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Abstract

We study the asymptotics of the spectral distribution for large empirical covariance matrices composed of independent Multifractal Random Walk processes. The asymptotic is taken as the observation lag shrinks to 0. In this setting, we show that there exists a limiting spectral distribution whose Stieltjes transform is uniquely characterized by equations which we specify.

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Contents

1. Introduction

Since the seminal work of Marčenko and Pastur [8] in 1967, there has been growing interest in studying the asymptotics of large empirical covariance matrices. These studies have found applications in many fields of science: physics, telecommunications, information theory and finance, etc... The main motivation of this work stems from finance: we refer to [9], [4] for a discussion on the applications of large empirical covariance matrices in finance and in particular in portfolio theory. In the context of finance, the study of the limiting spectral distribution of large empirical covariance matrices is of particular interest as it is a crucial statistical tool in identifying the different market modes (see again [9], [4] for a nice introduction to this topic). It is indeed very natural to try to identify common causes (or factors) that explain the dynamics of N quantities, which will be stock's prices in finance. We will denote by N the number of stocks and by T the number of time intervals where we observed prices of the N stocks. In this setting, the Marčenko Pastur paper enables to deal with the case where stock prices follow independent Brownian motions. More precisely, let us define a $N \times T$ matrix X_N such that $X_N(ij)$ is the realization of the return on the j-th time interval (of shrinking size $1/T$) of stock number i by:

$$X_N(ij) = B_i\left(\frac{j}{T}\right) - B_i\left(\frac{j-1}{T}\right) \tag{1.1}$$

where the B_i are i.i.d. standard Brownian motions. The empirical covariance matrix is now defined as $R_N = X_N X_N^t$. If $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of R_N, the empirical spectral distribution of the matrix R_N is the probability measure defined by:

$$\mu_{R_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}. \tag{1.2}$$

The Marčenko and Pastur result states that, in the limit of large matrices $N, T \to \infty$ with $N/T \to q \in (0, 1]$, the empirical spectral distribution μ_{R_N} weakly converges (almost surely) to a probability measure whose density $\rho(x)$ is defined by:

$$\rho(x) = \frac{1}{2\pi q} \frac{\sqrt{(\gamma_+ - x)(x - \gamma_-)}}{x} \mathbb{1}_{[\gamma_-, \gamma_+]} dx \tag{1.3}$$

where $\gamma_\pm = 1 + q \pm 2\sqrt{q}$.

Independently of the aforementionned work on random matrix theory, much work has been devoted to studying the statistics of financial stocks. It turns out that most financial assets (stocks, indices, etc...) possess universal features, called stylized facts. We refer to the review [5] for a discussion on this topic. On the other hand, many models have been proposed in the litterature that take into account these stylized facts. Among them, there has been growing interest in the Multifractal Random Walk (MRW) model introduced in [1]. The MRW is simply defined as:

$$X_t = B\left(M[0, t]\right) \tag{1.4}$$

where B is a standard Brownian motion and M is an independent multifractal random measure (MRM for short), see section 2.1 for a reminder of the construction/definition as well as (standard) notations used throughout the paper.

We thus aim at studying the large sample covariance matrices where the underlyings evolve independenly as Brownian motions with a time change, which can be thought of as a volatility process with memory (i.e. the volatility process is correlated in time). The main example of such processes we are interested in is the multifractal random walk but we will also consider other examples. More precisely, the matrix X_N can be defined, for $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant T$, by:

$$X_N(ij) = B^i_{M^i(0, \frac{j}{T})} - B^i_{M^i(0, \frac{j-1}{T})} = r_i(j) \tag{1.5}$$

where the B_i are i.i.d. Brownian motions and the M_i are i.i.d. multifractal random measures independent of the B_i.

The purpose of this work is to characterize, in the limit of large matrices ($N, T \to \infty$ with $N/T \to q \in (0, 1]$), the limit of the empirical spectral measure $\mu_{R_N} = 1/N \sum_{i=1}^N \delta_{\lambda_i}$ where $(\lambda_1, \dots, \lambda_N)$ are the eigenvalues of the empirical covariance matrix $R_N := X_N X_N^t$.

The next sections are organized as follows. In section 2, we remind the definition of MRW and introduce the main notations of the paper. In section 3, we state our main theorems which are characterizations of the limiting spectral measure of R_N through its Stieltjes transform for different type of underlying process X. Since these equations are tedious to invert, we leave the analysis of the underlying probability measure to a forthcoming study (we are working on this subject). The proofs appear in section 4 with some auxiliary lemmas proved in the appendix. The strategy of our proofs is classical among the random matrix litterature (the so-called resolvent method) as it relies on the Schur recursion formula for the Stieltjes tranform; in particular, we follow the approach of [3]. The main difficulty lies in handling the Stieltjes transforms in a multifractal setting.

2. Background, notations and main results

2.1 Reminder of the construction of MRM

To fix precisely the notations that we will use throughout the paper, we quickly remind the main steps of the construction of Multifractal Random Measures (MRM). The description is necessarily concise and the reader is referred to [1] for further details. In particular, we use the same notations as in [1] to facilitate the reading. We consider the characteristic function of an infinitely divisible random variable Z, which can be written as $\mathbb{E}[e^{ipZ}] = e^{\varphi(p)}$ where (Lévy-Khintchine's formula):

$$\varphi(p) = imp - \frac{1}{2}\gamma^2 p^2 + \int_{\mathbb{R}^*} (e^{ipx} - 1)\,\nu(dx) \tag{2.1}$$

and $\nu(dx)$ is a so-called Lévy measure (ie satisfying $\int_{\mathbb{R}^*} \min(1, x^2)\,\nu(dx) < +\infty$) together with the following additional assumption:

$$\int_{[-1,1]} |x|\,\nu(dx) < +\infty, \tag{2.2}$$

so that its characteristic function perfectly makes sense as written in (2.1). We also introduce the Laplace exponent ψ of Z by $\psi(p) = \varphi(-ip)$ for each p such that both terms of the equality make sense, and we assume that the following renormalization condition holds: $\psi(1) = 0$.

We further consider the half-space $S = \{(t, y); t \in \mathbb{R}, y \in \mathbb{R}_+^*\}$, with which we associate the measure (on the Borel σ-algebra $\mathcal{B}(S)$):

$$\theta(dt, dy) = y^{-2} dt\,dy. \tag{2.3}$$

Then we consider an independently scattered infinitely divisible random measure μ associated to (φ, θ) and distributed on S.

Then we define a process ω_ϵ for $\epsilon > 0$ by the following. Given a positive parameter τ, let us define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by:

$$f(r) = \begin{cases} r, & \text{if } r \leqslant \tau \\ \tau & \text{if } r \geqslant \tau \end{cases}$$

The cone-like subset $A_\epsilon(t)$ of S is defined by:

$$A_\epsilon(t) = \{(s, y) \in S; y \geqslant \epsilon, -f(y)/2 \leqslant s - t \leqslant f(y)/2\}. \tag{2.4}$$

We then define the stationary process $(\omega_\epsilon(t))_{t\in\mathbb{R}}$ by:

$$\omega_\epsilon(t) = \mu\left(A_\epsilon(t)\right). \tag{2.5}$$

The Radon measure M is then defined as the almost sure limit (in the sense of weak convergence of Radon measures) by:

$$M(A) = \lim_{\epsilon\to 0^+} M_\epsilon(A) = \lim_{\epsilon\to 0^+} \int_A e^{\omega_\epsilon(r)}\,dr$$

for any Lebesgue measurable subset $A \subset \mathbb{R}$. The convergence is ensured by the fact that the family $(M_\epsilon(A))_{\epsilon>0}$ is a right-continuous positive martingale. The structure exponent of M is defined by:

$$\forall p \geqslant 0, \quad \zeta(p) = p - \psi(p)$$

for all p such that the right-hand side makes sense. The measure M is different from 0 if and only if there exists $\epsilon > 0$ such that $\zeta(1+\epsilon) > 1$, (or equivalently $\psi'(1) < 1$). In that case, we have:

Theorem 2.1. *The measure M is stationary and satisfies the* **exact stochastic scale invariance property***: for any $\lambda \in]0,1]$,*

$$(M(\lambda A))_{A\subset B(0,\tau)} \overset{\text{law}}{=} (\lambda e^{\Omega_\lambda} M(A))_{A\subset B(0,\tau)},$$

where Ω_λ is an infinitely divisible random variable, independent of $(M(A))_{A\subset B(0,T)}$, the law of which is characterized by:

$$\mathbb{E}[e^{ip\Omega_\lambda}] = \lambda^{-\varphi(p)}.$$

2.2 Notations

Let N and $T := T(N)$ be two integers, the aim of this paper is to compute the empirical spectral measure of the matrix $R_N := X_N {}^t X_N$ as $N \to \infty$, where X_N is a $N \times T$ real matrix the entries of which are given by (1.5). Recall that the number N of sampled processes is supposed to be comparable with the sample size $T := T(N)$, and more precisely, we will suppose in the following that there exists a parameter $q \in]0,1]$ such that:

$$\lim_{N\to\infty} \frac{N}{T} = q. \tag{2.6}$$

We further set $\widetilde{R}_N := {}^t X_N X_N$, and if M is a symmetric real matrix, we will denote by μ_M the empirical spectral measure of M.

Define the $(T+N) \times (T+N)$ matrix B_N by:

$$B_N = \begin{pmatrix} 0 & {}^t X_N \\ X_N & 0 \end{pmatrix}.$$

We also define for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$A_N(z) = (zI_{T+N} - B_N) = \begin{pmatrix} zI_T & -{}^t X_N \\ -X_N & zI_N \end{pmatrix}.$$

Notice that

$$B_N^2 = \begin{pmatrix} \widetilde{R}_N & 0 \\ 0 & R_N \end{pmatrix}$$

and that the eigenvalues of $\widetilde{R}_N$ are those of R_N augmented with $T-N$ zero eigenvalues. We thus have:

$$\mu_{B_N^2} = 2\frac{N}{N+T}\mu_{R_N} + \frac{T-N}{N+T}\delta_0, \tag{2.7}$$

where δ_x stands for the Dirac mass at x. Combining this equality with the relation

$$\int f(x)\mu_{B_N^2}(dx) = \int f(x^2)\mu_{B_N}(dx) \tag{2.8}$$

true for all bounded continuous functions f on $\mathbb{R}$, we see that it is sufficient to study the weak convergence of the spectral measure of B_N for the study of the convergence of the spectral measure μ_{R_N}.

We will thus work on the (weak) convergence of the spectral measures μ_{B_N} and $\mathbb{E}[\mu_{B_N}]$ in the following. To that purpose, it is sufficient to prove the convergence of the Stieltjes transform of these two measures. Recall that, for a probability measure μ on $\mathbb{R}$, the Stieltjes transform G_μ of μ is defined, for all $z \in \mathbb{C} \setminus \mathbb{R}$, as:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\mu(dx). \tag{2.9}$$

and one can note that:

$$G_{\mu_{B_N}}(z) = \frac{1}{N+T}\mathrm{Trace}(G_N(z)), \tag{2.10}$$

where we have set:

$$G_N(z) = (A_N(z))^{-1}. \tag{2.11}$$

Hence, we have to investigate the convergence of the right-hand side of (2.10). Let us introduce the two following complex measures $L_N^{1,z}$ and $L_N^{2,z}$ such that, for all bounded and measurable function $f : [0,1] \to \mathbb{R}$:

$$L_N^{1,z}(f) = \frac{1}{T}\sum_{k=1}^{T} f\left(\frac{k}{T}\right) G_N(z)_{kk}$$

$$L_N^{2,z}(f) = \frac{1}{N}\sum_{k=1}^{N} f\left(\frac{k}{N}\right) G_N(z)_{k+T,k+T}$$

Clearly, we have the relation

$$\frac{1}{N+T}\mathrm{Trace}(G_N(z)) = \frac{T}{N+T}L_N^{1,z}([0,1]) + \frac{N}{N+T}L_N^{2,z}([0,1]) \tag{2.12}$$

so that it suffices to establish the convergence of the two complex measures $L_N^{1,z}$ and $L_N^{2,z}$.

3. Main results

3.1 Lognormal multifractal random walk

We first present our results when the process $X(t)$ is a lognormal multifractal random walk, i.e. $X(t) = B(M[0;t])$ where M is the MRM whose characteristic and structure exponent (see section 2.1) are respectively given by:

$$\varphi(q) = -i\frac{\gamma^2}{2}q - \frac{\gamma^2}{2}q^2,$$
$$\zeta(q) = (1 + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2.$$

We will make the assumption that the intermitency parameter γ^2 is small enough so as to overcome in our proofs the strong correlations of the model.

Assumption 3.1. *More precisely, let us suppose that:*

$$\gamma^2 < \frac{1}{3}. \tag{3.1}$$

Though we conjecture that our results hold as soon as the measure M is non degenerated, i.e. $\gamma^2 < 2$ (see [1]), Assumption 3.1 is largely sufficient to cover most practical applications. For instance, in financial applications or in the field of turbulence, γ^2 is found empirically around 2.10^{-2}.

We can now state our main result about the convergence of the empirical spectral measures and mean empirical spectral measures of the matrices B_N and R_N:

Theorem 3.2. *i) There exists a probability measure υ on $\mathbb{R}$ such that the two mean spectral measures $\mathbb{E}[\mu_{B_N}]$ and $\mathbb{E}[\mu_{R_N}]$ converge weakly respectively towards the two probability measures $\frac{2q}{1+q}\upsilon + \frac{1-q}{1+q}\delta_0$ and $\upsilon \circ (x^2)^{-1}$ as N goes to ∞, where $\upsilon \circ (x^2)^{-1}$ is the pushforward of the measure υ by the mapping $x \mapsto x^2$.*

ii) The two spectral measures μ_{B_N} and μ_{R_N} converge weakly in probability respectively to the two probability measures $\frac{2q}{1+q}\upsilon + \frac{1-q}{1+q}\delta_0$ and $\upsilon \circ (x^2)^{-1}$ as N goes to ∞. More precisely, for any bounded and continuous function f, $\int f(x)\mu_{R_N}(dx)$ converges in probability to $\int f(x)\upsilon \circ (x^2)^{-1}(dx)$.

iii) Let N_k be an increasing sequence of integers such that $\sum_{k=1}^{\infty} N_k^{-1} < +\infty$, then the two sequences $\mu_{B_{N_k}}$ and $\mu_{R_{N_k}}$ converge weakly almost surely to the two probability measures $\frac{2q}{1+q}\upsilon + \frac{1-q}{1+q}\delta_0$ and $\upsilon \circ (x^2)^{-1}$ as k goes to ∞.

Theorem 3.2 is implied by (2.10), (2.12) and by Theorem 3.3:

Theorem 3.3. *i) The measures $\mathbb{E}[L_N^{1,z}]$ and $\mathbb{E}[L_N^{2,z}]$ converge weakly towards two complex measures. More precisely, there exist a unique $\mu_z^2 \in \mathbb{C}$ and a unique bounded* *measurable function $K_z(x)$ over $[0,1]$ such that, for all bounded and continuous function f on $[0,1]$, we have respectively:*

$$\mathbb{E}\left[L_N^{1,z}(f)\right] \to_{N\to\infty} \int_0^1 K_z(x) f(x)\,dx,$$
$$\mathbb{E}\left[L_N^{2,z}(f)\right] \to_{N\to\infty} \mu_z^2 \int_0^1 f(x)\,dx.$$

ii) In addition, we have the following relation between $\mu_z^2 \in \mathbb{C}$ and $K_z(x)$:

$$\int_0^1 K_z(x)\,dx = q\mu_z^2 + \frac{1-q}{z} \tag{3.2}$$

iii) Furthermore, there exists a unique probability measure υ on $\mathbb{R}$ whose Stieltjes transform is μ_z^2, meaning that for all $z \in \mathbb{C}\setminus\mathbb{R}$,

$$\mu_z^2 = \int_{\mathbb{R}} \frac{\upsilon(dx)}{z-x}. \tag{3.3}$$

It is important to state a characterization of the probability measure υ: it is done by means of its Stieltjes transform μ_z^2:

Theorem 3.4. *The constant μ_z^2 and the bounded function $K_z(x)$ are uniquely determined for all $z \in \mathbb{C}\setminus\mathbb{R}$, by the following system of equations:*

$$\mu_z^2 = \mathbb{E}\left[\left(z - \int_0^1 K_z(t) M(dt)\right)^{-1}\right], \tag{3.4}$$

$$K_z(x) = \left(z - q\mathbb{E}\left[\left(z - \int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} K_z(t) M(dt)\right)^{-1}\right]\right)^{-1} \tag{3.5}$$

where M is the MRM with structure exponent $\zeta(q) = (1+\gamma^2/2)q - q^2\gamma^2/2$.

Let us notice that one can give a precise meaning to (3.5) for all $\gamma^2 \in [0,2[$. Indeed, we can define for all $x \in [0,1]$ and all continuous function f, the following almost sure limit as a definition:

$$\int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} f(t) M(dt) = \lim_{\eta\to 0} \int_{t\in[0,1];|t-x|>\eta} \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} f(t) M(dt) \tag{3.6}$$

One can also check with this definition that we have:

$$\int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} f(t) M(dt) = \lim_{\epsilon\to 0} \int_0^1 e^{cov(\omega_\epsilon(t),\omega_\epsilon(x))} f(t) e^{\omega_\epsilon(t)} dt$$

Conjecture 3.5. *With this extended definition, we conjecture that theorem 3.4 holds in the lognormal multifractal case for all $\gamma^2 \in [0,2[$ and thus that the limiting equations can be obtained by the ones of theorem 3.9 (see below) with $2W = \omega_\epsilon$ as $\epsilon \to 0$.*

3.2 General multifractal random walk

We now look at the more general case when the change of time is a measure M for which the function $\varphi(q)$ is given by (2.1) and the structure exponent by $\zeta(q) = q - \psi(q)$ with $\psi(q) = \varphi(-iq)$.

We still have to make an assumption to avoid the issue of strong correlations. In this more general setting, Assumption (3.1) becomes:

Assumption 3.6. *Assume that the structure exponent of the MRM satisfies the condition:*

$$\zeta(2) > 5 - 4\zeta'(1). \tag{3.7}$$

and that there exists $\delta > 0$ such that:

$$\zeta(2+\delta) > 1. \tag{3.8}$$

As in the previous section, we conjecture that our results hold as soon as the measure M is non degenerated, i.e. (see [1]) $\zeta(1+\epsilon) > 1$ for some $\epsilon > 0$.

Theorems 3.2 and 3.3 remain unchanged for this more general context. Theorem 3.4 becomes:

Theorem 3.7. *The constant μ_z^2 and the bounded function $K_z(x)$ are uniquely determined for all $z \in \mathbb{C} \setminus \mathbb{R}$, by the following system of equations:*

$$\mu_z^2 = \mathbb{E}\left[\left(z - \int_0^1 K_z(t)M(dt)\right)^{-1}\right], \tag{3.9}$$

$$K_z(x) = \left(z - q\mathbb{E}\left[\left(z - \int_0^1 \left(\frac{T}{|t-x|}\right)_+^{\kappa} K_z(t)Q(dt)\right)^{-1}\right]\right)^{-1} \tag{3.10}$$

with $\kappa = \psi(2)$ and where M is the MRM whose characteristic and structure exponent are respectively $\varphi(q), \zeta(q)$ and where the random Radon measure Q is defined, conditionally on M, as the almost sure weak limit as ϵ goes to 0 of the family of random measures $Q_\epsilon(dt) := e^{\overline{\omega}_\epsilon(t)}M(dt)$ where, for each $\epsilon > 0$, the random process $\overline{\omega}_\epsilon$ is independent of M and defined as $\overline{\omega}_\epsilon(t) = \overline{\mu}(A_\epsilon(t))$ where $\overline{\mu}$ is the independently scattered log infinitely divisible random measure associated to $(\bar{\varphi}, \theta(\cdot \cap A_0(x)))$ with:

$$\bar{\varphi}(p) = ip(\gamma^2 - \kappa) + \int_{\mathbb{R}} (e^{ipx} - 1)(e^x - 1)\nu(dx). \tag{3.11}$$

3.3 Lognormal random walk

Let us mention that one can easily adapt the methods used to prove the above theorems in the simpler case (lognormal case) where $X(t)$ is defined, for all $t \in [0;1]$, by:

$$X(t) = B\left(\int_0^t e^{2W(s)} ds\right), \tag{3.12}$$

where $(W(s))_{s\in[0;1]}$ is a stationary gaussian process with expectation m and stationary covariance kernel k. The normalisation will be chosen such that: $m = -k(0)$.

In this context, the entries of X_N are given, for $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant T$ by:

$$X_N(ij) = \frac{1}{\sqrt{T}} e^{W_i(\frac{j}{T})} B_j^i := r_i(j) \tag{3.13}$$

where the $(B_j^i)_{ij}$ are i.i.d standard centered Gaussian random variables and the W_i are i.i.d stationary Gaussian processes with expectation m and stationary covariance kernel k. Indeed, if one makes the following extra assumption:

Assumption 3.8. *Assume that for some constants $C > 0$ and $\beta > 0$, the covariance kernel k satisfies:*

$$\forall x \in \mathbb{R}, \quad |k(x) - k(0)| \leqslant C|x|^\beta.$$

With the same notations as in the previous section, we can now state the following theorem under assumption 3.8:

Theorem 3.9. *The system of equations for μ_z^2 and $K_z(x)$ becomes:*

$$\mu_z^2 = \mathbb{E}\left[\left(z - \int_0^1 K_z(t)e^{2W(t)}\,dt\right)^{-1}\right] \tag{3.14}$$

$$K_z(x) = \left(z - q\mathbb{E}\left[\left(z - \int_0^1 K_z(t)e^{4k(t-x)}e^{2W(t)}\,dt\right)^{-1}\right]\right)^{-1}. \tag{3.15}$$

where $(W(t))_{t\in[0;1]}$ is a stationary gaussian process with expectation m and stationary covariance kernel k.

4. Proofs of the main results

In this section, we give the proofs of theorems 3.2, 3.3 and 3.4. The proof of Theorem 3.7 is very similar and we will not explain it in every detail, except for the final part where we establish the second equation of the system in Theorem 3.7 verified by K_z. We will give the details for this part of the proof in the appendix. The proof of theorem 3.9 is an easy adaptation of our proofs for theorems 3.2, 3.3 and 3.4; it is left to the reader. Furthermore, the proofs are very similar when $q = 1$ or when $q < 1$. For the sake of clarity, we assume $T = N$ and hence $q = 1$ in the proofs that follow.

Hence, in the following, we will suppose (unless otherwise stated) that:

$$\begin{aligned}
\varphi(q) &= -iq\frac{\gamma^2}{2} + q^2\frac{\gamma^2}{2},\\
\psi(q) &= \varphi(-iq),\\
\zeta(q) &= (1 + \frac{\gamma^2}{2})q + q^2\frac{\gamma^2}{2},\\
\gamma^2 &< \frac{1}{3},
\end{aligned}$$

and M will be the MRM whose structure exponent is ζ (see section 2.1 for a reminder).

Our approach to show the convergence of $\mathbb{E}[L_N^{1,z}]$ and $\mathbb{E}[L_N^{2,z}]$ consists in proving tightness and characterizing uniquely the possible limit points. The classical Schur complement formula is our basic linear algebraic tool to study $\mathbb{E}[L_N^{1,z}]$ and $\mathbb{E}[L_N^{2,z}]$ recursively on the dimension N, as is usual when the resolvent method is used. The original part of our proof is that we apply the Schur complement formula two times in a row to find the second equation of the system in theorem 3.4 involving the limit point $K_z(x)$ of the measure $\mathbb{E}[L_N^{1,z}]$. We will also show that the limit points of the two complex measures $\mathbb{E}[L_N^{1,z}]$ and $\mathbb{E}[L_N^{2,z}]$ satisfy a fixed point system (written in theorem 3.4).

We begin by showing tightness.

4.1 Tightness of the complex measures $\mathbb{E}[L_N^{1,z}], \mathbb{E}[L_N^{2,z}]$ and limit points

Lemma 4.1. *The two families of complex measures $(\mathbb{E}[L_N^{i,z}])_{N\in\mathbb{N}}, i = 1, 2$ are tight and bounded in total variation.*

Proof. Let us present the proof for $(\mathbb{E}[L_N^{1,z}])_{N\in\mathbb{N}}$; the other proof is similar.

One has, for each $N \in \mathbb{N}$:

$$\mid \mathbb{E}[L_N^{1,z}] \mid [0,1] = \frac{1}{N}\sum_{k=1}^{N} \mid \mathbb{E}[G_N(z)_{kk}] \mid \ \leqslant \frac{1}{\mid \Im(z) \mid}, \tag{4.1}$$

and so the family of complex measures $(\mathbb{E}[L_N^{1,z}])_{N\in\mathbb{N}}$ is bounded in total variation. It is obviously tight since the support of all the complex measures in the family is included in $[0, 1]$, which is a compact set. $\square$

Using Prokhorov's theorem, we know that those two families of complex measures are sequentially compact in the space of complex Borel measure on $[0, 1]$ equipped with the topology of weak convergence. In particular, there exists a subsequence such that, for all bounded continuous function f, one has, when N goes to $+\infty$ along this subsequence:

$$\mathbb{E}\left[L_N^{1,z}(f)\right] \to \int_0^1 f(x)\mu_z^1(dx). \tag{4.2}$$

Lemma 4.2. *The complex measure $\mu_z^1(dx)$ has Lebesgue density; more precisely, there exists a bounded measurable function $K_z(x)$ such that:*

$$\mu_z^1(dx) = K_z(x)dx. \tag{4.3}$$

Proof. One has:

$$\left|\mathbb{E}\left[L_N^{1,z}(f)\right]\right| \leqslant \frac{1}{N}\sum_{k=1}^{N}|f(k/N)|\mathbb{E}\left[G_N(z)_{kk}\right] \tag{4.4}$$

$$\leqslant \frac{1}{|\Im(z)|}\frac{1}{N}\sum_{k=1}^{N}|f(k/N)| \tag{4.5}$$

Letting $N \to +\infty$ along a subsequence, one obtains:

$$\left|\int_0^1 f(x)\mu_z^1(dx)\right| \leqslant \frac{1}{|\Im(z)|}\int_0^1 |f(x)|dx. \tag{4.6}$$

This proves the lemma. □

Thus, there exists a subsequence such that, as N tends to $+\infty$ along this subsequence:

$$\mathbb{E}\left[L_N^{1,z}(f)\right] \to \int_0^1 f(x)K_z(x)dx. \tag{4.7}$$

Lemma 4.3. *There exists a subsequence and a constant $\mu_z^2 \in \mathbb{C}$ such that, as N goes to $+\infty$ along this subsequence:*

$$\mathbb{E}\left[L_N^{2,z}(f)\right] \to \mu_z^2\int_0^1 f(x)dx. \tag{4.8}$$

Proof. It is easy to see that the $G_N(z)_{kk}$, $k = N+1, \dots, N$ are identically distributed. In particular, these variables have the same mean $\mu_z^2(N)$. One has, for all N:

$$|\mu_z^2(N)| \leqslant \frac{1}{|\Im(z)|}. \tag{4.9}$$

So there exists a subsequence and a complex number μ_z^2 such that, as N goes to $+\infty$ along this subsequence, $\mu_z^2(N) \to \mu_z^2$. One thus obtains, as N goes to $+\infty$ along this subsequence:

$$\mathbb{E}\left[L_N^{2,z}(f)\right] \to \mu_z^2\int_0^1 f(x)dx. \tag{4.10}$$

□

Following the classical method as in [3], [2], [7], we will show in the following that the limit point μ_z^2 and $K_z(x)$ are defined uniquely and do not depend on the subsequence. We will first recall some preliminary results on resolvents.

4.2 Preliminary results on resolvents

We first recall the following standard and general result; the next lemmas of this section are also standard but are applied to our particular case.

Lemma 4.4. *Let A be a symmetric real valued matrix of size N. For $z \in \mathbb{C} \setminus \mathbb{R}$, let us denote by $G(z)$ the matrix*

$$G(z) = (z - A)^{-1}. \tag{4.11}$$

For $z \in \mathbb{C} \setminus \mathbb{R}$ and $k \in \{1, \dots, N\}$, we have

$$\Im(z)\Im(G(z)_{kk}) < 0 \quad \text{ and } |G(z)_{kk}| \leqslant \frac{1}{|\Im(z)|}. \tag{4.12}$$

In particular, if $F \subset \{1, \dots, N\}$ is a finite set and $(a_i)_{i \in F}$ a finite sequence of positive number, then:

$$\frac{\Im\Big(z - \sum_{i \in F} a_i G(z)_{ii}\Big)}{\Im(z)} \geqslant 1. \tag{4.13}$$

and we also have:

$$\frac{1}{\big|z - \sum_{i \in F} a_i G(z)_{ii}\big|} \leqslant \frac{1}{|\Im(z)|}. \tag{4.14}$$

Proof. Write $A = \bar{U}^t D U$ where D is a diagonal matrix with diagonal real entries $(\lambda_i)_{1 \leqslant i \leqslant N}$. Then

$$G(z)_{kk} = \sum_{i=1}^{N} |U_{ki}|^2 \frac{1}{z - \lambda_i}.$$

Since $\Re\Big(\frac{1}{z-\lambda_i}\Big) = \frac{\Re(z)-\lambda_i}{(\Re(z)-\lambda_i)^2+\Im(z)^2}$ and $\Im\Big(\frac{1}{z-\lambda_i}\Big) = \frac{-\Im(z)}{(\Re(z)-\lambda_i)^2+\Im(z)^2}$ the relation (4.12) follows. It is then straightforward to derive (4.13) from (4.12). ☐

For $i = 1, \dots, N$, let $X_N^{(i)} = (X_N(kl))_{k,l \neq i}$ be the matrix obtained from X_N by taking off the i-th column and row. Define, also for $i = 1, \dots, 2N$ the $(2N-1) \times (2N-1)$ matrix $A_N^{(i)}(z)$ obtained from $A_N(z)$ by taking off the i-th column and row. In particular, for $i = 1, \dots, N$,

$$A_N^{(N+i)}(z) = \begin{pmatrix} zI_N & -{}^tX_N^{(i)} \\ -X_N^{(i)} & zI_{N-1} \end{pmatrix},$$

For $i = 1, \dots, 2N$, set:

$$G_N^{(i)}(z) = (A_N^{(i)}(z))^{-1}. \tag{4.15}$$

Let now $\hat{X}_N^{(i)}$ denote the matrix X_N with the i-th column and row set to 0 and $\hat{A}_N^{(i)}(z)$ denote the matrix $A_N(z)$ with the i-th column and row set to 0 excepted the diagonal term. Again we have, for $i = 1, \dots, N$:

$$\hat{A}_N^{(N+i)}(z) = \begin{pmatrix} zI_N & -{}^t\hat{X}_N^{(i)} \\ -\hat{X}_N^{(i)} & zI_N \end{pmatrix},$$

For $i = 1, \dots, 2N$, set:

$$\hat{G}_N^{(i)}(z) = (\hat{A}_N^{(i)}(z))^{-1}. \tag{4.16}$$

In the paper, we will also use the terms $A_N^{(k,i)}(z), G_N^{(k,i)}(z), \hat{A}_N^{(k,i)}(z), \hat{A}_N^{(k,i)}(z)$. The double superscript just means that you make the operations described above to the rows and columns i and k.

Lemma 4.5. *For all $k \in \{1, \dots, N\}$ and all $t \neq N+k$, one has:*

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(N+k)}(z)_{tt}\right|\right] \leqslant \frac{1}{\sqrt{N}|\Im(z)|^2}. \tag{4.17}$$

Proof. Multiply the identity:

$$\hat{A}_N^{(N+k)}(z) - A_N(z) = \hat{A}_N^{(N+k)}(0) - A_N(0) \tag{4.18}$$

to the left by $G_N(z)$ and to the right by $\hat{G}_N^{(N+k)}(z)$ to obtain

$$G_N(z) - \hat{G}_N^{(N+k)}(z) = G_N(z)(\hat{A}_N^{(N+k)}(0) - A_N(0))\hat{G}_N^{(N+k)}(z). \tag{4.19}$$

Then one has:

$$G_N(z)_{tt} - \hat{G}_N^{(N+k)}(z)_{tt} = \Big(G_N(z)(\hat{A}_N^{(N+k)}(0) - A_N(0))\hat{G}_N^{(N+k)}(z)\Big)_{tt} \tag{4.20}$$

$$= \hat{G}_N^{(N+k)}(z)_{N+k,t} \sum_{i=1}^{N} G_N(z)_{ti} r_k(i) \tag{4.21}$$

$$+ G_N(z)_{t,N+k} \sum_{j=1}^{N} r_k(j) \hat{G}_N^{(N+k)}(z)_{jt} \tag{4.22}$$

$$= G_N(z)_{t,N+k} \sum_{j=1}^{N} r_k(j) \hat{G}_N^{(N+k)}(z)_{jt} \tag{4.23}$$

where we have noticed that, for all $t \neq N+k$, $\hat{G}_N^{(N+k)}(z)_{N+k,t} = 0$.

Therefore, we find that:

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(N+k)}(z)_{tt}\right|\right] \leqslant \mathbb{E}\left[|G_N(z)_{t,N+k}|^2\right]^{1/2} \mathbb{E}\left[\left|\sum_{j=1}^{N} r_k(j)\hat{G}_N^{(N+k)}(z)_{jt}\right|^2\right]^{1/2} \tag{4.24}$$

by Cauchy-Schwartz's inequality. Using then the independence of $r_k(j)$ and $\hat{G}_N^{(N+k)}(z)$, we get:

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(N+k)}(z)_{tt}\right|\right] \leqslant \mathbb{E}\left[|G_N(z)_{t,N+k}|^2\right]^{1/2} \mathbb{E}\left[r_k(1)^2\right]^{1/2} \mathbb{E}\left[\sum_{j=1}^{N}\left|\hat{G}_N^{(N+k)}(z)_{jt}\right|^2\right]^{1/2}$$

$$\leqslant \frac{1}{\sqrt{N}|\Im(z)|^2}.$$

The proof is complete. $\square$

Lemma 4.6. *There exists a constant $C > 0$ such that, for all $k \in \{1, \ldots, N\}$ and all $t \neq k$:*

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(k)}(z)_{tt}\right|\right] \leqslant \frac{C}{|\Im(z)|^2} \frac{1}{N^{\frac{1-\gamma^2}{4}}}. \tag{4.25}$$

Proof. Again, we start from the relation:

$$G_N(z) - \hat{G}_N^{(k)}(z) = G_N(z)(\hat{A}_N^{(k)}(0) - A_N(0))\hat{G}_N^{(k)}(z).$$

Thus we have

$$\begin{aligned} G_N(z)_{tt} - \hat{G}_N^{(k)}(z)_{tt} &= \Big(G_N(z)(\hat{A}_N^{(k)}(0) - A_N(0))\hat{G}_N^{(k)}(z)\Big)_{tt} &(4.26)\\ &= \hat{G}_N^{(k)}(z)_{k,t} \sum_{i=N+1}^{N} G_N(z)_{ti} r_i(k) &(4.27)\\ &+ G_N(z)_{t,k} \sum_{j=1}^{N+1} r_j(k)\hat{G}_N^{(k)}(z)_{jt} &(4.28)\\ &= G_N(z)_{t,k} \sum_{j=1}^{N+1} r_j(k)\hat{G}_N^{(k)}(z)_{jt} &(4.29) \end{aligned}$$

where we have noticed that, for all $t \neq k, \hat{G}_N^{(k)}(z)_{k,t} = 0$.

Therefore, we find that:

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(k)}(z)_{tt}\right|\right] \leqslant \mathbb{E}\left[|G_N(z)_{t,k}|^2\right]^{1/2} \mathbb{E}\left[\left|\sum_{j=1}^{N} r_j(k)\hat{G}_N^{(k)}(z)_{jt}\right|^2\right]^{1/2} \tag{4.30}$$

by Cauchy-Schwartz's inequality. We want to expand the square in the above expression. To that purpose, we first observe that, conditionnally to the M^i, the variables $(r_j(k))_j$ are independent from $\hat{G}_N^{(k)}(z)$ and centered. Hence we have for $j \neq j'$,

$$\mathbb{E}\left[r_j(k)r_{j'}(k)\hat{G}_N^{(k)}(z)_{jt}\hat{G}_N^{(k)}(z)_{j't}\right] = 0.$$

Thus we get:

$$\begin{aligned} \mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(k)}(z)_{tt}\right|\right] &\leqslant \mathbb{E}\left[|G_N(z)_{t,k}|^2\right]^{1/2} \left(\sum_{j=1}^{N+1} \mathbb{E}\left[r_j(k)^2 \left|\hat{G}_N^{(k)}(z)_{jt}\right|^2\right]\right)^{1/2}\\ &\leqslant \mathbb{E}\left[|G_N(z)_{t,k}|^2\right]^{1/2} \left(\sum_{j=1}^{N+1} \mathbb{E}\left[r_j(k)^4\right]^{1/2} \mathbb{E}\left[\left|\hat{G}_N^{(k)}(z)_{jt}\right|^4\right]^{1/2}\right)^{1/2}\\ &\leqslant \frac{\mathbb{E}[r_1(k)^4]^{1/4}}{|\Im(z)|} \left(\sum_{j=1}^{N+1} \mathbb{E}\left[\left|\hat{G}_N^{(k)}(z)_{jt}\right|^4\right]^{1/2}\right)^{1/2}\\ &\leqslant \frac{\mathbb{E}[r_1(k)^4]^{1/4}}{|\Im(z)|}(N+1)^{1/4} \left(\sum_{j=1}^{N+1} \mathbb{E}\left[\left|\hat{G}_N^{(k)}(z)_{jt}\right|^4\right]\right)^{1/4} \end{aligned}$$

Now we use the scaling properties of the MRM to obtain, for some positive constant C,

$$\mathbb{E}[r_j(k)^4] = 3\mathbb{E}[M(0, \frac{1}{N})^2] \leqslant CN^{-\zeta(2)}.$$

Furthermore, by using Lemma A.1 which assures that, almost surely:

$$\sum_{j=1}^{N+1} \left|\hat{G}_N^{(k)}(z)_{jt}\right|^2 \leqslant \frac{1}{|\Im(z)|^2} \tag{4.31}$$

and the fact that:

$$\sum_{j=1}^{N+1} \left|\hat{G}_N^{(k)}(z)_{jt}\right|^4 \leqslant \left(\sum_{j=1}^{N+1} \left|\hat{G}_N^{(k)}(z)_{jt}\right|^2\right)^2, \tag{4.32}$$

we finally obtain

$$\mathbb{E}\left[\left|G_N(z)_{tt} - \hat{G}_N^{(k)}(z)_{tt}\right|\right] \leqslant \frac{C}{|\Im(z)|^2}\left(\frac{1}{N}\right)^{\frac{\zeta(2)-1}{4}}.$$

It just remains to check that $\zeta(2) = 2 - \gamma^2$. $\square$

Lemma 4.7. *For each $k \in \{1, \ldots, 2N\}$, if $t \neq k$, then*

$$G_N^{(k)}(z)_{tt} = \hat{G}_N^{(k)}(z)_{tt}, \tag{4.33}$$

and if $t = k$, then $\hat{G}_N^{(k)}(z)_{k,k} = z^{-1}$.

Proof. It is straigthforward to see that the two matrices $G_N^{(k)}(z)$ and $\hat{G}_N^{(k)}(z)$ have the same eigenvalues except that $\hat{G}_N^{(k)}(z)$ has one more zero eigenvalue. In addition, the eigenvectors look also very similar since you can obtain $2N$ eigenvectors of $\hat{G}_N^{(k)}(z)$ by adding a zero entry to the eigenvectors of $G_N^{(k)}(z)$ (between the entries $k-1$ and k). The last eigenvector of $\hat{G}_N^{(k)}(z)$ is the vector of $\mathbb{R}^N$ for which all entries are zero except the entry number k.

Now observe that with $G_N^{(k)}(z) = U\mathrm{diag}(z-\lambda)U^*$ and $\hat{G}_N^{(k)}(z) = V\mathrm{diag}(z-\widetilde{\lambda})V^*$,

$$G_N^{(k)}(z)_{tt} = \sum_{i=1}^{2N} |u_{ti}|^2 \frac{1}{z-\lambda_i} \tag{4.34}$$

$$\hat{G}_N^{(k)}(z)_{tt} = \sum_{i=1}^{N} |v_{ti}|^2 \frac{1}{z-\widetilde{\lambda}_i}. \tag{4.35}$$

The result follows since, for $t \neq k$,

$$\sum_{i=1}^{2N-1} |u_{ti}|^2 \frac{1}{z-\lambda_i} = \sum_{i=1}^{2N} |v_{ti}|^2 \frac{1}{z-\widetilde{\lambda}_i} \tag{4.36}$$

and, for $t = k$, $\hat{G}_N^{(k)}(z)_{k,k} = z^{-1}$. $\square$

Lemma 4.8. *For all $z \in \mathbb{C}$ and Lebesgue almost every point $x \in [0,1]$, we have*

$$\Im(z)\Im(K_z(x)) \leqslant 0 \tag{4.37}$$

and

$$|\Im(K_z(x))| \leqslant \frac{1}{\Im(z)} \tag{4.38}$$

Proof. This is a straightforward consequence of Lemma 4.4. Indeed, we have for all positive continuous function f on $[0,1]$ and $N \in \mathbb{N}$:

$$\Im(z)\Im\Big(\int_0^1 f(x)\mathbb{E}[L_N^{1,z}](dx)\Big) \leqslant 0.$$

We pass to the limit as N goes to ∞ along some suitable subsequence and obtain:

$$\Im(z)\Im\Big(\int_0^1 f(x)K_z(x)\,dx\Big) \leqslant 0.$$

The result follows. $\square$

4.3 Concentration inequalities

This lemma is adpated to our case from Lemma 5.4 in [3].

Lemma 4.9. *Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function. For each $i \in \{1,2\}$, we have the following concentration results:*

$$\mathbb{E}\left[\mid L_N^{i,z}(f) - \mathbb{E}[L_N^{i,z}(f)] \mid^2\right] \leqslant \frac{8}{N}\frac{\|\, f \,\|_\infty^2}{\mid \Im z \mid^4}. \tag{4.39}$$

Proof. Define two functions F_N^1 and F_N^2 such that:

$$F_N^1\left(\left(X_{ij}^{(N)}\right)_{1\leqslant j\leqslant N+1}, 1\leqslant i\leqslant N\right) = \frac{1}{N}\sum_{k=1}^{N} f\left(\frac{k}{N}\right) G_N(z)_{kk} \tag{4.40}$$

$$F_N^2\left(\left(X_{ij}^{(N)}\right)_{1\leqslant j\leqslant N+1}, 1\leqslant i\leqslant N\right) = \frac{1}{N}\sum_{k=1}^{N+1} f\left(\frac{k}{N+1}\right) G_N(z)_{k+N,k+N} \tag{4.41}$$

We will prove the Lemma for $L_N^{1,z}$; the proof for $L_N^{2,z}$ is a straightforward adaptation.

Let, for $k \in \{1,\dots,N+1\}$,

$$\mathcal{F}_k = \sigma\left(\left(X_{ij}^{(N)}\right)_{1\leqslant j\leqslant N}, 1\leqslant i\leqslant k\right) \tag{4.42}$$

If P denotes the law of the vector $\left(X_{1j}^{(N)}\right)_{1 \leqslant j \leqslant N}$,

$$
\begin{aligned}
&\mathbb{E}\left[\mid F_N^1 - \mathbb{E}[F_N^1] \mid^2\right] \\
&= \sum_{i=0}^{N} \mathbb{E}\left[\mid \mathbb{E}[F_N^1 \mid \mathcal{F}_{i+1}] - \mathbb{E}[F_N^1 \mid \mathcal{F}_i] \mid^2\right] \\
&= \sum_{i=0}^{N} \int \mid \int \left(F_N(x_1, x_2, \ldots, x_{i+1}, y_{i+2}, \ldots, y_{N+1}) - F_N(x_1, x_2, \ldots, x_i, y_{i+1}, \ldots, y_{N+1})\right) dP^{\otimes N+1}(y) \mid^2 \\
&dP^{\otimes i+1}(x) \\
&\leqslant \sum_{i=0}^{N} \int \mid \int \left(F_N(x_1, x_2, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_{N+1}) - F_N(x_1, x_2, \ldots, x_i, y, x_{i+2}, \ldots, x_{N+1})\right) dP(y) \mid^2 \\
&dP^{\otimes N+1}(x) \\
&\leqslant \sum_{i=0}^{N} \sup_{\mathbb{R}^{(N+1)^2}} \| \nabla_{x_{i+1}} F_N \|^2 \int \| x - y \|^2 \, dP^{\otimes 2}(x, y).
\end{aligned}
$$

The quantity $\nabla_{x_{i+1}} F_N^1$ refers to the gradient of F_N^1 in the direction of the vector x_{i+1}.

If we consider a couple of processes $(\widetilde{B}^1, \widetilde{M}^1)$ independent from (B^1, M^1) with the same law, it is easy to see that:

$$
\begin{aligned}
\int \| x - y \|^2 \, dP \otimes dP(x, y) &= \sum_{j=1}^{N} \mathbb{E}\Big[(B^1_{M^1(0,\frac{j}{N})} - B^1_{M^1(0,\frac{j-1}{N})} - \widetilde{B}^1_{\widetilde{M}^1(0,\frac{j}{N})} + \widetilde{B}^1_{\widetilde{M}^1(0,\frac{j-1}{N})})^2\Big]. \\
&= 2 - 2\sum_{j=1}^{N} \mathbb{E}\Big[(B^1_{M^1(0,\frac{j}{N})} - B^1_{M^1(0,\frac{j-1}{N})})(\widetilde{B}^1_{\widetilde{M}^1(0,\frac{j}{N})} \widetilde{B}^1_{\widetilde{M}^1(0,\frac{j-1}{N})})\Big] \\
&= 2.
\end{aligned}
$$

In our case, we have, for $i \in \{1, \ldots, N+1\}, j \in \{1, \ldots, N\}$:

$$
\frac{\partial G_N(z)_{kk}}{\partial X_{ij}} = G_N(z)_{k,j} G_N(z)_{N+i,k} + G_N(z)_{k,N+i} G_N(z)_{j,k} \tag{4.43}
$$

Thus,

$$
\nabla_{x_{i+1}} F_N = \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{k}{N}\right) \nabla_{x_{i+1}} G_N(z)_{kk} \tag{4.44}
$$

It is now plain to compute:

$$
\| \nabla_{x_{i+1}} F_N \|^2 = \frac{1}{N^2} \sum_{j=1}^{N} \mid \left(G_N(z) D^1(f) G_N(z)\right)_{N+i+1,j} + \left(G_N(z) D^1(f) G_N(z)\right)_{j,N+i+1} \mid^2
$$

where $D^1(f)$ is the $(2N)$-dimensional diagonal matrix of entries:

$$D^1(f)_{kk} = f\left(\frac{k}{N}\right) 1_{\{1 \leqslant k \leqslant N\}}.$$

One thus has:

$$\begin{aligned} || \nabla_{x_{i+1}} F_N^1 ||^2 &= \frac{4}{N^2} \sum_{j=1}^{N} | \left(G_N(z) D^1(f) G_N(z)\right)_{N+i+1,j} |^2 \\ &\leqslant \frac{4}{N^2} \sum_{j=1}^{2N} | \left(G_N(z) D^1(f) G_N(z)\right)_{N+i+1,j} |^2 \\ &\leqslant \frac{4}{N^2} \frac{|| f ||_\infty^2}{| \Im z |^4}. \end{aligned}$$

where, in the last line, we used lemma A.1 and the fact that the matrix $G_N(z)D^1(f)G_N(z)$ has a spectral radius smaller than $|| f ||_\infty / | \Im z |^2$.

Finally,

$$\mathbb{E}\left[| F_N^1 - \mathbb{E}[F_N^1] |^2\right] \leqslant \frac{8}{N} \frac{|| f ||_\infty^2}{| \Im z |^4}. \tag{4.45}$$

□

We also prove the following lemma:

Lemma 4.10. *For all $\alpha > 1$ such that $\zeta(2\alpha) > 1$, we have*

$$\mathbb{E}\left[\left|\sum_{t=1}^{N} r_k(t)^2 \left(\hat{G}_N^{(N+k)}(z)_{tt} - \mathbb{E}[\hat{G}_N^{(N+k)}(z)_{tt}]\right)\right|\right] \leqslant \frac{C(\ln N)^2}{N^{\frac{\zeta(2\alpha)-1}{\alpha}} |\Im(z)|^4} \tag{4.46}$$

for some positive constant C independent from N, z, k.

Proof. Notice that $(r_k(t))_t$ and $\hat{G}_N^{(N+k)}(z)$ are independent. Hence, by conditioning with respect to the process $(r_k(t))_t$, we can argue along the same lines as in the previous lemma with $r_k(t)$ instead of $\frac{1}{N} f(\frac{t}{N})$ and we get the formula:

$$\mathbb{E}\left[\left|\sum_{t=1}^{N} r_k(t)^2 \left(\hat{G}_N^{(N+k)}(z)_{tt} - \mathbb{E}[\hat{G}_N^{(N+k)}(z)_{tt}]\right)\right|^2\right] \leqslant \frac{8}{|\Im(z)|^4} \mathbb{E}[\sup_t r_k(t)^4].$$

We conclude with Proposition B.1 in the appendix . □

In the following, we fix $\alpha > 1$ such that $\zeta(2\alpha) > 1$ (because of the expression of ζ and the inequality $\gamma^2 < 1/3$, it is clear that such a number exists).

4.4 The system verified by the limit point μ_z^2 and $K_z(x)$: first equation

From the Schur complement formula (see e.g. Lemma 4.2 in [3] for a reminder), one has for $k \in \{1, \dots, N\}$:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{s,t=1}^{N} r_k(s) r_k(t) G_N^{(N+k)}(z)_{st} \right]^{-1} \tag{4.47}$$

Using Lemma A.2, one can write:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} r_k(t)^2 G_N^{(N+k)}(z)_{tt} + \epsilon_{N,k}^1(z) \right]^{-1} \tag{4.48}$$

where $\epsilon_{N,k}^1(z)$ is a complex valued random variable for which there exists $C > 0$ such that for all $N \in \mathbb{N}$ and $1 \leqslant k \leqslant N$,

$$\mathbb{E}[|\epsilon_{N,k}^1(z)|^2] < \frac{C}{N^{1-\gamma^2}}. \tag{4.49}$$

By using Lemma 4.7, we can write:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} r_k(t)^2 \hat{G}_N^{(N+k)}(z)_{tt} + \epsilon_{N,k}^1(z) \right]^{-1}. \tag{4.50}$$

Lemma 4.10 applied to $\alpha > 1$ such that $\zeta(2\alpha) > 1$ yields:

$$\mathbb{E}\left[\left| \sum_{t=1}^{N} r_k(t)^2 \left(\hat{G}_N^{(N+k)}(z)_{tt} - \mathbb{E}[\hat{G}_N^{(N+k)}(z)_{tt}] \right) \right|^2 \right] \leqslant \frac{C(\ln N)^2}{N^{\frac{\zeta(2\alpha)-1}{\alpha}} |\Im(z)|^4}. \tag{4.51}$$

Thus, one can write:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} r_k(t)^2 \mathbb{E}\left[\hat{G}_N^{(N+k)}(z)_{tt} \right] + \epsilon_{N,k}^1(z) + \epsilon_{N,k}^2(z) \right]^{-1} \tag{4.52}$$

where $\epsilon_{N,k}^2(z)$ is a complex valued random variable such that for all $N \in \mathbb{N}$ and $1 \leqslant k \leqslant N+1$,

$$\mathbb{E}[|\epsilon_{N,k}^2(z)|^2] < \frac{C(\ln N)^2}{N^{\frac{\zeta(2\alpha)-1}{\alpha}} |\Im(z)|^4}. \tag{4.53}$$

In addition, using Lemma 4.5, we can show:

$$\mathbb{E}\left[\left|\sum_{t=1}^{N} r_k(t)^2 \left(\mathbb{E}\left[\hat{G}_N^{(N+k)}(z)_{tt} - G_N(z)_{tt}\right]\right)\right|\right] \tag{4.54}$$

$$\leqslant \sum_{t=1}^{N} \mathbb{E}[r_k(t)^2]\mathbb{E}\left[\left|\hat{G}_N^{(N+k)}(z)_{tt} - G_N(z)_{tt}\right|\right] \tag{4.55}$$

$$\leqslant \frac{1}{|\Im(z)|^2\sqrt{N}}. \tag{4.56}$$

It follows:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} r_k(t)^2 \mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon^1_{N,k}(z) + \epsilon^2_{N,k}(z) + \epsilon^3_{N,k}(z)\right]^{-1} \tag{4.57}$$

where $\epsilon^3_{N,k}(z)$ is a complex valued random variable such that for all $N \in \mathbb{N}$ and $1 \leqslant k \leqslant N+1$,

$$\mathbb{E}\left[|\epsilon^3_{N,k}(z)|\right] < \frac{1}{|\Im(z)|^2\sqrt{N}}. \tag{4.58}$$

Let us denote by I_N^t the interval $[\frac{t-1}{N}, \frac{t}{N}]$. Then we have:

Lemma 4.11. *The following inequality holds:*

$$\mathbb{E}\left[\left|\sum_{t=1}^{N}\left(r_k(t)^2 - M^k(I_N^t)\right)\mathbb{E}\left[G_N(z)_{tt}\right]\right|^2\right] \leqslant \frac{C}{N^{1-\gamma^2}|\Im(z)|^2}$$

for some positive constant C.

Proof. We expand the square and, because $r_k(t)$ and $r_k(t')$ are independent for $t \neq t'$ conditionally to M^k, we have:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\sum_{t=1}^{N}\Big(r_k(t)^2 - M^k(I_N^t)\Big)\mathbb{E}\left[G_N(z)_{tt}\right]\Big|^2\Big] \\
&\qquad = \sum_{t,t'=1}^{N}\mathbb{E}\Big[\Big(r_k(t)^2 - M^k(I_N^t)\Big)\Big(r_k(t')^2 - M^k(I_N^{t'})\Big)\mathbb{E}\left[G_N(z)_{tt}\right]\mathbb{E}\left[G_N(z)_{t't'}\right]\Big] \\
&\qquad = \sum_{t=1}^{N}\mathbb{E}\Big[\Big(r_k(t)^2 - M^k(I_N^t)\Big)^2\Big]\mathbb{E}\left[G_N(z)_{tt}\right]^2 \\
&\qquad = 2\sum_{t=1}^{N}\mathbb{E}\left[\left(M^k(I_N^t)\right)^2\right]\mathbb{E}\left[G_N(z)_{tt}\right]^2 \\
&\qquad \leqslant 2C\frac{N}{N^{\zeta(2)}|\Im(z)|^2}
\end{aligned}$$

∎

We can thus write

$$G_N(z)_{N+k,N+k} = \Bigg[z - \sum_{t=1}^{N} M^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] \tag{4.59}$$

$$+ \epsilon^1_{N,k}(z) + \epsilon^2_{N,k}(z) + \epsilon^3_{N,k}(z) + \epsilon^4_{N,k}(z)\Bigg]^{-1} \tag{4.60}$$

where $\epsilon^4_{N,k}(z)$ is a complex valued random variable such that for all $N \in \mathbb{N}$ and $1 \leqslant k \leqslant N+1$,

$$\mathbb{E}\left[|\epsilon^4_{N,k}(z)|^2\right] \leqslant \frac{C}{N^{\zeta(2)-1}|\Im(z)|^2}. \tag{4.61}$$

Set $\epsilon_{N,k}(z) = \epsilon^1_{N,k}(z) + \epsilon^2_{N,k}(z) + \epsilon^3_{N,k}(z) + \epsilon^4_{N,k}(z)$ and rewrite:

$$G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} M^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon_{N,k}(z)\right]^{-1} \tag{4.62}$$

We now need to introduce the truncated Radon measure $M^k_\epsilon(dx)$ with Lebesgue density $e^{\omega^k_\epsilon(x)}$ which converges almost surely as ϵ goes to 0, in the sense of weak convergence in the space of Radon measure, to the measure M^k (see section 2.1).

Lemma 4.12. *For $\epsilon > 0$, the following uniform bound holds:*

$$\sup_N \mathbb{E}\big[\big| \sum_{t=1}^{N} M^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] - \sum_{t=1}^{N} M^k_\epsilon(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]\big|^2\big] \leqslant \frac{C\epsilon^{1-\gamma^2}}{|\Im(z)|^2}.$$

Proof. We expand the square. Note that the covariance function ρ_ϵ of the process ω_ϵ increases as ϵ decreases to 0 and uniformly converges as $\epsilon \to 0$ towards $\ln_+ \frac{\tau}{|x|}$

over the complement of any ball centered at 0. Thus we have:

$$
\begin{aligned}
\sup_N \mathbb{E}\big[| \sum_{t=1}^{N} & M^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] - \sum_{t=1}^{N} M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] |^2\big] \\
&= \sup_N \sum_{t,t'=1}^{N} \mathbb{E}\big[(M^k(I_N^t) - M_\epsilon^k(I_N^t))(M^k(I_N^{t'}) - M_\epsilon^k(I_N^{t'}))\big]\mathbb{E}\left[G_N(z)_{tt}\right]\mathbb{E}\left[G_N(z)_{t't'}\right] \\
&= \sup_N \sum_{t,t'=1}^{N} \mathbb{E}\left[(M^k(I_N^t) - M_\epsilon^k(I_N^t))(M^k(I_N^{t'}) - M_\epsilon^k(I_N^{t'}))\right]\mathbb{E}\left[G_N(z)_{tt}\right]\mathbb{E}\left[G_N(z)_{t't'}\right] \\
&= \sup_N \sum_{t,t'=1}^{N} \left(\mathbb{E}\left[M^k(I_N^t)M^k(I_N^{t'})\right] - \mathbb{E}\left[M_\epsilon^k(I_N^t)M_\epsilon^k(I_N^{t'})\right]\right)\mathbb{E}\left[G_N(z)_{tt}\right]\mathbb{E}\left[G_N(z)_{t't'}\right] \\
&= \sup_N \sum_{t,t'=1}^{N} \mathbb{E}\left[G_N(z)_{tt}\right]\mathbb{E}\left[G_N(z)_{t't'}\right]\int_{I_N^t}\int_{I_N^{t'}} \left(e^{\psi(2)\ln_+ \frac{\tau}{|r-u|}} - e^{\psi(2)\rho_\epsilon(r-u)}\right) drdu \\
&\leqslant \frac{1}{|\Im(z)|^2}\int_0^1\int_0^1 \left(e^{\psi(2)\ln_+ \frac{\tau}{|r-u|}} - e^{\psi(2)\rho_\epsilon(r-u)}\right) drdu.
\end{aligned}
$$

where, in the fourth line, we used the fact that, if $\mathcal{F}_\epsilon$is the sigma field generated by the random variables $\mu(A), A \in \mathcal{B}(\{(t,y) : y \geqslant \epsilon\})$, then $\mathbb{E}[M^k(A)|\mathcal{F}_\epsilon] = M_\epsilon^k(A)$ for all borelian set A. A straightforward computation leads to the relation

$$
\rho_\epsilon(t) = \begin{cases} \ln \frac{\tau}{\epsilon} + 1 - \frac{|t|}{\epsilon} & \text{if } |t| \leqslant \epsilon \\ \ln \frac{\tau}{|t|} & \text{if } \epsilon \leqslant |t| \leqslant \tau \\ 0 & \text{if } \tau < |t| \end{cases} \tag{4.63}
$$

By using the expression of ρ_ϵ, it is then plain to obtain the desired bound. $\square$

We can thus write

$$
G_N(z)_{N+k,N+k} = \left[z - \sum_{t=1}^{N} M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon_{N,k}(z) + \delta(\epsilon, N, z)\right]^{-1}, \tag{4.64}
$$

where

$$
\sup_N \mathbb{E}[|\delta(\epsilon, N, z)|^2] \to 0 \quad \text{as } \epsilon \to 0, \tag{4.65}
$$

and also:

$$
\mathbb{E}\left[G_N(z)_{N+k,N+k}\right] = \mathbb{E}\left[\left[z - \sum_{t=1}^{N} M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon_{N,k}(z) + \delta(\epsilon, N, z)\right]^{-1}\right]. \tag{4.66}
$$

The next step is to study the convergence of the above quantity. Hence we prove (see the proof in the appendix):

Lemma 4.13. *The random variable $\sum_{t=1}^N M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]$ converges in probability as $N \to +\infty$ towards $\int_0^1 K_z(x)M_\epsilon^k(dx)$.*

We fix $\epsilon > 0$. For that ϵ, the family of random variables $(\delta(\epsilon, N, z))_N$ is bounded in L^2 so that it is tight. Even if it means extracting again a subsequence we assume that the couple $(\sum_{t=1}^N M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right], \delta(\epsilon, N, z))_N$ converges in law towards the couple $(\int_0^1 K_z(x)M_\epsilon^k(dx), Y_\epsilon)$. We remind the reader of (4.47) which implies that

$$\left| \left(z - \sum_{t=1}^N M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon_{N,k}(z) + \delta(\epsilon, N, z) \right)^{-1} \right| \leqslant \frac{1}{|\Im(z)|}.$$

The quantity $\left(z - \sum_{t=1}^N M_\epsilon^k(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right] + \epsilon_{N,k}(z) + \delta(\epsilon, N, z) \right)^{-1}$ is therefore bounded uniformly with respect to N, ϵ and converges in law towards

$$\left(z - \int_0^1 K_z(x)M_\epsilon^k(dx) + Y_\epsilon \right)^{-1}.$$

We deduce that the expectation of the former quantity converges as $\epsilon \to 0$ towards the expectation of the latter quantity. From (4.66), we deduce that

$$\mu_z^2 = \mathbb{E}\left[\left(z - \int_0^1 K_z(x)M_\epsilon^k(dx) + Y_\epsilon \right)^{-1} \right]. \tag{4.67}$$

Clearly, standard arguments prove that $\int_0^1 K_z(x)M_\epsilon^k(dx)$ converges almost surely towards $\int_0^1 K_z(x)M^k(dx)$ as $\epsilon \to 0$ (K_z is deterministic (see lemma 4.9), measurable and bounded) and, because of (4.65), Y_ϵ converges almost surely towards 0 as $\epsilon \to 0$. Again, because the quantity $\left(z - \int_0^1 K_z(x)M_\epsilon^k(dx) + Y_\epsilon \right)^{-1}$ is bounded uniformly with respect to ϵ, we deduce that:

$$\mu_z^2 = \mathbb{E}\left[\left(z - \int_0^1 K_z(x)M^k(dx) \right)^{-1} \right]. \tag{4.68}$$

4.5 Second equation

Now we turn our attention to the terms $G_N(z)_{kk}$ for $k \in \{1, \dots, N\}$. Again, by using the Schur complement formula, we can write, for $k \in \{1, \dots, N\}$:

$$G_N(z)_{kk} = \left[z - \sum_{i,j=1}^N r_i(k)r_j(k)G_N^{(k)}(z)_{N+i,N+j} \right]^{-1} \tag{4.69}$$

$$= \left[z - \sum_{i=1}^N r_i(k)^2 G_N^{(k)}(z)_{N+i,N+i} + \eta_{N,k}^1(z) \right]^{-1} \tag{4.70}$$

where, using Lemma A.3, $\eta^1_{N,k}(z)$ is a complex valued random variable for which there exists $c>0$ such that for all $N\in\mathbb{N}$ and $1\leqslant k\leqslant N$, $\mathbb{E}[|\eta^1_{N,k}(z)|^2]<c/N$.

With a further use of the Schur complement formula for the term $G_N^{(k)}(z)_{N+i,N+i}$, we obtain:

$$G_N(z)_{kk}=\left[z-\sum_{i=1}^N r_i(k)^2\left[z-\sum_{s,t\neq k}^N r_i(s)r_i(t)G_N^{(k,N+i)}(z)_{st}\right]^{-1}+\eta^1_{N,k}(z)\right]^{-1} \tag{4.71}$$

where $G_N^{(k,N+i)}(z)=A_N^{(k,N+i)}(z)^{-1}$. Note that $G_N^{(k,N+i)}(z)$ is independent of $(r_i(t))_{t=1,\dots,N}$. Using the same arguments as in the derivation of the first equation (in particular Lemmas A.2, 4.7, 4.10, B.1, 4.6 and 4.5), one can show that:

$$G_N(z)_{kk}=\left[z-\sum_{i=1}^N\frac{r_i(k)^2}{z-\sum_{t=1}^N M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]+\delta_{N,k,i}(z)}+\eta^1_{N,k}(z)\right]^{-1} \tag{4.72}$$

where $(\delta_{N,k,i}(z))_{1\leqslant i\leqslant N}$ are complex random variable such that

$$\mathbb{E}[|\delta_{N,k,i}(z)|]\leqslant\frac{C}{N^{\min(\frac{1-\gamma^2}{4},\frac{\zeta(2\alpha)-1}{\alpha})}} \tag{4.73}$$

for some positive constant C that does not depend on i,N and for $\alpha>1$ such that $\zeta(2\alpha)>1$.

Lemma 4.14. *One can write:*

$$G_N(z)_{kk}=\left[z-\sum_{i=1}^N\frac{r_i(k)^2}{z-\sum_{t=1}^N M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]}+\eta^1_{N,k}(z)+\eta^2_{N,k}(z)\right]^{-1} \tag{4.74}$$

where $\eta^2_{N,k}(z)$ is a random variable that tends to 0 in probability as N goes to ∞.

Proof. By using Lemma 4.4, we deduce that:

$$\begin{aligned}\sum_{i=1}^N\Big|&\frac{r_i(k)^2}{z-\sum_{t=1}^N M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]+\delta_{N,k,i}(z)}-\frac{r_i(k)^2}{z-\sum_{t=1}^N M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]}\Big|\\&\leqslant\frac{1}{|\Im(z)|^2}\sum_{i=1}^N r_i(k)^2\min(|\delta_{N,k,i}(z)|,2).\end{aligned} \tag{4.75}$$

We stress that the lemma is proved as soon as we can prove that the left-hand side in (4.75) converges in probability to 0. Hence it is enough to prove that

$$\mathbb{E}\left[\sum_{i=1}^N r_i(k)^2\min(|\delta_{N,k,i}(z)|,2)\right]$$

converges to 0 as N tends to ∞. By noticing that:

$$\delta_{N,k,i}(z) = \sum_{s,t\neq k}^{N} r_i(s)r_i(t)G_N^{(k,N+i)}(z)_{st} - \sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right], \tag{4.76}$$

it is straightforward to see that the variables $\Big(r_i(k)^2 \min(|\delta_{N,k,i}(z)|, 2)\Big)_{1\leqslant i\leqslant N+1}$ are identically distributed. Thus we have

$$\mathbb{E}\big[\sum_{i=1}^{N} r_i(k)^2 \min(|\delta_{N,k,i}(z)|, 2)\big] = N\mathbb{E}\big[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\big].$$

Then for all $A > 1$ and $\alpha > 0$, we have

$$\begin{aligned}
N\mathbb{E}\big[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\big] =& N\mathbb{E}\big[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\mathbb{1}_{\{Nr_1(k)^2\leqslant A\}}\big]\\
&+ N\mathbb{E}\big[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\mathbb{1}_{\{Nr_1(k)^2> A\}}\big]\\
\leqslant& A\mathbb{E}[|\delta_{N,k,1}(z)|] + 2\mathbb{E}\big[Nr_1(k)^2\mathbb{1}_{\{Nr_1(k)^2> A\}}\big]\\
\leqslant& \frac{AC}{N^{\frac{\zeta(2)-1}{4}}} + \frac{2}{A^\alpha}\mathbb{E}\big[N^{1+\alpha}r_1(k)^{2(\alpha+1)}\big]\\
=& \frac{AC}{N^{\frac{\zeta(2)-1}{4}}} + \frac{2N^{1+\alpha}}{A^\alpha}\mathbb{E}\big[M^1(0,\frac{1}{N})^{\alpha+1}\big]
\end{aligned}$$

By using the scale invariance property of the measure M^1, we have:

$$\mathbb{E}\left[M^1(0,1/N)^{\alpha+1}\right] = \frac{1}{N^{\zeta(1+\alpha)}}\mathbb{E}\left[M^1(0,1)^{\alpha+1}\right],$$

in such a way that

$$N\mathbb{E}\left[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\right] \leqslant \frac{AC}{N^{\frac{\zeta(2)-1}{4}}} + 2\mathbb{E}\left[M^1(0,1)^{\alpha+1}\right]\frac{N^{\psi(1+\alpha)}}{A^\alpha}. \tag{4.77}$$

Since $\zeta(2) > 5 - 4\zeta'(1)$ (this inequality is clear with $\zeta(q) = (1+\gamma^2/2)q + q^2\gamma^2/2$ and is due to our hypotheses of Assumption 3.7 in the more general case), we can choose $p > 0$ such that

$$\frac{\zeta(2)-1}{4} > p > 1 - \zeta'(1) = \psi'(1). \tag{4.78}$$

The mapping $\alpha \in]0,+\infty[\mapsto p\alpha - \psi(1+\alpha)$ reduces to 0 for $\alpha = 0$ and, because $p > \psi'(1)$, is strictly positive for $\alpha > 0$ small enough. So we choose $\alpha < 1$ such that $p\alpha - \psi(1+\alpha) > 0$ and we set $A = N^p$. We obtain:

$$N\mathbb{E}\big[r_1(k)^2 \min(|\delta_{N,k,1}(z)|, 2)\big] \leqslant \frac{C}{N^{\frac{\zeta(2)-1}{4}-p}} + 2^{2+\alpha}\mathbb{E}\big[M^1(0,T)^{\alpha+1}\big]\frac{1}{N^{\alpha p-\psi(1+\alpha)}}.$$

The result follows by letting $N \to \infty$ since $\min((\zeta(2)-1)/4 - p, \alpha p - \psi(1+\alpha)) > 0$. $\square$

Lemma 4.15. *There exists a constant $c > 0$, which does not depend on N, such that for each $N \in \mathbb{N}$:*

$$\mathbb{E}\Bigg[\Bigg|\sum_{i=1}^{N}\Bigg(\frac{r_i(k)^2}{z-\sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]} - \mathbb{E}\Bigg[\frac{r_i(k)^2}{z-\sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]}\Bigg]\Bigg)\Bigg|^2\Bigg] \leqslant \frac{c}{N^{1-\gamma^2}}.$$

Proof. The proof is straightforward using the fact that for $i \in \{1, \dots, N\}$, the random variables

$$\frac{r_i(k)^2}{z-\sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]} \tag{4.79}$$

are i.i.d. random variables and Lemma 4.4. $\square$

Therefore we can write

$$G_N(z)_{kk} = \left[z - \sum_{i=1}^{N}\frac{r_i(k)^2}{z-\sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\left[G_N(z)_{tt}\right]} + \eta_{N,k}^1(z) + \eta_{N,k}^2(z) + \eta_{N,k}^3(z)\right]^{-1} \tag{4.80}$$

with $\mathbb{E}[(\eta_{N,k}^3(z))^2] \leqslant \frac{c}{N^{1-\gamma^2}}$.

Now we can take the expectation in (4.80) to obtain

$$\begin{aligned}
&\mathbb{E}[L_N^{1,z}(f)]\\
&\quad= \frac{1}{N}\sum_{k=1}^{N} f(k/N)\mathbb{E}[G_N(z)_{kk}]\\
&\quad= \frac{1}{N}\sum_{k=1}^{N} f(k/N)\mathbb{E}\Big[\Big(z - \mathbb{E}\Big[\sum_{i=1}^{N}\frac{r_i(k)^2}{z-\sum_{t=1}^{N} M^i(I_N^t)\mathbb{E}\big[G_N(z)_{tt}\big]}\Big] + \eta_{N,k}(z)\Big)^{-1}\Big]\\
&\quad= \frac{1}{N}\sum_{k=1}^{N} f(k/N)\mathbb{E}\Big[\Big(z - N\mathbb{E}\Big[\frac{M\left[\frac{k-1}{N};\frac{k}{N}\right]}{z-\sum_{t=1}^{N} M(I_N^t)\mathbb{E}\big[G_N(z)_{tt}\big]}\Big] + \eta_{N,k}(z)\Big)^{-1}\Big]
\end{aligned}$$

with $\eta_{N,k}(z) = \eta_{N,k}^1(z)+\eta_N^2(z)+\eta_{N,k}^3(z)$. Then, by introducing the truncated measure M_ϵ and by using the Girsanov formula, we can approximate (uniformly in N) this last expression by:

$$\frac{1}{N}\sum_{k=1}^{N} f(k/N)\mathbb{E}\Big[\Big(z - N\mathbb{E}\Big[\frac{M_\epsilon\left[\frac{k-1}{N};\frac{k}{N}\right]}{z-\sum_{t=1}^{N} M_\epsilon(I_N^t)\mathbb{E}\big[G_N(z)_{tt}\big]}\Big]\Big)^{-1} + \hat{\delta}(N,k,z,\epsilon)\Big] \tag{4.81}$$

with $\sup_{N,k}\mathbb{E}[|\hat{\delta}(N,k,z,\epsilon)|^2]$ going to 0 when ϵ is going to 0. Along some appropriate subsequence, this latter quantity converges as $N \to +\infty$ to:

$$\int_0^1 f(x)\mathbb{E}\left[\left(z - \mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z-\int_0^1 K_z(r)\,M_\epsilon(dr)}\right]\right)^{-1} + Y^\epsilon\right] dx \tag{4.82}$$

where Y^ϵ is such that $\mathbb{E}[(Y^\epsilon)^2]$ converges to 0 when ϵ is going to 0. And, we thus obtain gathering the above arguments that:

$$\int_0^1 f(x)K_z(x)\,dx = \int_0^1 f(x)\mathbb{E}\left[\left(z - \mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z - \int_0^1 K_z(r)\,M_\epsilon(dr)}\right]\right)^{-1} + Y^\epsilon\right] dx. \tag{4.83}$$

It remains to pass to the limit as $\epsilon \to 0$ in that expression. This job is carried out with the help of a Girsanov type transform in Appendix C. □

4.6 Uniqueness of the solution to the system of equations

Let X be the space of bounded measurable functions $[0,1] \to \mathbb{C}$ endowed with the uniform norm defined for $f \in X$ by:

$$||f||_\infty = \sup_{x\in[0,1]} |f(x)|. \tag{4.84}$$

Define the operator $T : X \to X$ by setting, for $g \in X$ and for all $x \in [0,1]$:

$$Tg(x) = \frac{1}{z - q\mathbb{E}\left[\left(z - \int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} g(t)M(dt)\right)^{-1}\right]} \tag{4.85}$$

For $g, h \in X$ and for all $x \in [0,1]$, we have:

$$\begin{aligned}
|Tg(x) - Th(x)| &\leqslant \frac{q}{|\Im(z)|^4}\mathbb{E}\left[\int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} |g(t)-h(t)|M(dt)\right] \\
&\leqslant \frac{q}{|\Im(z)|^4}\mathbb{E}\left[\int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} M(dt)\right] ||g-h||_\infty \\
&\leqslant \frac{q}{|\Im(z)|^4} \int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} dt||g-h||_\infty.
\end{aligned}$$

Recall that $\gamma^2 < 1/3$, and thus it is easy to see that:

$$\sup_{x\in[0,1]} \int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} dt < +\infty \tag{4.86}$$

And we can deduce that there exists a positive constant C such that:

$$\sup_{x\in[0,1]} |Tg(x) - Th(x)| \leqslant \frac{C}{|\Im(z)|^4}||g-h||_\infty \tag{4.87}$$

If z is such that $C/|\Im(z)|^4 < 1$, the operator T is contracting and thus has a unique fixed point g in the Banach space X. We conclude that, for each z with $|\Im(z)|$ large enough, there exists a unique bounded function $K_z : [0,1] \to \mathbb{C}$ such that for all $x \in [0,1]$:

$$K_z(x) = \frac{1}{z - q\mathbb{E}\left[\left(z - \int_0^1 \left(\frac{\tau}{|t-x|}\right)_+^{\gamma^2} K_z(t) M(dt)\right)^{-1}\right]}. \tag{4.88}$$

Using the first equation, it is now plain to see that, for z such that $C/|\Im(z)|^4 < 1$, the constant μ_z^2 is uniquely defined by the system of equations (by the first equation, it is a function of the function K_z, which is uniquely defined for such z).

Now it remains to show that the limit point μ_z^2 is uniquely defined for all $z \in \mathbb{C}\backslash\mathbb{R}$. It will be easy to see using analycity arguments. Indeed, from the Montel theorem, every limit point μ_z^2 is holomorphic on the set $\mathbb{C}\backslash\mathbb{R}$ since it is the pointwise limit of a subsequence of the sequence of holomorphic functions $L_N^{1,z}([0,1])$ that are uniformly bounded on each compact set of $\mathbb{C}\backslash\mathbb{R}$ (see Lemma 4.4). Thus, μ_z^2 is uniquely defined for each $z \in \mathbb{C}\backslash\mathbb{R}$ by analytic extension (we have just seen that μ_z^2 is uniquely defined for a set of z with accumulation points).

The same argument holds for the unicity of the integral $\int_0^1 K_z(x)dx$. Indeed, every limit point $\int_0^1 K_z(x)dx$ is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ that has some prescribed value on the set $\{z \in \mathbb{C} \setminus \mathbb{R} : C/|\Im(z)|^4 < 1\}$, which has accumulation points.

4.7 Proof of Theorem 3.2, 3.3 and 3.4

Let us gather the above arguments to prove the main theorems.

Proof of theorem 3.4: it is a direct consequence of sections 4.4, 4.5 and 4.6.

Proof of theorem 3.3 i): The limit points $K_z(x)dx$ and $\mu_z^2 dx$ of the two complex measures $\mathbb{E}[L_N^{1,z}]$ and $\mathbb{E}[L_N^{2,z}]$ are uniquely defined because μ_z^2 and $K_z(x)$ satisfy a fixed point system of equations (we have just seen this in theorem 3.4).

Proof of theorem 3.3 iii): We need to prove that μ_z^2 is the Stieltjes transform of a probability measure υ. From [6], it suffices to prove that μ_z^2 is holomorphic over $\mathbb{C} \setminus \mathbb{R}$, maps $\{z \in \mathbb{C} \setminus \mathbb{R}; \Im(z) < 0\}$ to $\{z \in \mathbb{C} \setminus \mathbb{R}; \Im(z) > 0\}$ and that $\lim_{y\to\infty} iy\mu_{iy}^2 = 1$ ($y \in \mathbb{R}$). Let us check those properties. We have already seen in section 4.6 that μ_z^2 is holomorphic. From Lemma 4.4, μ^2 maps $\{z \in \mathbb{C} \setminus \mathbb{R}; \Im(z) < 0\}$ to $\{z \in \mathbb{C} \setminus \mathbb{R}; \Im(z) > 0\}$. Finally, from Theorem 3.4, we have

$$z\mu_z^2 = \mathbb{E}\left[\frac{1}{1 - z^{-1}\int_0^1 K_z(x)\, M(dx)}\right].$$

As $|K_z(x)| \leqslant |\Im(z)|^{-1}$, the term $\int_0^1 K_z(x)\, M(dx)/z$ converges pointwise towards 0 when $z = iy$ and $y \to \infty$. Furthermore, from Lemma 4.8, we have $\Im(z)\Im(K_z(x) \leqslant 0$

in such a way that $\left|z - \int_0^1 K_z(x)\, M(dx)\right|^{-1} \leqslant |\Im(z)|^{-1}$. Therefore

$$|\frac{z}{z - \int_0^1 K_z(x)\, M(dx)}| \leqslant 1$$

when z takes on the form $z = iy$ ($y \in \mathbb{R}$). The dominated convergence theorem then implies that $\lim_{y\to\infty} iy\mu^2_{iy} = 1$ and we can conclude μ^2 is indeed the Stieltjes transform of a (unique) probability measure υ.

Proof of theorem 3.2 i) and 3.3 ii) We observe that, for $z \in \mathbb{C} \setminus \mathbb{R}$:

$$A_N(z) \begin{pmatrix} zI_T & 0 \\ X_N & zI_N \end{pmatrix} = \begin{pmatrix} z^2 I_T - {}^tX_N X_N & -z^tX_N \\ 0 & z^2 I_N \end{pmatrix}. \tag{4.89}$$

Let us rewrite the matrix $G_N(z) = A_N(z)^{-1}$ under the form:

$$G_N(z) = \begin{pmatrix} G_1(z) & {}^tG_{1,2}(z) \\ G_{1,2}(z) & G_2(z) \end{pmatrix}, \tag{4.90}$$

where $G_1(z), G_{1,2}(z), G_2(z)$ are respectively of size $T \times T$, $N \times T$, $N \times N$.

By taking the inverse in the relation (4.89), we obtain:

$$\begin{pmatrix} I_T/z & 0 \\ -X_N/z^2 & I_N/z \end{pmatrix} \begin{pmatrix} G_1(z) & {}^tG_{1,2}(z) \\ G_{1,2}(z) & G_2(z) \end{pmatrix} = \begin{pmatrix} (z^2 I_T - {}^tX_N X_N)^{-1} & B \\ 0 & I_N/z^2 \end{pmatrix} \tag{4.91}$$

where $B = (z^2 I_T - {}^tX_N X_N)^{-1} {}^tX_N/z$.

It can be rewritten, using the fact that $-X_N G_1(z) + zG_{1,2}(z) = 0$ and $-X_N {}^tG_{1,2}(z) + zG_2(z) = I_N$, as:

$$\begin{pmatrix} G_1(z)/z & {}^tG_{1,2}(z)/z \\ 0 & I_N/z^2 \end{pmatrix} = \begin{pmatrix} (z^2 I_T - {}^tX_N X_N)^{-1} & B \\ 0 & I_N/z^2 \end{pmatrix} \tag{4.92}$$

Therefore, taking the trace we get:

$$\frac{1}{Tz} \sum_{k=1}^{T} G_N(z)_{kk} = \frac{1}{T} \mathrm{tr}(z^2 I_T - {}^tX_N X_N)^{-1}, \tag{4.93}$$

and, by using the fact that the eigenvalues of ${}^tX_N X_N$ are those of $X_N {}^tX_N$ augmented with $T - N$ zeros:

$$\frac{1}{Tz} \sum_{k=1}^{T} G_N(z)_{kk} = \frac{1}{T} \mathrm{tr}(z^2 I_N - X_N {}^tX_N)^{-1} + \frac{T-N}{Tz^2}. \tag{4.94}$$

Now, taking expectation and using theorem 3.3, we deduce:

$$\int_0^1 K_z(x)dx = qz \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\mathrm{tr}(z^2 I_N - X_N {}^tX_N)^{-1}\right] + \frac{1-q}{z} \tag{4.95}$$

Using the fact that (by (2.7)) the spectrum of B_N contains $2N$ eigenvalues which are the positive and negative square-roots of the spectrum of $R_N = {}^tX_N X_N$ plus $T-N$ zero eigenvalues and the fact that $1/(z-\lambda)+1/(z+\lambda) = 2z/(z^2-\lambda^2)$, we can see that:

$$\frac{1}{N+T}\sum_{k=1}^{N+T} G_N(z)_{kk} = \frac{2z}{N+T}\mathrm{tr}(z^2 I_N - X_N{}^tX_N)^{-1} + \frac{T-N}{T+N}\frac{1}{z} \tag{4.96}$$

Using the relation 2.12 and theorem 3.3, it is easy to see that:

$$\lim_{N\to+\infty}\frac{1}{N+T}\sum_{k=1}^{N+T}\mathbb{E}[G_N(z)_{kk}] = \frac{1}{1+q}\left(q\mu_z^2 + \int_0^1 K_z(x)dx\right) \tag{4.97}$$

Taking expectation in 4.96 and using (4.97), we get:

$$\frac{1}{1+q}\left(q\mu_z^2 + \int_0^1 K_z(x)dx\right) = \frac{2qz}{1+q}\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}(z^2 I_N - X_N{}^tX_N)^{-1}\right] \tag{4.98}$$

$$+\frac{1-q}{1+q}\frac{1}{z}. \tag{4.99}$$

From equations (4.95) and (4.98), we get the following relation:

$$\int_0^1 K_z(x)dx = q\mu_z^2 + \frac{1-q}{z}. \tag{4.100}$$

and theorem 3.3 ii). is proved.

With (4.100), (4.97) becomes:

$$\lim_{N\to+\infty}\frac{1}{N+T}\sum_{k=1}^{N+T}\mathbb{E}[G_N(z)_{kk}] = \frac{1}{1+q}\left(2q\mu_z^2 + \frac{1-q}{z}\right) \tag{4.101}$$

and, we note that the right hand side of (4.101) is the Stieltjes transform of the measure $2q/(1+q)\upsilon(dx) + (1-q)/(1+q)\delta_0(dx)$. Thus, the mean spectral measure $\mathbb{E}[\mu_{B_N}]$ converges weakly to the measure $2q/(1+q)\upsilon(dx) + (1-q)/(1+q)\delta_0(dx)$.

We have also:

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}(z^2 I_N - X_N{}^tX_N)^{-1}\right] = \frac{\mu_z^2}{z} \tag{4.102}$$

Again using the fact that, for all $x\in\mathbb{R}, 1/(z^2-x^2) = (1/(z-x)+1/(z+x))/(2z)$ and the fact that $\upsilon(dx)$ is a symmetric measure on $\mathbb{R}$ ($\upsilon(dx)$ is the weak limit of $\mathbb{E}\left[\mu_{B_N}\right]$, which is symmetric since the spectrum of B_N is symmetric with respect to 0 almost surely), we see that:

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{tr}(z^2 I_N - X_N{}^tX_N)^{-1}\right] = \frac{1}{z}\int_{\mathbb{R}}\frac{\upsilon(dx)}{z-x} \tag{4.103}$$

$$= \int_{\mathbb{R}}\frac{\upsilon\circ(x^2)^{-1}(dx)}{z^2-x}. \tag{4.104}$$

This implies that, for each $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\operatorname{tr}(zI_N - X_N{}^t X_N)^{-1}\right] = \int_{\mathbb{R}} \frac{\upsilon \circ (x^2)^{-1}(dx)}{z-x}. \tag{4.105}$$

and thus, the probability measure $\mathbb{E}[\mu_{R_N}]$ converges weakly to the measure $\upsilon \circ (x^2)^{-1}(dx)$.

Proof of theorem 3.2 ii): using relation (2.12) and lemma 4.9, it is plain to check that $\int_{\mathbb{R}}(z-x)^{-1}\mu_{B_N}(dx)$ converges in probability to the Stieltjes transform of the probability measure $2q/(1+q)\upsilon(dx)+(1-q)/(1+q)\delta_0(dx)$. This convergence holds for finite dimensional vectors $(\int_{\mathbb{R}}(z_i-x)^{-1}\mu_{B_N}(dx)), i=1,\dots,d)$ as well. Using the fact that the set of functions $\{(z-x)^{-1}, z\in\mathbb{C}\setminus\mathbb{R}\}$ is dense in the set $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ going to 0 at infinity, we can show, for each $f\in C_0(\mathbb{R})$, that $\int f(x)\mu_{B_N}(dx)$ converges in probability to $\int f(x)(2q/(1+q)\upsilon(dx)+(1-q)/(1+q)\delta_0(dx))$. But, since $\mu_{B_N}(\mathbb{R}) = 2q/(1+q)\upsilon(\mathbb{R}) + (1-q)/(1+q)\delta_0(\mathbb{R}) = 1$, this vague convergence can be strengthened in a weak convergence. With the relations $\mu_{B_N^2} = 2N/(N+T)\mu_{R_N} + (T-N)/(T+N)\delta_0$ and the fact that $\int f(x)\mu_{B_N^2}(dx) = \int f(x^2)\mu_{B_N}(dx)$, it is plain to conclude that μ_{R_N} converges weakly in probability to $\upsilon\circ(x^2)^{-1}(dx)$.

Proof of theorem 3.2 iii): again using relation (2.12) and lemma 4.9 together with Borel-Cantelli's lemma, one can show that the two spectral measures $\mu_{B_{N_k}}$ converges weakly almost surely to $2q/(1+q)\upsilon(dx)+(1-q)/(1+q)\delta_0(dx)$. It is then easy to deduce as before that $\mu_{R_{N_k}}$ converges weakly almost surely to $\upsilon\circ(x^2)^{-1}(dx)$.

A. Auxiliary lemmas

Lemma A.1. *Let A be a $n\times n$ complex matrix such that the Hermitian matrix $M = A\bar{A}^T$ has spectral radius λ_{max}. Then, for all i, we have:*

$$\sum_{j=1}^{n} \mid A_{ij} \mid^2 \leqslant \lambda_{max}. \tag{A.1}$$

Proof. It is straightforward to see that all the entries of M are, in modulus, smaller than λ_{max}. On the other hand, we have:

$$M_{ii} = \sum_{j=1}^{n} \mid A_{ij} \mid^2 .$$

and, thus:

$$\sum_{j=1}^{n} \mid A_{ij} \mid^2 \leqslant \lambda_{max}. \tag{A.2}$$

□

Lemma A.2. *There exists $C > 0$ such that for each $N \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$:*

$$\mathbb{E}\left[\left|\sum_{s\neq t}^{N} r_k(s) r_k(t) G_N^{(N+k)}(z)_{st}\right|^2\right] \leqslant \frac{C}{N^{1-\gamma^2}}.$$

Similarly, for each $N \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$, $i \in \{1, \ldots, N\}$, we have the following inequality concerning the conditional expectation with respect to M^i:

$$\mathbb{E}\left[\left|\sum_{s,t\neq k, s\neq t}^{N} r_i(s) r_i(t) G_N^{(k,N+i)}(z)_{st}\right|^2 |M^i\right] \leqslant \frac{C}{N^{1-\gamma^2}}.$$

Proof. We first expand the square and use the independence of $(r_k(s))_s$ from $G_N^{(N+k)}(z)$:

$$\mathbb{E}\left[\left|\sum_{s\neq t}^{N} r_k(s) r_k(t) G_N^{(N+k)}(z)_{st}\right|^2\right] = 2\sum_{s\neq t}^{N} \mathbb{E}\left[r_k(s)^2 r_k(t)^2\right] \mathbb{E}\left[\left|G_N^{(N+k)}(z)_{st}\right|^2\right]$$

Now we compute

$$\begin{aligned}
\mathbb{E}\left[r_k(s)^2 r_k(t)^2\right] &= \mathbb{E}\left[M^k(\frac{s-1}{N}, \frac{s}{N}) M^k(\frac{t-1}{N}, \frac{t}{N})\right] \\
&= \int_{\frac{s-1}{N}}^{\frac{s}{N}} \int_{\frac{t-1}{N}}^{\frac{t}{N}} \max\left(1, \frac{\tau}{|r-u|}\right)^{\psi(2)} drdu \\
&\leqslant \int_0^{\frac{1}{N}} \int_{\frac{1}{N}}^{\frac{2}{N}} \max\left(1, \frac{\tau}{|r-u|}\right)^{\psi(2)} drdu
\end{aligned}$$

We consider N large enough so as to make $2/N \leqslant \tau$. The above integral is then plain to compute and we get

$$\mathbb{E}\left[r_k(s)^2 r_k(t)^2\right] \leqslant \frac{\tau^{\psi(2)}(2^{2-\psi(2)} - 2)}{(1-\psi(2))(2-\psi(2))} \frac{1}{N^{2-\psi(2)}}. \tag{A.3}$$

Thus we have for some positive constant C

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{s\neq t}^{N} r_k(s) r_k(t) G_N^{(N+k)}(z)_{st}\right|^2\right] &\leqslant \frac{C}{N^{2-\psi(2)}} \sum_{s\neq t}^{N} \mathbb{E}\left[\left|G_N^{(N+k)}(z)_{st}\right|^2\right] \\
&\leqslant \frac{C}{N^{1-\psi(2)}} \frac{1}{|\Im(z)|^2},
\end{aligned}$$

where we have used the fact that almost surely:

$$\frac{1}{2N-1} \sum_{s,t\neq N+k}^{2N} \left|G_N^{(N+k)}(z)_{st}\right|^2 \leqslant \frac{1}{|\Im(z)|^2}.$$

It just remains to see that $\psi(2) = \gamma^2$. To prove the second relation, we follow the same argument by noticing that $(r_i(t))_t$ and $G_N^{(k,N+i)}(z)$ are independent conditionally to M^i. $\square$

Lemma A.3. *There exists some constant $c > 0$ such that for each $N \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$:*

$$\mathbb{E}\left[\left|\sum_{i\neq j}^{N} r_i(k) r_j(k) G_N^{(k)}(z)_{N+i,N+j}\right|^2\right] \leqslant \frac{c}{N}.$$

Proof. Again we expand the square and we use the fact that, conditionally to the $(M^i)_i$, the quantities $r_i(k), r_j(k), G_N^{(k)}(z)_{N+i,N+j}$ are independent and $r_i(k), r_j(k)$ are centered. Indeed, conditionally to the $(M^i)_i$, the variables $r_i(k), r_j(k), G_N^{(k)}(z)_{N+i,N+j}$ involve different increments of the Brownian motion. Thus we have

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{i\neq j}^{N} r_i(k) r_j(k) G_N^{(k)}(z)_{N+i,N+j}\right|^2\right] &= \sum_{i\neq j}^{N} \mathbb{E}\left[r_i(k)^2 r_j(k)^2\right] \mathbb{E}\left[\left|G_N^{(k)}(z)_{N+i,N+j}\right|^2\right] \\
&\leqslant \sum_{i\neq j}^{N} \mathbb{E}[r_i(k)^2]\mathbb{E}[r_j(k)^2]\mathbb{E}\left[\left|G_N^{(k)}(z)_{N+i,N+j}\right|^2\right] \\
&= N^{-2} \sum_{i\neq j}^{N} \mathbb{E}\left[\left|G_N^{(k)}(z)_{N+i,N+j}\right|^2\right] \\
&\leqslant \frac{c}{N},
\end{aligned}$$

where we have used the fact that almost surely:

$$\frac{1}{2N-1} \sum_{i,j\neq k}^{2N} \left|G_N^{(k)}(z)_{i,j}\right|^2 \leqslant \frac{1}{|\Im(z)|^2}.$$

$\square$

Proof of Lemma 4.13. We define the function $f_N^{k,\epsilon}$ on the interval $[0,1]$ by

$$f_N^{k,\epsilon}(x) = N M^{k,\epsilon}(I_N^t) \text{ if } x \in I_N^t.$$

Notice the relation:

$$\sum_{t=1}^{N} M_\epsilon^k(I_N^t) \mathbb{E}\left[G_N(z)_{tt}\right] = \int_0^1 f_N^{k,\epsilon}(r)\, d\mathbb{E}[L_N^{1,z}](dr).$$

Then, by stationarity, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^1 f_N^{k,\epsilon}(r)\, d\mathbb{E}[L_N^{1,z}](dr) - \int_0^1 e^{\omega_\epsilon^k(r)}\, d\mathbb{E}[L_N^{1,z}](dr)\right|\right] \\
&\leqslant \sum_{t=1}^{N} \mathbb{E}\left[\left|\int_{I_N^t} (f_N^{k,\epsilon}(r) - e^{\omega_\epsilon^k(r)})\, d\mathbb{E}[L_N^{1,z}](dr)\right|\right] \\
&\leqslant \frac{N}{|\Im(z)|} \sup_{r\in I_N^1} \mathbb{E}\left[\left|\int_{I_N^1} (e^{\omega_\epsilon^k(u)} - e^{\omega_\epsilon^k(r)})\, du\right|\right] \\
&\leqslant \frac{N}{|\Im(z)|} \sup_{r\in I_N^1} \int_{I_N^1} \mathbb{E}\left[\left|e^{\omega_\epsilon^k(u)} - e^{\omega_\epsilon^k(r)}\right|^2\right]^{1/2} du \\
&\leqslant \frac{N}{|\Im(z)|} \sup_{r\in I_N^1} \int_{I_N^1} \left(2e^{\psi(2)\rho_\epsilon(0)} - 2e^{\psi(2)\rho_\epsilon(r-u)}\right)^{1/2} du.
\end{aligned}$$

Because of the continuity of the function ρ_ϵ over $[0,1]$, we have

$$\mathbb{E}\Big[\Big|\int_0^1 f_N^{k,\epsilon}(r)\, d\mathbb{E}[L_N^{1,z}](dr) - \int_0^1 e^{\omega_\epsilon^k(r)}\, d\mathbb{E}[L_N^{1,z}](dr)\Big|\Big] \to 0 \quad \text{as } N\to\infty. \tag{A.4}$$

In a quite similar way, we can prove that

$$\mathbb{E}\Big[\Big|\int_0^1 e^{\omega_\epsilon^k}*\phi_p(r)\, d\mathbb{E}[L_N^{1,z}](dr) - \int_0^1 e^{\omega_\epsilon^k(r)}\, d\mathbb{E}[L_N^{1,z}](dr)\Big|\Big] \to 0 \quad \text{as } p\to\infty \text{ uniformly w.r.t. N} \tag{A.5}$$

and

$$\mathbb{E}\Big[\Big|\int_0^1 e^{\omega_\epsilon^k}*\phi_p(r) K_z(r)\, dr - \int_0^1 e^{\omega_\epsilon^k(r)} K_z(r)\, dr\Big|\Big] \to 0 \quad \text{as } p\to\infty \text{ uniformly w.r.t. N} \tag{A.6}$$

where $(\phi_p)_{p\in\mathbb{N}}$ is a regularizing sequence and $*$ stands for the convolution. Furthermore, for each fixed p and because of the weak convergence of $\mathbb{E}[L_N^{1,z}]$ towards $K_z(x)dx$, we have almost surely

$$\int_0^1 e^{\omega_\epsilon^k} * \phi_p(r)\, d\mathbb{E}[L_N^{1,z}](dr) \to \int_0^1 e^{\omega_\epsilon^k} * \phi_p(r) K_z(r)\, dr \quad \text{as } N\to\infty. \tag{A.7}$$

We prove the result by gathering (A.4) (A.5) (A.6) and (A.7). □

B. Sup of MRW

Here we prove

Proposition B.1. *We have for all $k = 1,\dots,N+1$*

$$\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4\right] \leqslant C\frac{(\ln N)^2}{N^{\frac{\zeta(2\alpha)-1}{\alpha}}}.$$

for some positive constant C.

Proof. To prove the result, we first prove

Lemma B.2. *There exists a constant C such that, if $(X_i)_{1 \leqslant i \leqslant N}$ are iid centered Gaussian random variables then:*

$$\mathbb{E}\left[\max_{1 \leqslant i \leqslant N} |X_i|^4\right] \leqslant C \max_{1 \leqslant i \leqslant N} \mathbb{E}[X_i^2]^2 (\ln N)^2.$$

Proof. By homogeneity, it suffices to assume that $\mathbb{E}[X_i^2] = 1$. Then we have for all $\delta \geqslant 0$

$$\begin{aligned}
\mathbb{E}\left[\max_{1 \leqslant i \leqslant N} |X_i|^4\right] &\leqslant \delta + N \int_\delta^\infty \mathbb{P}(|X_1|^4 > t) dt \\
&\leqslant \delta + 2N \int_\delta^\infty \mathbb{P}(X_1 > t^{1/4}) dt \\
&\leqslant \delta + \frac{2N}{\sqrt{2\pi}} \int_\delta^\infty e^{-\sqrt{t}} dt \\
&\leqslant \delta + \frac{4N}{\sqrt{2\pi}} \int_{\sqrt{\delta}}^\infty e^{-t} t dt \\
&\leqslant \delta + \frac{4N}{\sqrt{2\pi}} \left(\sqrt{\delta} e^{-\sqrt{\delta}} + e^{-\sqrt{\delta}}\right),
\end{aligned}$$

and this last expression can be made smaller than $C(\ln N)^2$ by choosing $\delta = (\ln N)^2$. □

We want apply the above lemma after conditioning with respect to the law of the MRM M^k:

$$\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4\right] = \mathbb{E}\left[\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4 | M^k\right]\right].$$

Notice then that, conditionally to $M^k(0, \frac{1}{N}) = x_1, \dots, M^k(\frac{N-1}{N}, 1) = x_N$, the vector $(r_k(1), \dots, r_k(N)$ has the same law as the increments of B: $(B_{x_1} - B_0, \dots, B_{x_N} - B_{x_{N-1}})$. By applying Lemma B.2, we deduce that

$$\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4 | M^k\right] \leqslant C(\ln N)^2 \max_{t=1,\dots,N} M^k\big(\frac{t-1}{N}, \frac{t}{N}\big)^2.$$

Thus we deduce

$$\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4\right] \leqslant C(\ln N)^2 \mathbb{E}\left[\left(\max_{t=1,\dots,N} M^k\big(\frac{t-1}{N}, \frac{t}{N}\big)\right)^2\right]. \tag{B.1}$$

Finally we have for all $\delta > 0$ and for $\alpha > 1$ such that $\zeta(2\alpha) > 1$:

$$\begin{aligned}
\mathbb{E}\left[\left(\max_{t=1,\dots,N} M^k\big(\frac{t-1}{N}, \frac{t}{N}\big)\right)^2\right] &\leqslant \delta + N \int_\delta^\infty \mathbb{P}\big(M^k\big(\frac{t-1}{N}, \frac{t}{N}\big)^2 > x\big)\, dx \\
&\leqslant \delta + N \int_\delta^\infty \frac{1}{x^\alpha} \mathbb{E}\big[M^k\big(\frac{t-1}{N}, \frac{t}{N}\big)^{2\alpha}\big]\, dx \\
&\leqslant \delta + C\delta^{1-\alpha} N^{1-\zeta(2\alpha)}
\end{aligned}$$

for some constant C only depending on α, τ and γ^2. Choose now $\delta = N^{\frac{1-\zeta(2\alpha)}{\alpha}}$ so as to get

$$\mathbb{E}\left[\sup_{t=1,\dots,N} r_k(t)^4\right] \leqslant (1+C)\frac{(\ln N)^2}{N^{\frac{\zeta(2\alpha)-1}{\alpha}}} \tag{B.2}$$

□

C. Girsanov transform

Lemma C.1. *Let μ be an independently scattered infinitely divisible random measure associated to (ψ, θ), where*

$$\forall q \in \mathbb{R}, \quad \psi(q) = mq + \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}} (e^{qz} - 1)\nu(dz),$$

$\psi(2) < +\infty$ and $\psi(1) = 0$. Let B be a bounded Borelian set. We define a new probability measure $\mathbb{P}_B$ (with expectation $\mathbb{E}_B$) by:

$$\forall A measurable\ set, \quad \mathbb{P}_B(A) = \mathbb{E}[\mathbb{1}_A e^{\mu(B)}].$$

Then, under $\mathbb{P}_B$, μ has the same law as $\mu+\mu_B$ where μ_B is an independently scattered infinitely divisible random measures independent of μ and is associated to (ψ_B, θ_B) given by

$$\begin{aligned}
\psi_B(q) &= q\sigma^2 + \int_{\mathbb{R}} (e^{qx} - 1)(e^x - 1)\nu(dx) \\
\theta_B(\cdot) &= \theta(\cdot \cap B).
\end{aligned}$$

Proof. It suffices to compute the joint distribution of p disjoint sets $A_1, \dots, A_p$. We have for any $\lambda_1, \dots, \lambda_p \in \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}_B\left[e^{\lambda_1\mu(A_1)+\dots+\lambda_p\mu(A_p)}\right] &= \mathbb{E}\left[e^{\lambda_1\mu(A_1)+\dots+\lambda_p\mu(A_p)+\mu(B)}\right] \\
&= \mathbb{E}\left[e^{\lambda_1\mu(A_1\backslash B)+\dots+\lambda_p\mu(A_p\backslash B)+\lambda_1\mu(A_1\cap B)+\dots+\lambda_p\mu(A_p\cap B)+\mu(B)}\right] \\
&= \mathbb{E}\left[e^{\lambda_1\mu(A_1\backslash B)+\dots+\lambda_p\mu(A_p\backslash B)+(\lambda_1+1)\mu(A_1\cap B)+\dots+(\lambda_p+1)\mu(A_p\cap B)+\mu(B\backslash\bigcup_{i=1}^n A_i)}\right] \\
&= \mathbb{E}\left[e^{\lambda_1\mu(A_1\backslash B)+\dots+\lambda_p\mu(A_p\backslash B)}\right]\mathbb{E}\left[e^{(\lambda_1+1)\mu(A_1\cap B)+\dots+(\lambda_p+1)\mu(A_p\cap B)}\right] \\
&= e^{\psi(\lambda_1)\theta(A_1\backslash B)+\dots+\psi(\lambda_p)\theta(A_p\backslash B)} e^{\psi(\lambda_1+1)\theta(A_1\cap B)+\dots+\psi(\lambda_p+1)\theta(A_p\cap B)} \\
&= e^{\psi(\lambda_1)\theta(A_1)+\dots+\psi(\lambda_p)\theta(A_p)} e^{(\psi(\lambda_1+1)-\psi(\lambda_1))\theta(A_1\cap B)+\dots+(\psi(\lambda_p+1)-\psi(\lambda_p))\theta(A_p\cap B)}.
\end{aligned}$$

Then it suffices to notice that:

$$\psi(q+1) - \psi(q) = m + \sigma^2 q + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^{(q+1)z} - e^{qz})\nu(dz)$$

and $\psi(1) = 0$. □

Lemma C.2. *If the process ω_ϵ is defined as $\omega_\epsilon(x) = \mu(A_\epsilon(x))$ where μ is an independently scattered random measure associated to (φ, θ) with $\varphi(q) = -iq\gamma^2/2 - q^2\gamma^2/2$ and θ given by 2.3, then:*

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z - \int_0^1 K_z(r) e^{\omega_\epsilon(r)} dr}\right] = \mathbb{E}\left[\left(z - \int_0^1 \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} K_z(r) M(dr)\right)^{-1}\right]$$

where M is the lognormal MRM.

Proof. One can check that $(\omega_\epsilon(x))_{x \in [0;1]}$ is a stationary gaussian process with covariance given by $\gamma^2 \rho_\epsilon(x-y)$. So, using Girsanov transform, we can write:

$$\mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z - \int_0^1 K_z(r) e^{\omega_\epsilon(r)} dr}\right] = \mathbb{E}\left[\left(z - \int_0^1 K_z(r) e^{\gamma^2 \rho_\epsilon(r-x)} e^{\omega_\epsilon(r)} dr\right)^{-1}\right]$$

We are interested in the limit when ϵ goes to 0 of this latter term, we thus approximate it with a simpler term:

$$\begin{aligned}
&\Bigg| \mathbb{E}\left[\left(z - \int_0^1 K_z(r) e^{\gamma^2 \rho_\epsilon(r-x)} e^{\omega_\epsilon(r)}\, dr\right)^{-1}\right] \\
&- \mathbb{E}\left[\left(z - \int_0^1 K_z(r) \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} e^{\omega_\epsilon(r)}\, dr\right)^{-1}\right] \Bigg| \\
&\leqslant \frac{1}{|\Im(z)|^2} \mathbb{E}\left[\int_0^1 |K_z(r)| e^{\omega_\epsilon(r)} \left| e^{\gamma^2 \rho_\epsilon(r-x)} - \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} \right| dr\right] \\
&\leqslant \frac{1}{|\Im(z)|^3} \int_0^1 \left| e^{\gamma^2 \rho_\epsilon(r-x)} - \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} \right| dr
\end{aligned} \tag{C.1}$$

where we have used Lemmas 4.4 and 4.8 and the normalization $\psi(1) = 0$.

Because $\gamma^2 < 1$, the dominated convergence theorem implies that C.1 converges to 0 when ϵ goes to 0.

We thus look at the limit when ϵ goes to 0 of the term:

$$\mathbb{E}\left[\left(z - \int_0^1 K_z(r) \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} e^{\omega_\epsilon(r)}\, dr\right)^{-1}\right].$$

The random variable

$$\int_0^1 K_z(r) \left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} M(dr)$$

is well defined and is finite almost surely since:

$$\mathbb{E}\left[\left|\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} M(dr)\right|\right] \leqslant \int_0^1 |K_z(r)|\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} dr < +\infty.$$

And thus, we can compute:

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(z-\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} e^{\omega_\epsilon(r)}\, dr\right)^{-1}\right]\right. \\
&\left.-\,\mathbb{E}\left[\left(z-\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} M(dr)\right)^{-1}\right]\right| \\
&\leqslant \frac{1}{|\Im(z)|^2}\mathbb{E}\left[\left|\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2}\left(e^{\omega_\epsilon(r)}dr - M(dr)\right)\right|\right],
\end{aligned}$$

and, for all $n\in\mathbb{N}$, this latter term is smaller than

$$\mathbb{E}\left[\left|\int_0^1 K_z(r)\left[\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} - \min\left(\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2}, n\right)\right] e^{\omega_\epsilon(r)}dr\right|\right] \tag{C.2}$$

$$+\,\mathbb{E}\left[\left|\int_0^1 K_z(r)\min\left(\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2}, n\right)\left(e^{\omega_\epsilon(r)}dr - M(dr)\right)\right|\right] \tag{C.3}$$

$$+\,\mathbb{E}\left[\left|\int_0^1 K_z(r)\left[\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} - \min\left(\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2}, n\right)\right] M(dr)\right|\right]. \tag{C.4}$$

The two quantities C.2 and C.4 are smaller than

$$\int_0^1 |K_z(r)|\left[\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2} - \min\left(\left(\frac{\tau}{|r-x|}\right)_+^{\gamma^2}, n\right)\right] dr \tag{C.5}$$

and thus converge to 0, uniformly in ϵ as n goes to infinity.

For a fixed n, the function $\min((\tau/|r-x|)_+^{\gamma^2}, n)$ is measurable and bounded and thus it is plain to see that, for a fixed n, the term C.3 goes to 0 when ϵ goes to 0.

The lemma follows gathering the above estimates. $\square$

Lemma C.3. *If the process ω_ϵ is defined as $\omega_\epsilon(x)=\mu(A_\epsilon(x))$ where μ is an independently scattered random measure associated to (φ,θ) where φ is given by (2.1),i.e.*

$$\varphi(q) = imq - \frac{\gamma^2}{2}q^2 + \int_{\mathbb{R}}(e^{iqx}-1)\nu(dx)$$

and where θ given by (2.3), then:

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z-\int_0^1 K_z(r)e^{\omega_\epsilon(r)}dr}\right] = \mathbb{E}\left[\left(z-\int_0^1\left(\frac{\tau}{|r-x|}\right)_+^{\kappa} K_z(r)Q(dr)\right)^{-1}\right]$$

with $\kappa = \gamma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx)$ and where the random Radon measure Q is defined, conditionally on a MRM denoted by M whose structure exponent is $\zeta(q) := q - \varphi(-iq)$, as the almost sure weak limit as ϵ goes to 0 of the family of random measures $Q_\epsilon(dt) := e^{\overline{\omega}_\epsilon(t)} M(dt)$ where, for each $\epsilon > 0$, the random process $\overline{\omega}_\epsilon$ is independent of M and defined as $\overline{\omega}_\epsilon(t) = \overline{\mu}(A_\epsilon(t))$ where $\overline{\mu}$ is the independently scattered log infinitely divisible random measure associated to $(\bar{\varphi}, \theta(\cdot \cap A_0(x)))$ where

$$\bar{\varphi}(p) = ip(\gamma^2 - \kappa) + \int_{\mathbb{R}} (e^{ipx} - 1)(e^x - 1)\nu(dx). \tag{C.6}$$

Proof. We want to apply Lemma C.1 to the process ω_ϵ. If we set $B = A_\epsilon(x)$, Lemma C.1 tells us that, under $\mathbb{P}_B$, the process ω_ϵ possesses the same law as the process

$$\omega_\epsilon^{(1)}(r) + \omega_\epsilon^{(2)}(r) \quad \text{with } \omega_\epsilon^{(1)}(r) = \mu^{(1)}(A_\epsilon(r)) \text{ and } \omega_\epsilon^{(2)}(r) = \mu^{(2)}(A_\epsilon(r)),$$

where $\mu_\epsilon^{(1)}, \mu_\epsilon^{(2)}$ are independent independently scattered log infinitely divisible random measures respectively associated to (φ, θ) and $(\varphi^{(2)}, \theta^{(2)})$ with:

$$\varphi^{(2)}(q) = i\gamma^2 q + \int_{\mathbb{R}} (e^{iqx} - 1)(e^x - 1)\nu(dx) \text{ and } \theta^{(2)}(\cdot) = \theta(\cdot \cap A_\epsilon(x)). \tag{C.7}$$

Define:

$$\kappa = \gamma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx), \overline{\varphi}(q) = \varphi^{(2)}(q) - iq\kappa, \overline{\psi}(q) = \overline{\varphi}(-iq). \tag{C.8}$$

Notice that $\overline{\psi}$ is then normalized so as to make $\overline{\psi}(1) = \overline{\psi}(0) = 0$. Let us define the process $\overline{\omega}_\epsilon$ by:

$$\overline{\omega}_\epsilon(r) = \omega_\epsilon^{(2)}(r) - \kappa\theta(A_\epsilon(r) \cap A_\epsilon(x)) = \omega_\epsilon^{(2)}(r) - \kappa\rho_\epsilon(r - x), \tag{C.9}$$

and notice that $\mathbb{E}[e^{iq\overline{\omega}_\epsilon(r)}] = e^{\overline{\varphi}(q)\rho_\epsilon(r-x)}$.

We can now apply Lemma C.1:

$$\mathbb{E}\left[\frac{e^{\omega_\epsilon(x)}}{z - \int_0^1 K_z(r) e^{\omega_\epsilon(r)} dr}\right] = \mathbb{E}\left[\left(z - \int_0^1 K_z(r) e^{\overline{\omega}_\epsilon(r) + \kappa\rho_\epsilon(r-x) + \omega_\epsilon(r)} dr\right)^{-1}\right]$$

We are interested in the limit when ϵ goes to 0 of this latter term, we thus approximate it with a simpler term:

$$\begin{aligned}
&\Bigg|\mathbb{E}\left[\left(z - \int_0^1 e^{\overline{\omega}_\epsilon(r) + \omega_\epsilon(r) + \kappa\rho_\epsilon(r-x)} K_z(r)\, dr\right)^{-1}\right] \\
&- \mathbb{E}\left[\left(z - \int_0^1 e^{\overline{\omega}_\epsilon(r) + \omega_\epsilon(r)} \left(\frac{\tau}{|r - x|}\right)_+^\kappa K_z(r)\, dr\right)^{-1}\right]\Bigg| \\
&\leqslant \frac{1}{|\Im(z)|^2} \mathbb{E}\left[\int_0^1 e^{\overline{\omega}_\epsilon(r) + \omega_\epsilon(r)} \left| e^{\kappa\rho_\epsilon(r-x)} - \left(\frac{\tau}{|r - x|}\right)_+^\kappa \right| |K_z(r)|\, dr\right] \\
&\leqslant \frac{1}{|\Im(z)|^3} \int_0^1 \left| e^{\kappa\rho_\epsilon(r-x)} - \left(\frac{\tau}{|r - x|}\right)_+^\kappa \right| dr
\end{aligned} \tag{C.10}$$

where we have used Lemmas 4.4 and 4.8, the normalizations $\overline{\psi}(1) = 0, \psi(1) = 0$ and the independence between $\overline{\omega}_\epsilon$ and ω_ϵ.

Let us show that $\kappa < 1$. Indeed, we have:

$$\begin{aligned}
\kappa &= \gamma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \\
&= \gamma^2 + \int_{\mathbb{R}} (e^{2x} - 1)\nu(dx) - 2\int_{\mathbb{R}} (e^x - 1)\nu(dx) \\
&= \gamma^2 + \int_{\mathbb{R}} (e^{2x} - 1)\nu(dx) + 2(m + \frac{1}{2}\gamma^2) \\
&= 2m + 2\gamma^2 + \int_{\mathbb{R}} (e^{2x} - 1)\nu(dx) \\
&= \psi(2)
\end{aligned}$$

where, in the third line, we used the fact that $\psi(1) = 0$ (which implies the relation $\int_{\mathbb{R}} (e^x - 1)\nu(dx) = -(m + \gamma^2/2)$). We will now show that $\psi(2)$ is strictly less than 1. It suffices to show that $\zeta(2) > 1$. Using the concavity of the function ζ, we have the inequality:

$$\frac{\zeta(2+\epsilon) - \zeta(1)}{1+\epsilon} < \zeta(2) - \zeta(1) \tag{C.11}$$

and with assumption 3.7, we see that $\zeta(2) - \zeta(1) = \zeta(2) - 1 > 0$. We can thus conclude that $\kappa < 1$.

Because $\kappa < 1$, the dominated convergence theorem implies that C.10 converges to 0 when ϵ goes to 0.

For each Borelian set A of $[0;1]$, the family $M_\epsilon(A) := \int_A e^{\omega_\epsilon(r)} dr, \epsilon > 0$ is a positive martingale with respect to ϵ and that it converges almost surely to $M(A)$. With the assumption 3.7 and in particular the condition $\zeta(2+\epsilon) > 1$, we can show (see [1] for a proof) that the family $(M_\epsilon(A))_{\epsilon>0}$ is in fact uniformly integrable. In particular, if we let $\mathcal{F}_\epsilon$ be the sigma field generated by the family of random variables $(\omega_\eta(r))_{\eta>\epsilon, r\in\mathbb{R}}$, we have the following almost sure equality:

$$\mathbb{E}\left[M(A)|\mathcal{F}_\epsilon\right] = M_\epsilon(A). \tag{C.12}$$

Conditionally to the random measure M, the family $P_\epsilon(A) := \int_A e^{\overline{\omega}_\epsilon(r)} M(dr), \epsilon > 0$ is also a positive martingale with respect to ϵ. Thus, $P_\epsilon(A)$ converges almost surely to a random variable that we will denote by $P(A)$. We know that this defines a random Radon measure P on $[0;1]$ and that the family of random Radon measures P_ϵ converges, when ϵ goes to 0, weakly almost surely to P in the space of Radon measures. Denote, conditionally to the random measure M, by $\mathbb{P}_M$ the law $\mathbb{P}[\cdot|M]$ and let us show that the family $(P_\epsilon([0;1]))_{\epsilon>0}$ is $\mathbb{P}_M$-uniformly integrable. Let $\overline{\delta}$ be such that $\overline{\psi}(1+\overline{\delta}) < +\infty$ (we can show, using the condition $\psi(2+\delta) < +\infty$, that that there exists such $\overline{\delta}$). We will show that the family $(P_\epsilon([0;1]))_{\epsilon>0}$ is uniformly bounded in $L^{1+\overline{\delta}}(\mathbb{P}_M)$. Indeed, conditionally to the random measure M:

$$\begin{aligned}\mathbb{E}_M\left[\left(\int_0^1 e^{\overline{\omega}_\epsilon(r)}M(dr)\right)^{1+\overline{\delta}}\right] &\leqslant \mathbb{E}_M\left[\int_0^1 e^{(1+\overline{\delta})\overline{\omega}_\epsilon(r)}M(dr)\right]M[0;1]^{\overline{\delta}}\\ &\leqslant \int_0^1 e^{\overline{\psi}(1+\overline{\delta})\rho_\epsilon(r-x)}M(dr)M[0;1]^{\overline{\delta}}\\ &\leqslant M[0;1]^{\overline{\delta}}e^{\overline{\psi}(1+\overline{\delta})}\int_0^1\left(\frac{\tau}{|r-x|}\right)_+^\kappa M(dr)<+\infty.\end{aligned}$$

The family $(P_\epsilon([0;1]))_{\epsilon>0}$ is therefore $\mathbb{P}_M$-uniformly integrable, in particular, $P_\epsilon([0;1])$ converges to $P([0;1])$ also in L^1, which implies that P is a non degenerated random measure. Moreover, denoting by $\overline{\mathcal{F}}_\epsilon$ the sigma field generated by the family of random variables $(\overline{\omega}_\eta(r))_{\eta>\epsilon,r\in\mathbb{R}}$, we have, almost surely, conditionally to M, for all Borelian set A of $[0;1]$:

$$\mathbb{E}_M\left[P(A)|\mathcal{F}_\epsilon\right]=P_\epsilon(A).$$

Now, as before, it is easy to see that the family $Q_\epsilon(A):=\int_A e^{\omega_\epsilon(r)+\overline{\omega}_\epsilon(r)}dr$, $\epsilon>0$ is also a positive martingale with respect to ϵ. Therefore, $Q_\epsilon(A)$ converges almost surely to a random variable that we will denote by $Q(A)$. This defines a random Radon measure Q and the family of random Radon measure Q_ϵ converges, as $\epsilon\to 0$, weakly almost surely to Q in the space of Radon measure. We want to show that the two random measures P and Q have the same law.

Gathering the above arguments, we can write, almost surely:

$$\begin{aligned}\mathbb{E}\left[P(A)|\sigma(\mathcal{F}_\epsilon,\overline{\mathcal{F}}_\epsilon)\right] &= \mathbb{E}\left[\mathbb{E}[P(A)|\overline{\mathcal{F}}_\epsilon]\right]\\ &= \mathbb{E}\left[\int_A e^{\overline{\omega}_\epsilon(r)}M(dr)|\mathcal{F}_\epsilon\right]\\ &= \int_A e^{\omega_\epsilon(r)+\overline{\omega}_\epsilon(r)}dr,\end{aligned}$$

and the latter quantity has the same law as $Q_\epsilon(A)$. Since the martingale $(\mathbb{E}[P(A)|\sigma(\mathcal{F}_\epsilon,\overline{\mathcal{F}}_\epsilon)])_{\epsilon>0}$ is uniformly integrable, we deduce that the family $(Q_\epsilon(A))_{\epsilon>0}$ is also uniformly integrable. Hence, both random variables $P(A)$ and $Q(A)$ have the same law. We can show easily that in fact the two random measures P and Q have the same law. In particular, Q is non degenerated.

It is now easy to see that, for all bounded and continuous function f, the two random variables $\int_\mathbb{R} f(r)P(dr)$ and $\int_\mathbb{R} f(r)Q(dr)$ have the same law. By regularizing the function $\left(\frac{\tau}{|r-x|}\right)_+^\kappa$ and with the dominated convergence theorem, we conclude as in the proof of lemma C.2 using the fact that $\kappa<1$ that:

$$\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^\kappa Q(dr)\overset{(law)}{=}\int_0^1 K_z(r)\left(\frac{\tau}{|r-x|}\right)_+^\kappa P(dr). \tag{C.13}$$

Gathering the above argument and letting ϵ go to 0 concludes the proof. □

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