|_{L^1}\ .$$

Since g_1 is in $\mathcal{C}^\infty_0(\mathbb{R}^d_\xi)$ we can apply the symbolic calculus

$$\left[\hat{\tau}_P^h, hg_1(\xi)\times\right] = \frac{h^2}{i}\left\{e^{i\sigma(P,X)}\,, g_1(\xi)\right\}^W(-hD_\xi,\xi) + \mathcal{O}\left(h^2\left\langle P\right\rangle^k\right)$$

where for some integer k,

$$\left\|\left\{e^{i\sigma(P,X)}\,, g_1(\xi)\times\right\}^W(-hD_\xi,\xi)\right\|_{\mathcal{L}(L^2_\xi)} = \left\langle P\right\rangle^k\mathcal{O}_{h\to 0}(1)\,,$$

which concludes the proof of the lemma. □

We can then estimate the three error terms $\Delta_{\{,\},j}$.

Proof of 1 in Proposition 6.19. It is a result of Proposition 6.13 and the estimate of

$$\left\|\left[\hat{\tau}_P^h\,, \chi_R\partial_t\omega_{h,\varepsilon,t}\times\right]\right\|_{\mathcal{L}(L^2_\xi)}$$

of the lemma. □

Proof of 2 in Proposition 6.19. The second error term can be expressed as

$$\Delta_{\{,\},2} = \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\Big[\hat{\boldsymbol{\rho}}_t^{app}\,\frac{1}{ih}\Big(\left[\hat{\tau}_P^h\,, \chi_R\partial_t\omega_{h,\varepsilon,t}\times\right] - \frac{h}{i}\left\{\hat{\tau}_P^h\,, \chi_R\xi^{\cdot 2}\right\}^W\Big)\Big]\,đP$$

so that the lemma and Proposition 6.17 give the estimation. □

Proof of 3 in Proposition 6.19. It is an application of Proposition 6.13. □

6.5. The collision terms m_- and m_+.

Proposition 6.25. *Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$ and $\rho \in \mathcal{L}^+_1 L^2_x$, $\mathrm{Tr}\,\rho \leq 1$ such that for some $R > 0$, $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$ and $\mathrm{Supp}\,\hat{\rho}(\xi,\xi') \subset B^2_R$. Then*

$$m_\pm = m(Q_{\pm,t}(b),t) + \Delta_\pm$$

and for any $\alpha \in [0,1[$, there are constants $\mu = \mu(d,\alpha) > 0$ and $C_{R,b,G,d,\alpha,\mu} > 0$, such that for $h^\alpha \leq \frac{th}{\varepsilon} \leq 1$,

$$|\Delta_\pm| \leq C_{R,b,G,\mu}\Big(\frac{ht}{\varepsilon} + h^\mu\Big)\,.$$

Definition 6.26. For $\zeta > 0$, $r \in \mathbb{R}$ and $P \in \mathbb{R}^{2d}_{p_x,p_\xi}$, set

$$\begin{aligned}
\kappa^\zeta(r) &= \frac{1}{\pi}\frac{\zeta}{r^2+\zeta^2}\,,\\
\mathfrak{c}(\xi) &= 2\pi\int_{\mathbb{R}^d_\eta} \hat{G}(\eta+\xi)\,\delta(\eta^{\cdot 2}-\xi^{\cdot 2})\,\mathrm{d}\eta\,,\\
\mathfrak{c}^\zeta(\xi) &= 2\pi\int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\,\kappa^\zeta(\eta^{\cdot 2}-2\xi.\eta)\,\mathrm{d}\eta\,,\\
\mathfrak{c}^\zeta_{P,t}(x,\xi) &= 2\pi\int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\,e^{i\sigma(P,(-2t\eta,-\eta))}\kappa^\zeta(\eta^{\cdot 2}-2\xi.\eta)\,\mathrm{d}\eta\,.
\end{aligned}$$

Associate with these functions the operators $Q_\pm$, $Q^\zeta_{\pm,t}$ defined by

$$\begin{aligned}
Q_-(b) &= \mathfrak{c}\,b\,,\\
Q^\zeta_-(b) &= \mathfrak{c}^\zeta\,b\,,\\
Q^\zeta_{+,t}b(x,\xi) &= \int_{\mathbb{R}^{2d}_P} \mathcal{F}^\sigma b(P)\,e^{i\sigma(P,X)}\,\mathfrak{c}^\zeta_{P,t}(x,\xi)đP\,,
\end{aligned}$$

for $b \in \mathcal{C}^\infty_0(\mathbb{R}^d_x \times \mathbb{R}^{d*}_\xi)$.

Proposition 6.27. *For $d \geq 3$, and $h^\alpha \leq \frac{ht}{\varepsilon} \leq 1$,*

$$\begin{aligned}
m_\pm &= \int_P \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \Gamma(e^{ip_x\cdot\varepsilon\eta})\,\mathscr{A}_{\pm,P}\right]đP\\
&= m(Q_{\pm,t}(b),t) + \sum_{k=1}^4 \Delta_{\pm,k}
\end{aligned}$$

with

- $|\Delta_{\pm,1}| \leq \frac{ht}{\varepsilon} C_d \max\{\|\hat{G}\|_{L^1}, \|G\|_{L^1}\}\,\|\mathcal{F}^\sigma b\|_{L^1_P}$,
- $|\Delta_{\pm,2}| \leq C_{\alpha,\beta,\nu,G,d} h^\nu$,
- $|\Delta_{\pm,3}| \leq \zeta^\gamma \mathcal{N}_{k(d)}(b)\,C_{d,G,C,\gamma}$ *for* $\gamma \in]0,1[$,
- $|\Delta_{\pm,4}| \leq \frac{ht}{\varepsilon}\left\|\mathcal{F}^\sigma\left(Q_{\pm,\frac{ht}{\varepsilon}}(b)\right)\right\|_{L^1_P}$

for some ν, $\beta > 0$ with $\zeta = h^\beta$.

This result will be proved in the next paragraphs by considering successively all the error terms. These error terms $\Delta_{\pm,j}$, $j = 1,\dots,4$ are given by the following approximation process (where we write shortly B^W for $B^W(-hD_\xi,\xi)$)

$$\begin{aligned}
m_\pm &= \int \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\mathscr{A}_{\pm,P}\right]đP\\
&= \int \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\,\mathscr{A}_{\pm,P}\right]đP + \Delta_{\pm,1}\\
&= \int \mathcal{F}^\sigma b(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\left(\mathfrak{c}^\zeta_{\pm,P}e^{i\sigma(P,\cdot)}\right)^W\right]đP + \sum_{j=1}^2 \Delta_{\pm,j}\\
&= \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t\left(Q^\zeta_{\pm,\frac{ht}{\varepsilon}}b\right)^W\right] + \sum_{j=1}^2 \Delta_{\pm,j}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(Q_{\pm,\frac{ht}{\varepsilon}}b\right)^W\right] + \sum_{j=1}^{3}\Delta_{\pm,j}\\
&= \int \mathcal{F}^\sigma\big(Q_{\pm,\frac{ht}{\varepsilon}}b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\hat{\tau}_P^h\right]\text{đ}P + \sum_{j=1}^{3}\Delta_{\pm,j}\\
&= \int \mathcal{F}^\sigma\big(Q_{\pm,\frac{ht}{\varepsilon}}b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat{\tau}_P^h\right]\text{đ}P + \sum_{j=1}^{4}\Delta_{\pm,j}\\
&= m\big(Q_{\pm,\frac{ht}{\varepsilon}}b,t\big) + \sum_{j=1}^{4}\Delta_{\pm,j}\,.
\end{aligned}
$$

The error terms $\Delta_{\pm,j}$ are thus given by

$$\Delta_{\pm,1} = \int \mathcal{F}^\sigma b\,(P)\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\Gamma(e^{ip_x\cdot\varepsilon\eta})-\mathrm{Id}\right)\mathscr{A}_{\pm,P}\right]\text{đ}P\,, \tag{6.10}$$

$$\Delta_{\pm,2} = \int \mathcal{F}^\sigma b\,(P)\,\mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\mathscr{A}_{\pm,P} - \left(\mathfrak{c}^\zeta_{\pm,P}e^{i\sigma(P,\cdot)}\right)^W\right)\right]\text{đ}P\,, \tag{6.11}$$

$$\Delta_{\pm,3} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(Q^\zeta_{\pm,\frac{ht}{\varepsilon}}b - Q_{\pm,\frac{ht}{\varepsilon}}b\right)^W\right]\,, \tag{6.12}$$

$$\Delta_{\pm,4} = \int \mathcal{F}^\sigma\big(Q_{\pm,\frac{ht}{\varepsilon}}b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\left(\mathrm{Id}-\Gamma\left(e^{ip_x\cdot\varepsilon\eta}\right)\right)\hat{\tau}_P^h\right]\text{đ}P\,, \tag{6.13}$$

since $\hat{\tau}_P^h = \left(e^{i\sigma(P,\cdot)}\right)^W$,

$$\left(Q^\zeta_{\pm,\frac{ht}{\varepsilon}}b\right)^W = \int_{\mathbb{R}^{2d}_P}\mathcal{F}^\sigma b(P)\left(\mathfrak{c}^\zeta_{\pm,P}e^{i\sigma(P,\cdot)}\right)^W\text{đ}P\,,$$

and the same relation holds without ζ and

$$\int_P \mathcal{F}^\sigma\big(Q_{\pm,\frac{ht}{\varepsilon}}b\big)(P)\ \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app}\,\Gamma(e^{ip_x\cdot\varepsilon\eta})\,\hat{\tau}_P^h\right]\text{đ}P = m\big(Q_{\pm,\frac{ht}{\varepsilon}}b,t\big)\,.$$

The term $\Delta_{\pm,4}$ can be estimated right away using Proposition 6.13.

6.5.1. *Computation of the operators $\mathscr{A}_{\pm,P}$.* We recall that the operators $\mathscr{A}_{\pm,P}$ and $\mathscr{A}_{-,j,P}\mathscr{A}_- = \mathscr{A}_{-,1} + \mathscr{A}_{-,2}$ are defined in Equations (6.5), (6.6), (6.7), (6.8), (6.9) by their kernels

$$
\begin{aligned}
\mathscr{A}_{-,P}(\xi_1,\xi_2) &= \mathscr{A}_{-,1,P}(\xi_1) + \mathscr{A}_{-,2,P}(\xi_2)\,,\\
ih\,\mathscr{A}_{-,j,P}(\xi_j) &= i\partial_t\left(\tfrac{1}{2}\big|z^{\xi_j}_{h,\varepsilon,t}\big|^2\right)\hat{\tau}_P^h(\xi_2,\xi_1)\,,\ j=1,2\,,\\
ih\,\mathscr{A}_{+,P}(\xi_1,\xi_2) &= i\partial_t\left[\varphi,p_x\right]_2\,\hat{\tau}_P^h(\xi_2,\xi_1)\,.
\end{aligned}
$$

Thus we need to compute $\partial_t(\frac{1}{2}|z^{\xi_j}_{h,\varepsilon,t}|^2)$ and $\partial_t\left[\varphi,p_x\right]_2$.

Lemma 6.28. *The time derivative of $\frac{1}{2}|z_{h,\varepsilon,t}|^2$ is given by*

$$\partial_t\left(\frac{1}{2}\left|z_{h,\varepsilon,t}\right|^2\right) = h\Re\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{is\left(\eta^{\cdot 2}-2\xi_j\cdot\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,.$$

Proof. For $z_t = -i\int_0^t e^{-i\frac{s}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_\cdot.\varepsilon\eta\right)} f_{h,\varepsilon}\,\mathrm{d}s$ with $f_{h,\varepsilon}(\eta) = \varepsilon^{d/2}\sqrt{\frac{h}{\varepsilon}}\hat{V}(-\varepsilon\eta)$,

$$\partial_t\left(\frac{1}{2}|z_t|^2\right) = \Re\int_{\mathbb{R}^d_\eta}\bar{z}_t\partial_t z_t$$

and $\partial_t z_t = -ie^{-i\frac{t}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_\cdot.\varepsilon\eta\right)} f_{h,\varepsilon}$. A simple computation gives

$$\begin{aligned}\partial_t\left(\frac{1}{2}|z_t|^2\right) &= \Re\int_{\mathbb{R}^d_\eta}\int_0^t e^{i\frac{t-s}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_\cdot.\varepsilon\eta\right)}|f_{h,\varepsilon}(\eta)|^2\,\mathrm{d}s\,\mathrm{d}\eta\\ &= h\Re\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{is\left(\eta_\cdot^{\cdot 2}-2\xi_\cdot.\eta\right)}\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\end{aligned}$$

which is the expected result. □

Lemma 6.29. *The time derivative of* $[\varphi, p_x]_2$ *is given by*

$$\begin{aligned}\partial_t[\varphi,p_x]_2 = \quad & h\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{ip_x\cdot\eta}e^{is\left(\eta_\cdot^{\cdot 2}-2\xi_1.\eta\right)}e^{-i\frac{t}{\varepsilon}\left(\eta_\cdot^{\cdot 2}-2\xi_2.\eta\right)}\,\mathrm{d}s\,\hat{G}(\eta)\,\mathrm{d}\eta\\ & + h\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{ip_x\cdot\eta}e^{i\frac{t}{\varepsilon}\left(\eta_\cdot^{\cdot 2}-2\xi_1.\eta\right)}e^{-is\left(\eta_\cdot^{\cdot 2}-2\xi_2.\eta\right)}\,\mathrm{d}s\,\hat{G}(\eta)\,\mathrm{d}\eta\,.\end{aligned}$$

Proof. Two analogous terms appear in this computation

$$\begin{aligned}\partial_t[\varphi,p_x]_2 &= \partial_t\left\langle z_t^{\xi_2}, e^{ip_x\cdot\varepsilon\eta}z_t^{\xi_1}\right\rangle_{L^2_\eta}\\ &= \left\langle z_t^{\xi_2}, e^{ip_x\cdot\varepsilon\eta}\partial_t z_t^{\xi_1}\right\rangle_{L^2_\eta} + \left\langle \partial_t z_t^{\xi_2}, e^{ip_x\cdot\varepsilon\eta}z_t^{\xi_1}\right\rangle_{L^2_\eta}.\end{aligned}$$

Consider the first one:

$$\begin{aligned}&\left\langle \partial_t z_t^{\xi_2}, e^{ip_x\cdot\varepsilon\eta}z_t^{\xi_1}\right\rangle_{L^2_\eta}\\ &\quad= \left\langle e^{i\frac{t}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_2.\varepsilon\eta\right)}f_{h,\varepsilon}, e^{ip_x\cdot\varepsilon\eta}\int_0^t e^{i\frac{s}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_1.\varepsilon\eta\right)}f_{h,\varepsilon}\,\mathrm{d}s\right\rangle\\ &\quad= \int_{\mathbb{R}^d_\eta}\int_0^t e^{ip_x\cdot\varepsilon\eta}e^{i\frac{s}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_1.\varepsilon\eta\right)}e^{-i\frac{t}{\varepsilon}\left(\varepsilon^2\eta_\cdot^{\cdot 2}-2\xi_2.\varepsilon\eta\right)}\,\mathrm{d}s\,|f_{h,\varepsilon}(\eta)|^2\,\mathrm{d}\eta\\ &\quad= h\int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon} e^{ip_x\cdot\eta}e^{is\left(\eta_\cdot^{\cdot 2}-2\xi_1.\eta\right)}e^{-i\frac{t}{\varepsilon}\left(\eta_\cdot^{\cdot 2}-2\xi_2.\eta\right)}\,\mathrm{d}s\hat{G}(\eta)\,\mathrm{d}\eta\,.\end{aligned}$$

With analogous computations we get the result for the second one. □

Proposition 6.30. *The operators* $\mathscr{A}_{-,j}$ *can be expressed as*

$$\begin{aligned}\mathscr{A}_{-,1,P} &= \int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon}\hat{\tau}_P^h\circ\Re\left(e^{is\left(\eta_\cdot^{\cdot 2}-2\xi.\eta\right)}\right)\times\,\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,,\\ \mathscr{A}_{-,2,P} &= \int_{\mathbb{R}^d_\eta}\int_0^{t/\varepsilon}\Re\left(e^{is\left(\eta_\cdot^{\cdot 2}-2\xi.\eta\right)}\right)\times\,\circ\hat{\tau}_P^h\,\hat{G}(\eta)\,\mathrm{d}s\,\mathrm{d}\eta\,.\end{aligned}$$

Proof. From the definition of $\mathscr{A}_{-,j,P}$ in terms of their kernel, we get

$$ih\,\mathscr{A}_{-,1,P} = i\hat{\tau}_P^h \circ \left[\partial_t\left(\frac{1}{2}\left|z_t^\xi\right|^2\right)\times\right],$$
$$ih\,\mathscr{A}_{-,2,P} = i\left[\partial_t\left(\frac{1}{2}\left|z_t^\xi\right|^2\right)\times\right] \circ \hat{\tau}_P^h\,.$$

Lemma 6.28 yields the result. □

Consider now the term $\mathscr{A}_+$.

Proposition 6.31. *The operators $\mathscr{A}_{+,j}$ can be expressed as*

$$\mathscr{A}_{+,1,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,\left(2\frac{t}{\varepsilon}\eta,\eta\right)\right)}\hat{\tau}_P^h \circ e^{-is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$
$$\mathscr{A}_{+,2,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,\left(2\frac{t}{\varepsilon}\eta,\eta\right)\right)}e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)} \circ \hat{\tau}_P^h\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,.$$

Proof. Lemma 6.29 allows us to write

$$\mathscr{A}_{+,1,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{ip_x\cdot\eta}e^{-i\frac{t}{\varepsilon}\left(\eta^{\cdot 2}-2\xi.\eta\right)} \circ \hat{\tau}_P^h \circ e^{is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$
$$\mathscr{A}_{+,2,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{ip_x\cdot\eta}e^{-is\left(\eta^{\cdot 2}-2\xi.\eta\right)} \circ \hat{\tau}_P^h \circ e^{i\frac{t}{\varepsilon}\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,.$$

Since

$$e^{2i\frac{t}{\varepsilon}\xi.\eta} \circ \hat{\tau}_P^h = e^{-2i\frac{t}{\varepsilon}p_\xi\eta}\hat{\tau}_P^h \circ e^{2i\frac{t}{\varepsilon}\xi.\eta}\,,$$
$$\hat{\tau}_P^h \circ e^{-2i\frac{t}{\varepsilon}2\xi.\eta} = e^{-2i\frac{t}{\varepsilon}p_\xi\eta}e^{-2i\frac{t}{\varepsilon}2\xi.\eta} \circ \hat{\tau}_P^h\,,$$

we get

$$\mathscr{A}_{+,1,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,\left(2\frac{t}{\varepsilon}\eta,\eta\right)\right)}\hat{\tau}_P^h \circ e^{-i\left(\frac{t}{\varepsilon}-s\right)\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s\,,$$
$$\mathscr{A}_{+,2,P} = \int_0^{t/\varepsilon}\int_{\mathbb{R}^d_\eta} e^{-i\sigma\left(P,\left(2\frac{t}{\varepsilon}\eta,\eta\right)\right)}e^{i\left(\frac{t}{\varepsilon}-s\right)\left(\eta^{\cdot 2}-2\xi.\eta\right)} \circ \hat{\tau}_P^h\,\hat{G}(\eta)\,\mathrm{d}\eta\,\mathrm{d}s$$

and with a change of variable we obtain the expected result. □

Thus we get six different terms (four for the $\mathscr{A}_-$ terms due to the real parts and two for the $\mathscr{A}_+$ terms) with a very similar structure. In order to avoid repeating analogous calculations several times we introduce the following notations.

Notation: Set (by writing shortly B^W for $B^W(-hD_\xi,\xi)$)

$$\mathscr{A}^1_{\vec{\mu}}(s) = \int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}\,\hat{\tau}_P^h \circ e^{-\mu_2 is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\mathrm{d}\eta\,, \tag{6.14}$$

$$\mathscr{B}^1_{\vec{\mu}}(s) = \int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}\,\hat{\tau}^h_{(p_x-\mu_2 2s\eta,p_\xi)}\,e^{-\mu_2 is\eta^{\cdot 2}}\,\mathrm{d}\eta\,, \tag{6.15}$$

$$\mathscr{C}^{1,\zeta}_{\vec{\mu}} = \int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\left(e^{\mu_1 i\tilde{\sigma}}e^{i\sigma(P,\cdot)}\right)^W\frac{\mathrm{d}\eta}{\zeta+\mu_2 i\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,, \tag{6.16}$$

$$\mathscr{A}^2_{\vec{\mu}}(s) = \int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\, e^{\mu_1 i\tilde{\sigma}} e^{\mu_2 is\left(\eta^{\cdot 2} - 2\xi.\eta\right)} \circ \hat{\tau}^h_P \,\mathrm{d}\eta\,, \tag{6.17}$$

$$\mathscr{B}^2_{\vec{\mu}}(s) = \int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\, e^{\mu_1 i\tilde{\sigma}}\, \hat{\tau}^h_{(p_x+\mu_2 2s\eta, p_\xi)}\, e^{\mu_2 is\eta^{\cdot 2}} \,\mathrm{d}\eta\,, \tag{6.18}$$

$$\mathscr{C}^{2,\zeta}_{\vec{\mu}} = \int_{\mathbb{R}^d_\eta} \hat{G}(\eta) \left(e^{\mu_1 i\tilde{\sigma}} e^{i\sigma(P,\cdot)}\right)^W \frac{\mathrm{d}\eta}{\zeta - \mu_2 i\left(\eta^{\cdot 2} - 2\xi.\eta\right)}\,, \tag{6.19}$$

with $\tilde{\sigma} = \sigma\left(P, \left(-2h\frac{t}{\varepsilon}\eta, -\eta\right)\right)$. The terms μ_1, μ_2 are chosen to adapt to the cases of the terms $m_\pm$.

More precisely, for $j = 1, 2$, the previous quantities become

$$\mathscr{A}_{-,j} = \frac{1}{2}\int_0^{t/\varepsilon} \left(\mathscr{A}^j_{0,1}(s) + \mathscr{A}^j_{0,-1}(s)\right) \mathrm{d}s\,,$$
$$\mathscr{A}_{+,j} = \int_0^{t/\varepsilon} \mathscr{A}^j_{1,1}(s)\,\mathrm{d}s\,.$$

We will first show that the operators $\mathscr{C}^\zeta_{\vec{\mu}}$ are good approximations of the operators $\mathscr{A}_{\vec{\mu}} = \int_0^{t/\varepsilon} \mathscr{A}_{\vec{\mu}}(s)\,\mathrm{d}s$ if the parameter ζ is well chosen. We use the operators $\int_0^{t/\varepsilon} \mathscr{B}_{\vec{\mu}}(s)\,\mathrm{d}s$ as an intermediate step.

Then we study the limit of the operators $\mathscr{C}^\zeta_{\vec{\mu}}$, with a distinction between the cases m_- and m_+.

6.5.2. *Estimate of the error terms $\Delta_{\pm,1}$.*

Proposition 6.32. *For $d \geq 3$, the inequality*

$$|\Delta_{\pm,1}| \leq \frac{ht}{\varepsilon} C_d \max\left\{\|\hat{G}\|_{L^1}, \|G\|_{L^1}\right\} \|\mathcal{F}^\sigma b\|_{L^1_P}$$

holds.

Proof. The term $\Delta_{\pm,1}$ was defined in Equation (6.10). This inequality follows from Propositions 6.13 and 6.33 below since $s \mapsto \min\{1, s^{-d/2}\}$ is integrable on $\mathbb{R}^+$ for $d \geq 3$. □

Proposition 6.33. *The families of operators $\mathscr{A}(s) = \mathscr{A}^j_{\vec{\mu}}(s)$ satisfy*

$$\|\mathscr{A}(s)\|_{\mathcal{L}(L^2_\xi)} \leq C_d \max\left\{\|\hat{G}\|_{L^1}, \|G\|_{L^1}\right\} \min\left\{1, s^{-d/2}\right\}.$$

Proof. By a uniform estimate of Equations (6.14), (6.17) we get

$$\left\|\mathscr{A}^j_{\vec{\mu}}(s)\right\|_{\mathcal{L}(L^2_\xi)} \leq C_d \|\hat{G}\|_{L^1}\,.$$

In order to obtain the part of the estimate with the dependence in s, we use the formula

$$\left\|\mathscr{A}^j_{\vec{\mu}}(s)\right\|_{\mathcal{L}(L^2_\xi)} = \sup_{\|\psi\|_{L^2_\xi} = \|\varphi\|_{L^2_\xi} = 1} \left|\left\langle \psi, \mathscr{A}^j_{\vec{\mu}}(s)\varphi \right\rangle\right|\,.$$

We can then compute, for ψ, $\varphi \in L^2_\xi$,

$$\begin{aligned}
&\left\langle \psi, \mathscr{A}^j_{\vec{\mu}}(s)\varphi \right\rangle \\
&\quad = \int_{\mathbb{R}^d_\eta} \left\langle \psi, \hat{G}(\eta)\, e^{i\mu_1\tilde{\sigma}}\, \hat{\tau}^h_P \circ e^{-\mu_2 is\left(\eta^{\cdot 2} - 2\xi.\eta\right)} \varphi \right\rangle_\xi \mathrm{d}\eta \\
&\quad = \int_{\mathbb{R}^d_\xi} \left\langle \hat{G}(\eta)\, \hat{\tau}^h_{-P}\psi(\xi), e^{\mu_1 i\tilde{\sigma}} e^{-\mu_2 is\left(\eta^{\cdot 2} - 2\xi.\eta\right)} \varphi(\xi) \right\rangle_\eta \mathrm{d}\xi \\
&\quad = \int_{\mathbb{R}^d_\theta} \left\langle \psi_\theta, \varphi_{\vec{\mu},\theta} \right\rangle_\xi \frac{1}{(2\pi)^d}\, \mathrm{d}\theta\,,
\end{aligned}$$

where we defined, for $\theta \in \mathbb{R}^d_\theta$,

$$\begin{aligned}
\varphi_{\vec{\mu},\theta} &= \int e^{i\theta\eta} e^{\mu_1 i\tilde{\sigma}} e^{-\mu_2 is\left(\eta^{\cdot 2} - 2\xi.\eta\right)} \varphi(\xi)\, \mathrm{d}\eta\,, \\
\psi_\theta &= \int e^{i\theta\eta}\, \hat{G}(\eta)\, \hat{\tau}^h_{-P}\, \psi(\xi)\, \mathrm{d}\eta\,.
\end{aligned}$$

We first compute

$$\varphi_{\vec{\mu},\theta}\left(\xi\right) = \left(\frac{\pi}{s}\right)^{d/2} e^{i\frac{\left(\theta + \mu_2 2s\xi + \mu_1\left(2hsp_\xi - p_x\right)\right)^2}{4\mu_2 s}} e^{i\frac{\pi}{4}d} \varphi\left(\xi\right)$$

where we used the formula

$$\int e^{-ix\eta} e^{-a\eta^2}\, \mathrm{d}\eta = \left(\frac{\pi}{a}\right)^{d/2} e^{-x^2/4a}$$

with $a = \mu_2 is$ and $x = -\left(\theta + \mu_2 2s\xi + \mu_1\left(2hsp_\xi - p_x\right)\right)$ and so for a fixed θ

$$\left\|\varphi_{\vec{\mu},\theta}\right\|_{L^2_\xi} \leq \left(\frac{\pi}{s}\right)^{d/2} \left\|\varphi\right\|_{L^2_\xi}\,.$$

We now observe that in $L^1(\mathbb{R}^d_\theta; \mathcal{L}(L^2_\xi))$

$$\left\| \int e^{i\theta\eta}\, \hat{G}(\eta)\, \hat{\tau}^h_{-P}\, \mathrm{d}\eta \right\|_{L^1(\mathbb{R}^d_\theta; \mathcal{L}(L^2_\xi))} \leq (2\pi)^d \left\|G\right\|_{L^1}$$

so that $\left\|\psi_\theta\right\|_{L^1(\mathbb{R}^d_\theta; L^2_\xi)} \leq C_d \left\|G\right\|_{L^1} \left\|\psi\right\|_{L^2_\xi}$. And finally

$$\begin{aligned}
\left|\left\langle \psi, \mathscr{A}_{\vec{\mu}}\left(s\right)\varphi \right\rangle\right| &\leq \frac{1}{(2\pi)^d} \left\|\psi_\theta\right\|_{L^1(\mathbb{R}^d_\theta; L^2_\xi)} \left\|\varphi_{\vec{\mu},\theta}\right\|_{L^\infty(\mathbb{R}^d_\theta; L^2_\xi)} \\
&\leq C_d \left\|G\right\|_{L^1} \left(\frac{\pi}{s}\right)^{d/2} \left\|\varphi\right\|_{L^2_\xi} \left\|\psi\right\|_{L^2_\xi}
\end{aligned}$$

and we obtain the desired result $\left\|\mathscr{A}_{\vec{\mu}}(s)\right\|_{\mathcal{L}(L^2_\xi)} \leq C_d \left\|G\right\|_{L^1} s^{-d/2}$. □

6.5.3. *Estimate of the error terms* $\Delta_{\pm,2}$.

Proposition 6.34. *Let* $\alpha \in]0,1]$. *There are constants* $\beta = \beta(d,\alpha) \in]0,1[$, $\nu = \nu(d,\alpha) \in]0,1[$ *and* $C = C(\alpha,\beta,\nu,d,G) > 0$ *such that, for* $h^\alpha \leq \frac{th}{\varepsilon} \leq 1$, *and* $\zeta = h^\beta$,

$$\left|\Delta_{\pm,2}\right| \leq \left\| \left\langle\cdot\right\rangle^k \mathcal{F}^\sigma b \right\|_{L^1} C h^\nu\,.$$

In order to prove this result we use Proposition 6.17 and thus control

$$\left\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s - \mathscr{C}^\zeta \right\|_{\mathcal{L}(L^2_\xi)}.$$

We first give an abstract result and then show that our cases fit within this framework.

Proposition 6.35. *For M, t, ε such that $1 \le M \le \frac{t}{\varepsilon}$. Suppose given $(\mathscr{A}(s))_{s\ge 0}$, $(\mathscr{B}(s))_{s\ge 0}$ and $(\mathscr{C}^\zeta)_{0<\zeta<1}$ three families of operators in $\mathcal{L}(L^2_\xi)$ (also dependent on h and $P = (p_x, p_\xi)$) such that for some constants $C_{\mathscr{A}}$, $C_{\mathscr{A},\mathscr{B}}$, $C_{\mathscr{B},\mathscr{C}}$, independent of $h, \varepsilon, t, P, M, \zeta$,*

1. $\|\mathscr{A}(s)\|_{\mathcal{L}(L^2_\xi)} \le C_{\mathscr{A}} \min\left\{1, s^{-d/2}\right\}$,
2. $\|\mathscr{A}(s) - \mathscr{B}(s)\|_{\mathcal{L}(L^2_\xi)} \le C_{\mathscr{A},\mathscr{B}} hs\,|p_\xi|$,
3. $r_{\zeta,M}(x,\xi) := \mathrm{Symb}^{Weyl}(\int_0^M \mathscr{B}(s)\,e^{-\zeta s}\,\mathrm{d}s - \mathscr{C}^\zeta)$ *satisfies for some* $k = k(d) \in \mathbb{N}$,

$$\sup_{|\alpha|\le k} \left\|\partial^\alpha_{x,\xi} r_{\zeta,M}\right\|_{L^\infty_{x,\xi}} \le C_{\mathscr{B},\mathscr{C}} \langle P\rangle^k \left(\frac{M}{\zeta}\right)^k e^{-\zeta M}.$$

Then, for $\zeta M \ge 1$,

1. $\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{d}{d-2} C_{\mathscr{A}}$,
2. $\| \int_0^{t/\varepsilon} \mathscr{A}(s)\,\mathrm{d}s - \int_0^M \mathscr{A}(s)\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{2}{d-2} C_{\mathscr{A}} M^{1-\frac{d}{2}}$,
3. *depending on d,*

$$\begin{aligned}
\| \int_0^M \mathscr{A}(s)\left(1 - e^{-\zeta s}\right)\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} &\le \frac{9}{2} C_{\mathscr{A}} \zeta^{1/2} \qquad \text{if } d = 3\\
&\le \frac{3}{2} C_{\mathscr{A}} \zeta\,|\log\zeta| \quad \text{if } d = 4\\
&\le C_{\mathscr{A}} \zeta \frac{d+2}{2d-4} \quad \text{if } d \ge 5\\
&\left(\le 5 C_{\mathscr{A}} \zeta^{1/2} \qquad \text{if } d \ge 3\right),
\end{aligned}$$

4. $\| \int_0^M (\mathscr{A}(s) - \mathscr{B}(s))\,e^{-\zeta s}\,\mathrm{d}s\|_{\mathcal{L}(L^2_\xi)} \le \frac{1}{2} C_{\mathscr{A},\mathscr{B}} h \zeta^{-2} |p_\xi|$,
5. *for some integer $k = k(d)$,*

$$\left\| \int_0^M \mathscr{B}(s)\,e^{-\zeta s}\,\mathrm{d}s - \mathscr{C}^\zeta \right\|_{\mathcal{L}(L^2_\xi)} \le C_{d,k'} C_{\mathscr{B},\mathscr{C}} \langle P\rangle^k \left(\frac{M}{\zeta}\right)^k e^{-\zeta M}.$$

6. *Let $\frac{ht}{\varepsilon} \ge h^\alpha$, $\zeta = h^\beta$ with $\beta \in \left]0, \frac{1}{2}\right[$ and $\beta + \alpha < 1$, and $\nu = \nu(d,\alpha,\beta) < \min\{(1-\alpha)\left(\frac{d}{2} - 1\right), \tilde\beta(d), 1-2\beta\}$ with $\tilde\beta(3,\beta) = \beta/2$, $\tilde\beta(4,\beta) = \beta$ and $\tilde\beta(d \ge 5, \beta) = \beta$ we have*

$$\left\| \int_0^{\frac{t}{\varepsilon}} \mathscr{A}(s)\,\mathrm{d}s - \mathscr{C}^\zeta \right\|_{\mathcal{L}(L^2_\xi)} \le C h^\nu$$

with $C = C(\nu, \alpha, \beta, C_{\mathscr{A}}, C_{\mathscr{A}\mathscr{B}}, C_{\mathscr{B}\mathscr{C}})$.

Proofs of 1 and 2. By integration of the first assumed estimate and using $1 \le M \le \frac{t}{\varepsilon}$ for *2.* □

Proof of 3. By integration of the first assumed estimate, using $1-e^{-\zeta s} \leq \zeta s$ for $\zeta s \leq 1$ and $1-e^{-\zeta s} \leq 1$ for $\zeta s \geq 1$,

$$\begin{aligned}\int_0^M \left(1-e^{-\zeta s}\right) \min\left\{1, s^{-d/2}\right\} \mathrm{d}s & \\ \leq \zeta \int_0^1 s \,\mathrm{d}s + \zeta \int_1^{1/\zeta} s^{1-\frac{d}{2}} \,\mathrm{d}s + \int_{1/\zeta}^{+\infty} s^{-d/2} \,\mathrm{d}s\,, & \qquad \text{if } d=3,4, \\ \leq \zeta \int_0^1 s \,\mathrm{d}s + \zeta \int_1^{+\infty} s^{1-\frac{d}{2}} \,\mathrm{d}s\,, & \qquad \text{if } d \geq 5\,,\end{aligned}$$

which brings the result. □

Proof of 4. We control $\|\mathscr{A}(s)-\mathscr{B}(s)\|_{\mathcal{L}(L^2_\xi)}$ using the second assumption and use

$$\int_0^M s e^{-\zeta s} \,\mathrm{d}s \leq \zeta^{-2} \int_0^{+\infty} u e^{-u} \,\mathrm{d}u\,.$$

□

Proof of 5. The known estimates for pseudo-differential operators give

$$\begin{aligned}\left\|r^W(-hD_\xi,\xi)\right\| &\leq C_{1,k} \sum_{|\alpha|\leq N_k} \left\|\partial^\alpha_{x,\xi} r\right\|_{L^\infty(\mathbb{R}^{2d})} \\ &\leq C_{2,k} \sup_{|\alpha|\leq N_k} \left\|\partial^\alpha_{x,\xi} r\right\|_{L^\infty(\mathbb{R}^{2d})}\,.\end{aligned}$$

This and the third hypothesis imply the result. □

Proof of 6. We would like to choose the (h-dependent) parameters M and ζ such that the quantity

$$M^{1-\frac{d}{2}} + \tilde{\zeta}\,(d,\zeta) + h\zeta^{-2} + \left(\frac{M}{\zeta}\right)^k e^{-\zeta M}\,,$$

with ($\tilde{\zeta}\,(3,\zeta)=\sqrt{\zeta}$, $\tilde{\zeta}\,(4,\zeta)=\zeta\,|\log\zeta|$, $\tilde{\zeta}\,(d\geq 5,\zeta)=\zeta$), is small when h tends to 0 and M not too big. We choose $hM=h^\alpha$ and $\zeta=h^\beta$ with $\beta+\alpha<1$, $\alpha,\beta>0$ so that the previous quantity is smaller than

$$h^{(1-\alpha)\left(\frac{d}{2}-1\right)} + h^{\tilde{\beta}(d,\beta)} + h^{1-2\beta} + h^{-k(1-\alpha+\beta)} \exp\left(-\left(h^{\beta+\alpha-1}\right)\right)$$

(with $\tilde{\beta}\,(3,\beta)=\beta/2$, $\tilde{\beta}\,(4,\beta)=\beta^-$ and $\tilde{\beta}\,(d\geq 5,\beta)=\beta$). In order to get a small quantity it suffices to require $\beta<\frac{1}{2}$. Then we get an error term whose size is controlled by $h^{\nu(d,\alpha,\beta)}$. □

Proposition 6.36. *The families of operators $\mathscr{A}(s)=\mathscr{A}^j_{\vec{\mu}}(s)$, $\mathscr{B}(s)=\mathscr{B}^j_{\vec{\mu}}(s)$ and $\mathscr{C}^\zeta=\mathscr{C}^{j,\zeta}_{\vec{\mu}}$ satisfy the hypotheses of Proposition 6.35 with*

$$C_{\mathscr{A}} = C_d \max\left\{\|\hat{G}\|_{L^1}, \|G\|_{L^1}\right\},\ C_{\mathscr{A},\mathscr{B}} = \left\||\cdot|\,\hat{G}\right\|_{L^1},\ C_{\mathscr{B},\mathscr{C}} = \left\|\langle\cdot\rangle^k \hat{G}\right\|_{L^1},$$

for some integer k.

Proof of 1. See Proposition 6.33. □

Proof of 2. We show the result for $\mathscr{A}^1_{\vec{\mu}}$ and $\mathscr{B}^1_{\vec{\mu}}$, the proof can be adapted to the case of $\mathscr{A}^2_{\vec{\mu}}$ and $\mathscr{B}^2_{\vec{\mu}}$. We observe that

$$\hat{\tau}^h_P \circ \left(e^{\mu_2 is 2\xi.\eta}\times\right) = e^{-\mu_2 is\eta h p_\xi}\hat{\tau}^h_{P-(\mu_2 2s\eta,0)}$$

and

$$\left(e^{i\sigma(P,X)}e^{\mu_2 is 2\xi.\eta}\right)^W(-hD_\xi,\xi) = \hat{\tau}^h_{(p_x-\mu_2 2s\eta,p_\xi)}\,.$$

Thus we obtain the estimation

$$\left\|\hat{\tau}^h_P \circ \left(e^{\mu_2 is 2\xi.\eta}\times\right) - \left(e^{i\sigma(P,X)}e^{\mu_2 is 2\xi.\eta}\right)^W(-hD_\xi,\xi)\right\|_{\mathcal{L}(L^2_\xi)} \leq hs\,|\eta|\,|p_\xi|$$

Since the Weyl symbol of $\mathscr{B}^1_{\vec{\mu}}(s)$ is

$$\frac{1}{2}\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{i\mu_1\tilde{\sigma}}e^{i\sigma(P,X)}e^{-\mu_2 is\left(\eta^{\cdot 2}-2\xi.\eta\right)}\,\mathrm{d}\eta$$

we finally get

$$\left\|\mathscr{A}^1_{\vec{\mu}}(s)-\mathscr{B}^1_{\vec{\mu}}(s)\right\|_{\mathcal{L}(L^2_\xi)} \leq hs\,|p_\xi|\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,|\eta|\,\mathrm{d}\eta$$

and this concludes the proof. □

Proof of 3. The Weyl symbol of $\int_0^M \mathscr{B}^1_{\vec{\mu}}(s)\,e^{-\zeta s}\,\mathrm{d}s$ is

$$\begin{aligned}
&\overset{Weyl}{\mathrm{Symb}}\int_0^M \mathscr{B}^1_{\vec{\mu}}(s)\,e^{-\zeta s}\,\mathrm{d}s\\
&\quad=\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}e^{i\sigma(P,X)}\left[\frac{e^{-\mu_2 is\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta s}}{-\mu_2 i\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta}\right]_0^M\mathrm{d}\eta\\
&\quad=\overset{Weyl}{\mathrm{Symb}}\,\mathscr{C}^{1,\zeta}_{\vec{\mu}}+r_{\zeta,M}
\end{aligned}$$

with

$$r_{\zeta,M}(x,\xi)=-\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,e^{\mu_1 i\tilde{\sigma}}e^{i\sigma(P,X)}\frac{e^{-\mu_2 iM\left(\eta^{\cdot 2}-2\xi.\eta\right)-\zeta M}}{\mu_2 i\left(\eta^{\cdot 2}-2\xi.\eta\right)+\zeta}\,\mathrm{d}\eta\,.$$

The remainder term $r_{\zeta,M}$ is in the symbol class $\mathcal{S}(1)$, and for $k=k(d)$, the operator norm $\left\|r^W_{\zeta,M}(-hD_\xi,\xi)\right\|_{\mathcal{L}(L^2_\xi)}$ can be controlled by

$$\sup_{|\alpha|\leq k}\left\|\partial^\alpha_{x,\xi}r_{\zeta,M}\right\|_{L^\infty_{x,\xi}}\,.$$

Thus we consider

$$\begin{aligned}
\left|\partial^\alpha_{x,\xi}r_{\zeta,M}(x,\xi)\right| &\leq \int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,\langle P\rangle^k\left(M\langle\eta\rangle\right)^k\frac{1}{\zeta^{k+1}}e^{-\zeta M}\,\mathrm{d}\eta\\
&\leq \langle P\rangle^k\left(\frac{M}{\zeta}\right)^{k+1}e^{-\zeta M}\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,\langle\eta\rangle^k\,\mathrm{d}\eta\,.
\end{aligned}$$

This yields the result

$$\left\|\int_0^M \mathscr{B}^1_{\vec{\mu}}(s)\,e^{-\zeta s}ds-\mathscr{C}^{1,\zeta}_{\vec{\mu}}\right\|_{\mathcal{L}(L^2_\xi)}\leq\langle P\rangle^k\left(\frac{M}{\zeta}\right)^k e^{-\zeta M}\int_{\mathbb{R}^d_\eta}\hat{G}(\eta)\,\langle\eta\rangle^k\,\mathrm{d}\eta\,,$$

for some $k = k(d)$. The same proof holds for $\mathscr{B}^2_{\vec{\mu}}(s)$ and $\mathscr{C}^{2,\zeta}_{\vec{\mu}}$. □

6.5.4. *Estimate of the error term* $\Delta_{-,3}$.

Proposition 6.37. *Let* $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ *with* $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$ *for some* $R > 1$. *Let* $\gamma \in]0,1[$. *There exists a constant* $C_{G,b,\gamma} > 0$ *such that, for all* $\zeta > 0$,

$$|\Delta_{-,3}| \leq \zeta^\gamma \mathcal{N}_k(b) C_{G,b,\gamma}$$

for some integer $k = k\,(d)$ *big enough.*

Proof. We recall that

$$\Delta_{-,3} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}^{app}_t \left(Q^\zeta_- b - Qb\right)^W (-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\right]$$

so that

$$\begin{aligned}|\Delta_{-,3}| &\leq \left\| \left(Q^\zeta_- b - Q_- b\right)^W (-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)) \right\|_{\mathcal{L}(L^2_\xi \otimes \Gamma L^2_\eta)} \\ &\leq C_{k,d} \mathcal{N}_k \left(Q^\zeta_- b - Q_- b\right)\end{aligned}$$

for some integer k big enough. By recalling $Q^\zeta_-(b) = \mathfrak{c}^\zeta\, b$ and $Q_-(b) = \mathfrak{c}\, b$ it is then sufficient to prove that

$$\sup_{|\alpha| \leq k} \sup_{\xi \in [R^{-1}, R]} \left|\partial^\alpha_\xi \left(\mathfrak{c}^\zeta - \mathfrak{c}\right)(\xi)\right| \leq C_{k,\gamma,G,R} \zeta^\gamma\,.$$

This is a consequence of the Lemma 6.38 below. □

Lemma 6.38. *For any integer* k *and* γ *in* $[0,1[$, *there exists a positive constant* $C_{k,\gamma,G,C}$ *such that for* $\zeta \in]0, \zeta_0[$

$$\sup_{|\alpha| \leq k} \sup_{|\xi| \in [R^{-1}, R]} \left|\partial^\alpha_\xi \left(\mathfrak{c}^\zeta - \mathfrak{c}\right)(\xi)\right| \leq C_{k,\gamma,G,R} \zeta^\gamma\,.$$

Proof. With κ^ζ, $\mathfrak{c}$, $\mathfrak{c}^\zeta$ introduced in Definition 6.26, $\mathfrak{c}^\zeta - \mathfrak{c}$ can be expressed as

$$\left(\mathfrak{c}^\zeta - \mathfrak{c}\right)(\xi) = \int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\, \kappa^\zeta(\eta^{\cdot 2} - 2\xi.\eta)\, \mathrm{d}\eta - \int_{\mathbb{R}^d_\eta} \hat{G}(\xi + \eta)\, \delta\big(\,|\eta|^2 - |\xi|^2\,\big)\, \mathrm{d}\eta\,.$$

We express the first integral as

$$\begin{aligned}\int_{\mathbb{R}^d_\eta} \hat{G}(\eta)\, \kappa^\zeta\left((\eta - \xi)^{\cdot 2} - \xi^{\cdot 2}\right) \mathrm{d}\eta &= \int_{S^{d-1}} \int_{\mathbb{R}_\rho} f_{\xi,\omega}\,(r)\, \kappa^\zeta(\xi^{\cdot 2} - r)\, \mathrm{d}r\, \mathrm{d}\omega \\ &= \int_{S^{d-1}} f_{\xi,\omega} * \kappa^\zeta(\xi^{\cdot 2})\, \mathrm{d}\omega\end{aligned}$$

and $f_{\xi,\omega}(r) := \frac{1}{2} r^{\frac{d-2}{2}} g(\xi + \sqrt{r}\omega)\, 1_{[0,+\infty[}(r)$. The partial derivative

$$\partial_{\xi_j} f_{\xi,\omega}(r) = \frac{1}{2} r^{\frac{d-2}{2}}\, \partial_{\xi_j} g(\xi + \sqrt{r}\omega)\, 1_{[0,+\infty[}(r)$$

has the same form as the function f_ξ. Then we observe that

$$\begin{aligned}&\partial_{\xi_j}\left(f_{\xi,\omega} * \kappa^\zeta - f_{\xi,\omega}\right)\left(\,|\xi|^2\,\right) \\ &\qquad = \left[(\partial_{\xi_j} f_{\xi,\omega}) * \kappa^\zeta - \partial_{\xi_j} f_{\xi,\omega}\right]\left(\,|\xi|^2\,\right) + \left[\partial_r (f_{\xi,\omega} * \kappa^\zeta - f_{\xi,\omega})\right]\left(\,|\xi|^2\,\right) 2\xi_j\end{aligned}$$

so that by doing successive derivations it suffices to deal only with quantities of the form

$$\partial_r^k \left(\partial_\xi^\beta f_{\xi,\omega} * \kappa^\zeta - \partial_\xi^\beta f_{\xi,\omega} \right)$$

which are in fact of the form $\partial_r^k \left(f * \kappa^\zeta - f \right)$ with f satisfying the hypotheses of Proposition D.2 uniformly in ω so that we get the expected control, by integration over ω. □

6.5.5. *Estimate of the error term* $\Delta_{+,3}$.

Remark 6.39. Throughout this section we will make definitions that are dependent on the value of $\frac{th}{\varepsilon}$. This will not be a problem as long as $\frac{th}{\varepsilon} \leq 1$ which will be satisfied with our choice of $\varepsilon = \varepsilon(h) \gg h$.

Proposition 6.40. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ with $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$ for some $R > 1$. Let $\gamma \in]0,1[$. There exists a constant $C_{G,R,\gamma} > 0$ such that, for all $\zeta > 0$,*

$$|\Delta_{+,3}| \leq \zeta^\gamma \, \mathcal{N}_k(b) \, C_{G,R,\gamma}$$

for some integer $k = k(d)$ big enough.

Proof. We recall that

$$\Delta_{+,3} = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}_t^{app} \left(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b \right)^W (-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)) \right]$$

so that

$$\begin{aligned} |\Delta_{+,3}| &\leq \left\| \left(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b \right)^W (-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)) \right\|_{\mathcal{L}(L^2_\xi \otimes \Gamma L^2_\eta)} \\ &\leq C_{k,d} \, \mathcal{N}_k\big(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b \big) \end{aligned}$$

for some integer $k = k(d)$ big enough.

Thus we boil down to prove that for any integer $k \geq 0$ there is a constant $C_{k,b,G,\gamma} > 0$ such that for any $\zeta > 0$

$$\mathcal{N}_k\Big(Q^\zeta_{+,\frac{ht}{\varepsilon}} b - Q_{+,\frac{ht}{\varepsilon}} b \Big) \leq C_{k,G,\gamma} \, \mathcal{N}_k(b) \, \zeta^\gamma \, .$$

But we have a convenient expression for $Q^\zeta_{+,\frac{ht}{\varepsilon}}$

$$\begin{aligned} Q^\zeta_{+,\frac{ht}{\varepsilon}} b(x,\xi) &= 2\pi \int_{\mathbb{R}^d_\eta} \hat{G}(\eta) \, b\big(x - 2\frac{t}{\varepsilon} h\eta, \xi - \eta\big) \, \kappa^\zeta(\eta^{\cdot 2} - 2\xi.\eta) \, \mathrm{d}\eta \\ &= 2\pi \int_{\mathbb{R}^d_\eta} \hat{G}(\xi - \eta) \, b\big(x - 2\frac{t}{\varepsilon} h\xi + 2\frac{t}{\varepsilon} h\eta, \eta\big) \, \kappa^\zeta(\eta^{\cdot 2} - 2\xi.\eta) \, \mathrm{d}\eta \\ &= \pi \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^+_r} \varphi_\omega(x,\xi,r) \, K^\zeta(r - \xi^{\cdot 2}) \, \mathrm{d}r \, \mathrm{d}\omega \, , \end{aligned}$$

with $\varphi_\omega(x,\xi,r) = 0$ for $r \leq 0$, and for $r \geq 0$,

$$\varphi_\omega(x,\xi,r) = \hat{G}(\xi - \sqrt{r}\omega) \, b\big(x - 2\frac{ht}{\varepsilon}\xi + 2\frac{ht}{\varepsilon}\sqrt{r}\omega, \sqrt{r}\omega\big) \, r^{d/2-1} \tag{6.20}$$

defined for $\omega \in \mathbb{S}^{d-1}$ and x, $\xi \in \mathbb{R}^d$. We also have a convenient expression for $Q_{+,\frac{ht}{\varepsilon}} b$ in terms of φ_ω,

$$Q_{+,\frac{ht}{\varepsilon}} b\,(x,\xi) = \pi \int_{\mathbb{S}^{d-1}_\omega} \varphi_\omega(x,\xi,\xi^{\cdot 2}) \, \mathrm{d}\omega \, .$$

The conclusion is then given by Lemma 6.41. □

Lemma 6.41. *For any $\gamma \in]0,1[$, uniformly in $\omega \in \mathbb{S}^{d-1}_\omega$,*

$$\mathcal{N}_k\Big(\int_{\mathbb{R}^+_r} \varphi_\omega(x,\xi,r)\, \kappa^\zeta(r-\xi^{\cdot 2})\, \mathrm{d}r - \varphi_\omega(x,\xi,\xi^{\cdot 2}) \Big) \leq C_{k,G,\gamma}\, \zeta^\gamma \,.$$

Proof. The integral can be expressed as a convolution product

$$\int_{\mathbb{R}_r} \varphi_\omega(x,\xi,r)\, \kappa^\zeta(r-\xi^{\cdot 2})\, \mathrm{d}r = \big(\varphi(x,\xi,\cdot) * \kappa^\zeta\big)(\xi^{\cdot 2}) \,.$$

Since the derivation behaves well with the difference, *i.e.*

$$\partial_x^\alpha \partial_\xi^\beta \left(\big(\varphi_\omega(x,\xi,\cdot) * \kappa^\zeta\big)(\xi^{\cdot 2}) - \varphi_\omega(x,\xi,\xi^{\cdot 2}) \right) = \sum_{\alpha',\beta',\gamma'} c_{\alpha',\beta',\gamma'} 2^{|\gamma'|} \xi^{\gamma'} \times$$
$$\Big[\Big(\big(\partial_x^{\alpha'} \partial_\xi^{\beta'} \partial_r^{\gamma'} \varphi_\omega\big)(x,\xi,\cdot) * \kappa^\zeta \Big)(\xi^{\cdot 2}) - \big(\partial_x^{\alpha'} \partial_\xi^{\beta'} \partial_r^{\gamma'} \varphi_\omega\big)(x,\xi,\xi^{\cdot 2}) \Big] \,,$$

it suffices to apply Proposition D.1. □

7. Comparisons of the measures of an observable at a mesoscopic scale for the original and approximated dynamics

Proposition 7.1. *Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$, $\rho \in \mathcal{L}_1 L^2_x$ and $t \geq 0$,*

$$m(b,\rho^\varepsilon_t) = \mathrm{Tr}\left[b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t \right] \,,$$
$$m(b,\rho^{\varepsilon,app}_t) = \mathrm{Tr}\left[b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}^{app}_t \right] \,.$$

Definition 7.2. Let $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$, $\rho \in \mathcal{L}_1 L^2_x$ a state, $t \geq 0$ and $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ we define

$$m(b,\rho,t,\chi) = \mathrm{Tr}\left[\chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}_t \right]$$
$$m^{app}(b,\rho,t,\chi) = \mathrm{Tr}\left[\chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \chi\big(\mathrm{d}\Gamma_\varepsilon(\eta)\big)\, \hat{\boldsymbol{\rho}}^{app}_t \right] \,.$$

Proposition 7.3. *Let b be a symbol in $\mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$ with positive values such that $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$ for some $R > 0$, $\rho \in \mathcal{L}^+_1 L^2_x$ with $\mathrm{Tr}\,\rho \leq 1$ and for $j = 1,2$, $\chi_j \in \mathcal{C}^\infty_0(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi_j(B_{M_j}) = \{1\}$ for $M_1 = 3R$ and with $\chi_2(\mathbb{R}^d - B_{R+1}) = \{0\}$. There is a constant C_{R,b,χ_1,χ_2} (which does not depend on ρ) such that*

$$m(b,\rho_t) - m^{app}\big(b, (\rho_{\chi_2})^{app}_t\big) \geq -\mathcal{E}_7$$

with $\mathcal{E}_7 = \mathcal{E}_{7.1} + \mathcal{E}_{7.2} + \mathcal{E}_{7.3}$ and $\rho_{\chi_2} = \chi_2(D_x)\, \rho\, \chi_2(D_x)$.

$$\mathcal{E}_7 = C_{R,b,\chi_1,\chi_2} \left(h + \left(\frac{ht}{\varepsilon}\right)^3 h^{-3/2} + \left(\frac{ht}{\varepsilon}\right)^4 h^{-2} + \mathcal{E}_6 \right) \,.$$

We shall prove it in three steps:

(1) $m(b,\rho_t) - m(b,\rho_{\chi_2},t,\chi_1) \geq -\mathcal{E}_{7.1}$,
$$\mathcal{E}_{7.1} = Ch \,,$$

(2) $m(b,\rho_{\chi_2},t,\chi_1) - m^{app}(b,\rho_{\chi_2},t,\chi_1) \geq -\mathcal{E}_{7.2}$,
$$\mathcal{E}_{7.2} = C_{b,R,\chi_1} \left(\left(\frac{ht}{\varepsilon}\right)^3 h^{-3/2} + \left(\frac{ht}{\varepsilon}\right)^4 h^{-2} \right) \,,$$

(3) $m^{app}(b,\rho_{\chi_2},t,\chi_1)-m(b,(\rho_{\chi_2})_t^{app})\geq -\mathcal{E}_{7.3}$,

$$\mathcal{E}_{7.3}=\mathcal{E}_6+Ch\,.$$

7.1. **Step 1: Introduction of cutoffs.** We introduce cutoff functions both on the state ρ and the Wick observable $b^W(-hD_\xi,\xi-d\Gamma_\varepsilon(\eta))$.

Proposition 7.4. *Let $b\in\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ non-negative such that $\mathrm{Supp}_\xi\, b\subset B_R$ for some $R>0$, $\rho\in\mathcal{L}_1^+L^2_x$, $\mathrm{Tr}\,\rho\leq 1$, and, for $j=1,2$, $\chi_j\in\mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$ and $\chi_j(B_{M_j})=\{1\}$ for some $M_j>0$. Then there is a constant C_{b,χ_1,χ_2} such that*

$$m(b,\rho_t)\geq m(b,\rho_{\chi_2},t,\chi_1)-\mathcal{E}_{7.1}$$

with $\mathcal{E}_{7.1}=C_{b,\chi_1,\chi_2}h$ and $\rho_{\chi_2}=\chi_2(D_x)\circ\rho\circ\chi_2(D_x)$.

Proof. Using the functional calculus for the self-adjoint operator $d\Gamma_\varepsilon(\eta)$ and since

$$\begin{aligned}b(x,\xi-\lambda)&\geq\chi_2(\xi)\,b(x,\xi-\lambda)\,\chi_1(\lambda)\,\chi_2(\xi)\\&\geq\chi_2(\xi)\,\sharp^h\,b(x,\xi-\lambda)\,\chi_1(\lambda)\,\sharp^h\,\chi_2(\xi)-C_{b,\chi_1,\chi_2}h\end{aligned}$$

holds uniformly in λ, we can write

$$\begin{aligned}&b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\\&\qquad\geq\chi_2(\xi)\circ b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\circ\chi_2(\xi)-C_{b,\chi_1,\chi_2}h\,.\end{aligned}$$

And thus

$$\begin{aligned}m(b,\rho_t)&=\mathrm{Tr}\left[b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\hat{\boldsymbol{\rho}}_t\right]\\&\geq\mathrm{Tr}\left[b^W\big(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\chi_1(\mathrm{d}\Gamma(\eta))\,\widehat{\boldsymbol{\rho}_{\boldsymbol{\chi_2}t}}\right]-C_{b,\chi_1,\chi_2}h\end{aligned}$$

since $[H_\varepsilon,\chi_2]=0$. □

7.2. **Step 2: Comparison between truncated solutions.**

Proposition 7.5. *Let b be a symbol in $\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ with positive values, $\rho\in\mathcal{L}_1^+L^2_x$, $\mathrm{Tr}\,\rho\leq 1$ and $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, and $\chi(B_M)=\{1\}$ for some $M>0$, then there is a constant $C_{G,b,\chi}$ such that*

$$|m(b,\rho,t,\chi)-m^{app}(b,\rho,t,\chi)|\leq\mathcal{E}_{7.2}$$

with $\mathcal{E}_{7.2}=C_{G,b,\chi}\left(\left(\frac{ht}{\varepsilon}\right)^3h^{-3/2}+\left(\frac{ht}{\varepsilon}\right)^4h^{-2}\right)$.

We will need the following number estimate.

Lemma 7.6. *Let $\hat\psi_0\in L^2_\xi$ be a normed vector. We have, for $\hat\Psi^\sharp_{h,\varepsilon,t}=\hat\Psi_{h,\varepsilon,t}=e^{-i\frac{t}{\varepsilon}\hat H}\hat\psi_0\otimes\Omega$ or $\hat\Psi^\sharp_{h,\varepsilon,t}=\hat\Psi^{app}_{h,\varepsilon,t}=e^{-i\frac{t}{\varepsilon}\hat H^{app}_{h,\varepsilon}}\hat\psi_0\otimes\Omega$,*

$$\left\|(\varepsilon+N_\varepsilon)^{1/2}\,\hat\Psi^\sharp_{h,\varepsilon,t}\right\|\leq C_d\left(\sqrt{\varepsilon}+\sqrt{\frac{t}{2}\frac{ht}{\varepsilon}\|\hat G\|_{L^1}}\right)\,.$$

Proof of the Lemma. Indeed let us define $\gamma_t=\|(\varepsilon+N_\varepsilon)^{1/2}\hat\Psi^\sharp_t\|$. Then

$$i\varepsilon\partial_t\left(\gamma_t^2\right)=\left\langle\hat\Psi^\sharp_t,[\Phi_\varepsilon(f_{h,\varepsilon}),N_\varepsilon]\,\hat\Psi^\sharp_t\right\rangle$$

with $f_{h,\varepsilon} = \sqrt{\frac{h}{\varepsilon}}\varepsilon^{d/2}\overline{\hat{V}(\varepsilon\eta)}$ since ξ and $\mathrm{d}\Gamma_\varepsilon(\eta)$ commute with N_ε. Using $N_\varepsilon = \mathrm{d}\Gamma_\varepsilon(1)$, we get

$$\begin{aligned}[a_\varepsilon(f_{h,\varepsilon}), \mathrm{d}\Gamma_\varepsilon(1)] &= i\partial_s \left[\Gamma(e^{i\varepsilon s})\, a_\varepsilon(f_{h,\varepsilon})\, \Gamma(e^{-i\varepsilon s})\right]\big|_{s=0} \\ &= a_\varepsilon(\varepsilon f_{h,\varepsilon})\,.\end{aligned}$$

The other term of the commutator can be computed analogously (but $a_\varepsilon(\cdot)$ is $\mathbb{C}$-antilinear whereas $a^*_\varepsilon(\cdot)$ is $\mathbb{C}$-linear). Introducing this relation into the differential equation and taking the modulus, we get

$$\left|i\varepsilon\partial_t\left(\gamma_t^2\right)\right| \leq \frac{1}{\sqrt{2}}\left\|\hat{\Psi}_t^\sharp\right\|\left(\left\|a_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}_t^\sharp\right\| + \left\|a^*_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}_t^\sharp\right\|\right)\,.$$

But

$$\begin{aligned}\left\|a_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}_t^\sharp\right\|^2 &\leq \|\varepsilon f_{h,\varepsilon}\|^2_{L^2_\xi}\left\langle\hat{\Psi}_t^\sharp, N_\varepsilon\hat{\Psi}_t^\sharp\right\rangle\,, \\ \left\|a^*_\varepsilon(\varepsilon f_{h,\varepsilon})\,\hat{\Psi}_t^\sharp\right\|^2 &\leq \|\varepsilon f_{h,\varepsilon}\|^2_{L^2_\xi}\left\langle\hat{\Psi}_t^\sharp, (\varepsilon + N_\varepsilon)\,\hat{\Psi}_t^\sharp\right\rangle\,.\end{aligned}$$

Using $\|\hat{G}\|_{L^1} = \frac{h}{\varepsilon}\|f_{h,\varepsilon}\|^2_{L^2_\xi}$, we finally get a differential inequality for the function γ_t

$$2\varepsilon\gamma_t\partial_t\gamma_t \leq \left|i\varepsilon\partial_t\left(\gamma_t^2\right)\right| \leq \sqrt{2\varepsilon h\|\hat{G}\|_{L^1}}\gamma_t\,.$$

Dividing by $2\varepsilon\gamma_t$ and integrating in time, we obtain the expected result

$$\gamma_t \leq \gamma_0 + t\sqrt{\frac{h}{2\varepsilon}\|\hat{G}\|_{L^1}}\,,$$

since $\gamma_0 = C_d\sqrt{\varepsilon}$. □

Set

$$b_\chi = b\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\,\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,. \tag{7.1}$$

We want to control the error when we consider $\mathrm{Tr}\,[b_\chi\,\boldsymbol{\rho}_t^{app}]$ instead of $\mathrm{Tr}\,[b_\chi\,\boldsymbol{\rho}_t]$ *i.e.* we want to control $\mathrm{Tr}\,[b_\chi\,u_t]$ with

$$u_t = \boldsymbol{\rho}_t - \boldsymbol{\rho}_t^{app}\,. \tag{7.2}$$

Since

$$\begin{aligned}i\varepsilon\partial_t\boldsymbol{\rho}_t &= [H_\varepsilon, \boldsymbol{\rho}_t] \\ i\varepsilon\partial_t\boldsymbol{\rho}_t^{app} &= [H_\varepsilon, \boldsymbol{\rho}_t^{app}] - [H_\varepsilon - H_\varepsilon^{app}, \boldsymbol{\rho}_t^{app}]\end{aligned}$$

the difference u_t is solution of the differential equation

$$\begin{aligned}i\varepsilon\partial_t u_t &= [H_\varepsilon, u_t] - \left[\mathrm{d}\Gamma_\varepsilon(\eta)^2 - \varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2), \boldsymbol{\rho}_t^{app}\right] \\ &= \left[\big(\xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)^2, u_t\right] + [\Phi_\varepsilon(f_{h,\varepsilon}), u_t] \\ &\quad - \left[\mathrm{d}\Gamma_\varepsilon(\eta)^2 - \varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2), \boldsymbol{\rho}_t^{app}\right]\end{aligned}$$

with initial data $u_{t=0}=0$. We thus get an integral expression for $\operatorname{Tr}[b_\chi\, u_t]$,

$$\begin{aligned}\operatorname{Tr}[b_\chi\, u_t] = &-\frac{i}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\mathrm{d}s\\ &+\frac{i}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2), \boldsymbol{\rho}_s^{app}\right]\right]\mathrm{d}s\\ &-\frac{i}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\Phi_\varepsilon(f_{h,\varepsilon}), u_s\right]\right]\mathrm{d}s\,.\end{aligned}$$

Remark 7.7. Let $\mathcal{H}$ be a Hilbert space. If A, $B\in\mathcal{L}(\mathcal{H})$ and $C\in\mathcal{L}_1(\mathcal{H})$, then

$$\operatorname{Tr}[A\,[B\,,C]] = \operatorname{Tr}[[A\,,B]\,C]\ .$$

Lemma 7.8. *There exists a constant C independent of χ such that for b_χ and u_t defined by Equations (7.1) and (7.2),*

(1) $\left|\frac{1}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_{h,\varepsilon,s}\right]\right]\mathrm{d}s\right| \le \frac{h}{\varepsilon}\int_0^t \|u_{h,\varepsilon,s}\|_{\mathcal{L}_1}\,\mathrm{d}s \le C\frac{h^2t^3}{\varepsilon^3}$,

(2) $\frac{1}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\mathrm{d}\Gamma_\varepsilon(\eta)^2-\varepsilon\,\mathrm{d}\Gamma_\varepsilon(\eta^2)\,, \boldsymbol{\rho}^{app}\right]\right]\mathrm{d}s = 0$,

(3) $\left|\frac{1}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\Phi_\varepsilon(f_{h,\varepsilon})\,, u_s\right]\right]\mathrm{d}s\right| \le C\frac{t^3h^{3/2}}{\varepsilon^{7/2}}\left(\sqrt{\varepsilon}+\sqrt{\frac{t}{2}}\sqrt{\frac{ht}{\varepsilon}}\right)$.

Proof of 1. Let us introduce $\chi_1\succ\chi$ in order to handle only bounded operators:

$$\begin{aligned}&\operatorname{Tr}\left[b_\chi\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\\ &\quad=\operatorname{Tr}\left[b_\chi\,\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\\ &\quad=\operatorname{Tr}\left[b_\chi\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\\ &\quad=\operatorname{Tr}\left[\left[b_\chi\,, \chi_1(\mathrm{d}\Gamma_\varepsilon(\eta))\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2\right]u_s\right]\\ &\quad=\operatorname{Tr}\left[\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,\frac{h}{i}\left\{b(x,\xi),\xi^2\right\}\left(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)u_s\right]\\ &\quad=\operatorname{Tr}\left[\frac{h}{i}\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,(2\xi.b)\left(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)u_s\right]\,.\end{aligned}$$

We can then estimate the initial trace by

$$\begin{aligned}&\left|\operatorname{Tr}\left[b_\chi\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\right|\\ &\quad\le h\left\|\chi(\mathrm{d}\Gamma_\varepsilon(\eta))\,(2\xi.b)\left(-hD_\xi,\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)\right\|_{\mathcal{L}L^2_\xi\otimes\Gamma L^2_\eta}\|u_s\|_{\mathcal{L}_1L^2_\xi\otimes\Gamma L^2_\eta}\\ &\quad\le Ch\,\|u_s\|_{\mathcal{L}_1L^2_\xi\otimes\Gamma L^2_\eta}\end{aligned}$$

and a time integration brings

$$\left|\frac{1}{\varepsilon}\int_0^t \operatorname{Tr}\left[b_\chi\left[\left(\xi-\mathrm{d}\Gamma_\varepsilon(\eta)\right)^2, u_s\right]\right]\mathrm{d}s\right| \le C\frac{h}{\varepsilon}\int_0^t \|u_s\|_{\mathcal{L}_1}\,\mathrm{d}s\,.$$

Then we use that both $\hat{\boldsymbol{\rho}}_t$ and $\hat{\boldsymbol{\rho}}_t^{app}$ have the same initial value $\rho_0\otimes\operatorname{proj}\Omega$ with $\rho_0=\sum_j\lambda_j\,|\psi_{0,j}\rangle\langle\psi_{0,j}|$, $\sum_j\lambda_j=\operatorname{Tr}\rho$, $\lambda_j\ge0$, $\|\psi_{0,j}\|=1$ to write

$$\boldsymbol{\rho}_t=\sum_j\lambda_j\,|\varphi_{t,j}\rangle\langle\varphi_{t,j}|\ ,\qquad \boldsymbol{\rho}_t^{app}=\sum_j\lambda_j\,|\varphi_{t,j}^{app}\rangle\langle\varphi_{t,j}^{app}|\ ,$$

and then $u_t = \sum_j \lambda_j \left(|\Psi_{t,j} - \Psi_{t,j}^{app}\rangle\langle\Psi_{t,j}| - |\Psi_{t,j}^{app}\rangle\langle\Psi_{t,j}^{app} - \Psi_{t,j}| \right)$ and

$$\|u_t\|_{\mathcal{L}_1 L^2_\xi} \leq 2 \sum_j \lambda_j \left\| \Psi_{t,j} - \Psi_{t,j}^{app} \right\| \leq C \frac{ht^2}{\varepsilon^2} \,.$$

This and the integral above yield the result. □

Proof of 2. Let $\chi_1 \succ \chi$,

$$\begin{aligned} &\mathrm{Tr}\left[b_\chi \left[\mathrm{d}\Gamma_\varepsilon(\eta)^2 - \varepsilon\, \mathrm{d}\Gamma_\varepsilon(\eta^2) \,, u_s \right] \right] \\ &\quad = \mathrm{Tr}\left[b_\chi \left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta)) \left(d\Gamma_\varepsilon(\eta)^2 - \varepsilon\, \mathrm{d}\Gamma_\varepsilon(\eta^2) \right) , u_s \right] \right] \\ &\quad = \mathrm{Tr}\left[\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta)) \left(\mathrm{d}\Gamma_\varepsilon(\eta)^2 - \varepsilon\, \mathrm{d}\Gamma_\varepsilon(\eta^2) \right) , b_\chi \right] u_s \right] \\ &\quad = 0 \end{aligned}$$

since

$$\left[\chi_1(\mathrm{d}\Gamma_\varepsilon(\eta)) \left(\mathrm{d}\Gamma_\varepsilon(\eta)^2 - \varepsilon\, \mathrm{d}\Gamma_\varepsilon(\eta^2) \right) , b_\chi \right] = 0 \,.$$ □

Proof of 3. We have, with $r_s = \hat\Psi_s - \hat\Psi_s^{app}$,

$$\mathrm{Tr}\left[b_\chi \left[\Phi_\varepsilon(f_{h,\varepsilon}) \,, u_s \right] \right] = \langle r_s | \left[b_\chi \,, \Phi_\varepsilon(f_{h,\varepsilon}) \right] | \hat\Psi_s \rangle + \langle \hat\Psi_s^{app} | \left[b_\chi \,, \Phi_\varepsilon(f_{h,\varepsilon}) \right] | r_s \rangle \,.$$

Taking the modulus we obtain

$$\begin{aligned} \left| \mathrm{Tr}\left[b_\chi \left[\Phi_\varepsilon(f_{h,\varepsilon}) \,, u_s \right] \right] \right| \leq C \, \|r_s\| \left\| \Phi_\varepsilon(f_{h,\varepsilon})\, \hat\Psi_s \right\| + \|r_s\| \left\| \Phi_\varepsilon(f_{h,\varepsilon})\, b_\chi \hat\Psi_s \right\| \\ + \|r_s\| \left\| \Phi_\varepsilon(f_{h,\varepsilon})\, b_\chi^* \hat\Psi_s^{app} \right\| + C \, \|r_s\| \left\| \Phi_\varepsilon(f_{h,\varepsilon})\, \hat\Psi_s^{app} \right\| \end{aligned}$$

and we observe that

$$\left\| \Phi_\varepsilon(f_{h,\varepsilon})\, \Psi_s^\sharp \right\| \leq \|f_{h,\varepsilon}\| \left\| (\varepsilon + N_\varepsilon)^{1/2}\, \hat\Psi_s^\sharp \right\|$$

and

$$\left\| \Phi_\varepsilon(f_{h,\varepsilon})\, b_\chi\, \hat\Psi_s^\sharp \right\|^2 \leq C \, \|f_{h,\varepsilon}\|^2 \left\| (\varepsilon + N_\varepsilon)^{1/2}\, \hat\Psi_s^\sharp \right\|^2$$

and thus

$$\left| \mathrm{Tr}\left[b_\chi \left[\Phi_\varepsilon(f_{h,\varepsilon}) \,, u_s \right] \right] \right| \leq C \, \|r_s\| \sqrt{\frac{h}{\varepsilon} \|\hat G\|_{L^1}} \left(\sqrt{\varepsilon} + \frac{s}{\sqrt{2}} \, \|f_{h,\varepsilon}\|_{L^2_\xi} \right)$$

by our number estimate. An integration then gives

$$\left| \frac{1}{\varepsilon} \int_0^t \mathrm{Tr}\left[b_\chi \left[\Phi_\varepsilon(f_{h,\varepsilon}) \,, u_s \right] \right] \mathrm{d}s \right| \leq C \frac{t^3 h^{3/2}}{\varepsilon^{7/2}} \|\hat G\|_{L^1}^{1/2} \left(\sqrt{\varepsilon} + t \sqrt{\frac{h}{2\varepsilon} \|\hat G\|_{L^1}} \right)$$

which is the expected estimate. □

7.3. **Step 3: Release of the truncation on the symbol.**

Proposition 7.9. *Let b be a symbol in $\mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ with positive values, such that $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$ for some $R > 1$, $\rho \in \mathcal{L}_1^+ L^2_x$, $\mathrm{Tr}\,\rho \leq 1$, with the support of $\hat\rho$ in B^2_{R+1} and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi(B_{3R}) = \{1\}$. There is a constant $C_{R,b,\chi}$ such that*

$$m^{app}(b,\rho,t,\chi) - m(b,\rho_t^{app}) \geq \mathcal{E}_{7.3}$$

with

$$\begin{aligned}\mathcal{E}_{7.3} &= \mathcal{E}_6 + C_{R,b,\chi} h \\ &= C\frac{ht}{\varepsilon}\left(\frac{ht}{\varepsilon} + h + \left[h\left(\frac{ht}{\varepsilon}\right)^{-1}\right]^{d/2-1} + h^{\nu(d,\alpha)} + h^{\gamma\beta(d,\alpha)}\right) + C_{r,b,\chi} h\,.\end{aligned}$$

Proof. We can restrict the proof to the case of $\rho = |\psi\rangle\langle\psi|$ with $\psi \in L^2_x$ since ρ is trace class, then $\hat{\boldsymbol{\rho}}_t = |\hat{\Psi}^{app}_t\rangle\langle\hat{\Psi}^{app}_t|$. We also define a positive symbol $b_1 \in \mathcal{C}^\infty_0(\mathbb{R}^d_\xi)$ such that $\operatorname{Supp} b_1 \subset [R^{-2}, R^2]$ and $b_1(\xi^2) \geq b(x,\xi)$. Then

$$\begin{aligned}&m(b, \rho^{app}_t) - m^{app}(b,\rho,t,\chi) \\ &= \operatorname{Tr}\left[b^W\big(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta)\big)\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,\hat{\boldsymbol{\rho}}_t\right] \\ &= \operatorname{Tr}\left[(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))^{1/2}\; b^W(-hD_\xi, \xi - \mathrm{d}\Gamma_\varepsilon(\eta))\,(1-\chi\left(\mathrm{d}\Gamma_\varepsilon\left(\eta\right)\right))^{1/2}\,\hat{\boldsymbol{\rho}}_t\right] \\ &\leq \operatorname{Tr}\left[b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)^2\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,\hat{\boldsymbol{\rho}}_t\right] + \mathcal{O}(h) \\ &= \operatorname{Tr}\left[b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)\hat{\boldsymbol{\rho}}_t\right] + \mathcal{O}(h)\end{aligned}$$

with $\hat{\Psi}^{app}_t(\xi) = 1_{[0,M]}(|\xi|)\,\hat{\Psi}^{app}_t(\xi)$ and $\operatorname{Supp} b_1 \subset [R^{-2}, R^2]$. Then we decompose

$$\begin{aligned}\hat{\Psi}^{app}_t &= 1_{[1/2R,2R]}(|\xi|)\,\hat{\Psi}^{app}_t + 1_{[0,M]\setminus[1/2R,2R]}(|\xi|)\,\hat{\Psi}^{app}_t \\ &= \hat{\Psi}^{app}_{t,1} + \hat{\Psi}^{app}_{t,2}\,.\end{aligned}$$

With $A = b^W_1\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big)\,(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,b^W_1\big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\big) \geq 0$ we have the estimate

$$\operatorname{Tr}\left[A\left|\hat{\Psi}^{app}_t\right\rangle\left\langle\hat{\Psi}^{app}_t\right|\right] \leq 2\operatorname{Tr}\left[A\left|\hat{\Psi}^{app}_{t,1}\right\rangle\left\langle\hat{\Psi}^{app}_{t,1}\right|\right] + 2\operatorname{Tr}\left[A\left|\hat{\Psi}^{app}_{t,2}\right\rangle\left\langle\hat{\Psi}^{app}_{t,2}\right|\right]\,.$$

For the first term,

$$\begin{aligned}&\operatorname{Tr}\left[b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,b^W_1\left((\xi - \mathrm{d}\Gamma(\eta))^{\cdot 2}\right)\left|\hat{\Psi}^{app}_{t,1}\right\rangle\left\langle\hat{\Psi}^{app}_{t,1}\right|\right] \\ &\quad= \operatorname{Tr}\Big[1_{[1/2R,2R]}(|\xi|)\,b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta))) \\ &\qquad\qquad b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)1_{[1/2R,2R]}(|\xi|)\,\left|\hat{\Psi}^{app}_{t,1}\right\rangle\left\langle\hat{\Psi}^{app}_{t,1}\right|\Big] \\ &\quad= 0\end{aligned}$$

since $|\xi| \in [1/2R, 2R]$, $|\xi - \mathrm{d}\Gamma_\varepsilon(\eta)| \leq R$ implies $|\mathrm{d}\Gamma_\varepsilon(\eta)| \leq 3R$ and $\chi(B_{3R}) = \{1\}$. For the second term,

$$\begin{aligned}&\operatorname{Tr}\left[b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\right)(1-\chi(\mathrm{d}\Gamma_\varepsilon(\eta)))\,b^W_1\left((\xi - \mathrm{d}\Gamma(\eta))^{\cdot 2}\right)\left|\hat{\Psi}^{app}_{t,2}\right\rangle\left\langle\hat{\Psi}^{app}_{t,2}\right|\right] \\ &\quad\leq \operatorname{Tr}\left[b^W_1\left((\xi - \mathrm{d}\Gamma_\varepsilon\left(\eta\right))^{\cdot 2}\right)^2\left|\hat{\Psi}^{app}_{t,2}\right\rangle\left\langle\hat{\Psi}^{app}_{t,2}\right|\right]\end{aligned}$$

since $1 - \chi(d\Gamma_\varepsilon(\eta)) \leq \mathrm{Id}$. Then we use the computation of the evolution of a symbol of $|\xi|^2$ in the case of the approximated equation as in Remark 6.2 to get

that, since $b_1 = b_1(|\xi|^2)$ it is unchanged under the evolution, and

$$\begin{aligned}
&\mathrm{Tr}\left[b_1^W\Big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\Big)^2 \left|\hat\Psi_{t,2}^{app}\right\rangle\left\langle\hat\Psi_{t,2}^{app}\right|\right]\\
&\quad\le \mathrm{Tr}\left[b_1^W\Big((\xi - \mathrm{d}\Gamma_\varepsilon(\eta))^{\cdot 2}\Big)^2 \left|\hat\psi_{0,2}\otimes\Omega\right\rangle\left\langle\hat\psi_{0,2}\otimes\Omega\right|\right] + \mathcal{E}_6\\
&\quad\le \mathrm{Tr}\left[b_1^W(\xi^{\cdot 2})^2 \left|\hat\psi_{0,2}\otimes\Omega\right\rangle\left\langle\hat\psi_{0,2}\otimes\Omega\right|\right] + \mathcal{E}_6\\
&\quad\le \mathcal{E}_6
\end{aligned}$$

since $\mathrm{Supp}\, b_1 \cap \mathrm{Supp}\, \hat\psi_{0,2} = \emptyset$. □

8. The derivation of the Boltzmann equation for the model

Proposition 8.1. *Let $b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}_{x,\xi})$ with $\mathrm{Supp}_\xi\, b \subset B_R - B_{1/R}$. Let ρ a state and $T > 0$.*

$$\liminf_{h\to 0}\left(m(b,\rho^h_{N,\Delta t}) - m\left(\mathcal{B}^T(T)\, b,\rho\right)\right) \ge 0$$

for a fixed $\alpha \in \left]\frac34, 1\right[$, $\Delta t = \Delta t(h) = h^\alpha$ and $N(h)\,\Delta t(h) = T$.

Proof. We define for $k \in \mathbb{N}$, $\Delta t > 0$, $b_{k,\Delta t} = \left(e^{\Delta t Q} e^{2\Delta t \xi.\partial_x}\right)^k b$. We begin by looking to one step of evolution with $e^{\Delta t Q} e^{2\Delta t\xi.\partial_x}$. □

Lemma 8.2. *With $b_t = e^{tQ}e^{2t\xi.\partial_x}b$, and the hypotheses of Proposition 8.1,*

$$\begin{aligned}
&m(b,\rho^h_{\Delta t}) - m(b_{\Delta t},\rho)\\
&\qquad\ge -C\big(h + h^{-3/2}(\Delta t)^3 + (\Delta t)^4 h^{-2} + \Delta t\big(\Delta t + h + (h/\Delta t)^{\frac{d}{2}-1} + h^\mu\big)\big)\,.
\end{aligned}$$

Proof. We recall that $\rho^h_{\Delta t} = \rho^\varepsilon_{\varepsilon\Delta t/h}$ so that with $\frac{ht}{\varepsilon} = \Delta t$, from Section 7,

$$\begin{aligned}
&m(b,\rho^h_{\Delta t}) - m\big(b,(\rho_{\chi_2})^{h,app}_{\Delta t}\big)\\
&\quad= m(b,\rho^\varepsilon_t) - m\big(b,(\rho_{\chi_2})^{\varepsilon,app}_t\big)\\
&\quad\ge -C\left(h + \left(\frac{ht}{\varepsilon}\right)^3 h^{-3/2} + \left(\frac{ht}{\varepsilon}\right)^4 h^{-2} + \frac{ht}{\varepsilon}\big(\frac{ht}{\varepsilon} + h + (\varepsilon/t)^{d/2-1} + h^\mu\big)\right)\\
&\quad\ge -C\Big(h + h^{-3/2}(\Delta t)^3 + (\Delta t)^4 h^{-2} + \Delta t\big(\Delta t + h + (h/\Delta t)^{d/2-1} + h^\mu\big)\Big)
\end{aligned}$$

and from Section 6 also used with $\frac{ht}{\varepsilon} = \Delta t$ we get

$$m\big(b,(\rho_{\chi_2})^{\varepsilon,app}_t\big) - m(b_t,\rho_{\chi_2}) \ge -\mathcal{E}_6 \ge -\mathcal{E}_7$$

and this term will be in particular controlled if we control the previous one. Finally from the conservation of the support in ξ of the symbol by the approximated Boltzmann equation we get

$$m(b_t,\rho_{\chi_2}) - m(b_t,\rho) \ge -\mathcal{O}(h^\infty)$$

for χ_2 a cutoff function chosen so that $\chi_2(B_R) = \{1\}$.

Thus we fix, for $j = 1, 2$, two cutoff functions $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R}^d_\lambda)$ with values in $[0,1]$, $\chi_j(B_{M_j}) = \{1\}$ for $M_1 = 3R$ and $M_2 = 1$ and with $\chi_2(\mathbb{R}^d - B_{R+1}) = \{0\}$.

Then we can iterate this $N(h)$ times and we get the estimation

$$m\Big(b, \rho^{\varepsilon}_{N(h),\varepsilon\Delta t/h}\Big) - m(b_{N,\Delta t}, \rho)$$
$$\geq -CN\Big(h + h^{-\frac{3}{2}}(\Delta t)^3 + (\Delta t)^4 h^{-2} + \Delta t\big(\sqrt{\Delta t} + h + (h/\Delta t)^{\frac{d}{2}-1} + h^{\mu}\big)\Big)$$

with $N\Delta t = T$ and $h^{\alpha} \leq \frac{ht}{\varepsilon} = \Delta t \leq 1$ for some $\alpha \in]1/2, 1[$. Thus we can choose $\Delta t = \frac{th}{\varepsilon} = h^{\alpha}$ and thus $N = Th^{-\alpha}$. Then we get the estimate

$$m\big(b, \rho_{N,\varepsilon\Delta t/h}\big) - m(b_{N,\Delta t}, \rho)$$
$$\geq -CTh^{-\alpha}\left(h + h^{3\alpha-3/2} + h^{4\alpha-2} + h^{\alpha}\left(h^{\alpha/2} + h + h^{(1-\alpha)(d/2-1)} + h^{\mu}\right)\right)$$
$$\geq -CTo_{h\to 0}(1)\,,$$

for $\alpha \in]\frac{3}{4}, 1[$. Finally it suffices to prove that

$$\lim_{h\to 0} m\big(b_{N(h),\Delta t(h)}, \rho\big) = m(b_T, \rho)$$

which is true since the estimates of Proposition 3.9 prove that, for some constant $C > 0$, $\|b_{N,\Delta t} - b_T\|_{\mathcal{L}L^2_x} \leq \frac{C}{N}$. □

Appendix A. Stochastics

We recall some results about Gaussian random fields that can be found in [25, 31].

Definition A.1. Let $(\Omega_{\mathbb{P}}, \mathcal{G}, \mathbb{P})$ be a probability space. A real-valued random field $(\mathcal{V}_{\omega}(x))_{(\omega,x)\in\Omega_{\mathbb{P}}\times\mathbb{R}^d}$ is a *Gaussian random field* if for all finite choices of $x_1, \ldots, x_k \in \mathbb{R}^d$, $(\mathcal{V}_{\omega}(x_1), \ldots, \mathcal{V}_{\omega}(x_k))$ is an $\mathbb{R}^k$ valued Gaussian random variable. To each such Gaussian process we can associate a *mean function* $\mu(x) = \mathbb{E}[\mathcal{V}(x)]$ $(x \in \mathbb{R}^d)$ and a *covariance function* $\Sigma(x, x') = \mathbb{E}[\mathcal{V}(x)\mathcal{V}(x')]$ $(x, x' \in \mathbb{R}^d)$. A Gaussian random field is *translation invariant* if its covariance function $\Sigma(x, x')$ only depends on the difference $x - x'$, *i.e.* if there is a function $G : \mathbb{R}^d \to \mathbb{R}$ such that $\Sigma(x, x') = G(x - x')$.

Definition A.2. A function $\Sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is *symmetric* if for all $x, x' \in \mathbb{R}^d$, $\Sigma(x, x') = \Sigma(x', x)$. It is *positive definite* if for all $x_1, \ldots, x_k \in \mathbb{R}^d$ and all $\xi_1, \ldots, \xi_k \in \mathbb{R}$,

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \xi_i \Sigma(x_i, x_j)\xi_j \geq 0\,.$$

A function $G : \mathbb{R}^d \to \mathbb{R}$ is *positive definite* if $\Sigma(x, x') = G(x - x')$ is positive definite.

Theorem A.3. *Given an arbitrary function $\mu : \mathbb{R}^d \to \mathbb{R}$, and a symmetric, positive definite function $\Sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, there exists a Gaussian random field $\mathcal{V}(x)$ with mean μ and covariance Σ.*

See [25] for a proof of Theorem A.3.

Theorem A.4 (Bochner)**.** *A function $G : \mathbb{R}^d \to \mathbb{R}$ is the Fourier transform of a positive bounded Borel measure on $\mathbb{R}^d$ if and only if it is continuous and positive definite.*

See [31] and the references therein for Bochner's theorem.

Theorem A.5 (Minlos). *A function $c : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is the Fourier transform*

$$c(f) = \int_{\mathcal{S}'(\mathbb{R}^d)} \exp(-i\langle f, T\rangle)\, \mathrm{d}\mu(T)$$

of a cylinder set measure μ on $\mathcal{S}'(\mathbb{R}^d)$ if and only if

1. *$c(0) = 1$,*
2. *$f \mapsto c\,(f)$ is continuous in the strong topology,*
3. *for any $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^d)$ and $z_1, \ldots, z_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} z_i z_j c(f_i - f_j) \geq 0\,.$$

See [31] and the references therein for Minlos' theorem.

Definition A.6. We consider the probability space $(\mathcal{S}'(\mathbb{R}^d), \mu)$ (the σ-algebra is the one generated by the cylinder sets) where μ is the measure obtained by Minlos' theorem with the positive definite function

$$c(f) = \exp\left(-\frac{\|f\|_{L^2}^2}{4}\right).$$

The *white noise* is the random variable on $\left(\mathcal{S}'\left(\mathbb{R}^d\right), \mu\right)$ with values in $\mathcal{S}'\left(\mathbb{R}^d\right)$ defined by $W_\omega = \omega$.

Remark A.7. Since $\|f\|_{L^2}^2 = \langle f(x_1), \delta(x_1 - x_2) f(x_2)\rangle$ we have (in a weak sense)

$$\mathbb{E}[W(x_1)W(x_2)] = \delta(x_1 - x_2)\,.$$

Proposition A.8. *Let $G : \mathbb{R}^d \to \mathbb{R}$ positive definite, such that $\hat{G} = |\hat{V}|^2$ with $\hat{V} \in \mathcal{S}(\mathbb{R}^d; \mathbb{R})$. The translation invariant centered Gaussian random field of covariance $G(x - x')$ is $\mathcal{V}_\omega = V * W_\omega$ where W_ω is the white noise.*

Remark A.9. Bochner's theorem justifies the form we choose for the function G as the positivity of the Fourier transform is natural for a covariance function.

Proof. After testing with elements in $\mathcal{S}(\mathbb{R}^d)$ the following calculations hold. The mean of $V * W_\omega(x)$ is zero:

$$\mathbb{E}[V * W_\omega(x)] = \int V(x - x_1)\, \mathbb{E}[W_\omega(x_1)]\, \mathrm{d}x_1 = 0$$

and its covariance is

$$\begin{aligned}
\mathbb{E}[\mathcal{V}(x)\, \mathcal{V}(x')] &= \mathbb{E}\big[\int V(x - x_1)\, W(x_1)\, V(x' - x_2)\, W(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2\big] \\
&= \int V(x - x_1)\, V(x' - x_1)\, \mathrm{d}x_1 \\
&= V * V(-\cdot)\,(x - x')
\end{aligned}$$

and $\mathcal{F}\,(V * V\,(-\cdot)) = |\hat{V}|^2$, so that we get the expected covariance. □

Appendix B. Semiclassical Measures

Semiclassical measures (and microlocal defect measures) have been studied among others in [8, 16, 17, 26]. We recall here some results. The first theorem can be found in [8].

Theorem B.1. *Let (u_k) be a sequence of L^2_x such that $u_k \rightharpoonup u$ weakly. For all real sequence (h_k) such that $h_k \to 0$, there exist a subsequence (u_{k_n}) of the sequence (u_k) and a measure $\mu \in \mathcal{M}_+(\mathbb{R}^{2d}_{x,\xi})$ such that for all $b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi})$,*

$$\lim_{n\to+\infty} \langle b(h_{k_n}x, D_x)u_{k_n}, u_{k_n}\rangle = \int_{\mathbb{R}^{2d}_{x,\xi}} b \, \mathrm{d}\mu \,.$$

Definition B.2. The measure μ above is called a *semiclassical measure* (or Wigner measure) associated with the sequence (u_k). If there is uniqueness of the "limit measure" the sequence (u_k) is said *pure* and we note $\{\mu\} = \mathcal{M}(u_k)$.

This result holds in the case of a family of states $(\rho_h)_{h\in]0,h_0]}$, *i.e.* $\rho_h \in \mathcal{L}^+_1 L^2_x$, $\operatorname{Tr}\rho_h = 1$.

Theorem B.3. *Let $(\rho_h)_{h\in]0,h_0]}$, $h_0 > 0$ be a family of states on L^2_x. There exist a sequence $h_k \to 0$ and a measure $\mu \in \mathcal{M}_+(\mathbb{R}^{2d}_{x,\xi})$ such that*

$$\forall b \in \mathcal{C}^\infty_0(\mathbb{R}^{2d}_{x,\xi}) \,, \quad \lim_{n\to+\infty} \operatorname{Tr}\left[b^W(h_{k_n}x, D_x)\rho_{h_k}\right] = \int_{\mathbb{R}^{2d}_{x,\xi}} b \, \mathrm{d}\mu \,.$$

Proof. We first take an arbitrary sequence (h_k) such that $h_k \to 0$. Then we can define positive numbers $(\lambda_{k,j})_{j,k\geq 0}$ and normed vectors $(u_{k,j})$ of L^2_x such that $\sum_j \lambda_{k,j} = 1$ and $\rho_{h_k} = \sum_j \lambda_{k,j} |u_{k,j}\rangle\langle u_{k_n,j}|$. We can extract from each sequence $(u_{k,j})_k$ a subsequence that converges weakly to a vector u_j ($\|u_j\| \leq 1$). A diagonal extraction enables those convergences to occur simultaneously. The sequence obtained this way is still denoted by $(u_{k,j})$.

Theorem B.1 applies to each sequence $(u_{k,j})_k$ and yields measures μ_j such that for well chosen subsequences h_{k_n}, $\lambda_{k_n,j} \to \lambda_j$ and

$$\lim_{n\to+\infty} \operatorname{Tr}\left[b^W\left(h_{k_n}x, D_x\right)|u_{k,j}\rangle\,\langle u_{k_n,j}|\right] = \int_{\mathbb{R}^{2d}_{x,\xi}} b \, \mathrm{d}\mu_j \,.$$

Again we apply a diagonal extraction argument to obtain these convergences simultaneously, and we stick with the notations $u_{k,j}$, $\lambda_{k,j}$ for the extracted objects. We observe that $\|u_j\| \leq 1$ and $\sum \lambda_j \leq 1$. We can thus sum these relations to get

$$\lim_{k\to+\infty} \sum_j \lambda_{k,j} \operatorname{Tr}\left[b^W\left(h_k x, D_x\right)|u_{k,j}\rangle\,\langle u_{k_n,j}|\right] = \int_{\mathbb{R}^{2d}_{x,\xi}} b \, \mathrm{d}\Big(\sum \lambda_j \mu_j\Big)$$

which is the expected result with $\mu = \sum_j \lambda_j \mu_j$. □

Appendix C. General results on semigroups

Some references about semigroups of operators in Banach spaces are [12, 9, 11].

In this appendix X represents a (real or complex) Banach space.

Definition C.1. A *strongly continuous semigroup* on X is a mapping $G : \mathbb{R}^+ \to \mathcal{L}(X)$, such that

(1) $\forall t, s \geq 0, \quad G(t+s) = G(t)\,G(s)\,,\, G(0) = I,$

(2) $G(\cdot)\,x$ is continuous for all $x \in X$.

The *infinitesimal generator* A of $G(\cdot)$ is defined by

$$D(A) = \left\{ x \in X \,,\, \exists \lim_{h\to 0^+} \frac{G(h)\,x - x}{h} \right\}, \qquad Ax = \lim_{h\to 0^+} \frac{G(h)\,x - x}{h}\,.$$

Proposition C.2. *Let G be a strongly continuous semigroup on X with infinitesimal generator $(A, D(A))$. Then $D(A)$ is dense in X and A is a closed operator.*

See [9] for a proof of Proposition C.2.

Notation C.3. For $M > 0$ and ω in $\mathbb{R}$, we denote by $\mathcal{G}(M,\omega)$ the set of all strongly continuous semigroups G such that

$$\forall t \geq 0, \quad \|G(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}\,.$$

Theorem C.4 (A perturbation result)**.** *Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous semigroup in $\mathcal{G}(M,\omega)$ and $B \in \mathcal{L}(X)$. Then $(A + B, D(A))$ is the infinitesimal generator of a strongly continuous semigroup in $\mathcal{G}(M, \omega + M\,\|B\|_{\mathcal{L}(X)})$.*

See [11, 12] for a proof of Theorem C.4.

Theorem C.5 (Trotter)**.** *Let A_j, $j = 1, \ldots, k$ be the infinitesimal generators of continuous semigroups $G_j \in \mathcal{G}(M_j, \omega_j)$. If $\cap_{j=1}^k D(A_j)$ is dense in X and*

$$\forall n \in \mathbb{N}, \quad \|(G_1(t)\,G_2(t)\cdots G_k(t))^n\|_{\mathcal{L}(X)} \leq M e^{n\omega t}$$

and if there exists p such that $\Re p > \omega$, $(pI - (A_1 + A_2 + \cdots + A_k))X$ is dense in X then

$$\overline{A_1 + A_2 + \cdots + A_k}$$

is the infinitesimal generator of a continuous semigroup in $\mathcal{G}\,(M,\omega)$.

In such a situation the semigroup G generated by $\overline{A_1 + A_2 + \cdots + A_k}$ satisfies

$$\forall x \in X, \quad G(t)x = \lim_{n\to\infty} \Big[G_1\Big(\frac{t}{n}\Big)\, G_2\Big(\frac{t}{n}\Big) \cdots G_k\Big(\frac{t}{n}\Big) \Big]^n x$$

with a uniform convergence on the bounded intervals $[0, T]$, with $T > 0$.

See [11] for a proof of Theorem C.5.

Appendix D. Lemmas about an approximate identity

For $\zeta > 0$, and $r \in \mathbb{R}$, let $\kappa^\zeta(r) = \frac{1}{\pi}\frac{\zeta}{r^2+\zeta^2}$.

Proposition D.1. *Let f be a function in the Schwartz class. Then for any $\gamma \in]0,1[$, a constant $C_\gamma > 0$ exists such that*

$$\forall \zeta > 0, \ \left\| f * \kappa^\zeta - f \right\|_{L^\infty} \leq \max\left\{ \|f\|_\infty \,, \|f'\|_\infty \right\} C_\gamma \zeta^\gamma\,.$$

Proof. We use the formula

$$f(r_0 + \zeta r) = f(r_0) + \zeta r \int_0^1 f'(r_0 + s\zeta r)\,\mathrm{d}s$$

so that we have both

$$|f(r_0 + \zeta r) - f(r_0)| \leq 2\,\|f\|_\infty\,,$$
$$|f(r_0 + \zeta r) - f(r_0)| \leq \|f'\|_\infty\,\zeta r\,,$$

and the interpolation of those two results gives, for $\gamma \in [0,1]$,

$$|f(r_0+\zeta r)-f(r_0)| \leq 2\max\left\{\|f\|_\infty, \|f'\|_\infty\right\}\zeta^\gamma |r|^\gamma .$$

So, for $\gamma \in [0,1[$,

$$\left|\int_{\mathbb{R}} [f(r_0+\zeta r)-f(r_0)]\,\frac{\mathrm{d}r}{r^2+1}\right| \leq \max\left\{\|f\|_\infty, \|f'\|_\infty\right\} C_\gamma \zeta^\gamma$$

which is the expected result. □

Proposition D.2. *Let $f:\mathbb{R}_r \to \mathbb{R}$ continuous, vanishing on $\mathbb{R}^-$, such that $f|_{\mathbb{R}_*^+}$ is in $\mathcal{C}^\infty(\mathbb{R}_*^+)$ and rapidly decreasing towards $+\infty$. Let $0<r_{\min}<r_{\max}$. Then, for any $\gamma \in]0,1[$, there is a constant $C_{f,\gamma}$ such that*

$$\left\| \partial_r^k \left[f*\kappa^\zeta - f\right]\big|_{[r_{\min},r_{\max}]}\right\|_{L^\infty} \leq C_\gamma \zeta^\gamma .$$

Proof. We choose A and Δr such that $0<A<\Delta r<r_{\min}/2$. Let $f_1=\chi_1 f$ and $f_2=(1-\chi_1)f$ with χ_1 a $\mathcal{C}^\infty$ decreasing function such that

$$\begin{aligned}\chi_1(r) &= 1 \quad \text{if} \quad r \leq A/2\\ &= 0 \quad \text{if} \quad A \leq r .\end{aligned}$$

Then $f=f_1+f_2$ and

$$f*\delta^\zeta = f_1 \underset{\mathcal{E}',\mathcal{C}^\infty}{*} \kappa^\zeta + f_2 \underset{\mathcal{S},L^1}{*} \kappa^\zeta .$$

The second term is the easiest to handle since $\partial_r^k\left(f_2*\kappa^\zeta\right) = \left(\partial_r^k f_2\right)*\kappa^\zeta$ and Proposition D.1 can be applied to get

$$\left\|\left(\partial_r^k f_2\right)*\kappa^\zeta - \pi\partial_r^k f_2\right\|_{L^\infty} \leq C_\gamma\left(\left\|f_2^{(k)}\right\|_\infty + \left\|f_2^{(k+1)}\right\|_\infty\right)\zeta^\gamma .$$

We now recall that we are only interested in $r\in[r_{\min},r_{\max}]$ with $0<r_{\min}<r_{\max}$ when evaluating $\partial_r^k\left(f*\kappa^\zeta\right)$. We insert another cutoff function $\chi_2 \in \mathcal{D}(\mathbb{R})$ such that

$$\begin{aligned}\chi_2(r) &= 0 \,\text{if}\, r \leq r_{\min}-2\Delta r\\ &= 1 \,\text{if}\, r_{\min}-\Delta r \leq r \leq r_{\max}+\Delta r\\ &= 0 \,\text{if}\, r_{\max}+2\Delta r \leq r .\end{aligned}$$

Then $f_1*\kappa^\zeta = f_1*\chi_2\kappa^\zeta + f_1*(1-\chi_2)\kappa^\zeta$ and our hypotheses on the supports give

$$\begin{aligned}\mathrm{Supp}\left\{f_1*(1-\chi_2)\kappa^\zeta\right\} &\subset \mathrm{Supp}\, f_1 + \mathrm{Supp}\,(1-\chi_2)\\ &\subset \mathbb{R}-[r_{\min}-\Delta r+A, r_{\max}+\Delta r] .\end{aligned}$$

Since $A<\Delta r$ we obtain $\left[f_1*(1-\chi_2)\kappa^\zeta\right]\big|_{[r_{\min},r_{\max}]}=0$ and we can restrict ourselves to the computation of

$$f_1 \underset{\mathcal{E}',\mathcal{C}_0^\infty}{*} \chi_2\kappa^\zeta .$$

More precisely we want to estimate

$$\left\|\partial_r^k\left(f_1 \underset{\mathcal{E}',\mathcal{C}_0^\infty}{*} \chi_2 u_\zeta\right)\Big|_{[r_{\min},r_{\max}]}\right\|_{L^\infty}$$

since $\chi_2\delta = 0$ and thus $f_1 \underset{\mathcal{E}',\mathcal{E}'}{*} \chi_2\delta = 0$. But the same considerations hold for the supports of the derivatives. Thus it is sufficient to observe that we have the control

$$\begin{aligned}\left\| f_1 \underset{L^1,\mathcal{C}_0^\infty}{*} \partial_r^k \left(\chi_2\kappa^\zeta\right) \right\|_{L^\infty} &\leq \|f_1\|_{L^1} \left\|\partial^k \left(\chi_2\kappa^\zeta\right)\right\|_{L^\infty} , \\ &\leq \|f_1\|_{L^1} C_{\chi_2} \sup_{r\geq r_{\min}-2\Delta r} \left|\partial^k\kappa^\zeta\right|\end{aligned}$$

where the sup can be controlled by $C\zeta$ with C only dependent on our choice of Δr and $r_{\min}$ as

$$2\partial^k\kappa^\zeta(r) = i^k k! \frac{-(ir-\zeta)^{k+1} + (ir+\zeta)^{k+1}}{(r^2+\zeta^2)^{k+1}} .$$

Consequently

$$\left\| \partial_r^k \Big[f_1 \underset{\mathcal{E}',\mathcal{C}_0^\infty}{*} \chi_2\kappa^\zeta - f_1 \underset{\mathcal{E}',\mathcal{E}'}{*} \chi_2\delta \Big] \Big|_{[r_{\min},r_{\max}]} \right\|_{L^\infty} \leq C\zeta$$

and this ends the proof. □

Appendix E. Formulae

E.1. **Symmetric Fock space.** For f, g in a complex Hilbert space $\mathcal{H}$,

- $a_\varepsilon(f) = \langle f, z\rangle^{Wick}$, $a_\varepsilon^*(f) = \langle z, f\rangle^{Wick}$,
- $[a_\varepsilon(f), a_\varepsilon(g)] = 0$, $[a_\varepsilon^*(f), a_\varepsilon^*(g)] = 0$, $[a_\varepsilon(f), a_\varepsilon^*(g)] = \varepsilon\langle f, g\rangle$,
- $\Phi_\varepsilon(f) = \big(a_\varepsilon(f) + a_\varepsilon^*(f)\big)/\sqrt{2}$,
- $W(f) = \exp i\Phi_\varepsilon(f)$, $W(f)W(g) = e^{-\frac{i\varepsilon}{2}\Im\langle f,g\rangle} W(f+g)$,
- $E(f) = W\left(\frac{\sqrt{2}}{i\varepsilon} f\right)|\Omega\rangle$.

E.2. **Fourier transforms.** Usual Fourier transform

For $u \in L^2_x$, $v \in L^2_\xi$,

$$\mathcal{F}_x u(\xi) = \int_{\mathbb{R}^d_x} e^{-ix.\xi}\, u(x)\,\mathrm{d}x \quad \text{and} \quad \mathcal{F}_x^{-1} v(x) = \int_{\mathbb{R}^d_\xi} e^{ix.\xi}\, v(\xi)\,đ\xi$$

with $đ\xi = \mathrm{d}\xi/(2\pi)^d$.

Symplectic Fourier transform

For $b \in L^2(\mathbb{R}^{2d}_P;\mathbb{C})$, $P = (p_x, p_\xi)$, $\sigma(P,P') = p_\xi.p'_x - p_x.p'_\xi$,

$$\mathcal{F}^\sigma b(P) = \int_{\mathbb{R}^{2d}_{x,\xi}} e^{-i\sigma(P,P')}\, b(P')đP' \quad \text{and} \quad \left(\mathcal{F}^\sigma\right)^{-1} = \mathcal{F}^\sigma$$

with $đP' = \mathrm{d}P'/(2\pi)^d$.

E.3. **Weyl quantization.**

- $\tau^h_{X'} = \left(e^{-i\sigma(\cdot,X')}\right)^W (hx, \mathrm{D}_x) = e^{-i\sigma(\cdot,X')^W(hx,\mathrm{D}_x)} = e^{i(\xi'\cdot hx - x'\cdot \mathrm{D}_x)}$
- $\hat\tau^h_{X'} = \mathcal{F}_x \tau^h_{X'} \mathcal{F}_x^{-1} = e^{-i(\xi'\cdot hD_\xi + x'\cdot\xi)}$
- $\tau^h_{X_1}\tau^h_{X_2} = e^{\frac{i}{2}h\sigma(X_1,X_2)}\tau^h_{X_1+X_2} = e^{ih\sigma(X_1,X_2)}\tau^h_{X_2}\tau^h_{X_1}$
- $\hat\tau^h_{X_1}\hat\tau^h_{X_2} = e^{\frac{i}{2}h\sigma(X_1,X_2)}\hat\tau^h_{X_1+X_2} = e^{ih\sigma(X_1,X_2)}\hat\tau^h_{X_2}\hat\tau^h_{X_1}$

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