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# EXISTENCE OF NODAL SOLUTIONS FOR DIRAC FIELD WITH FRACTIONAL NONLINEARITY

LOÏC LE TREUST

ABSTRACT. We prove by a shooting method the existence of infinitely many nodal solutions of the form  $\psi(x^0, x) = e^{-i\Omega x^0} \chi(x)$  for nonlinear Dirac equations:

$$i \sum_{\mu=0}^3 \gamma^\mu \partial_\mu \psi - m\psi - p|\bar{\psi}\psi|^{p-1}\psi = 0.$$

with  $m > 0$ ,  $p \in (0, 1)$  and  $\chi(x)$  compactly supported under some restrictive conditions over  $p$  and the frequency  $\Omega > m$ . We then study their behavior as  $p$  tends to zero to establish the link between these solutions and the M.I.T. bag model ones.

## CONTENTS

1. Introduction	1
2. Notations and preliminary results	3
2.1. Qualitative results	4
2.2. Regularization functions	6
2.3. Existence and local uniqueness results	7
3. The shooting method	8
3.1. Main results	8
3.2. Topological results	12
4. Existence of solution of (1.5)	13
5. M.I.T bag model	13
Appendix	16
References	18
Acknowledgment	18

## 1. INTRODUCTION

We study in this paper a class of relativistic models proposed by Mathieu and Saly [6, 5] that account for the internal structure of hadrons, that is how strong interaction forces bind quarks together. They recovered the M.I.T. bag model [4] as a limiting case.

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*Key words and phrases.* Nodal solutions, Shooting method, Winding number, M.I.T. bag model, Exited states.

We will look for solutions of the Dirac equation augmented by a nonlinear self-interacting term:

$$(1.1) \quad i \sum_{\mu=0}^3 \gamma^\mu \partial_\mu \psi - m\psi - F(\bar{\psi}\psi)\psi = 0.$$

The notations are the followings:  $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^4$ ,  $\partial_\mu \psi = \frac{\partial}{\partial x^\mu}$ ,  $0 \leq \mu \leq 3$ , with Einstein's convention for summation over  $\mu$ , and  $m > 0$ . We write  $\bar{\psi}\psi = (\gamma^0 \psi, \psi)$  where  $(.,.)$  is the usual scalar product and  $\gamma^\mu$  are the  $4 \times 4$  Pauli-Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ and } \gamma^\mu = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \text{ for } k = 1, 2, 3,$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We assume throughout this paper that

$$F(x) = \begin{cases} p|x|^{p-1} & \text{if } |x| > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with  $p \in (0, 1)$ . The solutions are sought among stationary states

$$(1.2) \quad \psi(x^0, x) = e^{-i\Omega x^0} \chi(x)$$

where  $x = (x^1, x^2, x^3) \in \mathbb{R}^3$  and whose frequency verifies  $\Omega > m$ , so that  $\chi$  is solution of:

$$(1.3) \quad i \sum_{k=1}^3 \gamma^k \partial_k \chi + \Omega \gamma^0 \chi - m\chi - F(\bar{\chi}\chi)\chi = 0.$$

Moreover, we restrict ourselves to functions given by the particular ansatz:

$$(1.4) \quad \chi(x) = \begin{pmatrix} v(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ iu(r) \begin{pmatrix} \cos \theta \\ \sin \theta e^{i\Phi} \end{pmatrix} \end{pmatrix}$$

where  $r = |x|$  and  $(\theta, \Phi)$  are the spherical coordinates in  $\mathbb{R}^3$  with  $\chi$  localized, that is :

$$\lim_{r \rightarrow \infty} (u, v) = 0.$$

Equation (1.3) then becomes a nonautonomous system of ordinary differential equations

$$(1.5) \quad \begin{cases} u' + \frac{2u}{r} & = v(-F(v^2 - u^2) - (m - \Omega)) \\ v' & = u(-F(v^2 - u^2) - (m + \Omega)). \end{cases}$$

For this to be well defined, we impose on  $u$  to be zero at zero and we consider the following Cauchy problem for  $x \in \mathbb{R}^+$ :

$$(1.6) \quad \begin{cases} (1.5) \\ (u(0), v(0)) = (0, x). \end{cases}$$

Notice that the nonlinearity  $F$  allows the zero map to be solution of (1.1), so that the solutions of (1.6) will be extended by zero when they hit zero. We choose  $x$  nonnegative due to the symmetry of the equations. For the sake of notation

simplicity, we will not write the  $p$  dependence unless it is necessary (for instance, we write  $(u, v)$ , (1.6) *etc* instead of  $(u_p, v_p)$ , (1.6) <sub>$p$</sub>  *etc*).

This phenomenological model has been proposed by Mathieu and Saly [6, 5]. They recovered the M.I.T. bag model [4] as a limiting case as  $p$  tends to zero and observed numerically a localization mechanism, indeed the solutions are compactly supported. Balabane, Cazenave and Vazquez [1] have shown by a shooting method the existence of a ground state for a more general class of nonlinearities and obtained a necessary and sufficient condition on them to give rise to compactly supported solutions.

Following [1], we adapt to our framework the shooting method of Balabane, Dolbeault and Ounaies [2] which established the existence of infinitely many compactly supported solutions for a sublinear elliptic equation with any given number of nodes. In their case, the lack of regularity of the nonlinearity in zero forces the solutions to be compactly supported. Our case is wilder since we have to deal with a lack of regularity not only in zero but in the whole set  $\{|u| = |v|\}$ . In this paper, we provide the first rigorous proof of the existence of localized nodal excited state solutions for the fractional model with  $p$  small studied in [5].

Afterwards, we study how these solutions are related to the M.I.T. bag model ones as it was first studied in [6].

Let us now state our main results:

**Theorem 1.1.** *There are  $\bar{p} \in (0, 1)$  and, for every  $p \in (0, \bar{p})$ , a constant  $\Omega_p > m$  such that if  $\Omega > \Omega_p$ , there exists an unbounded increasing sequence  $(x_k)_{k \in \mathbb{N}}$  of initial data such that for any  $k \in \mathbb{N}$ , the Cauchy problem (1.6) has a compactly supported solution which crosses the set  $\{(u, 0) | u \neq 0\}$  exactly  $k$  times.*

The following theorem establishes the close link between the fractional model and the M.I.T bag one.

**Theorem 1.2.** *For all  $k \in \mathbb{N}$ , there exist a finite number of points  $\{R^1, \dots, R^l\}$  with  $l \leq 2k + 1$ ,  $(u_0, v_0) \in \mathcal{C}^1(\mathbb{R}^+ \setminus \{R^1, \dots, R^l\})$  and a decreasing sequence  $(p_n)$  converging to zero such that :*

- (i)  $(u_{p_n}, v_{p_n})$  converges to  $(u_0, v_0)$  in  $L^2$  and uniformly on every compact interval of  $[0, R^l] \setminus \{R^1, \dots, R^l\}$  such that  $(u_0, v_0) \not\equiv 0$ .
- (ii)  $(u_0, v_0)$  is a solution of the free Dirac equation on  $[0, R^l] \setminus \{R^1, \dots, R^l\}$ :

$$\begin{cases} u' + \frac{2u}{r} = v(\Omega - m) \\ v' = -u(\Omega + m), \end{cases}$$

*discontinuous in  $R^1, \dots, R^l$ .*

- (iii)  $(u_0, v_0) \equiv 0$  on  $[R^l, \infty)$  and  $(v_0^2 - u_0^2)(R^i) = 0$ ,
- (iv)  $v_0^2 - u_0^2$  is positive on  $[0, R^1)$  and on exactly  $k$  intervals  $(R^i, R^{i+1})$ ,

where  $(u_p, v_p)$  is the solution of (1.6) <sub>$p$</sub>  found by theorem 1.1 with  $k$  nodes.

Let us notice that  $(u_0, v_0)$  is discontinuous at each bound of the  $k$  intervals of point (iv), so it has at least  $k$  discontinuities before  $R^l$ .

Indeed, in the case  $k = 0$ , the solution  $(u_0, v_0)$  is the ground state of the M.I.T. bag model as said [6] whereas the other nodal solutions are different from the ones derived by P. Mathieu in [5] (by lack of continuity).

Section 2 gives the framework of the proof of theorem 1.1 and some preliminary results. The core of the shooting method is given in the section 3 and the proof of

theorem 1.1 in section 4. We study the relation between the fractional model and the M.I.T. bag one in section 5.

## 2. NOTATIONS AND PRELIMINARY RESULTS

The singularity of the nonlinear term in equations (1.5) forces the solutions to be compactly supported as proved in [1] for the ground state. But it is the source of a lack of regularity when the trajectories cross the set  $\{|u| = |v|\}$ . So we have to consider solutions in the extended sense (see [3]). Indeed, our main contribution is to deal with the shooting method of [2] in a nonregular framework. This singularity introduces some additional difficulties since the main E.D.O. theorems [3] fail to show local uniqueness. Then we introduce a smooth function  $\varphi$  which vanishes in a neighborhood of  $\{|u| = |v|\}$  and whose values are in  $[0, 1]$  to regularize the system (1.5):

$$(2.1) \quad \begin{cases} u' + \frac{2u}{r}\varphi(u^2, v^2) & = v(-F(v^2 - u^2) - (m - \Omega)) \\ v' & = u(-F(v^2 - u^2) - (m + \Omega)). \end{cases}$$

In fact, this regularization does not give rise to regular solutions. But, since the system (2.1) is autonomous and hamiltonian near  $\{|u| = |v|\}$ , we get local existence and uniqueness while keeping qualitative properties. We give later (section 2.2) the rigorous definition of a family of regularization functions  $(\varphi_\epsilon)_\epsilon$  such that we will be able to pass to the limit to obtain the solutions of theorem 1.1.

Let us now define more precisely how we extend the notion of solution:

**Definition 2.1.**  $(u, v)$  is a solution of the system (2.1) if there exist two real numbers  $0 \leq R_1 < R_2 \leq \infty$  such that the following holds:

- (i)  $(u, v) \in W^{1,1}([R_1, R_2]) \cap C^1(\{r \in (R_1, R_2) | u^2 \neq v^2\})$
- (ii)  $u(0) = 0$  and  $v(0) > 0$  if  $R_1 = 0$
- (iii)  $\forall r \in [R_1, R_2], (u, v)(r) \neq 0$
- (iv)  $(u, v)$  is a solution of (2.1).

Notice that if  $\varphi(u^2, v^2) \equiv 1$ , then  $(u, v)$  is a solution of the system (1.5).

We remark that conditions (i-iv) ensure that  $(u, v)$  is a solution in the classical sense on  $\{r \in (R_1, R_2) | u^2 \neq v^2\}$ , (ii) guaranties the absence of singularity at the origin and we can assume  $v(0) > 0$  in (ii) due to the symmetries of (2.1).

When a solution  $(u, v)$  hits 0 at a finite radius, we extend it by 0 for the remaining values of  $r$ , so that the global function  $\psi$  induced (1.2-1.4) is solution of (1.1). We note  $x = v(0)$  the parameter of the shooting method and we will sometimes write  $(u_x, v_x)$ ,  $R_x$  etc instead of  $(u, v)$ ,  $R$  etc when we want to emphasize the solution dependance on  $x$ .

**2.1. Qualitative results.** Let us introduce two continuous functions on  $\mathbb{R}^2$ , the energy:

$$H(u^2, v^2) = -\frac{1}{2}(v^2 - u^2)|v^2 - u^2|^{p-1} + \frac{\Omega - m}{2}v^2 + \frac{\Omega + m}{2}u^2$$

and for  $\epsilon \in (0, m)$ ,

$$\begin{aligned} H_\epsilon(u^2, v^2) &= H(u^2, v^2) - (\Omega - \epsilon)v^2 \\ &= -(v^2 - u^2) \left( \frac{1}{2} |v^2 - u^2|^{p-1} + \frac{\Omega + m}{2} \right) + \epsilon v^2. \end{aligned}$$

Following [1], the next lemma gives us one of the main qualitative results, that is the decay of the energy. Let us assume in the sequel that  $(u, v)$  is a solution of (2.1) for some  $\varphi$  fixed.

**Lemma 2.2.** *We have:*

$$\frac{d}{dr}H(u^2, v^2)(r) = -\frac{2u^2\varphi(u^2, v^2)}{r}(p|v^2 - u^2|^{p-1} + (m + \Omega)),$$

so  $r \mapsto H(u^2, v^2)$  is nonincreasing.

*Proof.* We have :

$$\begin{aligned} \frac{d}{dr}H(u^2, v^2)(r) &= -p(v'v - u'u)|v^2 - u^2|^{p-1} + v'v(\Omega - m) + u'u(\Omega + m) \\ &= v'(u' + \frac{2u}{r}\varphi(u^2, v^2)) - u'v' \\ &= v' \left( \frac{2u}{r}\varphi(u^2, v^2) \right) \\ &= -\frac{2u^2\varphi(u^2, v^2)}{r}(p|v^2 - u^2|^{p-1} + (m + \Omega)). \end{aligned}$$

□

Let us study now the energy level shapes (figure 1) which thus play a key role in our study.

**Lemma 2.3.** *Let  $\varepsilon \in [0, m)$ ,*

(i) *there exist two constants  $A > 0$  and  $B > 0$  such that:*

$$H(u^2, v^2) \geq A(u^2 + v^2) - B \text{ for all } (u, v) \in \mathbb{R}^2;$$

(ii) *the set  $H^{-1}(-\infty, C)$  is bounded for every  $C$ ;*

(iii) *for  $\varepsilon > 0$ ,  $H_\varepsilon^{-1}(\{0\})$  is a connected unbounded curve in  $\{u^2 < v^2\} \cup \{0\}$ , such that  $H_\varepsilon^{-1}(\{0\}) \cap H^{-1}(\{0\}) = \{0\}$ ;*

(iv)  *$H_0^{-1}(\{0\}) = \{u^2 = v^2\}$ ;*

(v) *for all  $\gamma > 0$ , there exists  $C_\gamma > 0$  and  $D_\gamma > 0$  such that for all  $(u, v) \in \mathbb{R}^2$ :*

$$H(u^2, v^2) \geq \gamma \Rightarrow \begin{cases} C_\gamma H(u^2, v^2) \geq u^2. \\ u^2 + v^2 \geq D_\gamma. \end{cases}$$

Moreover,  $\gamma \mapsto C_\gamma$  is nonincreasing,  $\lim_{\gamma \rightarrow 0} C_\gamma = \infty$ ,  $\lim_{p \rightarrow 0} C_\gamma = 1/\Omega$ . Let us write  $a = (\Omega - m)^{\frac{1}{2(1-p)}} = \sup\{v | \exists u; H(u^2, v^2) = 0\}$ .

The proof is a straightforward calculation and is postponed in the appendix.

The second main qualitative result is the following lemma:

**Lemma 2.4.** *Let  $\theta(r) = \arctan(\frac{u}{v})(r)$ . We have:*

$$\begin{aligned} \theta(r)' &= \frac{pH(u^2, v^2) + (1-p)(v^2(\Omega - m)) + u^2(\Omega + m) - 2uv\varphi(u^2, v^2)/r}{u^2 + v^2} \\ &\geq (1-p)(\Omega - m) - \varphi(u^2, v^2)/r \text{ if } H(u^2, v^2)(r) \geq 0. \end{aligned}$$

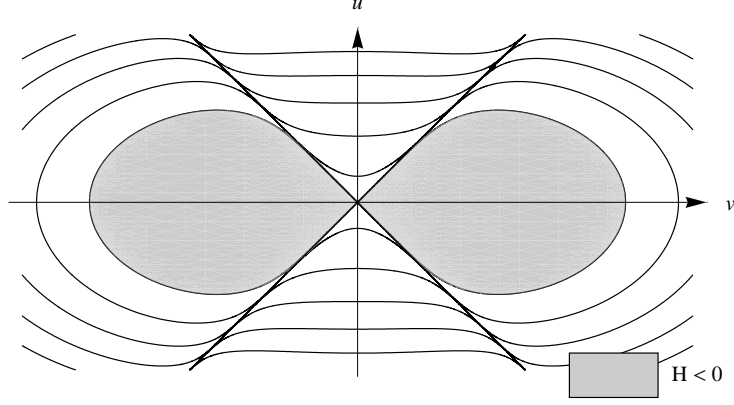


FIGURE 1. Energy levels

*Proof.* We have if  $H(u^2, v^2)(r) \geq 0$ :

$$\begin{aligned}
\theta(r)' &= \frac{u'v - v'u}{u^2 + v^2} \\
&= \frac{-p(v^2 - u^2)|v^2 - u^2|^{p-1} + v^2(\Omega - m) + u^2(\Omega + m) - 2uv\varphi(u^2, v^2)/r}{u^2 + v^2} \\
&= \frac{pH(u^2, v^2) + (1-p)(v^2(\Omega - m) + u^2(\Omega + m)) - 2uv\varphi(u^2, v^2)/r}{u^2 + v^2} \\
&\geq \frac{(1-p)(\Omega - m)(v^2 + u^2) - 2uv\varphi(u^2, v^2)/r}{u^2 + v^2} \\
&\geq (1-p)(\Omega - m) - \varphi(u^2, v^2)/r.
\end{aligned}$$

□

In the next lemma, we give uniform estimations far enough from  $r = 0$  and from the first radius of the crossing of the set  $\{u^2 = v^2\}$  to get a lower bound on  $\theta'$ .

**Lemma 2.5.** *There exist a unique  $\bar{p} \in (0, 1)$  and, for all  $0 < q < \bar{p}$ ,  $\Omega_q > m$  such that if  $\Omega > \Omega_q$ , then there are  $r_0 > 0$ ,  $\alpha > 0$  and we have for all  $0 < p \leq q$ :*

- (i)  $r_0 > \frac{1}{(\Omega - m)(1 - p)}$ ,
- (ii)  $v_{x,p}^2 - u_{x,p}^2 \geq \alpha x^2$  for all  $r \in [0, r_0)$
- (iii) there exists  $\Theta > 0$  such that if  $r > r_0$  and  $H_p(u_{x,p}^2, v_{x,p}^2)(r) \geq 0$  then  $\theta_{x,p}(r)' \geq \Theta$ ;

for every solution  $(u_{x,p}, v_{x,p})$  of (1.6)<sub>p</sub>. ( $\bar{p} \simeq 0.0173622$ )

For  $\zeta > 0$  and  $p \in (0, \bar{p} - \zeta]$ ,  $r_0$ ,  $\Theta$ ,  $\alpha$  and  $\Omega_p$  may be chosen independent on  $p$ .

The proof is a straightforward calculus similar to the one of [1] and is postponed in the appendix. We fix  $p < \bar{p}$  and  $\Omega > \Omega_p$  in the sequel.

**2.2. Regularization functions.** We are now able to construct the regularization functions. Let us define for  $\varepsilon \in (0, m)$  fixed, the sets (figure 2):

$$\begin{aligned}
\mathcal{R}_\varepsilon^1 &= \{(u^2, v^2) \in \mathbb{R}^2 : |v^2 - u^2| \leq \alpha a^2, H_\varepsilon(u^2, v^2) \geq 0, H_\varepsilon(v^2, u^2) \geq 0\} \\
\mathcal{R}_\varepsilon^2 &= \{(u^2, v^2) \in \mathbb{R}^2 : |v^2 - u^2| \leq \alpha a^2/2, H_{\frac{\varepsilon}{2}}(u^2, v^2) \geq 0, H_{\frac{\varepsilon}{2}}(v^2, u^2) \geq 0\},
\end{aligned}$$

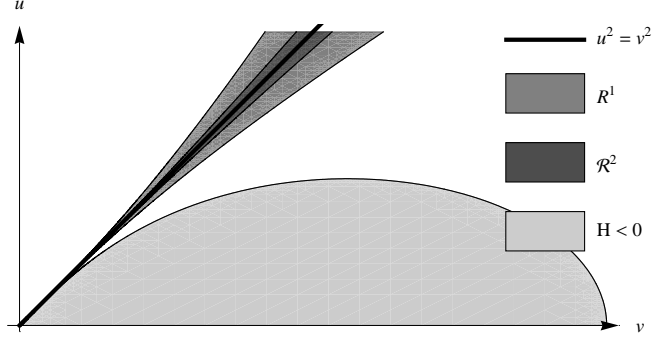


FIGURE 2. Regularization area

where  $a$  and  $\alpha$  are the constants of lemmas 2.3–2.5.

We choose for  $\varphi_\varepsilon$  a smooth function with values in  $[0, 1]$  such that

$$\varphi_\varepsilon \equiv \begin{cases} 0 & \text{on } \mathcal{R}_\varepsilon^2 \\ 1 & \text{on } (\mathcal{R}_\varepsilon^1)^c. \end{cases}$$

Moreover, we have  $\cap_{\varepsilon>0} \mathcal{R}_\varepsilon^1 = \{u^2 = v^2\}$  because of lemma 2.3. Notice that for every  $x \geq a$ ,  $(u_x, v_x)(r_0)$  does not belong to  $\mathcal{R}_\varepsilon^1$  by lemma 2.5.

**2.3. Existence and local uniqueness results.** Let us fix in this part  $\varepsilon \in (0, m)$ . For the sake of simplicity, we remove  $\varepsilon$  from the notation.

**Lemma 2.6.** *Consider the initial value problem for  $r_1 \geq 0$ ,  $(u_1, v_1) \in \mathbb{R}^2 \setminus \{0\}$ :*

$$\begin{cases} (2.1) \\ (u, v)(r_1) = (u_1, v_1), \end{cases}$$

*such that  $u_1 = 0$  if  $r_1 = 0$ . Then there exists a unique local solution.*

*Proof.* The Cauchy-Lipschitz theorem shows this result provided that  $r_1 \neq 0$  and  $u_1^2 \neq v_1^2$ . The contraction mapping argument sketched in [1] ensures the result for  $r_1 = 0$ . So the point is when  $u_1^2 = v_1^2$ . In this case, the system (2.1) reduces itself into the autonomous hamiltonian one:

$$\begin{cases} u' = v(-p|v^2 - u^2|^{p-1} - (m - \Omega)) \\ v' = u(-p|v^2 - u^2|^{p-1} - (m + \Omega)). \end{cases}$$

Because of lemma 2.2, any such solution has a constant energy. Moreover lemma 2.4 and the energy level shapes (figure 1) show that these solutions hit the set  $\{(u, v) | u^2 = v^2\} \setminus \{0\}$  as long as  $H(u^2, v^2) > 0$ . This sets the existence and uniqueness on the left. The same argument is still valid on the right.  $\square$

**Lemma 2.7.** *For every initial condition  $(u, v) = (0, x)$ , there exists a unique maximal solution of the problem (2.1) on an interval  $[0, R_x)$  and  $\|(u, v)\| \leq x$  on  $[0, R_x)$ . If  $R_x < \infty$ , the solution hits 0 at  $R_x$ .*

*Proof.* There exists a unique maximal solution of the problem (2.1) on an interval  $[0, R_x)$  because of lemma 2.6. The decay of the energy 2.2 and energy level shapes show that  $\|(u, v)\| \leq x$  on  $[0, R_x)$ . Lemma 2.6 gives the latter assertion.  $\square$



As we look for localized solutions, we extend those such that  $R_x < \infty$  by 0 so that every  $x > 0$  gives rise to a function defined on  $\mathbb{R}^+$ .

### 3. THE SHOOTING METHOD

We set for every  $\gamma \geq 0$ ,

$$\rho_x(\gamma) := \sup\{r > 0 : H(u_x^2, v_x^2) \geq \gamma\},$$

and the winding number

$$N_x(a, b) := \#\{r \in (a, b) | v_x(r) = 0\}.$$

We will write  $N_x(b)$  instead of  $N_x(0, b)$ . The core of the shooting method will be the study of the following sets which are introduced in [2]:

**Definition 3.1.**

$$\begin{aligned} A_k &:= \{x > 0 \mid \lim_{r \rightarrow \infty} H(u_x^2, v_x^2)(r) < 0, (u_x, v_x)(r) \neq 0 \forall r \geq 0, N_x(\infty) = k\} \\ I_k &:= \{x > 0 \mid \lim_{r \rightarrow \rho_x(0)} (u_x^2, v_x^2)(r) = 0, N_x(\rho_x(0)) = k\}. \end{aligned}$$

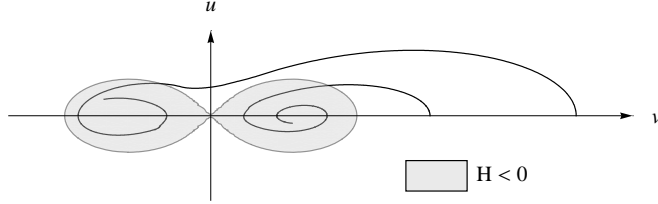


FIGURE 3. Solutions belonging to  $A_0$  and  $A_1$ .

These sets are well defined because of 2.7. The key idea is that the trajectories wind around the zero energy level and cross it in finite time. Some of them fall into the negative energy domain and form the  $A_k$  sets, these cannot converge to zero so do not yield to localized solutions of (2.1). Others converge to zero, these are the sought ones.

#### 3.1. Main results.

**Lemma 3.2.** *Let us assume that  $r_0 < \rho_x(0)$  and let  $r_0 \leq a < b \leq \rho_x(0)$ , then:*

$$N_x(a, b) \geq \lfloor \frac{\Theta}{\pi}(b - a) \rfloor$$

*Proof.* By lemma 2.5, we get

$$\theta_x(b) - \theta_x(a) = \int_a^b \theta_x(s)' ds \geq \Theta(b - a),$$

so that  $N_x(a, b) \geq \lfloor \frac{\Theta}{\pi}(b - a) \rfloor$ . □

Let  $\gamma > 0$  such that  $q := 2(1 - p)(\Omega + m)C_\gamma - 1 > 0$ , such a  $\gamma$  exists because of lemma 2.3.

**Lemma 3.3.** *Let us assume that  $\rho_x(\gamma) > 0$ , then we have:*

$$\frac{d}{dr} \left( r^{2(\Omega+m)C_\gamma} H(u_x^2, v_x^2) \right)^{1-p} \geq -2p(1-p)C_\gamma^p \rho(\gamma)^q |1 - 2\sin^2 \theta_x|^{p-1}.$$

*Proof.* For the sake of clarity, we remove here the  $x$  subscripts. By lemma 2.2, we have for  $r \in (0, \rho(\gamma))$ :

$$\begin{aligned} \frac{d}{dr} H(u^2, v^2) &\geq -\frac{2u^2}{r} (p|v^2 - u^2|^{p-1} + (m + \Omega)) \\ &\geq -\frac{2C_\gamma(\Omega + m)H(u^2, v^2)}{r} - \frac{2pu^2}{r(u^2 + v^2)^{1-p}} |1 - 2\sin^2 \theta|^{p-1} \\ &\geq -\frac{2C_\gamma(\Omega + m)H(u^2, v^2)}{r} - \frac{2pC_\gamma^p H(u^2, v^2)^p |1 - 2\sin^2 \theta|^{p-1}}{r}, \end{aligned}$$

so,

$$\begin{aligned} \frac{d}{dr} \left( r^{2(\Omega+m)C_\gamma} H(u^2, v^2) \right) &\geq -2pC_\gamma^p H(u^2, v^2)^p |1 - 2\sin^2 \theta|^{p-1} r^{2(\Omega+m)C_\gamma-1} \\ &\geq -2pC_\gamma^p |1 - 2\sin^2 \theta|^{p-1} r^q (r^{2(\Omega+m)C_\gamma} H(u^2, v^2))^p \\ &\geq -2pC_\gamma^p |1 - 2\sin^2 \theta|^{p-1} \rho(\gamma)^q (r^{2(\Omega+m)C_\gamma} H(u^2, v^2))^p, \end{aligned}$$

this gives us the result.  $\square$

**Proposition 3.4.** *We have  $\lim_{x \rightarrow \infty} N_x(r_0, \rho_x(\gamma)) = \infty$ .*

*Proof.* For  $x$  big enough, we have  $H(u_x^2, v_x^2)(r_0) > \gamma$  by the second point of lemma 2.5. The previous lemma ensures that:

$$\begin{aligned} &\left( \rho_x(\gamma)^{2(\Omega+m)C_\gamma} \gamma \right)^{1-p} - \left( r_0^{2(\Omega+m)C_\gamma} H(u_x^2, v_x^2)(r_0) \right)^{1-p} \\ &\geq -2p(1-p)C_\gamma^p \rho_x(\gamma)^q \int_{r_0}^{\rho_x(\gamma)} |1 - 2\sin^2 \theta_x(s)|^{p-1} ds \\ &\geq -\frac{2p(1-p)C_\gamma^p \rho_x(\gamma)^q}{\Theta} \int_{r_0}^{\rho_x(\gamma)} |1 - 2\sin^2 \theta_x(s)|^{p-1} \theta'_x(s) ds \\ &\geq -\frac{2p(1-p)C_\gamma^p \rho_x(\gamma)^q}{\Theta} \int_{\theta_x(r_0)}^{\theta_x(\rho_x(\gamma))} |1 - 2\sin^2 \theta|^{p-1} d\theta \\ &\geq -\frac{2p(1-p)C_\gamma^p \rho_x(\gamma)^q}{\Theta} (N_x(r_0, \rho_x(\gamma)) + 2) \int_0^\pi |1 - 2\sin^2 \theta|^{p-1} d\theta, \end{aligned}$$

since  $p \in (0, 1)$ ,  $\int_0^\pi |1 - 2\sin^2 \theta|^{p-1} d\theta < \infty$ . Lemma 2.5 ensures that

$$\lim_{x \rightarrow \infty} H(u_x^2, v_x^2)(r_0) = \infty,$$

so lemma 3.2 and the previous inequality give the result.  $\square$

**Proposition 3.5.** *For all  $k \in \mathbb{N}$ , there exists  $\sigma > 0$  such that if  $N_x(\bar{R}) = k$  and  $u_x^2(\bar{R}) + v_x^2(\bar{R}) < \sigma^2$  for some  $x$  and  $\bar{R}$ , then  $x$  belongs to  $A_k \cup I_k \cup A_{k+1}$ . The parameter  $\sigma$  may be chosen independent on  $\varepsilon$ .*

*Proof.* The proof is based on the one of the proposition 3 from [2] with a slightly more complicated trapping region. The decay of the energy makes the result obvious if  $H(u_x^2, v_x^2)(\bar{R}) \leq 0$ . By symmetry, we will just show it in the case

$$(u_x^2, v_x^2)(\bar{R}) \in \{u > 0\} \cup \{H > 0\} \text{ and } x > a.$$

Let  $(M_1, M_2)$  be the unique point of

$$H^{-1}(\{0\}) \cap \{v > u > 0\} \cap \{v^2 - u^2 = (\frac{\Omega - m}{p})^{\frac{1}{p-1}}\},$$

$(M_1/2, v_1)$  be the unique point of  $\partial\mathcal{R}_\varepsilon^1 \cap \{u > v > 0\}$ . We set

$$K = H((M_1/2)^2, v_1^2) > 0.$$

The parameter  $\sigma$  is chosen such that:

$$0 < \sigma < \frac{M_1^3}{4M_2(\frac{\pi(k+2)}{\Theta} + r_0)},$$

and  $B(0, \sigma) \subset H^{-1}(-\infty, K) \cap \{0 < u < M_1/2\}$ . Let

$$\mathcal{D} = \{0 < H < K\} \cap \{0 < u < M_1\},$$

(see figure 4). If  $(u, v)$  exits  $\mathcal{D}$  crossing the boundary at  $\{H = 0\}$ , we have the result. We now prove that this is the only possible way to exit  $\mathcal{D}$ .

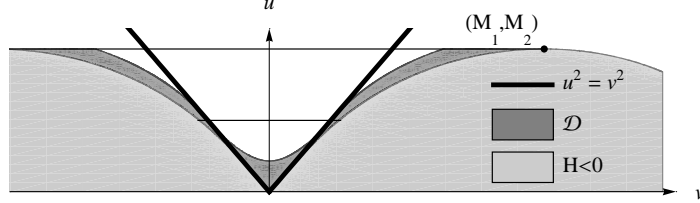


FIGURE 4. Trapping region

Let us assume by contradiction that  $(u, v)$  do not cross the boundary at  $\{H = 0\}$ , then by the decay of  $H$ ,  $(u, v)$  must exit  $\mathcal{D}$  at  $\{u = M_1\}$ . Let  $R'' > \bar{R}$  be the value of  $r$  at the crossing, then  $H(u^2, v^2)(R'') > 0$ . Since  $x > a$ , lemma 2.5 ensures that  $r_0 < \bar{R}$ . By lemma 3.2, we have:

$$\begin{aligned} k = N(\bar{R}) &\leq N(R'') \leq k + 1, \\ N(R'') &\geq N(r_0, R'') \geq \lfloor \frac{\Theta}{\pi}(R'' - r_0) \rfloor, \end{aligned}$$

so

$$R'' \leq \frac{\pi(k+2)}{\Theta} + r_0.$$

Let  $R' = \sup\{r \in (\bar{R}, R'') \mid u < M_1/2\}$ . By the decay of the  $H$ , we have  $\varphi_\varepsilon \equiv 1$  on  $(R', R'')$ , so

$$\begin{aligned} H(R') - H(R'') &= \int_{R'}^{R''} \frac{2u^2}{r} (p|u^2 - v^2|^{p-1} + (m + \Omega)) dr \\ &\geq \frac{M_1^2}{2(\frac{\pi(k+2)}{\Theta} + r_0)} \int_{R'}^{R''} (p|u^2 - v^2|^{p-1} + (m + \Omega)) dr, \end{aligned}$$

moreover,

$$\begin{aligned} M_1/2 = u(R'') - u(R') &= \int_{R'}^{R''} -v(p|u^2 - v^2|^{p-1} + (m + \Omega)) - \frac{2u}{r} dr \\ &\leq M_2 \int_{R'}^{R''} (p|u^2 - v^2|^{p-1} + (m + \Omega)) dr, \end{aligned}$$

since  $v \leq 0$  and  $u \geq 0$  on  $(R', R'')$ . Thus, we have:

$$\begin{aligned} H(R'') &\leq H(R') - \frac{M_1^3}{4M_2(\frac{\pi(k+2)}{\sigma} + r_0)} \\ &\leq H(\bar{R}) - \sigma \leq 0. \end{aligned}$$

This is the wanted contradiction. Indeed, in the nonregularized case, we have that

$$M_1^2 = \left( \frac{\Omega - m}{p} \right)^{\left( \frac{p}{p-1} \right)} \left( \frac{1-p}{2\Omega} \right),$$

and we can show that all the above constants can be chosen independent on  $\varepsilon$ .  $\square$

**Lemma 3.6.** *Let us assume that there exist  $y > 0$  and  $\bar{R} > r_0$  such that*

$$\|(u_x, v_x)(r)\| \leq y \text{ and } H(u_x^2, v_x^2) > 0,$$

for all  $r \in [r_0, \bar{R}]$  where  $\|\cdot\|$  is the euclidean norm. Then for all  $s \in (1, \frac{1}{1-p})$ , there exists  $C > 0$  such that for all  $r_0 < r_1 < r_2 < \bar{R}$ :

$$\|u_x^2\|_{W^{1,s}(r_1, r_2)}, \|v_x^2\|_{W^{1,s}(r_1, r_2)}, \|u_x v_x\|_{W^{1,s}(r_1, r_2)} \leq C(N(r_1, r_2) + (r_2 - r_1) + 1),$$

$C$  does not depend  $\varepsilon$  and  $x$ .

*Proof.* For the sake of clarity, we remove here the  $x$  subscripts. We have:

$$\frac{du^2}{dr} = 2uv(-p|u^2 - v^2|^{p-1} + (\Omega - m)) - \frac{2u^2\varphi(u^2, v^2)}{r},$$

so, for  $r_0 < r_1 < r_2 < \bar{R}$ ,

$$\begin{aligned} \int_{r_1}^{r_2} \left| \frac{du^2}{dr} \right|^s dr &\leq C \left( \int_{r_1}^{r_2} |1 - 2\sin^2 \theta|^{s(p-1)} dr + (r_2 - r_1) \right) \\ &\leq C(N(r_1, r_2) + (r_2 - r_1) + 2), \end{aligned}$$

because of lemma 2.5 and  $s(p-1) > -1$ . The same proof works as well for  $v^2$  and  $uv$ .  $\square$

**Lemma 3.7.** *Let  $(\bar{u}, \bar{v})$  be a solution of (2.1) such that  $\|(\bar{u}, \bar{v})\| \neq 0$  on  $[r_1, r_2]$ . For all  $\eta > 0$ , there exists  $\delta > 0$ , such that if  $(u, v)$  is also a solution of (2.1) with  $\|(u, v)(r_1) - (\bar{u}, \bar{v})(r_1)\| < \delta$ , we have  $\|(u, v)(r) - (\bar{u}, \bar{v})(r)\| < \eta$  on  $[r_1, r_2]$ .*

*Proof.* The proof is the same as in [2] with the estimation of lemma 3.6.  $\square$

**Lemma 3.8.** *For all  $x > 0$ , we have:*

- (i) either  $N_x(\infty) < \infty$ ,
- (ii) or  $N_x(\infty) = \infty$ ,  $\lim_{r \rightarrow \infty} H(u_x^2, v_x^2) = 0$  and  $(u_x, v_x) \neq 0$  on  $(0, \infty)$ .

*Proof.* For the sake of clarity, we remove here the  $x$  subscripts. We recall that  $R$  is the latest radius before the solution hits 0.

(a) If  $R < \infty$ , then  $N(\infty) < \infty$ .

(b) Let us assume now that there exists  $r_1 \geq 0$  such that

$$H(u^2, v^2)(r_1) \leq 0 \text{ and } (u, v)(r_1) \neq 0.$$

Lemma 2.2 ensures that  $H(u^2, v^2)(r) < 0$  for a  $r > r_1$ , so  $R = \infty$  by lemma 2.7.  $(u, v)$  will not cross the set  $\{v = 0\}$  anymore and  $N(\infty) < \infty$ .

(c) Let us assume that there exists  $\gamma > 0$  such that  $H(u^2, v^2) \geq \gamma$  on  $(0, R)$ , then  $R = \infty$ . Let  $(r_n)$  be a sequence in  $(r_0, \infty)$  such that  $\lim_{n \rightarrow \infty} r_n = \infty$ . Because of lemma 2.4,  $\theta'$  is bounded from above for  $r > r_0$ . So the same proof as in lemma 3.2 ensures that there exists  $C > 0$  such that for all  $r_0 < r_1 < r_2$ ,  $N(r_1, r_2) \leq C(r_2 - r_1)$ . Lemma 3.6 shows then that for all  $r_0 < r_1$ , all  $0 < l$  and all  $s \in (1, \frac{1}{1-p})$ ,  $(u^2(\cdot + r_n))$ ,  $(v^2(\cdot + r_n))$  and  $(uv(\cdot + r_n))$  are bounded in  $W^{1,s}(r_1, r_1 + l)$ . Up to the extraction of a subsequence  $(n_k)$ , there exist  $U, V, W \in W^{1,s}(r_1, r_1 + l)$  such that:

$$\begin{aligned} U_k &:= (u^2(\cdot + r_{n_k})) \xrightarrow[k \rightarrow \infty]{} U \\ V_k &:= (v^2(\cdot + r_{n_k})) \xrightarrow[k \rightarrow \infty]{} V \\ W_k &:= (uv(\cdot + r_{n_k})) \xrightarrow[k \rightarrow \infty]{} W, \end{aligned}$$

in  $\mathcal{C}^0([r_1, r_1 + l])$ . By Fatou Lemma, we have:

$$0 = \liminf_{k \rightarrow \infty} \int_{r_1}^{r_1+l} \frac{U_k \varphi(U_k, V_k)}{r} (p|U_k - V_k|^{p-1} + (\Omega + m)) dr \geq \int_{r_1}^{r_1+l} \frac{U \varphi(U, V)}{r} (p|U - V|^{p-1} + (\Omega + m)) dr,$$

so,  $U \varphi(U, V) \equiv 0$ . We can then construct a solution  $(\bar{u}, \bar{v})$  of the autonomous system associated with (1.5) taking care of the sign of  $W$  such that  $\bar{u}^2 = U$  and  $\bar{v}^2 = V$ . This solution verifies  $H(\bar{u}^2, \bar{v}^2) \geq \gamma$ , its winding (lemma 2.4) contradicts  $U \varphi(U, V) \equiv 0$ , so that this case is impossible.

(d) The remaining case is the one of the second point of the lemma. □

**3.2. Topological results.** We are now able to give the properties of the  $A_k$  and  $I_k$  sets as in [2].

**Lemma 3.9.** *For all  $k \in \mathbb{N}$ ,*

- (i)  $A_k$  is an open set,
- (ii)  $A_k$  is bounded,
- (iii)  $I_k$  is bounded,
- (iv)  $\sup A_k \in I_{k-1} \cup I_k$ ,
- (v)  $\sup I_k \in I_k$ ,
- (vi) if  $x \in I_k$  then there exists a neighborhood  $V$  of  $x$  such that  $V \subset A_k \cup I_k \cup A_{k+1}$ .

*Proof.* We just prove here that  $\sup A_k$  belongs to  $\cup I_n$ . The same proof as in [2] shows the result with the help of the lemmas of the previous part.

Let  $x = \sup A_k$ , then we have  $x \notin \cup A_n$ . Let us assume that  $x \notin \cup I_n$ , then setting  $\bar{R} > \frac{\Theta(2+k)}{\pi} + r_0$ , we have  $H(u_x^2, v_x^2)(\bar{R}) > 0$  because of lemma 3.8 whereas there exists  $y \in A_k$  as close to  $x$  as we want such that  $H(u_y^2, v_u^2)(\bar{R}) < 0$ . This contradicts the flow continuity (lemma 3.7). So  $x$  belongs to  $\cup I_n$ . □

We get the key lemmas of [2], so that we have shown the following result:

**Proposition 3.10.**  $\forall \varepsilon \in (0, m)$ , there exists for all  $k \in \mathbb{N}$  a solution  $(u_x, v_x)$  of (2.1) such that:

- (i)  $R_x < \infty$ , i.e.  $(u_x, v_x)(R_x) = 0$ ,
- (ii)  $N_x(0, R_x) = k$ .

#### 4. EXISTENCE OF SOLUTION OF (1.5)

We give here the proof of theorem 1.1.

*Proof.* Let us fix  $k \in \mathbb{N}$ . We write here  $\varepsilon$  superscripts to emphasize the  $\varepsilon$  dependence of the solutions. Let  $(u_x^\varepsilon, v_x^\varepsilon)$  be a solution of (2.1) such that  $N_x^\varepsilon(\infty) = k$  with  $\varphi_\varepsilon$  defined in section 2.2. We have :

$$R_x^\varepsilon = \rho_x^\varepsilon(0) \leq \frac{(k+1)\pi}{\Theta} + r_0 = \bar{R}.$$

Let  $\gamma > 0$  such that  $q := 2(1-p)(\Omega+m)C_\gamma - 1 > 0$ , either  $H((u_x^\varepsilon)^2, (v_x^\varepsilon)^2)(r_0)$  is smaller than  $\gamma$  or not. In that case, we have because of lemma 3.3:

$$\begin{aligned} ((R_x^\varepsilon)^{2(\Omega+m)C_\gamma\gamma})^{1-p} - \left( r_0^{2(\Omega+m)C_\gamma} H((u_x^\varepsilon)^2, (v_x^\varepsilon)^2)(r_0) \right)^{1-p} \geq \\ -2p(1-p)C_\gamma^p (R_x^\varepsilon)^q \int_{r_0}^{R_x^\varepsilon} |1 - 2\sin^2 \theta|^{p-1} dr. \end{aligned}$$

this gives us an uniform bound on  $H((u_x^\varepsilon)^2, (v_x^\varepsilon)^2)(r_0)$  which depends neither on  $\varepsilon$  nor on  $x$ . By the decay of the  $H$ , we obtain a uniform bound on  $(u_x^\varepsilon, v_x^\varepsilon)$  on  $[r_0, \bar{R}]$  and thus an uniform Sobolev bound on  $((u_x^\varepsilon)^2, (v_x^\varepsilon)^2, u_x^\varepsilon v_x^\varepsilon)$  because of lemma 3.6. Following the same idea as in 3.8, we obtain the uniform convergence of a subsequence in  $\mathcal{C}^0([r_0, \bar{R}])$  to some functions  $U, V, W$  which allow us to construct a solution  $(u, v)$  of (1.5) such that  $(u, v)(\bar{R}) = 0$ . Let  $F(x) = (u_x, v_x)(r_0)$ , where  $(u_x, v_x)$  is a solution of (1.5).  $F$  is a bijective continuous map from  $[a, \infty)$  into  $F([a, \infty))$ . We have constructed the regularization system so that

$$\varphi_\varepsilon((u_x^\varepsilon)^2, (v_x^\varepsilon)^2)(r_0) = 1.$$

Since  $F([a, \infty))$  is closed,  $(u, v)(r_0) \in F([a, \infty))$  and there exists  $x > a$  such that  $(u_x, v_x)(r_0) = (u, v)(r_0)$ . This allow us to construct a solution of (1.5).

It just remains us to show that the winding number is preserved during the extraction. Let  $c_+ \in H^{-1}(\mathbb{R}_*^-) \cap \{(u, v) | u < 0 < v\}$  and  $\delta > 0$  such that  $B(c_+, \delta) \subset H^{-1}(\mathbb{R}_*^-) \cap \{(u, v) | u < 0 < v\}$ . We write  $c_- = -c_+$ . For every  $\varepsilon \geq 0$ , joining  $(u_x^\varepsilon, v_x^\varepsilon)(r_0)$  and 0, we construct the closed curve  $\gamma^\varepsilon$ . Setting:

$$N(\gamma) = -\frac{1}{2i\pi} \int_\gamma \left( \frac{1}{z - c_+} + \frac{1}{z + c_+} \right),$$

Lebesgue theorem shows that  $N(\gamma^\varepsilon) = k$  converge to  $N(\gamma^0)$  so that we have the result.  $\square$

#### 5. M.I.T BAG MODEL

Let us now study the link between the fractional model and the M.I.T. bag one. For any  $k \in \mathbb{N}$  fixed,  $(u_p, v_p)$  is the solution found by theorem 1.1 with  $k$  nodes and  $R_p$  is the radius at which it hits 0. We give here the proof of theorem 1.2.

*Proof.* We have by lemma 3.2 for every  $p$  small enough:

$$R_p \leq \frac{(k+1)\pi}{\Theta} = \bar{R}.$$

Lemma 2.5 ensures that  $\bar{R}$  does not depend on  $p$  provided that  $p \in (0, \bar{p} - \zeta]$ .

**Lemma 5.1.** *There exist  $g \in \mathcal{C}^0([r_0, \bar{R}])$ ,  $(u_0, v_0) \in \mathcal{C}^0(\{|g| > 0\})$ , a nonincreasing nonnegative function  $H_0$  and a zero convergent decreasing sequence  $(p_n)$  such that:*

- (i)  $(v_{p_n}^2 - u_{p_n}^2)$  converges uniformly to  $g$  in  $\mathcal{C}^0([r_0, \bar{R}])$ ,
  - (ii)  $(u_{p_n}, v_{p_n})$  converges uniformly to  $(u_0, v_0)$  on every compact interval of  $\{|g| > 0\}$  and  $v_0^2 - u_0^2 = g$ ,
  - (iii)  $H_{p_n}(u_{p_n}^2, v_{p_n}^2)$  converges a.e. to  $H_0$ ,
  - (iv)  $H_0 = -\frac{1}{2}\text{sgn}(v_0^2 - u_0^2) + \frac{\Omega-m}{2}v_0^2 + \frac{\Omega+m}{2}u_0^2$  on  $\{|g| > 0\}$ .
- $(u_0, v_0)$  is a solution of the free Dirac equation.

*Proof.* Let  $\gamma > 0$ . Notice that  $\lim_{p \rightarrow 0} 2(1-p)(\Omega+m)C_\gamma > 1$  by lemma 2.3. This allows us to do the same argument as in the proof of the previous section to show that  $(H_p(u_p^2, v_p^2))_p$  is bounded on  $[r_0, \bar{R}]$  uniformly in  $p$ . We claim that  $(u_p, v_p)_p$  is bounded on  $[r_0, \bar{R}]$  uniformly in  $p$  too. Let us assume by contradiction that  $(u_{p_n}, v_{p_n})$  is not bounded. Up to a subsequence also written  $(p_n)_n$ , there exists  $(r_{p_n})_n \in [r_0, \bar{R}]^{\mathbb{N}}$  such that  $(u_{p_n}, v_{p_n})(r_{p_n}) =: (u_n, v_n)$  is not bounded. He have:

$$\begin{aligned} (\Omega+m)u_n^2 + (\Omega-m)v_n^2 &\xrightarrow{n \rightarrow \infty} \infty, \\ H_{p_n}(u_n^2, v_n^2) &\leq C, \end{aligned}$$

so that

$$(v_n^2 - u_n^2)|v_n^2 - u_n^2|^{p_n-1} \xrightarrow{n \rightarrow \infty} \infty,$$

$v_n^2 - u_n^2$  is then nonnegative for  $n$  big enough and  $v_n^2 - u_n^2 \xrightarrow{n \rightarrow \infty} \infty$ , thus

$$H_{p_n}(u_n^2, v_n^2) = -\frac{1}{2}|v_n^2 - u_n^2|^{p_n} + \frac{\Omega-m}{2}(v_n^2 - u_n^2) + \Omega u_n^2 \xrightarrow{n \rightarrow \infty} \infty.$$

This is the wanted contradiction. Moreover, we have

$$\frac{d}{dr}(v_{p_n}^2 - u_{p_n}^2)(r) = 4 \left( \frac{u_{p_n}^2}{r} - \Omega u_{p_n} v_{p_n} \right),$$

so that  $v_{p_n}^2 - u_{p_n}^2$  is equicontinuous and bounded on  $[r_0, \bar{R}]$ . Ascoli's theorem shows the first point. On every compact interval of  $\{|g| > 0\}$ ,  $(u_{p_n}, v_{p_n})$  is also equicontinuous and bounded. Ascoli's theorem and a diagonal argument give us the second one. The remaining is immediate.  $\square$

**Lemma 5.2.** *Let  $(u, v) \in \mathbb{R}^2$ , then*

- (i) *For all  $\varepsilon > 0$ , there exists  $C > 0$  such that for all  $p \in (0, 1)$ , if  $v^2 - u^2 > \varepsilon$  and  $H_p(u^2, v^2) \geq 0$  then  $u^2 > C$  and  $v^2 > C$ .*
- (ii) *For all  $\gamma > 1/2$  there exists  $C > 0$  such that if  $p \in (0, 1)$  and  $(u, v) \in \mathbb{R}^2$  verifies  $u^2 - v^2 > 0$  and  $H_p(u^2, v^2) > \gamma$  then  $u^2 > C$  and  $v^2 > C$ .*

The proof is a straightforward calculus. Let  $R_p^i$  be the  $i$ -th crossing of the set  $\{u^2 = v^2\}$ ,  $\tilde{R}_p^i$  the  $i$ -th crossing of the set  $\{uv = 0\}$  for the solution  $(u_p, v_p)$ , where  $i$  belongs to  $\{1, \dots, 2k\}$ . Up to extraction, there exist  $(R_0^i)_i, (\tilde{R}_0^i)_i$  such that:

$$\begin{aligned} R_{p_n}^i &\xrightarrow{n \rightarrow \infty} R_0^i \text{ and } g(R_0^i) = 0, \\ \tilde{R}_{p_n}^i &\xrightarrow{n \rightarrow \infty} \tilde{R}_0^i, \end{aligned}$$

and  $R_0^1 \leq \tilde{R}_0^1 \leq R_0^2 \leq \dots \leq \tilde{R}_0^{2k}$ .

**Lemma 5.3.** *For all  $i$  even,  $\emptyset \neq (R_0^i, R_0^{i+1}) \subset \{g > 0\}$  and  $\emptyset \neq (r_0, R_0^i) \subset \{g > 0\}$ .*

*Proof.* By lemma 2.3, we have  $a_p = (\Omega - m)^{\frac{1}{2(1-p)}} = \sup\{v | \exists u; H_p(u^2, v^2) = 0\}$ , and  $\lim_{p \rightarrow 0} a_p = \sqrt{\Omega - m}$ . Thus, for all  $i$  even,  $(v_{p_n}^2 - u_{p_n}^2)(\tilde{R}_{p_n}^i) \geq (\Omega - m)/2$  for  $n$

big enough so that  $g(\tilde{R}_0^i) \geq (\Omega - m)/2$ . This ensures that  $R_0^i < \tilde{R}_0^i < R_0^{i+1}$ . We claim that  $(R_0^i, R_0^{i+1}) \subset \{g > 0\}$ . Let  $r^+ \in (\tilde{R}_0^i, R_0^{i+1})$ , we have

$$\begin{aligned} \theta_{p_n}(R_{p_n}^{i+1}) - \theta_{p_n}(r^+) &\geq \Theta(R_{p_n}^{i+1} - r^+), \\ H_{p_n}(u_{p_n}^2, u_{p_n}^2)(r^+) &\geq 0, \\ v_{p_n}^2(r^+) - u_{p_n}^2(r^+) &> 0, \end{aligned}$$

so that  $\exists C > 0$  such that  $u_{p_n}^2(r^+) > C$  and  $v_{p_n}^2(r^+) > C$  for all  $n$  by lemma 5.2 and  $\liminf_{n \rightarrow \infty} v_{p_n}^2(r^+) - u_{p_n}^2(r^+) = g(r^+)$  is also positive thus  $r^+ \in \{|g| > 0\}$ . The same argument works as well for  $r^- \in (R_0^i, \tilde{R}_0^i)$  and  $r_0$ . This gives us the lemma.  $\square$

**Lemma 5.4.** *For  $i$  odd, we have either*

$$(i) \emptyset \neq (R_0^i, R_0^{i+1}) \subset \{|g| > 0\}$$

or

$$(ii) \lim_{n \rightarrow \infty} \int_{R_{p_n}^i}^{R_{p_n}^{i+1}} u_{p_n}^2 ds = 0.$$

*Proof.* By symmetry, we can assume that for  $r \in (R_{p_n}^i, \tilde{R}_{p_n}^i)$ ,  $u_{p_n}(r) > v_{p_n}(r) > 0$ . We have:

$$\begin{aligned} \frac{d}{dr} (v_{p_n}^2 - u_{p_n}^2) &= 4u_{p_n}^2 (1/r - \Omega) + 4\Omega(u_{p_n}^2 - u_{p_n}v_{p_n}) \\ &\leq 4u_{p_n}^2 (1/r - \Omega) + 4\Omega(u_{p_n}^2 - v_{p_n}^2), \end{aligned}$$

so

$$\frac{d}{dr} e^{4\Omega r} (v_{p_n}^2 - u_{p_n}^2) \leq 4e^{4\Omega r} u_{p_n}^2 (1/r - \Omega).$$

Hence, for any  $r \in (R_{p_n}^i, \tilde{R}_{p_n}^i)$ , we have:

$$(v_{p_n}^2 - u_{p_n}^2)(r) \leq 4(1/r_0 - \Omega)e^{-4\Omega \bar{R}} \int_{R_{p_n}^i}^r u_{p_n}^2 ds.$$

Then we have either  $\limsup_{n \rightarrow \infty} \int_{R_{p_n}^i}^{\tilde{R}_{p_n}^i} u_{p_n}^2 ds$  is positive, then  $\tilde{R}_0^i > R_0^i$  and  $\tilde{R}_0^i \in$

$\{|g| > 0\}$  or  $\lim_{n \rightarrow \infty} \int_{R_{p_n}^i}^{\tilde{R}_{p_n}^i} u_{p_n}^2 ds = 0$ . In the latter case, we have either  $\tilde{R}_0^i = R_0^i$

or  $\lim_{n \rightarrow \infty} \int_{R_0^i}^{\tilde{R}_0^i} u_{p_n}^2 ds = 0$ .

Let us assume now that  $\tilde{R}_0^i > R_0^i$  and  $\tilde{R}_0^i \in \{|g| > 0\}$ . We write  $r_- \in (R_0^i, \tilde{R}_0^i)$  the latest radius at which  $g = 0$ . Then,  $(u_0, v_0)$  is a nonzero free Dirac solution on  $(r_-, \tilde{R}_0^i)$ . Up to a subsequence,  $H_{p_n}(u_{p_n}^2, v_{p_n}^2)(\tilde{R}_{p_n}^i)$  converges to  $(\frac{1}{2} + \frac{\Omega - m}{2}v_0^2 + \frac{\Omega + m}{2}u_0^2)(\tilde{R}_0^i) = \gamma > 1/2$ . Lemma 5.2 ensures that there exists  $C > 0$  such that



$u_{p_n}^2 > C$  and  $v_{p_n}^2 > C$  on  $[R_{p_n}^i, \tilde{R}_{p_n}^i]$ . We then have that  $\int_{R_{p_n}^i}^{R_{p_n}^i+r} u_{p_n}^2 ds$  for every  $r > 0$  small enough and  $r^- = R_0^i$ . This shows that:

$$\emptyset \neq (R_0^i, \tilde{R}_0^i) \subset \{|g| > 0\}.$$

The proof on the right hand side of  $\tilde{R}_0^i$  follows the same ideas.  $\square$

Let us now consider the following mapping:

$$\begin{aligned} \Gamma : \quad & (-\bar{p}/2, \bar{p}/2) \times (\sqrt{\Omega - m}/2, \infty) \rightarrow \mathbb{R}^3 \\ & (p, x) \mapsto (p, u_{p,x}(r_0), v_{p,x}(r_0)), \end{aligned}$$

where  $(u_{p,x}, v_{p,x})$  is the solution of the problem (1.6)<sub>p</sub> with  $x$  as initial condition.  $\Gamma$  is an injective immersion because of the flow regularity and the uniqueness in the fixed point theorem. We claim that the image  $\mathcal{I}$  of  $\Gamma$  is a embedded sub-manifold of  $\mathbb{R}^3$ . For, let us assume by contradiction that  $\mathcal{I}$  is not. There exist a sequence  $(p_n, x_n)$ ,  $(p, x)$  and  $\varepsilon > 0$  such that :

$$\begin{aligned} \|(p_n, x_n) - (p, x)\| &\geq \varepsilon \\ \|\Gamma(p_n, x_n) - \Gamma(p, x)\| &\xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

We get  $p_n \xrightarrow[n \rightarrow \infty]{} p$ . Moreover,  $\Gamma$  can be extended into an injective mapping on  $[-\bar{p}/2, \bar{p}/2] \times [\sqrt{\Omega - m}/2, \infty)$  therefore  $(x_n)$  must tend to infinity since the existence of an accumulation point contradicts injectivity. As it may be shown that lemma 2.5 is still true for  $p \in [-\bar{p}/2, \bar{p}/2]$ , we have:

$$v_{p_n, x_n}^2(r_0) \geq v_{p_n, x_n}^2(r_0) - u_{p_n, x_n}^2(r_0) \geq \alpha x_n^2.$$

Thus  $(\Gamma(p_n, x_n))$  cannot be bounded which is the wanted contradiction. So  $\Gamma$  is a diffeomorphism from  $(-\bar{p}/2, \bar{p}/2) \times (\sqrt{\Omega - m}/2, \infty)$  to  $\mathcal{I}$ . Therefore, there exists a unique  $x > 0$  such that  $(u_{0,x}, v_{0,x})(r_0) = (u_0, v_0)(r_0)$  with the notations of lemma 5.1 and the convergence of  $(u_{p_n}, v_{p_n})$  to  $(u_{0,x}, v_{0,x})$  is uniform on  $[0, r_0]$ . Thus we get the theorem.  $\square$

We have been unable to describe completely the behavior of the limit function. In particular, the evolution of the energy while the  $(u^0, v^0) = 0$  remains an open problem.

## APPENDIX

We give here the proof of lemma 2.3.

*Proof.* The proof of the first four points is given in [1]. It remains to show the last one. We set  $C_\gamma = \sup\{u^2 \mid \exists v, H(u^2, v^2) = \gamma\}/\gamma$ . We have for all  $(u, v) \in \mathbb{R}^2$ :

$$H(u^2, v^2) = \gamma \Rightarrow C_\gamma H(u^2, v^2) \geq u^2.$$

Since  $H^{-1}(\{\gamma\})$  is compact, there exists  $(u, v)$  such that  $H(u^2, v^2) = \gamma$  and  $\gamma C_\gamma = u^2$ . By the implicit function theorem, we can show that

$$\gamma = H(u^2, v^2) = \frac{\left(\frac{\Omega - m}{p}\right)^{\frac{p}{p-1}}}{2} (p-1) + \Omega u^2$$

so that,

$$C_\gamma = \frac{1}{\Omega} \left( 1 + \frac{(1-p) \left( \frac{\Omega-m}{p} \right)^{\frac{p}{p-1}}}{2\gamma} \right),$$

$\gamma \mapsto C_\gamma$  is nonincreasing, this gives us the result for  $C_\gamma$ . The proof for  $D_\gamma$  follows the same ideas.  $\square$

The following is the proof of lemma 2.5.

*Proof.* Let us assume that  $(u_{x,p}, v_{x,p})$  is a solution of  $(1.5)_p$ . We begin as in [1]. Let  $r_0 > \frac{1}{\Omega}$ ,  $\bar{R}_x = \sup\{r > 0 | v_x^2 - u_x^2 > 0\}$  and  $S_x = \min(\bar{R}_x, r_0)$ . For  $r \in (0, S_x)$ , we have because of  $(1.5)_p$ :

$$\begin{aligned} \frac{d}{dr}(v_x^2 - u_x^2) &= 2(v_x' v_x - u_x' u_x) \\ &= 4\left(\frac{u_x^2}{r} - \Omega u_x v_x\right) \\ &\geq 4\Omega(u_x^2 - v_x^2) - 4\left(\Omega - \frac{1}{r_0}\right)u_x^2 \\ &\geq 4\Omega(u_x^2 - v_x^2) - 4\left(\Omega - \frac{1}{r_0}\right)x^2. \end{aligned}$$

We get:

$$\frac{d}{dr}(e^{4\Omega r}(v_x^2 - u_x^2)) + 4\left(\Omega - \frac{1}{r_0}\right)x^2 e^{4\Omega r} \geq 0 \text{ on } (0, S_x).$$

and

$$v_x^2 - u_x^2 \geq x^2 \left( e^{-4\Omega r_0} \left( 1 + \frac{\left(\Omega - \frac{1}{r_0}\right)}{\Omega} \right) - \frac{\left(\Omega - \frac{1}{r_0}\right)}{\Omega} \right) \text{ on } [0, S_x).$$

Now let us set for  $(\Omega, p) \in (m, \infty) \times (0, 1)$ :

$$g : (\Omega, p) \mapsto \exp\left(-\frac{4\Omega}{(\Omega-m)(1-p)}\right) \left( 1 + \frac{p\Omega + (1-p)m}{\Omega} \right) - \frac{p\Omega + (1-p)m}{\Omega},$$

and

$$f : p \in (0, 1) \mapsto e^{-\frac{4}{1-p}}(1+p) - p.$$

On the one hand,  $\Omega \mapsto g(\Omega, p)$  is increasing and

$$\lim_{\Omega \rightarrow \infty} g(\Omega, p) = f(p), \quad \lim_{\Omega \rightarrow m} g(\Omega, p) = -1.$$

On the other hand,  $f$  is decreasing and

$$\lim_{p \rightarrow 0} f(p) = e^{-4} > 0, \quad \lim_{p \rightarrow 1} f(p) = -1.$$

Thus, there exists a unique  $\bar{p} \in (0, 1)$  such that

$$\forall p \in (0, \bar{p}), \quad f(p) > f(\bar{p}) = 0,$$

and for such a  $p$ , a unique  $\Omega_p > m$  such that

$$\forall \Omega > \Omega_p, \quad g(\Omega, p) > g(\Omega_p, p) = 0.$$

Finally, for  $0 < p < q < \bar{p}$ , we have for all  $\Omega > \Omega_q > m$ ,

$$\frac{1}{(\Omega-m)(1-p)} < \frac{1}{(\Omega-m)(1-q)} =: r_0$$

$\alpha := g(\Omega, q) > g(\Omega_q, q) = 0$ . Then,

$$v_{x,p}^2 - u_{x,p}^2 \geq \alpha x^2 \text{ for all } r \in [0, S_x),$$

this shows that  $S_x = r_0$  and thus we get the first two points of the lemma. The latter one is an easy consequence of lemma 2.4 and no constant depends on  $p$  provided that  $p \in (0, \bar{p} - \zeta]$  for some  $\zeta > 0$ .  $\square$

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