HAL open science

Geometry of warped products

Abdelghani Zeghib

To cite this version:

Abdelghani Zeghib. Geometry of warped products. 1999. hal-00605624

HAL Id: hal-00605624

https://hal.science/hal-00605624v1

Submitted on 2 Jul 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GEOMETRY OF WARPED PRODUCTS

ABDELGHANI ZEGHIB

ABSTRACT. This is a survey on the geometry of warped products, without, or essentially with only soft, calculation. Somewhere in the paper, the goal was to give a synthetic account since existing approaches are rather analytic. Somewhere else, we have interpreted statements, especially by means of a physical terminology. This is essentially heuristic, but we think it might be helpful in both directions, that is, in going from a synthetic geometrical language to a relativistic one, and vice-versa.

CONTENTS

1. Introduction 1
2. Local warped products 5
3. Characterization of local warped products 7
4. Transverse geometry of foliations 8
5. Isometric actions of Lie groups 9
6. Geodesics. Maupertuis' principle 11
7. Examples. Exact solutions 13
8. Big-bangs in anti-physical warped products 18
9. Proof of Theorem 8.3 19
References 24

1. INTRODUCTION

The warped product is a construction in the class of pseudo-Riemannian manifolds generalizing the direct product, and defined as follows. Let (L, h) and (N, g) be two pseudo-Riemannian manifolds and $w : L \to \mathbb{R}^+ - \{0\}$ a *warping* function. The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the metric $h \bigoplus wg$. The metric on M will be usually denoted by $\langle , \rangle$. Here, we will be especially interested in the case where M is Lorentzian (a spacetime) and sometimes Riemannian.

Previous works. There are several references on warped products, we mention a few: [2, 4, 8, 10, 21, 24]. Some of them are, like the present one, surveys, but, in

Date: July 2, 2011.

general, they originate from different points of view. The author met the subject while working on [29] [1].

Terminology. Usually, M is seen as a bundle over L (the *basis*) with *fiber* N. This point of view is surely justified. However, it turns out that one sometimes needs to project onto N. Indeed, the distortion of the structure comes from the transverse structure of the foliation determined by the factor L, the study of which involves projecting on N (see §2). Here, motivated by the analogy with a group theoretic situation (justified by 1.1), and to emphasize its importance, we will call N the **normal** factor of the warped product.

Let us introduce another useful terminology in this text. A warped product $M = L \times_w N$ is called a (global) **generalized Robertson-Walker** (**GRW** in short) space, provided N is a pseudo-Riemannian manifold of constant curvature (see [26] for another use of this terminology). Recall that classical Robertson-Walker spaces correspond to the case where N is a Riemannian manifold of constant curvature, and L is an interval of $\mathbb{R}$ endowed with the metric $-dt^2$. These Lorentz spacetimes model an expanding universe.

Interests. The warped product construction has at least two interesting properties. Firstly, it has a practical interest, since it gives sophisticated examples from simple ones: calculation on warped products is easy (but non-trivial). Secondly, having a large symmetry group generally involve a warped product structure. Actually, being "simple" and having a large symmetry group, are criterion of beauty. Therefore, imposing a warped product structure is somewhat a formulation of a philosophical and an aesthetical principle.

1.1. **Two fundamental extension facts.** As in the case of direct products, warped products enjoy the two following properties:

- Dynamical property: extension of isometries.
- Geometric (static) property: extension of geodesic submanifolds.

In the present article, we will specially investigate the first point. (We hope to consider the second one in a subsequent paper).

Let $f : N \to N$ be a diffeomorphism. Consider the *trivial* (or product) extension:

$$\bar{f} : (x, y) \in L \times N \to (x, f(y)) \in L \times N$$

With the notations above, we have $\bar{f}^*(h \bigoplus wg) = h \bigoplus wf^*g$. In particular:

Trivial isometric extension 1.1. *The trivial extension $\bar{f}$ is an isometry of $L \times_w N$ iff f is an isometry of N.*

Warped products are reminiscent of semi-direct products in the category of groups, the factor N playing the role of the normal subgroup. Indeed, $\mathrm{Isom}(N)$ is a normal subgroup of $\mathrm{Isom}^{\times}(L \times_w N)$, which designs the group of isometries of $L \times_w N$ preserving the topological product. This justifies calling N the **normal**

1. The present article exists in fact since 1999, it was destined to be published in the proceeding of a conference on pseudo-Riemannian geometry. I came back recently around the subject and discovered interest of some people who quote it; that is why I estimated it is worthwhile to revive this paper.

factor of the warped product.

The following is the second extension fact which will be proved in §4.

Geodesic extension 1.2. *Let $M = L \times_w N$ be a warped product, and S a submanifold of N. Then S is geodesic in N iff $L \times S$ is geodesic in M.*

As a corollary, we obtain that a warped product has many non-trivial (i.e. with dim > 1) geodesic submanifolds. This is the starting point of rigidity of GRW spaces.

1.2. **Content and around the article.** The article contains personnel points of view rather than a standard survey on warped products. One fact which seems to be new in our approach here, is to consider local warped product structures, a notion which belongs to the domain of foliations. This leads us in this paper to fix some known and used characterizations (but sometimes difficult to find in literature) of foliations with some transverse or tangential geometric structures (geodesic, umbilical, transversally pseudo-Riemannian...).

In another direction, one may also consider analytic pseudo-Riemannian manifolds, with a somewhere defined warped product structure, i.e. admitting an open set which is a warped product. In the direct (non-warped) product case, an analytic continuation is easily defined in the whole universal cover. (The reason is that we get parallel plane fields which we extend by parallel transport).

This is no longer true in the warped case. Firstly, in general, there is no mean to "extend analytically" (somewhere defined) foliations, since this is not uniquely defined even in the simply connected case, and also, because this would at most give rise to singular objects.

In the case of a somewhere defined warped product structure, we have a kind of a "rigid geometric structure", and one may use it as a model. One then considers points admitting charts isometric to it. We will meet in §9 a situation where the technical realization of this idea works well.

Actually, one solves Einstein equations (i.e. spacetimes with some geometry) in charts, which are, thanks to reasonable symmetry hypotheses, endowed with a warped product structure. One, in general, observes singularity of the metric written in these co-ordinates systems. It is usual to call such "singularities" **inessential**. From our point of view, they are still singularities, but for the warped product structure. So, it is an intererstingand natural problem to study the behaviour of analytic extension of somewhere defined warped product: their degenerations (horizons!) and their regenerations (but in a different physical nature). That is a question that the present article would suggest to consider and study in a systematic way, however, we do here only a few in the particular case developed in §9.

1.3. **Preliminary examples.**

1.3.1. *Polar coordinates.* This example illustrates how the presence of a warped product structure is related to symmetry, and how then, it is useful, as are the polar coordinates. Let us start with M^n a Riemannian manifold, and let $x \in M$. Locally $M - \{x\}$ is isometric to $\mathbb{R}^+ \times S^{n-1}$, endowed with a metric $g = dr^2 \bigoplus g_r$, where g_r is a metric on S^{n-1}. Observe that $O(n)$ acts naturally by $(A.(r,u)) \to (r, A(u))$.

FACT **1.3.** *Polar coordinates determine a warped product, that is, there is a metric g on S^{n-1} and a function $w(r)$ such that $g_r = w(r)g$, iff, the natural action of $O(n)$ is isometric. It then follows that g is, up to a multiplicative factor, the canonical metric on S^{n-1}, and that all the 2-planes at x have the same sectional curvature.*

Proof. Assume we have a warped product. In order to prove that the $O(n)$-action is isometric, it suffices to show that it is isometric on each sphere $S_r = \{r\} \times S^{n-1}$. Let $A \in O(n)$. All these spheres are homothetic, and the metric distortion of A is the same on all of them. But this distortion tends to 1 when $r \to 0$. Therefore, A has distortion 1 on each S_r, that is A acts isometrically. The remaining part of the fact is standard.

□

For example, polar coordinates determine a warped product in the case of constant curvature Riemannian spaces, the Euclidean case corresponds to $\mathbb{R}^+ \times_{r^2} S^{n-1}$.

The previous fact generalizes to pseudo-Riemannian manifolds. More precisely, the polar coordinates at a point x of a pseudo-Riemannian manifold $M^{p,q}$ of type (p, q), give rise to a warped product structure, iff, the natural action of $O(p, q)$ is isometric. Let us call x in this case, a point of *complete symmetry*. All the non-degenerate 2-planes at such a point have the same sectional curvature.

In particular, if all the points of M are points of complete symmetry, then, M has a constant curvature. It is then natural to ask if there are non-trivial, i.e. with non constant curvature, examples of pseudo-Riemannian manifolds with at least one point of complete symmetry. An averaging method works to give examples, in the Riemannian case, since $O(n)$ is compact. In the other cases, the "spheres" become complicated, and a large isotropy group at some point, may create extra symmetry elsewhere. However, nontrivial examples do exist, for instance, any Lorentz metric on $\mathbb{R}^2$ of the form $F(xy)dxdy$, where F is a positive real function defined on an interval containing 0, admits $(0, 0)$ as a point of complete symmetry. (The metric is defined on an open subset of $\mathbb{R}^2$ delimited by hyperbolas $xy =$ constant). A celebrated example of this form is the Kruskal plane (see for instance [21]).

More generally, in any dimension, one may consider Lorentz metrics of the form $g = F(q)q$ where q is a Lorentz form. The origin is a point of complete symmetry for g. Let us however that the situation becomes really rigid if one asks for many points of complete symmetry.

1.3.2. *Riemannian symmetric spaces.* We find the representation of the hyperbolic (Riemannian) space $\mathbb{H}^n$ as the warped product $\mathbb{R} \times_{e^t} \mathbb{R}^{n-1}$, to be the nicest model of it (here $\mathbb{R}$ and $\mathbb{R}^{n-1}$ are Euclidean). One amuzing fact coming from the theory of geodesics in warped products, is how geodesics of the hyperbolic plane are related to solutions of mechanical systems $x'' = ce^{-x}$ (c is a constant) (see §6.6). Of course the interest here is not to analytically solve this equation, but rather to see how it can be solved geometricaly.

Remark 1.4 (Generalization)**.** *The situation of more general Riemannian symmetric space is more subtle. It involves "multi-warped products". This means that we have (L, h), and (N, g), endowed with $T_1, \dots T_k$ supplementary subbundles of*

TN ($= T_1 \oplus \ldots \oplus T_k$), with restriction of the metric denoted g_i. We also have warping functions $w_1, \ldots, w_k$ defined on L, and construct from all, the metric $w_1 g_1 \oplus \ldots \oplus w_k g_k$. All Riemannian symmetric spaces (e.g. $\mathsf{SL}(n,\mathbb{R})/\mathsf{SO}(n)$) admit such a representation. The geometry of such "multi-warped products" is quite delicate, at least more than the somewhat usual definition in the literature, where the plane fields T_i are assumed to be integrable. However, it is the non-integrable case that covers the case of symmetric spaces. We think it is worthwhile investigating this generalization.

2. Local warped products

A pseudo-Riemannian manifold which is a warped product is in particular a global topological product. This is so restrictive (for instance for physical applications) and we are led to localize the notion of warped products as follows.

Definition 2.1. *Let M be a pseudo-Riemannian manifold. A local warped product structure on M is a pair $(\mathcal{L}, \mathcal{N})$ of transversal foliations, such that the metric on adapted flow-boxes is a warped product. More precisely, for any point of M there is a neighborhood U, and a warped product pseudo-Riemannian manifold $L \times_w N$, and an isometry $\phi : U \to L \times_w N$, sending the foliation $\mathcal{L}$ (resp. $\mathcal{N}$) to the foliation of $L \times N$ determined by the factor L (resp. N).*

A local warped product is called a local GRW structure if the factor $\mathcal{N}$ has a constant curvature (i.e. each leaf of $\mathcal{N}$ is a pseudo-Riemannian manifold of constant curvature).

2.1. Geometry of submanifolds.

In the sequel, we will investigate conditions on a pair of foliations $(\mathcal{L}, \mathcal{N})$ in order to determine a local warped structure. For this, let M be a pseudo-Riemannian manifold, and S a non-degenerate submanifold of M, that is the metric restricted to T_xS is non-degenerate for any $x \in S$. Recall that the **second fundamental form**, also called the **shape tensor**, at x is a bilinear map: $II_x : T_xS \times T_xS \to N_x$, where N_x is the normal space of T_xS, which measures how S is far from being geodesic (II is well defined because of the non-degeneracy hypothesis).

The submanifold S is **umbilic** if for any $x \in S$, II_x has the form $II_x = \langle , \rangle n_x$, where n_x is some normal vector to T_xS. In this case, the vector field (along S) $x \to n_x$ is called the **shape** vector field. (the terminology force field is also pertinent as may be seen from Theorem 6.3).

The (totally) **geodesic submanifolds** correspond to the case $n_x = 0$, for all $x \in S$.

We will also need the following notion: S is said to be **spherical**, if it is umbilic, and furthermore, the shape vector field $x \in S \to n_x$, is parallel (along S).

When we consider umbilic submanifolds, we will always assume that they have dimension > 1. Indeed, every 1-dimensional submanifold is umbilic (but need not to be spherical).

• Let us recall the geodesic invariance characteristic property of geodesic submaniflods. Let $x \in S$, $u \in T_xS$, and let $\gamma :]-\epsilon, +\epsilon[\to M$ be the geodesic in M determined by u. If S is geodesic, then the image of γ is contained in S, for ϵ sufficiently small. This fact is true also when S is umbilic, if in addition u is *isotropic*.

This is a remarkable rigidity fact in pseudo-Riemannian geometry, which has no counterpart in Riemannian geometry.

Example 2.2. Take M to be the pseudo-Euclidean space of type (p, q), i.e. $\mathbb{R}^{p+q}$ endowed with a pseudo-Euclidean form Q of type (p, q).

A (connected) geodesic hypersurface is an open set of an affine hyperplane. The (connected) umbilic hypersurfaces are contained in hyperquardrics $Q(x - O) = c$, where $O \in \mathbb{R}^{p,q}$ and c is a constant (the proof is formally the same as in the Euclidean case). One can verify that such a hyperquadric is ruled, that is, it contains the isotropic lines which are somewhere tangent to it.

In general, an umbilic submanifold is the intersection of a hyperquadric with an affine plane of some dimension.

In particular, one sees in the case of pseudo-Euclidean spaces, that umbilic submanifolds are spherical. This is true for all pseudo-Riemannian manifolds of constant curvature, but not true in the general case.

2.2. **Tangential geometry of foliations.** (See for instance [5, 25, 27] for more details). A foliation $\mathcal{F}$ is called geodesic, umbilic or spherical, if its leaves are geodesic, umbilic or spherical, respectively.

Let X be a vector field defined on an open subset $U \subset M$. We say that X is a ($\mathcal{F}$-) **normal foliated** vector field, if X is orthogonal to $\mathcal{F}$, and its local flow ϕ^t preserves $\mathcal{F}$, i.e. it sends a leaf of $\mathcal{F}$ to a leaf of $\mathcal{F}$ (everything is restricted to U).

As in the case of an umbilic submanifold, an umbilic foliation $\mathcal{F}$ has a **shape** vector field $\overrightarrow{n}$ defined by the relation $II = \langle , \rangle \overrightarrow{n}$, where II is the shape tensor.

Lemma 2.3. *Let $\mathcal{F}$ be a non-degenerate foliation of a pseudo-Riemannian manifold $(M, \langle , \rangle)$. Let f denote the first fundamental form of $\mathcal{F}$, that is the tensor which vanishes on $T\mathcal{F}^{\perp}$ and equals $\langle , \rangle$ on $T\mathcal{F}$, and denote by $II : T\mathcal{F} \times T\mathcal{F} \to T\mathcal{F}^{\perp}$ the second fundamental form.*

Let X be a normal foliated vector field, then the Lie derivative $L_X f$ satisfies:

$$(L_X f)(u, v) = -2\langle II(u, v), X\rangle,$$

for all $u, v \in T\mathcal{F}$. (In other words, if ϕ^t is the (local) flow of X, then, at any x, $\frac{\partial}{\partial t}(\phi^t_* f)_x|_{t=0} = -2\langle II_x(., .), X\rangle$).

Proof. Let u and v be two vector fields tangent to $\mathcal{F}$ which commute with X. Then by definition $(L_X f)(u, v) = X.f(u, v)$, which also equals $X.\langle u, v\rangle$. Now, $X.\langle u, v\rangle = \langle \nabla_X u, v\rangle + \langle u, \nabla_X v\rangle$. By commutation, this becomes $\langle \nabla_u X, v\rangle + \langle u, \nabla_v X\rangle$. Since $\langle X, v\rangle = \langle u, X\rangle = 0$, $(L_X f)(u, v) = -\langle X, \nabla_u v\rangle - \langle X, \nabla_v u\rangle$, and so by definition of II, we have: $(L_X f)(u, v) = -2\langle II(u, v), X\rangle$

□

Corollary 2.4. *If $\mathcal{F}$ is geodesic (resp. umbilic) then the flow of X maps isometrically (resp. conformally) a leaf of $\mathcal{F}$ onto a leaf of $\mathcal{F}$.*

Conversely, if the flow of any normal foliated vector field maps isometrically (resp. conformally) leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$, then $\mathcal{F}$is geodesic (resp. umbilic).

Proof. The proof is just the translation, with the above notation, of the fact that the flow ϕ^t maps isometrically (resp. conformally) leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$, into the equation: $\phi^t_* f = f$ (resp. $\phi^t_* f = af$ for some scalar function a).

□

Note however, that there is no a such characterization for spherical foliations. For example, any (local) umbilic foliation of the Euclidean space is spherical, as it is just a foliation by round spheres. The flow of a normal foliated vector field maps conformally a sphere to a sphere, but not more, for example not necessarily homothetically.

3. Characterization of local warped products

The following theorem is due to S. Hiepko, but with a different proof, and especially with a purely "analytic" formulation. We said in a previous version of this article, that this analytic formulation could explain why the article of Hiepko [15] seems to be not sufficiently known in the literature. Afterwards, we discover the work [23] by R. Ponge and H. Reckziegel, which contains a geometric approach.

Theorem 3.1. *Let $(M, \langle , \rangle)$ be a pseudo-Riemannian manifold endowed with a pair $(\mathcal{L}, \mathcal{N})$ of non-degenerate foliations. This determines a local warped product structure with $\mathcal{N}$ as a normal factor, iff, the foliations are orthogonal, $\mathcal{L}$ is geodesic, and $\mathcal{N}$ is* spherical.

Proof. Let $\mathcal{L}$ and $\mathcal{N}$ be two orthogonal foliations. Locally, at a topological level, we may suppose that $M = L \times N$, and that the foliations $\mathcal{L}$ and $\mathcal{N}$ correspond to those determined by the factors L and N. Let (x, y) be a fixed point in $L \times N$. The metric on M at (x, y) has the form $h_{(x,y)} \bigoplus f_{(x,y)}$, where $h_{(x,y)}$ (resp. $f_{(x,y)}$) is a metric on $L \times \{y\}$ (resp. on $\{x\} \times N$). Note that a normal foliated vector field for $\mathcal{L}$ is just a vector field of the form $X(x, y) = (0, \bar{X}(y))$, where $\bar{X}$ is a vector field on N, and similarly for $\mathcal{N}$.

By Corollary 2.4, $\mathcal{L}$ is geodesic, iff $h_{(x,y)} = h_y$. In the same way, $\mathcal{N}$ is umbilic, iff there is a function $w(x, y)$ such that $f_{(x,y)} = w(x, y) f_x$. Therefore, the fact that $\mathcal{L}$ is geodesic and $\mathcal{N}$ is umbilic, is equivalent to that the metric $\langle , \rangle$ of M is a **twisted product** $h \bigoplus wg$, where h and g are metrics on L and N respectively, and w is a function on $L \times N$.

By choosing a point (x_0, y_0), we may suppose that $g = f_{(x_0,y_0)}$, and hence $w(x_0, y) = 1$, for all $y \in N$.

The fact that this metric is a warped product means exactly that w is a function of x alone. Therefore, the statement of the theorem reduces now to the equivalence between the two facts, w being constant along $\mathcal{N}$, and $\mathcal{N}$ being spherical.

To check this, let $\bar{X}$ and $\bar{Y}$ be two vector fields on L and N, respectively, and let X and Y be the corresponding vector fields on M, which are normal foliated relatively to $\mathcal{N}$ and $\mathcal{L}$, respectively.

Since $\mathcal{N}$ is umbilic, $II = f\overrightarrow{n}$, where f and II are the first and second fundamental forms for $\mathcal{N}$ respectively, and $\overrightarrow{n}$ is its shape vector field.

We have, $Y\langle \overrightarrow{n}, X\rangle = \langle \nabla_Y \overrightarrow{n}, X\rangle + \langle \overrightarrow{n}, \nabla_Y X\rangle = \langle \nabla_Y \overrightarrow{n}, X\rangle + \langle \overrightarrow{n}, \nabla_X Y\rangle$, since X and Y commute.

Since $\mathcal{L}$ is geodesic, $\nabla_X Y$ is orthogonal to $\mathcal{L}$, in particular, $\langle \overrightarrow{n}, \nabla_X Y\rangle = 0$. It then follows that $Y\langle \overrightarrow{n}, X\rangle = \langle \nabla_Y \overrightarrow{n}, X\rangle$.

Lemma 2.3 says that $X.w = -2\langle \overrightarrow{n}, X\rangle$, and hence $Y.(X.w) = \langle \nabla_Y \overrightarrow{n}, X\rangle$. By definition, $\mathcal{N}$ is spherical iff $\langle \nabla_Y \overrightarrow{n}, X\rangle = 0$, for all X and Y, which is thus equivalent to $Y.(X.w) = 0$. This last equality, applied to a fixed Y, and an arbitrary X, means that $Y.w$ is a function of y only, say $Y.w = a(y)$. But, since $w(x_0, y) = 1$, it follows that $Y.w = 0$. Applying this to an arbitrary Y, leads to the fact that w does not depend on y, which in turn means that the metric is a warped product. □

4. Transverse geometry of foliations

Theorem 3.1 is expressed by means of tangential properties of foliations, i.e. by those of individual leaves. Sometimes, it is also interesting to consider the transverse structure of these foliations, i.e. the properties of their holonomy maps (see for instance [19] as a reference about such notions). These holonomy maps are especially easy to realize, for a foliation $\mathcal{F}$, when the orthogonal $T\mathcal{F}^\perp$ is integrable, that is, when it determines a foliation say $\mathcal{F}^\perp$. The holonomy maps of $\mathcal{F}$ are thus just the local diffeomorphisms between leaves of $\mathcal{F}^\perp$, obtained by integrating $\mathcal{F}^\perp$-normal foliated vector fields (see §2.2 for their definition).

The foliation $\mathcal{F}$ is said to be *transversally pseudo-Riemannian* if its holonomy preserves the pseudo-Riemannian metric on $T\mathcal{F}^\perp$. Similarly one defines the fact that $\mathcal{F}$ is transversally conformal (resp. transversally homothetic). Using this language, the previous developments imply straightforwardly the following fact.

FACT **4.1.** *A pair $(\mathcal{L}, \mathcal{N})$ determines a local warped product structure, iff $\mathcal{L}$ is transversally homothetic and $\mathcal{N}$ is transversally pseudo-Riemannian.*

In general (i.e. in a not necessarily warped product situation), we have the following duality between tangential and transverse structures of foliations.

FACT **4.2.** *Let $\mathcal{F}$ be a foliation admitting an orthogonal foliation $\mathcal{F}^\perp$. Then $\mathcal{F}$ is geodesic (resp. umbilic) iff $\mathcal{F}^\perp$ is transversally pseudo-Riemannian (resp. conformal), that is more precisely, the holonomy maps of the foliation $\mathcal{F}^\perp$, seen as local diffeomorphisms between leaves of $\mathcal{F}$, preserve the metric (resp. the conformal structure) induced on these leaves (of $\mathcal{F}$).*

4.1. **Proof of Fact 1.2.** Let S be a submanifold of N, and $M = L \times_w N$. In order to prove the equivalence, S a geodesic submanifold in $N \Longleftrightarrow L \times S$ a geodesic submanifold in M, it suffices to consider the case where the dimension of S is 1, i.e. S a (non-parameterized) geodesic (curve). Indeed the general case reduces to the 1-dimensional one by considering geodesic (curves) of S.

To simplify let us suppose that N is Riemannian, the general case needs only more notations.

A geodesic such as S can be locally extended to a 1-dimensional foliation $\mathcal{F}$ with an orthogonal foliation $\mathcal{F}^\perp$. To see this, take a hypersurface $S^\perp \subset N$ which is somewhere orthogonal to S, then the leaves of $\mathcal{F}^\perp$ are the parallel hypersurfaces of $S^\perp$. More precisely, they are the levels of the distance function $x \to a(x) = d(x, S^\perp)$. The leaves of $\mathcal{F}$ are the trajectories of ∇a, the gradient of a. Thus $\mathcal{F}^\perp$ is a transversally pseudo-Riemannian foliation of N. By taking the product of the leaves of $\mathcal{F}$ with L, one may define $L \times \mathcal{F}$ as a foliation of M. The orthogonal foliation $(L \times \mathcal{F})^\perp$ of $L \times \mathcal{F}$ is naturally identified with $\mathcal{F}^\perp$ (the leaf of $(x, y) \in$

$L \times N$ is $\{x\} \times \mathcal{F}_y^{\perp}$). From the form of the warped product metric, one sees that, like $\mathcal{F}^{\perp}$, $(L \times \mathcal{F})^{\perp}$ is a transversally pseudo-Riemannian foliation. Therefore, $L \times \mathcal{F}$ is a geodesic foliation, and in particular $L \times S$ is geodesic in M.

The implication, $L \times S$ geodesic in $M \Longrightarrow S$ geodesic in N, is in fact easier than its converse that we have just proved. Indeed, if ∇ is the connection on N, and X, Y are vector fields tangent to S, then $\nabla_X Y$ is tangent to $L \times S$ (since it is geodesic), and hence its orthogonal projection on N is tangent to S, that is, S is geodesic in N.

Remark 4.3. *Although, we are not interested here in global aspects, let us mention that there are many works about the structure of geodesic, umbilic, transversally Riemannian, transversally conformal foliations on compact manifolds. As an example, we may quote the references* [6, 9, 19].

5. Isometric actions of Lie groups

(Local) isometric actions of Lie groups on pseudo-Riemannian manifolds generally give rise to a warped product structure. In some sense, this phenomenon is the converse of the trivial isometric extension Fact 1.1. The following statement may be used to settle a warped product structure in many situations. It unifies and generalizes most of the existing results on the subject (see for instance [7, 11, 13, 16, 17, 22]).

Theorem 5.1. *Let G be a Lie group acting (locally) isometrically on a pseudo-Riemannian manifold M. Suppose that the orbits have a constant dimension and thus determine a foliation $\mathcal{N}$.*

Suppose further that the leaves of $\mathcal{N}$ are non-degenerate, and that the isotropy group in G of any $x \in M$, acts absolutely irreducibly *on $T_x\mathcal{N}$, i.e. its complexified representation is irreducible.*

Suppose finally that the orthogonal of $\mathcal{N}$ is integrable, say it is tangent to a foliation $\mathcal{L}$. Then $(\mathcal{L}, \mathcal{N})$ determines a local warped product structure, with $\mathcal{N}$ as a normal factor.

Proof. The question is local, and so we can suppose the situation is topologically trivial. For two nearby leaves N_1 and N_2, there is a projection $p : N_1 \to N_2$, defined by: $p(x)$ is the unique point of the intersection of $\mathcal{L}_x$ $(= \mathcal{N}_x^{\perp})$ with N_2 (for $x \in N_1$). This projection commutes with the action of G. The pull back by p of the metric on T_yN_2 (at $y = p(x)$) is another metric on T_xN_1, invariant by G_x. The fact that G_x is absolutely irreducible just implies that the two metrics are proportional. Therefore p is conformal. But since p commutes with the (transitive) G-action on N_1 and N_2, p must be homothetic.

It is easy to relate the projection p to the transverse holonomy of $\mathcal{L}$ (as developed in §4), proving that $\mathcal{L}$ is transversally homothetic. It is equally straightforward to relate the transverse holonomy of $\mathcal{N}$ to the G-action, and deducing that $\mathcal{N}$ is transversally pseudo-Riemannian, and therefore $(\mathcal{L}, \mathcal{N})$ determines a local warped product structure (by Fact 4.1).

□

Example 5.2. The absolute irreducibility hypothesis cannot be relaxed to an ordinary irreducibility one. To see this let N be a Lie group, and let G be the product $N \times N$ acting on N by the left and the right, that is $(\gamma_1, \gamma_2)x = \gamma_1^{-1} x \gamma_2$. The isotropy group of this action at the point 1, is nothing but the adjoint action of N on itself. It is irreducible (resp. absolutely irreducible) iff N is a simple (resp. an absolutely simple) Lie group (by definition). In the case N is simple but non absolutely simple, e.g. $\mathsf{SL}(2, \mathbb{C})$, the isotropy action preserves exactly (up to linear combination) two non-degenerate quadratic forms, those given by the real and the imaginary parts of the Killing form of N, seen as a complex group. This gives two G-invariant non-proportional metrics α and β on N.

Let (L, h) be another pseudo-Riemannian manifold, and let $f : L \to \mathbb{R}$ be a real function. Endow $L \times N$ with the metric $h \bigoplus (f\alpha + \beta)$. This is not a warped product.

The following result describes an example of a situation where the hypotheses of Theorem 5.1 are satisfied.

Theorem 5.3. *Let G be a Lie group acting (locally) isometrically on a pseudo-Riemannian manifold M. Suppose that the orbits are non-degenerate having a constant dimension and so determine a foliation $\mathcal{N}$.*

Suppose that the isotropy group in G of any $x \in M$, acts absolutely irreducibly on $T_x\mathcal{N}$, and that the metric on the orthogonal of $\mathcal{N}$ is definite (positive or negative), and in opposite the metric on $\mathcal{N}$ is non-definite. Then, the orthogonal of $\mathcal{N}$ is integrable, and the action determines a local warped product.

Proof. The warped product structure will follow from Theorem 5.1 once we show that the orthogonal of $\mathcal{N}$ is integrable. We will in fact prove this integrability, under the hypothesis that the isotropy is irreducible (not necessarily absolutely irreducible). Consider $\alpha : T\mathcal{N}^\perp \times T\mathcal{N}^\perp \to T\mathcal{N}$ the bilinear form (obstruction to the integrability of $T\mathcal{N}^\perp$) $\alpha(u, v) =$ the projection on $T\mathcal{N}$ of the bracket $[u, v]$, where u and v are vector fields on M with values in $T\mathcal{N}^\perp$. Let $x \in M$, and consider the subset A_x of $T_x\mathcal{N}$ which consists of the elements $\alpha(u, v)$, for u and v of length ≤ 1. This set is compact, and is invariant by the isotropy group G_x. Since α is equivariant, G_x acts precompactly on A_x since it acts so on $T_x\mathcal{N}^\perp$. It then follows that G_x acts precompactly on the linear space B_x generated by A_x. If $A_x = 0$, $\alpha = 0$, and we are done, if not $B_x = T_x\mathcal{N}$ by irreducibility. Thus, G_x preserves a positive scalar product on $T_x\mathcal{N}$. But, it also preserves the initial non-definite pseudo-Riemannian product. Polarize this latter with respect to the invariant positive scalar product, we get a diagonalizable endomorphism, that has both positive and negative eigenvalues since the pseudo-Riemannian product is non-definite. This contradicts the irreducibility.

□

A similar argument yields the following useful fact.

FACT **5.4.** *Let $\mathsf{SO}(3)$ act isometrically on a 4-Lorentz manifold with 2-dimensional orbits. Then, this determines a local warped product structure, with a local model $L \times_w S^2$ or $L \times_w \mathbb{R}P^2$. (One may exclude the projective plane case by a suitable orientability hypothesis).*

6. Geodesics. Maupertuis' principle

The goal here is to understand the geodesics of a warped product $M = L \times_w N$. Let $\gamma(t) = (x(t), y(t))$ be such a geodesic.

Fact 1.2 implies that $y(t)$ is a (non-parametrized) geodesic in N. To see this, let S be a (1-dimensional) geodesic of N such that $\gamma(t)$ is somewhere tangent to $L \times S$. Fact 1.2 says that $L \times S$ is geodesic in M, and therefore contains the whole of $\gamma(t)$, which thus projects onto an open subset of the geodesic S.

Now, it remains to draw equations, and especially to interpret them, for $x(t)$, and also determine the parameterization of $y(t)$. Here, the idea is to replace M by $L \times_w S$, which transforms the problem to a simpler one, that is the case where N has dimension 1 (since $L \times_w S$ is geodesic in M, we do not need the rest of M to understand a geodesic contained in $L \times_w S$!).

Clairaut first integral. The previous discussion allows one to restrict the study to warped products of the type $L \times_w (\mathbb{R}, c_0 dy^2)$, where y denotes the canonical coordinate on $\mathbb{R}$, and c_0 is -1, $+1$ or 0. Of course, the case $c_0 = 0$, i.e. when the non-parameterized geodesic $y(t)$ is lightlike, does not really correspond to a pseudo-Riemannian structure, so, let us assume $c_0 \neq 0$.

Actually, the geodesic S above in not necessarily complete, that is, it is not parameterized by $\mathbb{R}$ but just by an open subset of it. However, our discussion here is local in nature, so to simplify notation, we will assume S complete.

The isometric action of (the group) $\mathbb{R}$ on $(\mathbb{R}, c_0 dy^2)$ extends to an isometric flow on $L \times_w (\mathbb{R}, c_0 dy^2)$ (by Fact 1.1).

The so called Clairaut first integral (for the geodesic flow on the tangent bundle of $L \times_w (\mathbb{R}, c_0 dy^2)$) means here that $\langle y'(t), \partial/\partial y\rangle$ is constant, say, it equals c_1 (remember $\gamma(t) = (x(t), y(t)$ is our geodesic). Since $y'(t)$ and $\partial/\partial y$ are collinear, it follows that:

$$\langle y'(t), y'(t)\rangle = \frac{\langle y(t)', \partial/\partial y\rangle^2}{\langle \partial/\partial y, \partial/\partial y\rangle} = (\frac{c_1^2}{c_0})\frac{1}{w(x(t))}$$

In dimension 2, that is, $\dim L = 1$, the Clairaut integral together with the energy integral: $\langle \gamma'(t), \gamma'(t)\rangle =$ constant, suffice to understand completely the geodesics. The remaining developments concern the case $\dim N \geq 2$.

The shape vector field. The distortion of the warped product structure, i.e. the obstruction to being a direct product is encoded in ∇w, the gradient of w (with respect to the metric of L).

Obviously, the fact that the foliation $\mathcal{N}$ (i.e. that with leaves $\{x\} \times N$) be geodesic is also an obstruction for the warped product to be direct. The following fact is a quantitative version of this obstruction.

Fact **6.1.** *The shape vector field $\overrightarrow{n}$ of $\mathcal{N}$ is a $\mathcal{N}$-foliated vector field. More exactly (identifying TM with $TL \times TN$):*

$$\overrightarrow{n}(x, y) = \frac{-1}{2}(\frac{\nabla w(x)}{w}, 0)$$

Proof. With the notations of Lemma 2.3, we have $f = wg$, and thus (by definition of $\overrightarrow{n}$) $L_X wg = -2\overrightarrow{n} f$, and on the other hand $L_X(wg) = (X.w)g = \frac{X.w}{w} wg$.

□

(Observe that $\frac{\nabla w}{w}$ is well defined even for local warped product structures).

Projection onto L. We consider the case where $x(t)$ and $y(t)$ are regular curves, since the question is local and the other cases are quite easier. Therefore, these curves generate a surface, $(x(t), y(s))$, whose tangent bundle is generated by the natural frame (X, Y). Since X and Y commute, we have $\nabla_{X+Y}(X+Y) = \nabla_X X + 2\nabla_X Y + \nabla_Y Y$.

Since $\mathcal{L}$ is geodesic, $\nabla_X Y$ is tangent to $\mathcal{N}$ (indeed, if Z is tangent to $\mathcal{L}$, then, $\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0$, because $\mathcal{L}$ is geodesic). By definition, the projection of $\nabla_Y Y$ on $T\mathcal{L}$ equals $\langle Y, Y\rangle \overrightarrow{n}$. From this and Fact 6.1, we deduce that the projection of $\nabla_{X+Y}(X+Y)$ on $T\mathcal{L}$ equals $\nabla_X X - (1/2)\langle Y, Y\rangle \frac{\nabla w}{w}$, which must vanish in the geodesic case. Replacing $\langle Y, Y\rangle$ $(= \langle y'(t), y'(t)\rangle)$ by its expression above, we obtain:

$$\nabla_X X = (\frac{1}{2}\frac{c_1^2}{c_0})\frac{1}{w(x(t))}\frac{\nabla w}{w} = (-\frac{1}{2}\frac{c_1^2}{c_0})(\nabla\frac{1}{w})(x(t))$$

This proves the following.

FACT **6.2.** *The projections onto L of the geodesics of $L \times_w (\mathbb{R}, c_0 dy^2)$ are exactly the trajectories of the mechanical systems on L with potentials $\frac{c}{w}$, i.e. curves on L satisfying an equation of the form:*

$$x'' = -\nabla(\frac{c}{w})(x)$$

where c runs over $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) if $c_0 > 0$ (resp. if $c_0 < 0$).

From this, we deduce the following fact for a general N.

Theorem 6.3. *[Maupertuis' principle] The projections onto L of the geodesics of $M = L \times_w N$ are exactly the trajectories of the mechanical systems on L with potentials $\frac{c}{w}$, for $c \in \mathbb{R}$ if the metric on N is non-definite, for $c \in \mathbb{R}^+$ if the metric on N is positive definite, and for $c \in \mathbb{R}^-$ if the metric on N is negative definite.*

Equations. In the case where $y(t)$ is not lightlike, its parameterization is fully determined, whenever $x(t)$ is known, by using the first integral $\langle y'(t), \partial/\partial y\rangle = c_1$. Indeed one can identify $y'(t)$ with $y'(t)\partial/\partial y$, and thus with help of the notation above, $y'(t) = \frac{c_1}{\langle \partial/\partial y, \partial/\partial y\rangle(x(t))} = \frac{c_1}{c_0 w(x(t))}$.

There is no analogous equation in the case where $y(t)$ is lightlike. Let us derive the general equation in another way which covers the lightlike case. From the calculation before Fact 6.2, we have $\nabla_Y Y + 2\nabla_Y X = 0$. Now, for all Z tangent to $\mathcal{N}$, $\langle \nabla_Y X, Z\rangle = -\langle X, \nabla_Y Z\rangle = -\langle X, \overrightarrow{n}\rangle\langle Y, Z\rangle$. Therefore, $\nabla_Y X = -\langle X, \overrightarrow{n}\rangle Y = \frac{dw(X)}{2w} Y$, which proves:

FACT **6.4.** *The curve $y(t)$ has a geodesic support, and its parameterization is coupled with the companion curve $x(t)$ by means of the equation:*

$$y'' = -\frac{\partial}{\partial t}(\log w)(x(t))y'$$

Mechanics on M. The previous discussion relates the geodesics of M to trajectories of mechanical systems on L. Let us now start with a mechanical system $\gamma'' = -\nabla V$ on $M = L \times_w N$ itself. We assume that the potential V is constant on the leaves $\{x\} \times N$, and thus may be identified with a function on L. Essentially, by the same arguments, one proves.

Proposition 6.5. *Consider on $M = L \times_w N$, the equation $\gamma'' = -\nabla V$, where V is a function on L. Then, the projections of its trajectories on N are non-parameterized geodesics of N, and their projections on L are trajectories of mechanical systems on L with potentials of the form $V + \frac{c}{w}$, where c runs over $\mathbb{R}$ if the metric on N is nondefinite, and otherwise, c has the same sign as the metric of N.*

Corollary 6.6. *If $M = L \times_w N$, has dimension 2, i.e. L and N are locally isometric to $\mathbb{R}, \pm dt^2)$, then, the trajectories of the equation $\gamma'' = -\nabla V$, where V is a function on L are completely determined by means of:*

i) their projection on L satisfy $x'' = V + \frac{c}{w}$, and

ii) they satisfy the the energy conservation law $\langle \gamma'(t), \gamma'(t) \rangle + V(\gamma(t)) = constant$

Example 6.7. This applies in particular to solve the geodesic equation on the hyperbolic plane $\mathbb{H}^2 = \mathbb{R} \times_{e^t} \mathbb{R}$.

7. Examples. Exact solutions

In the sequel, we will in particular consider examples of warped product structures on exact solutions, i.e. explicit 4-Lorentz manifolds with an explicit Einstein tensor (for details, one may for instance consult [14], [18], [21]...). In fact, warped products are omnipresent in cosmological models, because of their simplicity and symmetry advantages, as explained in the introduction. However, the most important use of warped product is in formulating expanding universes. This needs the warped product to be of "physical" type. Let us formulate precisely what we mean by this.

Definition 7.1. *We say that a warped product structure on a Lorentz manifold $M = L \times_w N$ is* **physical** *if the metrics on both the factors L and N are definite (one positive and the other negative). Otherwise, the warped product structure is called* **anti physical**. *The same definitions apply for local warped products and GRW structures.*

Equivalently, the warped product is physical if N is spacelike or locally isometric to $(\mathbb{R}, -dt^2)$. The dynamical counterpart in the first case, i.e. when N is spacelike, is that of a universe in expansion (§7.1), and in opposite, a warped product structure for which N is locally isometric to $(\mathbb{R}, -dt^2)$, corresponds to a static universe (§7.2).

The warped product $M = L \times_w N$ is anti physical iff one of the factors L or N is Lorentzian.

7.1. **Expanding universes: classical Robertson-Walker spacetimes.** Here, L is an interval $(I, -dt^2)$, and N a 3-Riemannian manifold of constant curvature. Recall that the -energy tensor satisfies (or say, it is defined by) the Einstein equation: $T = (1/8\pi)(Ric - \frac{1}{2}Rg)$ (Ric and R are respectively the Ricci and scalar curvature of $\langle , \rangle$). Here, it has the form of a perfect fluid: $T = (\mu + p)\omega \bigotimes \omega + p\langle , \rangle$, where $\omega =$ the dual 1-form of $\frac{\partial}{\partial t}$ (with respect to the metric $\langle , \rangle$) and the functions μ (energy density) and p (pressure), are determined by the warping function w (by means of the Einstein equation). In fact, the condition that N has a constant curvature is exactly needed to get a perfect fluid.

7.2. **Static universes.** Not only expanding universes involve a warped product structure, but also the static ones, which are defined as those spacetimes having non-singular timelike Killing fields with an integrable orthogonal distribution. The fact that this gives a local warped structure with the trajectories of the given Killing field as a normal foliation, is a special elementary case of Theorem 5.1

Conversely, by the isometric extension Fact 1.1, a warped product $M = L \times_w N$, with N locally isometric to $(\mathbb{R}, -dt^2)$ (essentially N is an interval endowed with a negative metric) is static. Note however that a local warped product with a normal factor locally isometric to $(\mathbb{R}, -dt^2)$ is not necessarily static, since there is an ambiguity in defining a global Killing field as desired. The natural notion that can be considered here is that of a locally static spacetime, which will thus be equivalent to having a local warped product structure with a normal factor locally isometric to $(\mathbb{R}, -dt^2)$.

7.2.1. *A naive gravitational model.* Consider the warped product $M = (\mathbb{R}^3$, Euclidean) $\times_r (\mathbb{R}, -dt^2)$, where $w = r : \mathbb{R}^3 \to \mathbb{R}$ is the radius function. (The warped product metric is non-degenerate only for $r \neq 0$, so more exactly, M equals $(\mathbb{R}^3 - \{0\}) \times_r \mathbb{R}$).

From Theorem 6.3, the projection of the geodesics of M are the trajectories of the mechanical systems on the Euclidean space $\mathbb{R}^3$, with potentials of the form $V = c/r$, where c is a non-positive constant. By this, one obtains in particular the trajectories of the Newtonian potential $V = -1/r$.

In fact, this process gives a (naive) relativistic static model $L \times_w (\mathbb{R}, -dt^2)$ associated to any negative potential $V = -1/w : L \to \mathbb{R}^-$ on a Riemannian manifold L.

One flaw of such a model is that it is not characteristic of the initial potential V, since it cannot distinguish between the potentials cV for different (non-negative) constants c, and it recovers in particular the geometry of L, for $c = 0$. In fact, except for exceptional cases, two warped products $L \times_w \mathbb{R}$ and $L \times_{cw} \mathbb{R}$ are isometric by means of the *unique* mapping $(x, t) \to (x, ct)$, which acts as a time dilation. Therefore, the model would be specific of the potential if one introduces an extra structure breaking time dilations.

It seems interesting to investigate some features of these spaces, especially from the viewpoint of being perfect fluids.

"Newtonian spacetimes" (see for instance [18], §12) were introduced by E. Cartan for the goal of making geodesic the dynamics under a mechanical system derived from a potential. The structure there is that of an affine connection, which is poor, compared to the Lorentz structure here.

We think it is worthwhile investigating a synthesis of all the approaches to geodesibility processes of dynamical systems.

7.3. **Polar coordinates.** The polar coordinates at 0 endow the Minkowski space $(\mathbb{R}^{n,1}, \langle , \rangle)$ with a warped product structure defined away from the light cone $\{x/\langle x, x\rangle = 0\}$. Inside the cone, the structure is physical, with a normal factor homothetic to the hyperbolic space $\mathbb{H}^n$, and outside the cone, the structure is anti-physical, with a normal factor homothetic to the de Sitter space $\{x/\langle x, x\rangle = +1\}$.

7.4. **Spaces of constant curvature.** (See for instance [28] for some facts on this subject). The spaces of constant curvature are already "simple", but one may need for some calculations to write them as (non-trivial) warped products, for instance polar coordinates on these spaces give rise to warped product structures defined on some open sets.

Recall that for these spaces, umbilic submanifolds (with dimension ≥ 2) are spherical, and also have constant curvature. In particular, a warped product structure in this case is a GRW structure. (In dimension 4, and if the normal factor is spacelike, one obtains a classical Robertson-Walker structure, §7.1. The perfect fluid has in this case constant density and pressure).

One can prove the following fact which classifies the warped products in this setting. (See [26] for a study of global warped products of physical type).

FACT **7.2.** *Let N be an umbilic (non-degenerate) submanifold in a space of constant curvature X. Consider the foliation $\mathcal{L}$, defined on a neighborhood $\mathcal{O}(N)$ of N, having as leaves the geodesic submanifolds orthogonal to N.*

Then, the orthogonal distribution of $\mathcal{L}$ is integrable, say it is tangent to a foliation $\mathcal{N}$. Moreover, $(\mathcal{L}, \mathcal{N})$ determines a GRW structure.

Furthermore, $\mathcal{N}$ is the orbit foliation of the isometric action of a natural subgroup $T(N)$ of Isom(X) preserving N. In the case where N is a geodesic submanifold, $T(N)$ is the group generated by the transvections along the geodesics of N. (A transvection along a geodesic is an isometry which induces parallel translation along it).

7.5. **Schwarzschild spacetime.** The building of Schwarzschild spacetime gives an excellent example of how various warped product structures may be involved. We will essentially study it from this point of view. This spacetime models a relativistic one body universe (a star). Its construction is accomplished by translating the physical content into geometrical structures, and making at each stage "necessary" topological simplifying assumptions.

The spatial isotropy around the star leads to the first geometric structure, formulated by the fact that $\mathsf{SO}(3)$ acts isometrically with 2-dimensional orbits. From Fact 5.4, we get a local warped product of the type $L \times_w S^2$ (one excludes the $\mathbb{R}P^2$-case by an appropriate orientability extra hypothesis). One then makes the topological simplifying hypothesis that the warped product is global.

This warped product (in particular the function w) is canonical (it has a physical meaning) and is in particular compatible with the additional structures.

The second geometrical hypothesis on the spacetime is that it is static (which in fact leads to another local warped product structure with a normal factor locally isometric to $(\mathbb{R}, -dt^2)$).

The compatibility between structures, implies, essentially, that the surface L itself is static. Thus (after topological simplification) L is a warped product $(\mathbb{R}, g) \times_v (\mathbb{R}, -dt^2)$. (where g is some metric on $\mathbb{R}$).

By compatibility, the warping function w is invariant by the Killing timelike field $\frac{\partial}{\partial t}$ on L. Its gradient is thus tangent to the first factor $\mathbb{R}$ of L. Another topological simplification consists in assuming that w is regular, namely, $r = \sqrt{w}$ is a global coordinate function on $\mathbb{R}$ (the first factor of L). We write the metric on this factor as $g = g(r)dr^2$ (g is now a function on $\mathbb{R}$).

The metric on the spacetime has thus the form $g(r)dr^2 - v(r)dt^2 + r^2 d\sigma^2$ ($d\sigma^2$ is the canonical metric on S^2).

The third geometrical hypothesis is that the spacetime is empty (a vacuum), i.e. Ricci flat, leading to differential relations on the functions g and v. They imply that $g = \frac{1}{1-(2m/r)}$, and that vg equals a constant (here one has to perform some computation). This last constant must equal to 1, by the fourth geometrical hypothesis saying that the spacetime is asymptotically Minkowskian.

We have therefore, $L =]2m, +\infty[\times \mathbb{R}$, endowed with the metric:

$$\frac{1}{1-(2m/r)} dr^2 - (1-(2m/r))dt^2$$

The warped product $L \times_{r^2} S^2$ is called the *Schwarzschild exterior* spacetime.

It is natural to ask if other solutions exist without our topological simplification hypotheses. This is essentially equivalent to ask if the spacetime $L \times_{r^2} S^2$ admits non-trivial extensions. One easily sees that no such *static* extensions exist. However non-trivial analytic (and thus Ricci flat) extensions actually exist. They (essentially) correspond to analytic extensions of the Lorentz surface L.

The obvious one is given by adding $L^- =]0, 2m[\times \mathbb{R}$, endowed with the metric defined by the same formula. The warped product $L^- \times_{r^2} S^2$ is called the *Schwarzschild black hole*.

It has been observed (firstly by Lemaître, see for instance [18]) that the metric on $L \cup L^-$ admits an analytic extension to all $]0, +\infty[\times \mathbb{R}$.

Next, a larger extension $\hat{L}$, which turns out to be "maximal", was discovered by Kruskal. It can be described, at a "topological level" as follows. Endow $\mathbb{R}^2$ with coordinates (x, y) and a Lorentz scalar product (at 0) $dxdy$. Then, $\hat{L}$ is the part of $\mathbb{R}^2$ defined by an inequality $xy > c(m)$, where $c(m)$ is a negative constant. The metric has the form $F(xy)dxdy$, where $F :]c(m), +\infty[\to \mathbb{R}$ is an analytic real function which tends to ∞ at $c(m)$. (It turns out that a coordinate system where the metric has this form is unique up to a linear diagonal transformation.)

From the form of the metric, the flow $\phi^s(x, y) = (e^s x, e^{-s} y)$ acts isometrically on $\hat{L}$. This corresponds to the analytic extension of the Killing field $\frac{\partial}{\partial t}$ defined on L.

The time function t on L has the form $t(x,y) = a \ln \frac{x}{y}$, where a is a constant (which depends on the coordinate system).

The radius function r looks like a Lorentz radius, indeed it has the form, $r(x,y) = b(xy) + 2m$, for some analytic function $b : [c(m), +\infty[\to [-2m, +\infty[$, with $b(0) = 0$. (A natural Lorentz radius for $(\mathbb{R}^2, dxdy)$ is $\sqrt{|xy|}$).

Our initial surface L is identified with the positive quadrant $x, y > 0$.

The warped product structure (determined by the flow ϕ^s on $\hat{L} - \{xy = 0\}$) is physical on $xy > 0$, and anti-physical on $xy < 0$. In fact, this structure is conformal to that determined by the polar coordinates on $(\mathbb{R}^2, dxdy)$ (§§1.3.1, and 7.3).

7.5.1. *Geodesic foliations.* The factor $\hat{L}$ determines a geodesic foliation of the Kruskal spacetime $\hat{L} \times_{r^2} S^2$.

The static structure (on $L \times_{r^2} S^2$) determines a geodesic foliation $\mathcal{F}$ with leaves $t =$ constant, or equivalently $\frac{x}{y} =$ constant. Thus a leaf has the form: $F = R \times S^2$, where $R \subset \hat{L}$ is a ray emanating from 0.

This foliation extends to $(\hat{L} - 0) \times S^2$ (and to the whole Kruskal spacetime $\hat{L} \times_{r^2} S^2$, as a singular geodesic foliation).

The causal character of a leaf F is the same as that of the ray R. In particular, lightlike leaves correspond to lightlike rays, i.e. the coordinate axis.

7.5.2. *Geodesics.* To determine all the geodesics of $L \times_{r^2} S^2$, one uses Theorem 6.3 which reduces the problem to the calculation of the trajectories of mechanical systems on the surface L defined by the potentials $\frac{c}{r^2}$.

Now, since L itself is a warped product, one applies Corollary 6.6 to solve mechanical systems with potentials $\frac{c}{r^2}$ over it. This reduces to use the energy conservation, and solve the mechanical systems with potentials $c_1 \frac{1}{1-(2m/r)} + c_2 \frac{1}{r^2}$ on $(\mathbb{R}, \frac{1}{1-(2m/r)} dt^2)$.

Proposition 6.5 applies to these potentials (considering L as a warped product), which allows one to fully explicit the geodesics.

7.6. **Motivations for anti-physical warped products.** We think there is no reason to be troubled by anti-physical warped products. The adjective anti-physical must not suggest that they are "non physical", but rather that they are "mirror transform" of physical ones (to be found?). This clearly happens in the case of polar coordinates in the Minkowski space, where one sees how the anti-physical part of the GRW structure is dual to the physical one (§7.3). A similar duality holds between the interior and the exterior of the Schwarzshild spacetime. The exterior is static, by the existence of a timelike Killing field, which becomes spacelike in the interior. The interior of a black hole is anti-physical.

Let us enumerate further (physical) motivations of anti-physical warped products:

• With respect to the goal of constructing simple exact solutions, the calculations are formally the same, in the physical as well as in the the anti-physical cases. So, one may calculate, and forget that it is an anti-physical warped product!

• As was said before, the abundance of symmetries leads to a warped product structure, but actually, large symmetry groups involve anti-physical warped products. For example, non-proper isometry groups lead to an anti-physical warped product (see for instance [29]). Roughly speaking, non-proper means that the stabilizers are non-compact. Let us however say that only few exact solutions have non-proper isometry groups. It seems that this is the case, only for spaces of constant curvature and some gravitational plane waves.

• Finally, it seems interesting to formulate a complexification trick which exchanges anti-physical by physical structures. The very naive idea starts by considering a Riemannian analytic submanifold V in the Euclidean space $\mathbb{R}^N$, taking its "complexification" $V^{\mathbb{C}}$ and then inducing on it the holomorphic metric of $\mathbb{C}^N$, which as a real metric is pseudo-Riemannian. (The complexification is defined only locally but one may approximate by algebraic objects in order to get a global thing, see for instance [12] for related questions).

8. Big-bangs in anti-physical warped products

Consider the example of polar coordinates around 0 in the Minkowski space $\mathbb{R}^{n,1}$ (§7.3). When an interior point approaches the light cone (and especially 0), the warping function collapses, and the warped product structure disappears. However, the spacetime itself persists, beyond this "false big-bang". It seems interesting to know situations where a "true big bang" (i.e. a disappearing of the spacetime) must follow from a disappearing of the warped product structure. The results below provide an example of such a situation, but let us before try to give a more precise definition.

Definition 8.1. *Let $(\mathcal{L}, \mathcal{N})$ be a warped product structure on a pseudo-Riemannian manifold M. We say that it has an* **inessential big-bang** *if there is an isometric embedding of M in another pseudo-Riemannian manifold M', as an open* **proper** *subset, such that the shape vector field $\overrightarrow{n}$ of $\mathcal{N}$, is non-bounded in some compact subset of M'.*

In other words, we see M as an open subset of M', then, an inessential big-bang holds if there is a compact K in M' such that $\overrightarrow{n}$ is not bounded on $K \cap M$. (Observe that one may speak of bounded vector fields on compact sets without any reference to metrics). We have the following result.

Theorem 8.2. *An analytic anti-physical GRW structure with non-positively curved normal factor, has no inessential big-bangs.*

Let us give a purely mathematical essentially equivalent statement.

Theorem 8.3. *(*[29]*) Let M be an analytic Lorentz manifold such that some open subset U of M is isometric to a warped product $L \times_w N$, where N (is Lorentzian and) has a constant non-positive curvature (i.e. N is locally isometric to the Minkowski or the anti de Sitter spaces). Then, every point of M has a neighborhood isometric to a warped product of the same type. More precisely, if M is simply connected, then the warped product structure on U extends to a local warped product (of the same type) on M.*

Let us give another formulation in the vein of detecting singularities of a spacetime from that of a warped product structure on it.

Corollary 8.4. *Let M be a simply connected manifold, and U an open subset of M endowed with an analytic Lorentz metric g. Suppose that (U, g) is a warped product as above, and let x be a point in the boundary of U. If the warping function w tends to ∞ or 0 near x, then (not only the warped product structure, but also) the Lorentz metric g does not extend analytically near x.*

Remark 8.5. The case of polar coordinates on the Minkowski space shows that the hypotheses that the GRW structure is anti-physical and the normal factor of non-positive curvature are necessary.

In the sequel, we will give the proof of Theorem 8.3, and also details on the tools behind it, especially about lightlike Killing fields.

9. Proof of Theorem 8.3

9.1. **Beginning.**

9.1.1. *Trivial extension.* Let $\mathcal{E}_c^{d+1}$ denote the simply connected complete Lorentz space of constant curvature c (see for instance [28] for more details).

In the case $c \neq 0$, we assume that $d > 1$, that is, the dimension of the space is ≥ 3. In fact, in dimension 2, the sign of the curvature is irrelevant.

Let $U = L \times_w N$ be as in the statement of Theorem 8.3. By hypothesis N is locally isometric to $\mathcal{E}_c^{d+1}$ for some $c \leq 0$. We can restrict U so that N becomes identified to an open subset of $\mathcal{E}_c^{d+1}$.

By the trivial extension of isometries, Fact 1.1, Isom (N) acts on U. However, because N is a "small" open subset of $\mathcal{E}_c^{d+1}$, Isom(N) may be dramatically small, and for this, it is better to consider infinitesimal isometries, i.e. Killing vector fields. Indeed, like isometries, Killing vector fields of N, trivially extend to U. Now the Killing algebra of N (i.e. the algebra of Killing fields) is the same as that of $\mathcal{E}_c^{d+1}$ which we denote by $\mathcal{G}_c^{d+1}$. Therefore there is an **infinitesimal action** of $\mathcal{G}_c^{d+1}$ on U, i.e. a homomorphism which for $X \in \mathcal{G}_c^{d+1}$ associates an element $\bar{X}$ of the Killing algebra of U.

Note that, for our purpose, only the sign of c is relevant, that is we can assume $c = -1$, whenever $c < 0$.

Recall that $\mathcal{G}_0^{d+1}$, the Killing Lie algebra of the Minkowski space $\mathcal{E}_0^{d+1}$, is isomorphic to a semi-direct product $\mathbb{R}^{d+1} \rtimes o(1, d)$, and that the Killing Lie algebra of the anti de Sitter space $\mathcal{E}_{-1}^{d+1}$ is $\mathcal{G}_{-1}^{d+1} = o(2, d)$

9.1.2. *Analytic extension.* Henceforth, we will assume that M is simply connected and analytic (it suffices just to pass to the universal covering). A classical result [20] states that an analytic Killing field defined on an open subset extends as a Killing field to the whole of M.

By individual extension of Killing fields, we get an infinitesimal analytic isometric action of $\mathcal{G}$ on the whole of M.

However, this action does not a priori determine a regular foliation, namely, the dimension of the orbits is not necessarily constant.

Let us first observe that the analyticity implies that $d+1$, i.e. the dimension of the orbits of the points of U, is the generic dimension of orbits, that is, the dimension is everywhere $\leq d+1$. Indeed, if $X_1, \ldots, X_{d+1} \in \mathcal{G}$, then $\bar{X}(x) \wedge \ldots \wedge \bar{X}_{d+2}(x) = 0$ for $x \in U$, and hence everywhere (of course, we implicitly assume that all our spaces here are connected).

Proposition 9.1. *Let $\mathcal{G} = \mathcal{G}_c^{d+1}$ act infinitesimally isometrically on a Lorentz manifold M (here c is not assumed to be ≤ 0), with a generic orbit dimension $\leq d+1$. assume that all (the restrictions of the metrics on) the orbits are non-degenerate. In the case $c > 0$, assume further that at least one orbit is of Lorentzian type. Then, the $\mathcal{G}$-action determines a regular (i.e. with constant dimension) foliation, which is the normal foliation of a GRW structure.*

Proof. Observe that an orbit is a $\mathcal{G}$-locally homogeneous space. So, the proof of the proposition follows from Theorem 5.3 and from the following classical fact.

FACT **9.2.** *If a pseudo-Riemannian manifold of dimension $\leq d$, has a Killing algebra of the same dimension as that of a pseudo-Riemannian manifold of constant curvature and dimension d, then this manifold is necessarily of dimension d and has the same constant curvature.*

Proof. Recall that all the orthogonal algebras $o(p,q)$, with $p+q = d'$ have the same dimension, which equals in particular dim $o(d')$. Let x be a point of the given pseudo-Riemannian manifold. Its stabilizer algebra can be identified to a subalgebra of some $o(p,q)$, with $p+q \leq d$. But by hypothesis, this stabilizer has a dimension $\geq$ dim $o(d)$. It follows that $p+q = d$, and that the stabilizer is $o(p,q)$ itself. One deduces, in particular, that the dimension of the manifold equals d. To check that the curvature is constant, one observes that $O(p,q)$ acts transitively on the space of spacelike 2-planes at x. □

□

9.2. **Lightlike Killing fields.** The following notion will be useful.

Definition 9.3. *A Killing field X on a pseudo-Riemannian manifold is called geodesic (resp. lightlike) if $\nabla_X X = 0$ (resp. $\langle X, X\rangle = 0$).*

FACT **9.4.** *A Killing field X is geodesic iff, it has geodesic orbits, iff, it has constant length (i.e. $\langle X, X\rangle$ is constant). In particular a lightlike Killing field is geodesic.*

Proof. Let ∇ denote the Levi-Civita connection. Recall that a Killing field X is characterized by the fact that ∇X is antisymmetric, that is, $\langle \nabla_Y X, Z\rangle + \langle Y, \nabla_Z X\rangle = 0$, for any vector fields Y and Z. In particular, $\langle \nabla_X X, Y\rangle + \langle \nabla_Y X, X\rangle = 0$, and hence, $\langle \nabla_X X, Y\rangle = -(1/2)Y.\langle X, X\rangle$. Therefore, $\nabla_X X = 0$ is equivalent to that $\langle X, X\rangle$ is constant.

□

9.2.1. *Singularities.* A geodesic Killing field with a somewhere non-vanishing length is non-singular (since it has a constant length). This fact extends to lightlike Killing fields on Lorentzian manifolds.

Indeed, near a singularity, the situation looks like that of the Minkowskian case. In this case, the Killing field preserves (i.e. is tangent to) the "spheres" around the singularity, but some of these spheres are spacelike, contradiction!

As it is seen in this sketch of proof, the fact actually extends to non-spacelike Killing fields, i.e. $\langle X, X\rangle \leq 0$:

FACT **9.5.** *(*[3]*, see also* [1] *and* [30]*) A non-trivial non-spacelike Killing field on a Lorentz manifold is singularity free.*

9.2.2. *Curvature.*

FACT **9.6.** *Let X be a geodesic Killing field, then, for any vector Y,*

$$\langle R(X,Y)X, Y\rangle = \langle \nabla_Y X, \nabla_Y X\rangle$$

If M is Lorentzian *or* Riemannian*, and X is non-spacelike (i.e. $\langle X, X\rangle \leq 0$), then $\langle R(X,Y)X, Y\rangle \geq 0$. In particular, $Ric(X,X) \geq 0$, with equality (i.e. everywhere $Ric(X,X) = 0$), iff, the direction of X is parallel.*

In the case M is lorentzian and X is lightlike, the curvature of any non-degenerate 2-plane containing X is ≤ 0.

Proof. Let γ be a geodesic tangent to Y. Consider the surface S_γ obtained by saturating γ by the flow of X, i.e. if ϕ^t is the flow of X, then $S_\gamma = \cup_t \phi^t(\gamma)$ (here we assume that X is transversal to γ).

Take a geodesic parameterization of γ, and continue to denote by Y, the vector field on S_γ, obtained first, by parallel translating along γ, and then, saturating by ϕ^t (along S_γ).

We have: X and Y commute, $\nabla_Y Y = 0$, and $\nabla_X X = 0$ (since X is a geodesic Killing field). It remains to estimate $\nabla_X Y$ $(= \nabla_Y X)$. We have $0 = Y\langle X, X\rangle = 2\langle \nabla_Y X, X\rangle$ (since $\langle X, X\rangle$ is constant by Fact 9.4) and $X\langle Y, Y\rangle = 2\langle \nabla_X Y, Y\rangle$, since by construction $\langle Y, Y\rangle$ is constant along S_γ. Therefore, $\nabla_X Y$ $(= \nabla_Y X)$ is orthogonal to S_γ.

One may restrict consideration to the case where S_γ is non-degenerate, since, if not, one may approximate S_γ by non-degenerate S_{γ_n}, by choosing an appropriate sequence of geodesics γ_n.

The previous calculation implies that S_γ is intrinsically flat, since the orthogonal projection of the ambient connection vanishes (all the covariant derivatives obtained from X and Y are orthogonal to S_γ)..

The curvature equality follows from the Gauß equation.

Now, $\nabla_X Y$ is orthogonal to X, and hence it is spacelike whenever X is non-spacelike and M is Riemannian or Lorentzian.

Recall that $Ric(X,X)$ equals the trace of the linear endomorphism $Y \to A(Y) = R(X,Y)X$. Now, $\langle A(Y), Y\rangle \geq 0$ implies that trace$(A) \geq 0$, and it is also straightforward to see that if $Ric(X,X) = 0$, then $\nabla_Y X 0$ is isotropic for all Y. We have in addition that $\langle \nabla_Y X, X\rangle = 0$, and hence $\nabla_Y X$ is proportional to X. This is exactly the analytic translation of the fact that the direction field determined by X is parallel.

Finally, the sectional curvature of the plane generated by X and Y is $\frac{\langle R(X,Y)X,Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$, which has the opposite sign of $\langle R(X,Y)X, Y\rangle$.

□

9.2.3. *The constant curvature case.* Let $\mathbb{R}^{p,q}$ denote $\mathbb{R}^n$ ($n = p + q$), endowed with the standard form $Q = -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots x_n^2$, of signature (p, q). A Killing field X on $\mathbb{R}^{p,q}$ is of the form $x \to Ax + a$, where $a \in \mathbb{R}^n$, and $A \in o(p, q)$. Recall that $A \in o(p, q)$, iff, $AJ + JA^* = 0$, where

$$J = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$$

We have, $\nabla_X X = A^2$, and hence, X is geodesic, iff, $A^2 = 0$.

In the Lorentzian case (i.e. the Minkowski space), $p = 1$, the equation $A^2 = 0$, has no non-trivial solution, that is, if $A \in o(1, p)$, and $A^2 = 0$, then, $A = 0$. One may show this by a straightforward calculation, or by applying Fact 9.6 to $S^{1,p}(+1)$, which will be defined below. It follows that a geodesic Killing field is parallel, i.e. it has the form $X : x \to a$, and it is lightlike if furthermore a is isotropic.

In the non-Lorentzian case, non-trivial solutions of $A^2 = 0$ exist. Let us consider the case of $\mathbb{R}^{2,2}$. The standard form Q is equivalent to $Q' = dxdz + dydt$. Consider $\phi^s(x, y, z, t) = (x, y, z + sx, t + sy)$. This is a one-parameter group of orthogonal transformations of Q'. Its infinitesimal generator:

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies $B^2 = 0$. Thus, a conjugate A of B belongs to $o(2, 2)$ and satisfies $A^2 = 0$ A standard argument shows to that $o(2, 2)$ is in fact generated by elements satisfying the equation $A^2 = 0$. By the same argument one proves:

FACT **9.7.** *For $p \geq 2$, $q \geq 2$, $o(p, q)$ is generated (as a linear space) by its elements satisfying $A^2 = 0$. (Note that the condition on p and q is equivalent to that $o(p, q)$ has real rank ≥ 2).*

Consider $X_c = S^{p,q}(c) = \{x/Q(x, x) = c\}$. Then, for $c \neq 0$, X_c is non-degenerate, and the metric on it has signature $(p, q - 1)$ if $c > 0$, and signature $(p - 1, q)$ if $c < 0$. It has curvature $\frac{1}{c}$, and Killing algebra $o(p, q)$. The universal pseudo-Riemannian space of the same signature and curvature, is a cyclic (maybe trivial) covering of X_c. The Killing algebra of the universal cover is the same as that of X_c (see [28]).

A Killing field $A \in o(p, q)$ is geodesic (with respect to X_c), iff, $A^2 = \lambda I$, for some constant λ. It is lightlike, iff, $A^2 = 0$.

For example, in the Riemannian case, i.e. $p = 0$, solutions of $A^2 = \lambda I$ in $o(n)$ exist exactly if n is even, which give Hopf fibrations on odd dimensional spheres.

For the Lorentz case, we have, with the previous notations, $\mathcal{E}_c^{d+1} = S^{1,d+1}(c)$, if $c > 0$, and $\mathcal{E}_c^{d+1}$ is the universal cover of $S^{2,d}(c)$, if $c < 0$.

In particular, a solution of $A^2 = 0$ in $o(1, p)$ corresponds to a lightlike Killing field on the de Sitter space ($= \mathcal{E}_c^{d+1} = S^{1,d+1}(c)$). But, since this latter space is

Lorentzian and has positive curvature, such a non-trivial Killing field does not exist by Fact 9.6. Summarizing:

FACT **9.8.** *The de Sitter space has no lightlike (or even geodesic) Killing fields.*

A lightlike Killing field on the Minkowski space is parallel with isotropic translation vector.

The Killing algebra of the anti de Sitter space is generated, as a linear space, by its lightlike Killing fields.

9.3. **End of the proof of Theorem 8.3.** Observe that if $X \in \mathcal{G}_c^{d+1}$ is lightlike, as a Killing field on $\mathcal{E}_c^{d+1}$, then its trivial extension $\bar{X}$, is a lightlike Killing field on M.

Suppose by contradiction that there is a degenerate orbit N_0 of the $\mathcal{G}_c^{d+1}$-action.

From §9.1.2, N_0 has dimension $\leq d+1$. Observe first that $\dim N_0 > 0$, since lightlike Killing fields are singularity free.

The metric on N_0 is positive non-definite, with kernel of dimension 1 (since the metric on M is Lorentzian). This determines a 1-dimensional foliation $\mathcal{F}$, called the **characteristic** foliation of N_0. The tangent direction of $\mathcal{F}$ is the unique isotropic direction tangent to N_0. It then follows that if X is a lightlike Killing field, then the restriction of $\bar{X}$ to N_1 is tangent to $\mathcal{F}$ (equivalently, the flow of such a Killing field preserves individually the leaves of $\mathcal{F}$). Therefore, from Fact 9.4, the leaves of $\mathcal{F}$ are lightlike geodesics (in M).

The anti de Sitter case. In the case $c < 0$, $\mathcal{G}_c^{d+1}$ is generated by lightlike Killing fields, and hence $\mathcal{G}_c^{d+1}$ itself preserves individually the leaves of $\mathcal{F}$. Thus, by definition, N_0 has dimension 1. However, it is known that there is no $\mathcal{G}_c^{d+1}$-homogeneous space of dimension 1. This is particularly easy to see in the present situation. Indeed, here, $\mathcal{G}_c^{d+1}$ preserves the affine structure of the lightlike geodesic N_0, and hence $\mathcal{G}_c^{d+1}$ embeds in the Lie algebra of the affine group of $\mathbb{R}$, which is impossible.

The flat case. If N_0 has dimension 1, we get a contradiction as in the anti de Sitter case. If not (i.e. $\dim N_0 > 1$), consider the (local) quotient space $Q = N_0/\mathcal{F}$. (The global quotient does not necessarily exist, but because we deal with infinitesimal actions, we can restrict everything to a small open subset of M). The $\mathcal{G}_0^{d+1}$-action on N_0 factors through a faithful action of $o(1,d)$ $(= \mathcal{G}_0^{d+1}/\mathbb{R}^{d+1})$ on Q.

Observe that Q inherits a natural Riemannian metric. Indeed, the Lorentz metric restricted to N_0 is positive degenerate, with kernel $T\mathcal{F}$. But $\mathcal{F}$ is parameterized by any lightlike field $X \in \mathcal{G}_0^{d+1}$ (this is the meaning of the fact that the flow of $\bar{X}$ preserves individually the leaves of $\mathcal{F}$). Therefore the projection of this metric on Q is well defined.

This metric is invariant by the $o(1,d)$-action. As in the proof of Fact 9.2, since $\dim Q \leq d$, we have $\dim Q = d$, and furthermore, Q has constant curvature. Also, we recognize from the list of Killing algebras of constant curvature manifolds that Q has constant negative curvature, i.e. Q is a hyperbolic space.

It then follows that $\dim N_0 = d$, and in particular that the orbits of $\mathcal{G}_c^{d+1}$ determine a regular foliation near N_0.

Now, the contradiction lies in the fact that Q is hyperbolic, but the analogous quotient for generic leaves of the $\mathcal{G}_0^{d+1}$-action, are flat. More precisely, let $X \in \mathbb{R}^{d+1} \subset \mathcal{G}_0^{d+1}$ be a translation timelike Killing field. Consider M' the (local) space of orbits of X (instead of the whole of M, we take a small open subset intersecting N_0, where everything is topologically trivial). The $\mathcal{G}_0^{d+1}$-orbit foliation projects to a foliation $\mathcal{G}'$ of M'. For example, Q is a leaf of $\mathcal{G}'$ which is just the projection of N_0. In fact, as in the case of Q, the projection of the metric on the $\mathcal{G}_0^{d+1}$-orbits endows the leaves of $\mathcal{G}'$ with a Riemannian metric. Now, a generic leaf of $\mathcal{G}'$ is (locally) isometric to the quotient of the Minkowski space $\mathbb{R}^{d,1}$ by a timelike translation flow, which is thus a Euclidean space (of dimension d). But the leaf Q is hyperbolic which contradicts the obvious continuity (in fact the analyticity) of the leafwise metric of $\mathcal{G}'$. $\diamondsuit$

REFERENCES

[1] S. Adams, G. Stuck, The isometry group of a compact Lorentz manifold. Invent. Math. 129 (1997) 239–261.

[2] S. Alexander, R. Bishop, Warped products of Hadamard spaces. Manuscripta Math. 96 (1998) 487–505.

[3] J. Beem, P. Ehrlich, S. Markvorsen, Timelike isometries and Killing fields. Geom. Dedicata 26 (1988) 247–258.

[4] J. Beem, P. Ehrlich, T. Powell, Warped product manifolds in relativity. Selected studies: physics-astrophysics, mathematics, history of science, 41–56, North-Holland, Amsterdam-New York, 1982.

[5] R. Blumenthal, J. Hebda, de Rham decomposition theorems for foliated manifolds. Ann. Inst. Fourier (Grenoble) 33 (1983) 183–198.

[6] M. Brunella, E. Ghys, Umbilical foliations and transversely holomorphic flows. J. Differential Geom. 41 (1995), no. 1, 1 - 19.

[7] E. García-Río, D. Kupeli, A local characterization of Schwarzschild and Reissner metrics. Gen. Relativity Gravitation 26 (1994) 1233–1248.

[8] J. Carot, J. da Costa, On the geometry of warped spacetimes. Classical Quantum Gravity 10 (1993) 461–482.

[9] Y. Carrière, E. Ghys, Feuilletages totalement géodésiques. An. Acad. Brasil. Ciênc. 53 (1981), no. 3, 427- 432.

[10] H. Gauchman, On warped decomposition of Riemannian manifolds. Tensor 32 (1978) 65–68.

[11] S. Harris, A characterization of Robertson-Walker spaces by null sectional curvature. Gen. Relativity Gravitation 17 (1985) 493–498.

[12] P. Heinzner, A. Huckleberry, F. Loose, Kählerian extensions of the symplectic reduction. J. Reine Angew. Math. 455 (1994) 123–140.

[13] H. Karcher, Hermann Infinitesimale Charakterisierung von Friedmann-Universen. Arch. Math. (Basel) 38 (1982) 58–64.

[14] S. Hawking, G. Ellis, The large scale structure of space-time. Cambridge Monographs on Mathematical Physics 1. Cambridge University Press, London-New York, 1973.

[15] S. Hiepko, Eine innere Kennzeichnung der verzerrten Produkte. Math. Ann. 241 (1979) 209–215.

[16] D. Lappas, Locally warped products arising from certain group actions. Japan. J. Math. (N.S.) 20 (1994) 365–371.

[17] H. Matsuda, On Lorentz manifolds with abundant isometries. Tsukuba J. Math. 13 (1989) 113–129.

[18] Ch. Misner, Ch. Thorne, J. Wheeler, Gravitation. W. H. Freeman and Co. 1973.

[19] P. Molino, Riemannian foliations. Progress in Mathematics, 73. BirkhŁuser Boston, Inc. 1988.

[20] K. Nomizu, On local and global existence of Killing vector fields. Ann. of Math. 72 (1960) 105–120.

[21] B. O'Neill, Semi-Riemannian geometry, With applications to relativity. Pure and Applied Mathematics, 103. Academic Press, Inc., 1983.

[22] F. Podestà, Cohomogeneity one Riemannian manifolds and Killing fields. Differential Geom. Appl. 5 (1995) 311–320.

[23] R. Ponge; H. Reckziegel, Twisted products in pseudo-Riemannian geometry. Geom. Dedicata 48 (1993), no. 1, 15–25.

[24] M. Prvanović, On warped product manifolds. Conference "Filomat '94" (Niš, 1994). Filomat No. 9, part 2 (1995) 169–185.

[25] V. Rovenskii, Foliations on Riemannian manifolds and submanifolds. BirkhŁuser Boston, Inc., 1998.

[26] M. Sánchez, On the geometry of generalized Robertson-Walker spacetimes: geodesics. Gen. Relativity Gravitation 30 (1998) 915–932.

[27] Ph. Tondeur, Geometry of foliations. Monographs in Mathematics, 90. BirkhŁuser Verlag, Basel, 1997.

[28] J. Wolf, Spaces of constant curvature. McGraw-Hill Book Co., New York-London-Sydney 1967.

[29] A. Zeghib, Isometry groups and geodesic foliations of Lorentz manifolds. Part II: Geometry of analytic Lorentz manifolds with large isometry groups. GAFA, 9 (1999) 823-854.

[30] A. Zeghib, On affine actions of Lie groups. Math. Z. 227 (1998) 245–262.

UMPA, CNRS, ÉCOLE NORMALE SUPÉRIEURE DE LYON
46, ALLÉE D'ITALIE 69364 LYON CEDEX 07, FRANCE
E-mail address: abdelghani.zeghib@ens-lyon.fr
http://www.umpa.ens-lyon.fr/~zeghib/