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Christine Laurent-Thiébaut

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Abstract

We study the stability of the embeddability of compact 2-concave CR manifolds in complex manifolds under small horizontal perturbations of the CR structure.

The study of local and global embeddability of CR manifolds in complex manifolds has occupied a large number of mathematicians in the last forty years. Most of the results concern the case of strictly pseudoconvex CR manifolds of hypersurface type, very few is known in the other cases.

The stability of the embeddability property of a CR manifold M was first studied by N. Tanaka, [13], for strictly pseudoconvex CR manifolds of hypersurface type and real dimension greater or equal to 5 embedded in some $\mathbb{C}^N$. Few years later R. S. Hamilton, [5] et [6], was interested in the stability of the embeddability property for hypersurfaces provided the perturbation of the original CR structure is the restriction of a perturbation of a complex structure on some complex manifold X of which M is the boundary. It was proved in [8] and [7] that this last condition is satisfied for 2-concave hypersurfaces with a pertubation preserving the contact structure.

Here we consider CR manifolds of higher codimension type with mixed Levi signature and we are interested in the stability of the embeddability of such CR manifolds in complex manifolds under small perturbations of the CR structure preserving the complex tangent bundle.

Our main result is the following theorem:

Theorem 1. *Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, which is smoothly embeddable as a CR manifold in a complex manifold X. Assume that $(\mathbb{M}, H_{0,1}\mathbb{M})$ is either 2-concave and generically embeddable or $(\mathbb{M}, H_{0,1}\mathbb{M})$ satisfies both conditions Y(1) and Y(2). Let $\mathcal{E}_0$ be a smooth (generic if $(\mathbb{M}, H_{0,1}\mathbb{M})$ is 2-concave) embedding from $\mathbb{M}$ into X and denote by M_0 the image of $\mathbb{M}$ by the embedding $\mathcal{E}_0$. Assume that the $\overline{\partial}_b$-group of cohomology $H^{0,1}(M_0, T_{1,0}X_{|_{M_0}}) = 0$, and let $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$ defined by a $(0,1)$-form $\Phi \in \mathcal{C}^{l+2}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \geq 1$.*

UJF-Grenoble 1, Institut Fourier, Grenoble, F-38041, France
CNRS UMR 5582, Institut Fourier, Saint-Martin d'Hères, F-38402, France
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Then there exists a positive real number δ such that if $\|\Phi\|_{l+2} < \delta$, then the CR manifold $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ is embeddable in X as a CR submanifold of class $\mathcal{C}^l$.

Note that in the hypersurface case, by a theorem of Gray [4] saying all contact structures on a compact manifold near a fixed contact structure are equivalent, the assumption on the perturbation to be horizontal is not a true restriction.

Since strictly pseudoconvex CR manifolds of hypersurface type and of real dimension greater or equal to 7 satisfies both conditions Y(1) and Y(2), Theorem 1 precises the regularity in Tanaka's result on stability of embeddability in the case of real dimension greater or equal to 7.

Note also that, if the complex manifold X is some euclidean space $\mathbb{C}^n$, the cohomological condition $H^{0,1}(M_0, T_{1,0}X_{|_{M_0}}) = 0$ in Theorem 1 is reduced to $H^{0,1}(M_0) = 0$. Since, for a compact CR manifold satistying Y(1), $H^{0,1}(M_0)$ is finite dimensional, it follows from [3] that $H^{0,1}(M_0) = 0$, if $(\mathbb{M}, H_{0,1}\mathbb{M})$ is generically embedded in $\mathbb{C}^n$, so the cohomological condition $H^{0,1}(M_0, T_{1,0}X_{|_{M_0}}) = 0$ is always satisfied when $X = \mathbb{C}^n$ and $\mathcal{E}_0$ is generic.

Hence we get from Theorem 1 that, if $\widehat{H}_{0,1}M$ is a small perturbation of class $\mathcal{C}^{l+2}$ of the CR structure of a strictly pseudoconvex real hypersurface M of $\mathbb{C}^n$, $n \geq 4$, then the CR manifold $(M, \widehat{H}_{0,1}M)$ is embeddable in $\mathbb{C}^n$ as a CR submanifold of class $\mathcal{C}^l$.

In the case of mixed Levi signature, Theorem 1 was proved by P. L. Polyakov in [11] under the stronger hypothesis that $\mathbb{M}$ is 3-concave and generically embeddable, but with a larger loss of regularity. We have to notice that, compare to our situation, Polyakov has no restriction on the kind of the perturbation of the CR structure, but since in the 3-concave case both conditions $Y(1)$ and $Y(2)$ are satisfied, we do not need the generic embeddability of the initial CR structure. Our hypothesis of horizontality of the perturbation allows us to work with anisotropic Hölder spaces and to avoid a Nash-Moser process in the proof of the theorem.

The paper is organized as follows:

Section 1 consists in the description of the general setting and in the definitions of the the main objects used in this paper.

Section 2 is devoted to the proof of Theorem 1. We first remark that the problem can be reduced to the solvability of some tangential Cauchy-Riemann equation for the perturbed structure. Using global homotopy formulas with good estimates we are lead to a fixed point theorem, which gives the solution.

1 CR Structures

Let $\mathbb{M}$ be a $\mathcal{C}^l$-smooth, $l \geq 1$, paracompact differential manifold, we denote by $T\mathbb{M}$ the tangent bundle of $\mathbb{M}$ and by $T_{\mathbb{C}}\mathbb{M} = \mathbb{C} \otimes T\mathbb{M}$ the complexified tangent bundle.

Definition 1.1. An *almost CR structure* on $\mathbb{M}$ is a subbundle $H_{0,1}\mathbb{M}$ of $T_{\mathbb{C}}\mathbb{M}$ such that $H_{0,1}\mathbb{M} \cap \overline{H_{0,1}\mathbb{M}} = \{0\}$.

If the almost CR structure is integrable, i.e. for all $Z, W \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ then $[Z, W] \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$, then it is called a *CR structure*.

If $H_{0,1}\mathbb{M}$ is a CR structure, the pair $(\mathbb{M}, H_{0,1}\mathbb{M})$ is called an *abstract CR manifold*.

The *CR dimension* of $\mathbb{M}$ is defined by CR-dim $\mathbb{M} = \mathrm{rk}_{\mathbb{C}}\ \mathrm{H}_{0,1}\mathbb{M}$.

We set $H_{1,0}\mathbb{M} = \overline{H_{0,1}\mathbb{M}}$ and we denote by $H^{0,1}\mathbb{M}$ the dual bundle $(H_{0,1}\mathbb{M})^*$ of $H_{0,1}\mathbb{M}$.

The annihilator $H^0\mathbb{M}$ of $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ in $T^*_{\mathbb{C}}\mathbb{M}$ is called the *characteristic bundle* of $\mathbb{M}$. Given $p \in \mathbb{M}$, $\omega \in H^0_p\mathbb{M}$ and $X, Y \in H_p\mathbb{M}$, we choose $\widetilde{\omega} \in \Gamma(\mathbb{M}, H^0\mathbb{M})$ and $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathbb{M}, H\mathbb{M})$ with $\widetilde{\omega}_p = \omega$, $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$. Then $d\widetilde{\omega}(X, Y) = -\omega([X, Y])$. Therefore we can associate to each $\omega \in H^0_p\mathbb{M}$ an hermitian form

$$L_\omega(X) = -i\omega([X, \overline{X}]) \tag{1.1}$$

on $H_p\mathbb{M}$. This is called the *Levi form* of $\mathbb{M}$ at $\omega \in H^0_p\mathbb{M}$.

In the study of the $\overline{\partial}_b$-complex two important geometric conditions were introduced for CR manifolds of real dimension $2n - k$ and CR-dimension $n - k$. The first one by Kohn in the hypersurface case, $k = 1$, the condition Y(q), the second one by Henkin in codimension k, $k \geq 1$, the q-concavity.

A CR manifold $\mathbb{M}$ satisfies Kohn's condition $Y(q)$ at a point $p \in \mathbb{M}$ for some $0 \leq q \leq n - 1$, if the Levi form of $\mathbb{M}$ at p has at least $\max(n - q, q + 1)$ eigenvalues of the same sign or at least $\min(n - q, q + 1)$ eigenvalues of opposite signs.

A CR manifold $\mathbb{M}$ is said to be *q-concave* at $p \in \mathbb{M}$ for some $0 \leq q \leq n - k$, if the Levi form L_ω at $\omega \in H^0_p\mathbb{M}$ has at least q negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H^0_p\mathbb{M}$.

In [12] the condition Y(q) is extended to arbitrary codimension.

Definition 1.2. An abstract CR manifold is said to satisfy *condition Y(q)* for some $1 \leq q \leq n-k$ at $p \in \mathbb{M}$ if the Levi form L_ω at $\omega \in H^0_p\mathbb{M}$ has at least $n-k-q+1$ positive eigenvalues or at least $q + 1$ negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H^0_p\mathbb{M}$.

Note that in the hypersurface type case, i.e. $k = 1$, this condition is equivalent to the classical condition Y(q) of Kohn for hypersurfaces and in particular if the CR structure is strictly peudoconvex, i.e. the Levi form is positive definite or negative definite, condition Y(q) holds for all $1 \leq q < n-1$. Moreover, if $\mathbb{M}$ is q-concave at $p \in \mathbb{M}$, then $q \leq (n-k)/2$ and condition Y(r) is satisfied at $p \in \mathbb{M}$ for any $0 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$.

Let $\Lambda^{0,q}\mathbb{M} = \bigwedge^q(H^{0,1}\mathbb{M})$, then $\Omega^{0,q}_s(\mathbb{M}) = \Gamma^s(\mathbb{M}, \Lambda^{0,q}\mathbb{M})$ is called the space of $(0, q)$-forms of class $\mathcal{C}^s$, $0 \leq s \leq l$ on $\mathbb{M}$.

If the almost CR structure is a CR structure, i.e. if it is integrable, and if $s \geq 1$, then we can define an operator

$$\overline{\partial}_b \ : \ \Omega^{0,q}_s(\mathbb{M}) \to \Omega^{0,q+1}_{s-1}(\mathbb{M}) \tag{1.2}$$

called the *tangential Cauchy-Riemann operator* by setting $\overline{\partial}_b f = df_{|_{H_{0,1}\mathbb{M}\times\cdots\times H_{0,1}\mathbb{M}}}$. It satisfies $\overline{\partial}_b \circ \overline{\partial}_b = 0$.

Definition 1.3. Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract CR manifold, X be a complex manifold and $F : \mathbb{M} \to X$ be an embedding of class $\mathcal{C}^l$, then F is called a *CR embedding* if $dF(H_{0,1}\mathbb{M})$ is a subbundle of the bundle $T_{0,1}X$ of the holomorphic vertor fields of X and $dF(H_{0,1}\mathbb{M}) = T_{0,1}X \cap T_{\mathbb{C}}F(\mathbb{M})$.

Let F be a CR embedding of an abstract CR manifold into a complex manifold X and set $M = F(\mathbb{M})$, then M is a CR manifold with the CR structure $H_{0,1}M = T_{0,1}X \cap T_{\mathbb{C}}M$.

Let U be a coordinate domain in X, then $F_{|_{F^{-1}(U)}} = (f_1, \ldots, f_N)$, with $N = \dim_{\mathbb{C}}\mathrm{X}$, and F is a CR embedding if and only if, for all $1 \leq j \leq N$, $\overline{\partial}_b f_j = 0$.

A CR embedding is called *generic* if $\dim_{\mathbb{C}}\mathrm{X} - \mathrm{rk}_{\mathbb{C}}\mathrm{H}_{0,1}\mathrm{M} = \mathrm{codim}_{\mathbb{R}}\mathrm{M}$.

Definition 1.4. An almost CR structure $\widehat{H}_{0,1}\mathbb{M}$ on $\mathbb{M}$ is said an *horizontal perturbation* of the CR structure $H_{0,1}\mathbb{M}$ if $\widehat{H}_{0,1}\mathbb{M}$ can be represented as a graph in the complex tangent bundle $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$, which means that there exists $\Phi \in \mathcal{C}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$ such that

$$\widehat{H}_{0,1}\mathbb{M} = \{\overline{W} \in T_{\mathbb{C}}\mathbb{M} \mid \overline{W} = \overline{Z} - \Phi(\overline{Z}), \overline{Z} \in H_{0,1}\mathbb{M}\}. \tag{1.3}$$

The horizontal perturbation $\widehat{H}_{0,1}\mathbb{M}$ of the CR structure $H_{0,1}\mathbb{M}$ will be integrable if the form Φ satisfies some relation $P(\Phi) = 0$ (cf. [1]).

Assume $\mathbb{M}$ is an abstract CR manifold and $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of the original CR structure $H_{0,1}\mathbb{M}$ on $\mathbb{M}$. If $\overline{\partial}_b^{\Phi}$ denotes the tangential Cauchy-Riemann operator associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$, then we have

$$\overline{\partial}_b^{\Phi} = \overline{\partial}_b - \Phi \lrcorner d = \overline{\partial}_b - \Phi \lrcorner \partial_b, \tag{1.4}$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator associated to the original CR structure $H_{0,1}\mathbb{M}$ and ∂_b is defined by $\partial_b f = df_{|_{H_{1,0}\mathbb{M}}}$ for $f \in \Omega_s^{0,q}(\mathbb{M})$.

Note that if $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$, the space $H\widehat{\mathbb{M}} = H_{1,0}\widehat{\mathbb{M}} \oplus H_{0,1}\widehat{\mathbb{M}}$ coincides with the space $H\mathbb{M}$ and consequently the two abstract CR manifolds $\mathbb{M}$ and $\widehat{\mathbb{M}}$ have the same characteristic bundle and hence the same Levi form. This implies in particular that if $\mathbb{M}$ satisfies condition Y(q) at each point, then $\widehat{\mathbb{M}}$ satisfies also condition Y(q) at each point and that if $\mathbb{M}$ is q-concave then $\widehat{\mathbb{M}}$ is also q-concave.

2 Embedding of small horizontal perturbations in the case of $\mathbb{C}^n$

Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$ and $\mathcal{E}_0 : \mathbb{M} \to M_0 \subset X$ be a $\mathcal{C}^\infty$-smooth CR embedding in a complex manifold X, then M_0 is a compact CR submanifold of X of class $\mathcal{C}^\infty$ with the CR structure $H_{0,1}M_0 = d\mathcal{E}_0(H_{0,1}\mathbb{M}) = T_{\mathbb{C}}M_0 \cap T_{0,1}X$ and the tangential Cauchy-Riemann operator $\overline{\partial}_b$.

Let $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$, we are looking for an embedding $\mathcal{E} : \mathbb{M} \to M \subset X$ of class $\mathcal{C}^l$, $l \geq 1$, such that $d\mathcal{E}(\widehat{H}_{0,1}\mathbb{M}) = T_{\mathbb{C}}M \cap T_{0,1}X$, i.e. $\mathcal{E}$ is a CR embedding.

Set $\widehat{H}_{0,1}M_0 = d\mathcal{E}_0(\widehat{H}_{0,1}\mathbb{M})$, as $\mathcal{E}_0$ is a CR embedding then $\widehat{H}_{0,1}M_0$ is an horizontal perturbation of $H_{0,1}M_0$ and consequently it is defined by a $(0,1)$-form $\Phi \in \Omega_k^{0,1}(H_{1,0}M_0)$, $k \geq 1$. We denote by $\overline{\partial}_b^{\Phi}$ the associated tangential Cauchy-Riemann operator.

We will consider only *small* (the sense will be precised later) perturbations of the original structure, thus it is reasonable to assume that the diffeomorphism $F = \mathcal{E} \circ \mathcal{E}_0^{-1} : M_o \subset X \to M \subset X$ is close to identity.

We equipped the manifold X with some Riemannian metric (for example, if $X = \mathbb{C}P^n$, take the Fubini-Study metric). The idea is to look for some F in the subset of the restrictions to M_0 of $\mathcal{C}^l$-diffeomorphisms of X parametrized by sections of the vector bundle TX by mean of the exponential map. Let us consider the following diagram where U is a neighborhood of the zero section and σ a section of TX over M_0

$$\begin{array}{ccc} U \subset TX & \stackrel{\exp}{\longrightarrow} & X \times X \\ (x, \sigma(x)) & \longmapsto & (x, F(x)) = \exp_x \sigma(x) \\ \uparrow & & \downarrow \\ M_0 & \longrightarrow & X \\ x & \longmapsto & F(x) \end{array}$$

In fact $\mathcal{E}$ will be a CR embedding if and only if $\overline{\partial}_b^{\Phi} F = 0$, which means that we have to find a section σ of TX over M_0 such that the image of the new CR structure $\widehat{H}_{0,1} M_0$ by the map $\exp \circ \sigma$ is contained in $T_{0,1}(X \times X)$.

As the new CR structure $\widehat{H}_{0,1} M_0$ is an horizontal perturbation of the initial one, it is natural to look for a section σ of the bundle $T_{1,0} X_{|_{M_0}}$. Such a section is completely determined by its coordinates in a holomorphic frame over a point x, which can be expressed by mean of $\sigma \lrcorner d(Id) = \sigma \lrcorner \partial Id$, where Id denotes any coordinate map from a neighborhood of x in X into $\mathbb{C}^n$. This means that if $Id(x) = (z_1, \ldots, z_n)$, then $\sigma(x) = \sum_{k=1}^n (\sigma(z_1, \ldots, z_n) \lrcorner dz_k) \frac{\partial}{\partial z_k}$ and in case of change of coordinates with $Id(x) = (Z_1, \ldots, Z_n)$, with $Z_k = f_k(z_1, \ldots, z_n)$, we get

$$\sigma(Z_1, \ldots, Z_n) \lrcorner dZ_k = \sigma(z_1, \ldots, z_n) \lrcorner df_k(z_1, \ldots, z_n)$$

for each $k = 1, \ldots, n$.

Note that the differential of the map exp at a point is given by the map $(u, \xi) \mapsto (u, u + \xi)$, so if σ solves the problem, then it results from a calculation in coordinates that σ must satisfy $\overline{\partial}_b^{\Phi}(\sigma \lrcorner d(Id)) = \Phi \lrcorner d(Id)$. Moreover since Id is an holomophic map, $\Phi \lrcorner d(Id) = -\overline{\partial}_b^{\Phi} Id$ and consequently $\overline{\partial}_b^{\Phi}(\Phi \lrcorner d(Id)) = -\overline{\partial}_b^{\Phi}(\overline{\partial}_b^{\Phi} Id) = 0$, because the new CR structure $\widehat{H}_{0,1} M_0$ is integrable.

Finally all this means that our problem is reduced to the following $\overline{\partial}_b$-problem for sections of the bundle $T_{1,0} X_{|_{M_0}}$:

$$\overline{\partial}_b^{\Phi} \sigma = \Phi, \quad \text{with} \quad \overline{\partial}_b^{\Phi} \Phi = 0. \tag{2.1}$$

.

2.1 Reduction to a fixed point theorem

Let E be a CR bundle over M_0 which satisfies $H^{0,1}(M_0, E) = 0$ and g be a $\overline{\partial}_b^{\Phi}$-closed $(0,1)$-form in $\mathcal{C}_{0,1}^{l+2}(M_0, E)$ and let us consider the following equation:

$$\overline{\partial}_b^{\Phi} v = g. \tag{2.2}$$

By definition of the $\overline{\partial}_b^{\Phi}$-operator, the equation $\overline{\partial}_b^{\Phi} v = g$ is equivalent to the equation

$$\overline{\partial}_b v = g + \Phi \lrcorner \partial_b v. \tag{2.3}$$

A natural tool to solve such an equation is a global homotopy formula for the $\overline{\partial}_b$-operator with good estimates.

Assume $\mathbb{M}$ is 2-concave, then M_0 is also 2-concave and we may apply the results in [2] and [10] on local estimates and global homotopy formulas for the tangential Cauchy-Riemann operator or assume $\mathbb{M}$ satisfies condition Y(q) for $0 \leq q \leq 2$, then we can use the results in [12].

In [2] the following result is proved

Proposition 2.1. *Let M_0 be a 2-concave CR generic submanifold of X of class $\mathcal{C}^\infty$. For each point in M_0, there exist a neighborhood U and linear operators*

$$T_r \ : \ \mathcal{C}^0_{n,r}(M_0) \to \mathcal{C}^0_{n,r-1}(U), \quad 1 \leq r \leq 2$$

with the following two properties :

(i) For all $l \in \mathbb{N}$ and $1 \leq r \leq 2$,

$$T_r(\mathcal{C}^l_{n,r}(M_0)) \subset \mathcal{C}^{l+1/2}_{n,r-1}(\overline{U})$$

and T_r is continuous as an operator between $\mathcal{C}^l_{n,r}(M_0)$ and $\mathcal{C}^{l+1/2}_{n,r-1}(\overline{U})$.

(ii) If $f \in \mathcal{C}^1_{n,r}(M_0)$, $0 \leq r \leq 1$, has compact support in U, then, on U,

$$f = \begin{cases} T_1 \overline{\partial}_b f & \text{if } r = 0\,, \\ \overline{\partial}_b T_1 f + T_2 \overline{\partial}_b f & \text{if } r = 1\,. \end{cases} \tag{2.4}$$

and in [10] we have derived from the previous proposition a global homotopy formula by mean of a functional analytic construction.

Theorem 2.2. *Let E be an holomorphic vector bundle over X and M_0 be a compact 2-concave generic CR submanifold of X of class $\mathcal{C}^\infty$ such that $H^{0,1}(M_0, E) = 0$. Then there exist continuous linear operators*

$$A_r \ : \ \mathcal{C}^0_{0,r}(M_0, E) \to \mathcal{C}^0_{0,r-1}(M_0, E), \quad 1 \leq r \leq 2$$

such that

(i) For all $l \in \mathbb{N}$ and $1 \leq r \leq 2$,

$$A_r(\mathcal{C}^l_{0,r}(M_0, E)) \subset \mathcal{C}^{l+1/2}_{0,r-1}(M_0, E)$$

and A_r is continuous as an operator between $\mathcal{C}^l_{0,r}(M_0, E)$ and $\mathcal{C}^{l+1/2}_{0,r-1}(M_0, E)$.

(ii) For all $f \in \mathcal{C}^1_{0,1}(M_0, E)$

$$f = \overline{\partial}_b A_1 f + A_2 \overline{\partial}_b f \tag{2.5}$$

On an other hand, a stong Hodge decomposition theorem is proved in [12] for abstract CR manifold satisfying some Y(q) condition.

Theorem 2.3. *Let $\mathbb{M}$ be a compact CR manifold. Suppose that the Levi form of $\mathbb{M}$ satisfies condition Y(1), then there exists a compact operator $G_b : L^2_{0,1}(\mathbb{M}) \to Dom(\Box_b)$ such that*

(i) For all $s \in \mathbb{N}$, G_b is continuous from $W^s_{0,1}(\mathbb{M})$ into $W^{s+1}_{0,1}(\mathbb{M})$.

(ii) for any $f \in L^2_{0,1}(\mathbb{M})$

$$f = \overline{\partial}_b \overline{\partial}_b^* G_b f + \overline{\partial}_b^* \overline{\partial}_b G_b f + H_b f,$$

where H_b is the orthogonal projection on the harmonic $(0,1)$-forms.

(iii) $G_b H_b = H_b G_b = 0$, $G_b \Box_b = \Box_b G_b = I - H_b$ on $Dom(\Box_b)$ and if G_b is also defined on $L^2_{0,2}(\mathbb{M})$ (respectively $L^2_{0,0}(\mathbb{M})$), $G_b \overline{\partial}_b = \overline{\partial}_b G_b$ on $Dom(\overline{\partial}_b)$ (respectiely $G_b \overline{\partial}_b^* = \overline{\partial}_b^* G_b$ on $Dom(\overline{\partial}_b^*)$).

When $\mathbb{M}$ satisfies condition Y(q) for $1 \leq q \leq 2$, which is the case in particular if $\mathbb{M}$ is 3-concave or if $\mathbb{M}$ is stictly pseudoconvex of hypersurface type and of real dimension greater or equal to 7, global homotopy formulas can be deduced from Theorem 2.3.

Corollary 2.4. *Let $\mathbb{M}$ be a compact CR manifold of class $\mathcal{C}^\infty$ satisfying condition Y(q) for $1 \leq q \leq 2$. Then there exist continuous linear operators*

$$A_r : L^2_{0,r}(\mathbb{M}) \to L^2_{0,r-1}(\mathbb{M}), \quad 1 \leq r \leq 2$$

and a continuous linear projection H_1 in $L^2_{0,1}(\mathbb{M})$ with $\operatorname{Im} H_1 \subset \mathcal{Z}^\infty_{0,1}(\mathbb{M})$ and $\dim \operatorname{Im} H_1 < \infty$ such that

(i) For all $s \in \mathbb{N}$ and $1 \leq r \leq 2$,

$$A_r(W^s_{0,r}(\mathbb{M})) \subset W^{s+1/2}_{0,r-1}(\mathbb{M})$$

and A_r is continuous as an operator between $W^s_{0,r}(\mathbb{M})$ and $W^{s+1/2}_{0,r-1}(\mathbb{M})$.

(ii) For all $f \in \mathcal{C}^1_{0,1}(\mathbb{M}, E)$

$$f = \overline{\partial}_b A_1 f + A_2 \overline{\partial}_b f + H_1 f \tag{2.6}$$

Proof. If the Levi form of $\mathbb{M}$ satisfies condition Y(q) for $0 \leq q \leq 2$, then G_b is defined for $(0,1)$-forms and $(0,2)$-forms and $G_b \overline{\partial}_b = \overline{\partial}_b G_b$, so if we set $A_r = \overline{\partial}_b^* G_b$ for $r = 1, 2$ we get the homotopy formula (2.6).

The continuity properties of the A_r's in the Sobolev spaces are proved in [12]. □

In fact we need better estimates than the previous ones to reduce the solvability of our equation 2.3 to a fixed point theorem.

Assume there exist Banach spaces $\mathcal{B}^l(M_0)$, $l \in \mathbb{N}$, with the following properties :

(i) $\mathcal{C}^{l+2}(M_0) \subset \mathcal{B}^{2l+1}(M_0) \subset \mathcal{B}^{2l}(M_0) \subset \mathcal{C}^l(M_0)$;

(ii) $\cap_{l \in \mathbb{N}} \mathcal{B}^l(M_0) = \mathcal{C}^\infty(M_0)$;

(iii) $\mathcal{B}^l(M_0)$ is invariant under horizontal perturbations of the CR structure

(iv) If $f \in \mathcal{B}^l(M_0)$, $l \geq 1$, $X_{\mathbb{C}} f \in \mathcal{B}^{l-1}(M_0)$ when $X_{\mathbb{C}}$ is a complex vector field tangent to M_0

and that the previous operators A_r, $r = 1, 2$ are linear continuous operators from $\mathcal{B}^l_{0,r}(M_0, E)$ into $\mathcal{B}^{l+1}_{0,r-1}(M_0, E)$

Since M_0 is 2-concave and generic or satisfies both conditions Y(1) and Y(2) and E satisfies $H^{0,1}(M_0, E) = 0$, under this additional hypothesis, if v is a solution of (2.2), then $\overline{\partial}_b(g + \Phi \lrcorner \partial_b v) = 0$ and by (2.5) or (2.6)

$$\overline{\partial}_b(A_1(g + \Phi \lrcorner \partial_b v)) = g + \Phi \lrcorner \partial_b v.$$

Assume Φ is of class $\mathcal{C}^{l+2}$, then the map

$$\begin{aligned} \Theta \ : \ \mathcal{B}^{2l+1}(M_0, E) &\to \mathcal{B}^{2l+1}(M_0, E) \\ v &\mapsto A_1 g + A_1(\Phi \lrcorner \partial_b v)\,. \end{aligned}$$

is continuous, and the fixed points of Θ are good candidates to be solutions of (2.2).

2.2 A fixed point theorem

In this section we assume that all the assumptions of the previous section are satisfied.

Let δ_0 such that, if $\|\Phi\|_{l+2} < \delta_0$, then the norm of the bounded endomorphism $A_1 \circ \Phi \lrcorner \partial_b$ of $\mathcal{B}^{2l+1}(M_0, E)$ is equal to $\epsilon_0 < 1$. We shall prove that, if $\|\Phi\|_{l+2} < \delta_0$, the map Θ admits a unique fixed point, which is a solution of the equation $\overline{\partial}^{\Phi}_b v = g$.

Consider first the uniqueness of the fixed point. Assume v_1 and v_2 are two fixed points of Θ, then

$$\begin{aligned} v_1 = \Theta(v_1) &= A_1 g + A_1(\Phi \lrcorner \partial_b v_1) \\ v_2 = \Theta(v_2) &= A_1 g + A_1(\Phi \lrcorner \partial_b v_2). \end{aligned}$$

This implies

$$v_1 - v_2 = A_1\big(\Phi \lrcorner \partial_b (v_1 - v_2)\big)$$

and, by the hypothesis on Φ,

$$\|v_1 - v_2\|_{\mathcal{B}^{2l+1}} < \|v_1 - v_2\|_{\mathcal{B}^{2l+1}}$$

or $v_1 = v_2$ and hence $v_1 = v_2$.

For the existence we proceed by iteration. We set $v_0 = \Theta(0) = A_1(g)$ and, for $n \geq 0$, $v_{n+1} = \Theta(v_n)$. Then for $n \geq 0$, we get

$$v_{n+1} - v_n = A_1(\Phi \lrcorner \partial_b (v_n - v_{n-1})).$$

Therefore, if $\|\Phi\|_{l+2} < \delta_0$, the sequence $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{B}^{2l+1}(M_0, E)$ and hence converges to a form v, moreover by continuity of the map Θ, v satisfies $\Theta(v) = v$.

It remains to prove that v is a solution of (2.2). Since $H^{0,1}(M_0, E) = 0$, it follows from (2.5) or (2.6) and from the definition of the sequence $(v_n)_{n \in \mathbb{N}}$ that

$$g - \overline{\partial}^{\Phi}_b v_{n+1} = \Phi \lrcorner \partial_b (v_{n+1} - v_n) + A_2 \overline{\partial}_b (g + \Phi \lrcorner \partial_b v_n)$$

and since

$$\begin{aligned}\overline{\partial}_b(g+\Phi\lrcorner\partial_b v_n) &= \overline{\partial}_b g - \overline{\partial}_b(\overline{\partial}_b - \Phi\lrcorner\partial_b)v_n \\ &= \overline{\partial}_b g - \overline{\partial}_b(\overline{\partial}_b^{\Phi} v_n) \\ &= \overline{\partial}_b g - (\overline{\partial}_b^{\Phi} + \Phi\lrcorner\partial_b)(\overline{\partial}_b^{\Phi} v_n) \\ &= \overline{\partial}_b g - \Phi\lrcorner\partial_b(\overline{\partial}_b^{\Phi} v_n), \quad \text{since } (\overline{\partial}_b^{\Phi})^2 = 0 \\ &= \Phi\lrcorner\partial_b(g - \overline{\partial}_b^{\Phi} v_n), \quad \text{since } \overline{\partial}_b^{\Phi} g = 0,\end{aligned}$$

we get

$$g - \overline{\partial}_b^{\Phi} v_{n+1} = \Phi\lrcorner\partial_b(v_{n+1} - v_n) + A_2(\Phi\lrcorner\partial_b(g - \overline{\partial}_b^{\Phi} v_n)). \tag{2.7}$$

Note that since $g \in \mathcal{C}^{l+2}_{0,1}(M_0, E)$ and Φ is of class $\mathcal{C}^{l+2}$, then $\overline{\partial}_b v_0 = g - A_2(\Phi\lrcorner\partial_b g)$ belongs to $\mathcal{B}^{2l+1}_{0,1}(M_0, E)$ and consequently $\overline{\partial}_b v_0 - \Phi\lrcorner\partial_b v_0 = \overline{\partial}_b^{\Phi} v_o$ is in $\mathcal{B}^{2l}_{0,1}(M_0, E)$.

Thus by (2.7), it follows by induction that $\overline{\partial}_b^{\Phi} v_n \in \mathcal{B}^{2l}_{0,1}(M_0, E)$ for all $n \in \mathbb{N}$ and we have the estimate

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^{2l}} \leq \|\Phi\lrcorner\partial_b\|\|(v_{n+1} - v_n)\|_{\mathcal{B}^{2l+1}} + \|A_2 \circ \Phi\lrcorner\partial_b\|\|g - \overline{\partial}_b^{\Phi} v_n\|_{\mathcal{B}^{2l}}. \tag{2.8}$$

Let δ such that if $\|\Phi\|_{l+2} < \delta$, then the maximum of the norm of the bounded endomorphisms $A_r \circ \Phi\lrcorner\partial_b$, $r = 1, 2$, of $\mathcal{B}^{2l+1}_{0,r-1}(M_0, E)$ is equal to $\epsilon < 1$. Assume $\|\Phi\|_{l+2} < \delta$, then by induction we get

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^{2l}} \leq (n+1)\epsilon^{n+1}\|\Phi\lrcorner\partial_b\|\|v_0\|_{\mathcal{B}^{2l+1}} + \epsilon^{n+1}\|g - \overline{\partial}_b^{\Phi} v_0\|_{\mathcal{B}^{2l}}. \tag{2.9}$$

But $g - \overline{\partial}_b^{\Phi} v_0 = \Phi\lrcorner\partial_b A_1 g + A_2(\Phi\lrcorner\partial_b g)$ and hence $\|g - \overline{\partial}_b^{\Phi} v_0\|_{\mathcal{B}^{2l}} \leq \|\Phi\lrcorner\partial_b\|\|A_1 g\|_{\mathcal{B}^{2l+1}} + \epsilon\|g\|_{\mathcal{B}^{2l}}$. This implies

$$\|g - \overline{\partial}_b^{\Phi} v_{n+1}\|_{\mathcal{B}^{2l}} \leq (n+2)\epsilon^{n+1}\|\Phi\lrcorner\partial_b\|\|A_1\|\|g\|_{\mathcal{B}^{2l}} + \epsilon^{n+2}\|g\|_{\mathcal{B}^{2l}}. \tag{2.10}$$

Since $\epsilon < 1$, the righthand side of (2.10) tends to zero, when n tends to infinity and by continuity of the operator $\overline{\partial}_b^{\Phi}$ from $\mathcal{B}^{2l+1}(M_0, E)$ into $\mathcal{B}^{2l-1}_{0,1}(M_0, E)$, we get that v is a solution of (2.2).

2.3 Solution of the embedding problem

In the setting of the beginning of section 2, let us consider the following anisotropic Hölder spaces of functions:

- $\mathcal{A}^{\alpha}(M_0)$, $0 < \alpha < 1$, is the set of continuous functions on M_0 which are in $\mathcal{C}^{\alpha/2}(M_0)$.

- $\mathcal{A}^{1+\alpha}(M_0)$, $0 < \alpha < 1$, is the set of functions f such that $f \in \mathcal{C}^{(1+\alpha)/2}(M_0)$ and $X_{\mathbb{C}} f \in \mathcal{C}^{\alpha/2}(M_0)$, for all complex tangent vector fields $X_{\mathbb{C}}$ to M_0. Set

$$\|f\|_{A\alpha} = \|f\|_{(1+\alpha)/2} + \sup_{\|X_{\mathbb{C}}\| \leq 1} \|X_{\mathbb{C}} f\|_{\alpha/2} \tag{2.11}$$

- $\mathcal{A}^{l+\alpha}(M_0)$, $l \geq 2$, $0 < \alpha < 1$, is the set of functions f of class $\mathcal{C}^{[l/2]}$ such that $Xf \in \mathcal{A}^{l-2+\alpha}(M_0)$, for all tangent vector fields X to M and $X_{\mathbb{C}} f \in \mathcal{A}^{l-1+\alpha}(M_0)$, for all complex tangent vector fields $X_{\mathbb{C}}$ to M_0.

Fix some $0 < \alpha < 1$ and set $\mathcal{B}^l(M_0) = \mathcal{A}^{l+\alpha}(M_0)$. This sequence $(\mathcal{B}^l(M_0), l \in \mathbb{N})$ is a sequence of Banach spaces which satisfies properties (i) to (iv) listed in section 2.1

Moreover it is proved in [9] that the operators A_r, $r = 1, 2$, from Theorem 2.2 are linear continuous operators between the anisotropic Hölder spaces $\mathcal{B}^l_{0,r}(M_0, E)$ and $\mathcal{B}^{l+1}_{0,r-1}(M_0, E)$.

One can also consider the anisotropic Hölder spaces introduced by Folland and Stein when they studied the tangential Cauchy-Riemann complex on the Heisenberg group and more generally on strictly pseudoconvex CR manifolds.

Let M be a generic CR manifold of class $\mathcal{C}^\infty$ of real dimension $2n$ and CR dimension $2n-k$ and D be a relatively compact domain in M. Let $X_1, \ldots, X_{2n-2k}$ be a real basis of HM. A $\mathcal{C}^1$ curve $\gamma : [0, r] \to M$ is called admissible if for every $t \in [0, r]$,

$$\frac{d\gamma}{dt}(t) = \sum_{j=1}^{2n-2k} c_j(t) X_j(\gamma(t))$$

where $\sum |c_j(t)|^2 \leq 1$.

The Folland-Stein anisotropic Hölder spaces $\Gamma^{p+\alpha}(\overline{D} \cap M)$ are defined in the following way:

- $\Gamma^\alpha(\overline{D} \cap M)$, $0 < \alpha < 1$, is the set of continuous fonctions in $\overline{D} \cap M$ such that if for every $x_0 \in \overline{D} \cap M$

$$\sup_{\gamma(.)} \frac{|f(\gamma(t) - f(x_0)|}{|t|^\alpha} < \infty$$

for any admissible complex tangent curve γ through x_0.

- $\Gamma^{p+\alpha}(\overline{D} \cap M)$, $p \geq 1$, $0 < \alpha < 1$, is the set of continuous fonctions in M such that $X_{\mathbb{C}} f \in \Gamma^{p-1+\alpha}(\overline{D} \cap M)$, for all complex tangent vector fields $X_{\mathbb{C}}$ to M.

Fix some $0 < \alpha < 1$ the sequence $(\Gamma^{p+\alpha}, p \in \mathbb{N})$ is also a sequence of Banach spaces which satisfies properties (i) to (iv) listed in section 2.1

Continuity properties for the operators A_r and B_r, $r = 1, 2$, defined in Theorem 2.2 and 2.4 are proved in [9] and [12]. More precisely, for all $p \in \mathbb{N}$ and $0 < \alpha < 1$, the operators A_r, $r = 1, 2$, from Theorem 2.2 and Corollary 2.4 are continuous from $\Gamma^{p+\alpha}_{0,r}(M)$ into $\Gamma^{p+1+\alpha}_{0,r-1}(M)$.

We can apply now the method developed in section 2.2 with $E = T_{1,0}X_{|_{M_0}}$ to solve the equation

$$\overline{\partial}^\Phi_b \sigma = \Phi$$

and a unique solution is given by a fixed point of the map Θ if $\|\Phi\|_{l+2}$ is sufficiently small, and then $\mathcal{E} = F \circ \mathcal{E}_0$, with F defined by $F(x) = \exp_x \sigma(x)$, is the embedding we are looking for, which ends the proof of Theorem 1.

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