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The inverted pendulum, interface phonons and optic Tamm states

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The propagation of waves in periodic media is related to the parametric oscillators. We transpose the possibility that a parametric pendulum oscillates in the vicinity of its unstable equilibrium positions to the case of waves in lossless unidimensional periodic media. This concept formally applies to any kind of wave. We apply and develop it to the case of phonons in realizable structures and evidence new classes of phonons. Discussing the case of electromagnetic waves, we show that our concept is related to optic Tamm states one but extends it to periodic Optic Tamm state.

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Waves in periodic media have been the focus of interest of many scientists and have found a multitude of applications in different branches of physics: electromagnetic (EM) waves in photonic crystals [1], phonon (elastic wave) in crystalline solids or in phononic crystal [2], electron wave functions in crystalline solids [3] or electronic super-lattice [4]. The periodicity of the media yields the existence of band gaps (BG) in which the amplitude of the wave exponentially varies, hence corresponding to unphysical states in infinite media. Outside of these gaps, waves are extended and periodic modes. In the presence of localized defects, localized modes, including surface or interface modes have been evidenced.

In a recent paper [5], we have exploited the equivalence between the physic of the parametric oscillator and that of the propagations of waves in a lossless unidimensional infinite periodic (LUIP) medium. In this manuscript, we reveal the potentiality to transpose the striking possibility for an oscillator to oscillate in the vicinity of an unstable equilibrium position using a parametric excitation (in the inverted pendulum experiment, for instance) to the case of waves in LUIP media. We evidence unexpected types of waves. This idea is totally general and can formally apply to any kind of waves.

Before exposing our concept and applying it, we first explicit the equivalence between the parametric oscillator and the propagations of waves in LUIP media.

Main features of the parametric oscillator can be derived from the fixed point stability of a parametrically and sinusoidally excited pendulum (for instance, in the case of Fig. 1a) governed by the Mathieu equation:

$$\frac{d^2\phi}{d\tilde{t}^2} + [\eta_0 + 2\alpha \cos(2\tilde{t})]\phi = 0 \quad (1)$$

with $\tilde{t} = \omega_e t/2$, $\eta_0 = \pm 4\omega_0^2/\omega_e^2$ and $\alpha = -2z_0/l$, $\omega_0^2 = g/l$. where ϕ , l , g , ω_e and z_0 are respectively the angle and length of the pendulum, the standard gravity and the excitation frequency and amplitude. η_0 is positive around the fixed point $\vec{\Phi} = (\phi, \frac{d\phi}{dt}) = (0, 0)$ and negative around $\vec{\Phi} = (\pi, 0)$. Depending on the parame-

ters η_0 and α , solutions of Eq. (1) are either periodic or unbounded. Unbounded solutions are oscillating with an exponentially varying amplitude and correspond to the parametric resonance of the oscillator. Fig. 1b reports the phase diagram of this equation: using the Floquet theory, it is deduced from the eigenvalues of \mathbf{R}_0^π with $\mathbf{R}_{\tilde{t}_0}^{\tilde{t}}$ the resolvente (propagator) of Eq. (1): $\vec{\Phi}(\tilde{t}) = \mathbf{R}_{\tilde{t}_0}^{\tilde{t}} \vec{\Phi}(\tilde{t}_0)$. A remarkable feature of the parametric oscillator is the

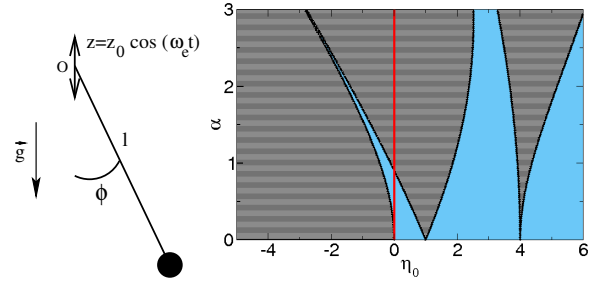


FIG. 1. a) Sketch of a parametrically and sinusoidally excited pendulum b) Phase diagram of Eq. (1). Solutions are respectively periodic and unbounded in blue and dashed regions. The vertical red line $\eta_0 = 0$ is a guide to the eyes.

possibility to find periodic solutions in the vicinity of the unstable fixed point $\vec{\Phi} = (\pi, 0)$ of the pendulum. Fig. 1b exhibits in the phase space $\eta_0 < 0$ periodic solutions: a striking example is the inverted pendulum experiment [6].

Considering now the propagation of waves in LUIP and focusing on two examples, EM and elastic waves, we reveal the equivalence between the parametric oscillator and the propagation of waves in a LUIP medium.

The time Fourier transform $\vec{E}(z, \omega)$ of the electric field $\vec{\mathcal{E}}(z, t) = \vec{E}(z, \omega)e^{i\omega t}$ of an EM wave in a LUIP medium with normal incidence is solution of:

$$\frac{d^2\vec{E}}{dz^2}(z, \omega) + p(z)\vec{E}(z, \omega) = \vec{0} \quad (2)$$

Where $p(z) = \frac{\epsilon(z)\omega^2}{c^2}$ and $\epsilon(z)$ and c are the relative dielectric permittivity and the light speed in vacuum.

For a longitudinal elastic wave with normal incidence, the time Fourier transform of the projected wave equation of the phonon displacement field $\vec{U}(z, t) = \mathfrak{U}(z, \omega)e^{i\omega t}\vec{z}$ writes using the linear elasticity theory:

$$C(z)\frac{d^2\mathfrak{U}}{dz^2}(z, \omega) + \frac{dC}{dz}(z)\mathfrak{U}_z(z, \omega) + \rho(z)\omega^2\mathfrak{U}(z, \omega) = 0 \quad (3)$$

where C and $\rho(z)$ are the elastic coefficient c_{zzzz} and the volumic mass. Eq. (3) does not seem equivalent to Eq. (2) though as suggested both equations describe the same physic. A mathematical transformation clarifies this point: setting $\mathfrak{U}(z, \omega) = Q(z, \omega)u(z, \omega)$ with $Q(z, \omega)$ satisfying $2C\frac{dQ}{dz} + \frac{dC}{dz}Q = 0$, $u(z, \omega)$ is solution of:

$$\frac{d^2u}{dz^2}(z, \omega) + p(z)u(z, \omega) = 0 \quad (4)$$

Where $p(z) = \frac{\rho\omega^2}{C} + \frac{1}{2}\frac{d^2C}{dz^2} - \frac{1}{4C^2}[\frac{dC}{dz}]^2$ is a generally positive quantity. Eq. (2) and (4) are actually equivalent. In a LUIP medium, the function $p(z)$ is real and periodic (period λ) and Eq. (2)(Eq. (4)) is a Hill equation, the equation of a parametric oscillator. Considering a sinusoidal variation of $p(z)$ or limiting the Fourier series of $p(z) \approx p_0 + p_1 \cos(\frac{2\pi z}{\lambda})$, we recover the Mathieu equation

$$\frac{d^2\phi}{d\tilde{z}^2} + [\eta_0 + 2\alpha \cos(2\tilde{z})]\phi = 0 \quad (5)$$

Where $\eta_0 = \frac{p_0\lambda^2}{\pi^2}$, $\alpha = \frac{p_1\lambda^2}{2\pi^2}$, $\tilde{z} = \frac{\pi z}{\lambda}$, and ϕ designs either $\vec{E}(z, \omega)$ from Eq. (2) or $u(z, \omega)$ from Eq. (4).

Eq. (1) for the parametric oscillator and (5) for waves in LUIP media are equivalent: they have thus the same solutions [7]. Periodic solutions of Eq. (5) correspond to extended waves whereas unbounded ones, to waves in the BGs of the LUIP medium.

In this study, we address the following questions: Do the periodic solutions in the phase space $\eta_0 < 0$ of Fig. (1)b exist for waves in real, eventually advanced materials? if yes to what kind of waves do they correspond?

In Eq. (2), in a lossless material, $p(z)$ is either positive (noticeably in dielectric media) or negative (noticeably in metals below their plasma frequency). The propagation of EM waves in a periodic metallic layered system below the metals plasma frequencies seems a straightforward transposition of the inverted pendulum cases: the negative values of the relative dielectric permittivity warrants a negative value of η_0 in Eq. (5). And indeed, calculating the dispersion diagram of a theoretical lossless periodic metallic layered system, we have evidenced some periodic EM waves that propagate below the metals plasma frequencies. However, such study is only theoretical: in real metals, $p(z)$ in Eq. (2) is a complex quantity whose imaginary part is related to the absorption. A parametric excitation cannot affect the dissipation. Investigating bi-metallic structures based on real common pure metals [8], the absorption dominates and screens the effect we wish to evidence.

To overcome the absorption drawback of metals, we will investigate in the following weakly absorbing materials i.e. dielectric materials and neglect their absorption. The dielectric permittivity in Eq. (2) is thus real and positive i.e. a case equivalent to the propagation of the elastic waves: $p(z) > 0 \forall z$ in Eq. (4).

Waves in the BGs are oscillating with an exponentially varying amplitude: this amplitude hence varies as the angle of a free pendulum closed to its unstable equilibrium position. The amplitude of the EM waves or elastic waves in BGs behaves similarly to waves in a negative dielectric permittivity medium or in a hypothetical imaginary sound speed medium i.e. mimics a negative effective η_0 in Eq. 5 with ϕ being the amplitude of the wave.

To model the inverted pendulum stabilization mechanism, we propose to use a medium displaying two periodicities: a first one that would create a BG (in the absence of the second periodicity) and mimic a negative effective η_0 , and a second one that parametrically creates some periodic solutions inside this BG.

Eq. (5) is formally replaced by the following modified Mathieu equation involving two parametric excitations:

$$\frac{d\phi}{d\tilde{z}^2} + (\eta_0 + 2\alpha \cos(2\tilde{z}) + \beta \cos(k_e\tilde{z} + \varphi))\phi = 0 \quad (6)$$

Where η_0 is positive (since we assume $p(z) > 0$). $\cos(k_e\tilde{z})$ is the periodicity supposed to create some periodic solutions inside the BG induced by $\cos(2\tilde{z})$ (with $\beta = 0$). Eq. (6) has been studied in the general case [9, 10]: actually, for some set of parameters η_0, α and $\beta \neq 0$, Eq. (6) has periodic solutions that would be unbounded if $\beta = 0$.

However, rather than discussing on the solutions of this theoretical equation, we propose to apply the previous concept to the propagation of phonons in an experimentally realizable structure. Such structure needs to involve two periodicities: we propose to use a Super-SuperLattice (SSL) (see Fig. 2a): a L -periodic structure whose elementary unit cell is composed of two superlattices (SL) SL1 and SL2: respectively, $10 + x$ periods of a 5.65 nm (10 monolayers(ML))/2.26 nm (4ML) GaAs/AlAs SL (period L_1) with $x = 0.5$ and 10 periods of a 11.3 nm (20ML)/4.52 nm (8ML) GaAs/AlAs SL (period L_2) [11]. Considering only wave vectors with normal incidence, Fig. 2b report the dispersion diagrams of the SSL, SL1 and SL2 and, Fig. 2c a zoom in the frequency range 0.26 THz-0.34 THz, corresponding to a BG region of both SL1 and SL2. These dispersion diagrams are calculated from the transfer matrix [12], a representation of the resolvente of the wave equations in these systems. The x-axis of Fig. 2b and c reports the Block wave vector normalized by $k_f = \frac{\pi}{L}$. In Fig. 2b, the multiple foldings and the mini-gaps created by the periodicity L of the SSL appear. More interestingly, the SSL phase diagram shows some phonons (blue point curves in Fig. 2 c) referred in the following as **interface acoustic phonons** in the overlapping region of SL1 and SL2

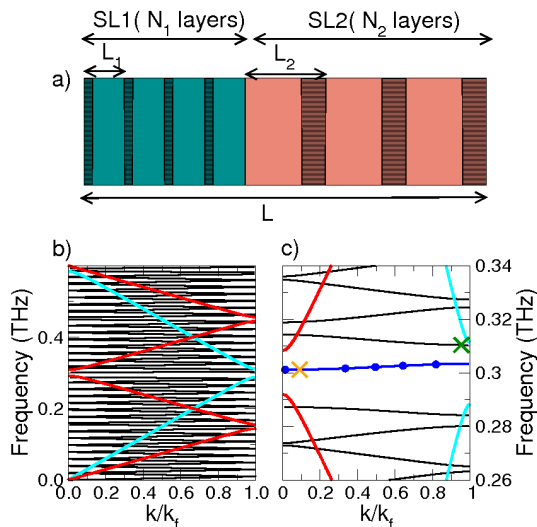


FIG. 2. a) Sketch of the elementary cell of a SSL. The background colors are respectively cyan and red in the SL1 and SL2 regions, dashed regions underline the stratified structure in each SL. b) and c) Dispersion diagram of the SSL (black), SL1 (cyan) and SL2 (red) between 0-0.6 THz and 0.26-0.34 THz. In c), the blue point curve identifies the acoustic interface modes band in the SSL. The orange and green crosses mention the mode of the SSL at $\nu = 0.3012$ THz plotted in Fig. 3 and at $\nu = 0.3104$ THz.

BGs. Fig. 3 reports the displacement fields $u(z)$, solutions of the wave equation in the SSL for $\nu = 0.3012$ THz, an interface acoustic phonons identified by an orange cross in Fig. 2c). Though in the BGs of both SL1 and SL2, this solution describes a periodic mode that can thus propagate in an infinite medium. It is formed by an oscillating function whose envelop is alternatively exponentially increasing and decreasing. The Fourier analysis of this mode evidences three characteristic lengths: a short wavelength (roughly $2L_1 = L_2$) oscillation exhibited in Fig. 3b, an intermediate length $\approx L$, related to the exponential variation of the wave amplitude and, a long $\gg L$ characteristic length related to the Bloch wave vector of this mode. An orange curve emphasizes this latter oscillation in Fig. 3a. Its wavelength corresponds to the Bloch wave vector of this mode reported in Fig. 2c.

These interface acoustic modes precisely correspond to some periodic solutions of the parametric oscillator in the phase space $\eta_0 < 0$ in Fig. 1b).

We believe that the physical mechanism described here also leads to the existence of interface optical phonon [13] in SLs: the atomic potential induces a fast periodicity (term $\cos(2\tilde{z})$ in Eq. (6)) responsible of the BG between acoustic and optic phonons, and the SL periodicity plays the role of a parametric stabilizing excitation (term $\cos(k_e\tilde{z})$ in Eq. (6)). The common description of interface optical phonon involves a frequency dependent dielectric constant that accounts for the fast periodicity [14].

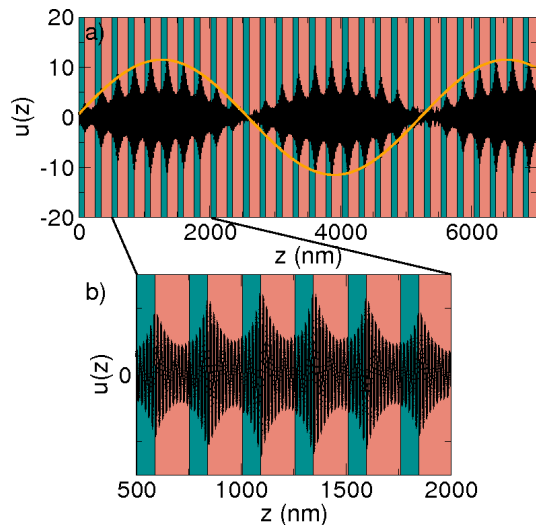


FIG. 3. Displacement fields $u(z)$ solutions of the wave equation in the SSL for $\nu = 0.3012$ THz between 0-7000nm a) and 500-2000 nm b). The background colors are respectively cyan and red in the SL1 and SL2 regions. An orange curve, a guide to the eye in Fig.a emphasizes the oscillations at the Bloch wave vector.

Besides the interface acoustic modes and as in the case of optic modes, we have also evidenced from Fig. 2b some **confined acoustic phonons**. Their frequency belongs to only one of the BG of SL1 or SL2: for instance the mode at $\nu = 0.3104$ THz identified by a green cross in Fig. 2b, in the BG of SL1 but outside the BG of SL2. These confined acoustic phonons correspond to some periodic solutions of the parametric oscillator either in the phase space $\eta_0 < 0$ or $\eta_0 > 0$ in Fig. 1b). A detailed discussion will be reported elsewhere.

Beyond the existence of the interface acoustic modes, we now show that it is possible to fully control their frequencies from the engineering of the SSL, and more precisely by using fractional number of the period of SL1 or SL2. We consider a SSL formed by $10 + x$ periods of SL1 and 10 periods of SL2 and vary x from 0 to 1. Fig.4 shows a band frequency (blue point curve in Fig. 4) that continuously shifts from above to below the overlapping region of SL1 and SL2 BGs. Hence, by judiciously choosing the value of x , some periodic SSL modes in the gaps of SL1 and/or SL2 at arbitrarily frequencies can be produced.

As already mentioned the concept described here is totally general and can formally apply to any kind wave. We now discuss this possibility for EM waves. The propagation of an EM wave in a structure approaching the one of a SSL has been considered and is related to the existence of optic Tamm state(OTS). Kavokin et al. [15] have considered the propagation of EM waves across two different semi-infinite Photonic Crystals(PC) PC1 and PC2 with a common interface and with overlapping stop bands (or band gap). By assuring the continuity of the

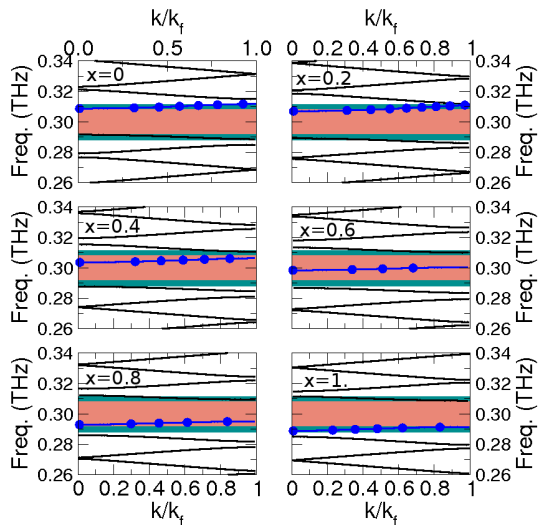


FIG. 4. Dispersion diagram of the SSL between 0.26-0.34 THz as a function of x . The BGs of SL1 and SL2 are represented by the cyan+red and red regions.

in-plane components of the electric and magnetic fields at the interface between the two PCs, they evidence some modes, called OTS in the overlapping region of both PC1 and PC2 stop bands. Experimental evidences of OTS have been given recently (for instance [16]).

Similarly to the work performed for phonons, we have calculated the phase diagram of a SuperPhotonic Crystal (SPC) whose unit cell is composed by $(5+0.4)$ periods of a PC Ta_2O_5/SiO_2 101.6 nm/149.2 nm and 5 periods of a PC $Bi:YIG/SiO_2$ 86 nm/138 nm [16] [17]. We have exhibited a band of periodic modes of energy $E \simeq 1.5eV$ in the overlapping region of BG of both PCs. The corresponding modes, referred in the following as periodic OTS are similar to the interface acoustic modes depicted in Fig. 3. Moreover, as expected, the transmission spectra of a finite SPC (with several unit cells) exhibits transmission peaks corresponding to these modes.

The concept described in this paper not only relates OTS to the physic of the parametric oscillator, but also extends the idea of Kavokin et al. showing the possibility to simultaneously assure the continuity conditions at all interfaces of an infinite succession of different finite PCs.

As a conclusion, we would like to underline the intrinsic relation between the propagation of wave in LUIP media and the physics of the parametric oscillator. Especially, we have shown how we can transpose the possibility to stabilize an inverse parametric pendulum to the case of waves. This concept is very general and can formally apply to any kind of waves: while it is well known that the periodicity induces some forbidden band frequency, a periodicity can also stabilize solutions that were expected to be unbounded. We have chosen here to develop this concept to the case of elastic waves evidencing some new

classes of phonons in realizable structures. The case of EM waves has been discussed: we have evidenced the relation between the inverse parametric pendulum and the OTS and have extended their existence to periodic OTS. This work opens numerous possibilities of applications of this concept. Due to the high localized amplitude of interface type modes, exhibited devices (SSL or SPC) are expected to be useful in the investigation of non-linear effects or in the improvements of electron-phonon or electron-photon couplings. Moreover, they can also be used to elaborate advanced interferential filters.

Finally, numerous perspectives to this work can be thought about: one can consider to apply the present concept to any kind of waves: electronic, spin, capillary waves. . . While our study has focused on normal incident waves in lossless materials, it needs to be extended to oblique wave incidence and to investigate the effect of absorption. Though for simplicity we have considered an unidimensional system, the generalization of the described concept to 2 and 3-dimensional devices might be considered.

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