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UNIQUENESS OF TRACES ON LOG-POLYHOMOGENEOUS PDOs

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(June 28, 2011)

Dedicated to Alan Carey, on the occasion of his 60th birthday

Abstract

We show how to derive the uniqueness of graded or ordinary traces on some algebras of log-polyhomogeneous pseudodifferential operators (PDOs) from the uniqueness of their restriction to classical pseudodifferential ones.

Keywords and phrases: log-polyhomogeneous pseudodifferential operators, traces.

1. Introduction

We consider a closed connected riemannian manifold M of finite dimension n and a finite rank hermitian vector bundle E over M. A pseudodifferential operator acting on smooth sections of E is called classical (or polyhomogeneous) ([12]) if locally its symbol is classical, i.e. it admits an asymptotic expansion in positively homogeneous components.
A pseudodifferential operator L acting on smooth sections of E is called log-polyhomogeneous if locally its symbol has the form

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$$a_k(x,\xi)\log^k|\xi| + a_{k-1}(x,\xi)\log^{k-1}|\xi| + \cdots + a_0(x,\xi), \tag{1}$$

where $k \in \mathbb{N}$ and $a_0, \cdots, a_k$ are classical symbols. We call the integer k the *log-degree* of L. We denote by

- $\mathcal{L}$ the algebra of log-polyhomogeneous PDOs acting on smooth sections of E
- $C\ell$ the subalgebra of classical PDOs of $\mathcal{L}$
- Q an admissible classical PDO of positive order such that $\mathrm{Log}Q$ exists
- $\mathcal{A}_{C\ell}$ a subalgebra of $C\ell$ such that $[\mathcal{A}_{C\ell},\ \mathrm{Log}Q] \subset \mathcal{A}_{C\ell}$ (in general we only have that the commutator of a classical PDO and a logarithm is a classical PDO, hence we only have $[\mathcal{A}_{C\ell}, \mathrm{Log}Q] \subset C\ell$).
- $\mathcal{A}$ the subalgebra of $\mathcal{L}$ generated by $\mathrm{Log}Q$ and $\mathcal{A}_{C\ell}$.

The assumption on $\mathcal{A}_{C\ell}$ implies the following fundamental decomposition of $\mathcal{A}$ (see Lemma **??** Paragraph 2)

$$\mathcal{A} = \bigoplus_{k=0}^{+\infty} \mathcal{A}_{C\ell}\ \mathrm{Log}^k Q. \tag{2}$$

We assume that $\mathcal{A}_{C\ell}$ does not only contain smoothing operators. Otherwise, $\mathcal{A}_{C\ell}$ and $\mathcal{A}$ are algebras of smoothing operators and V. Guillemin ([4]) has shown that the L^2 trace is the unique trace on such algebras.

On an algebra, we say that a linear form τ is a trace if for any operators A and B in the algebra, we have $\tau(AB) = \tau(BA)$ i.e. τ vanishes on commutators. On a graded algebra $\mathcal{B} = \bigoplus_{k\geq 0} \mathcal{B}^k$, a graded trace is a sequence $(\tau_k)_{k\in\mathbb{N}}$ of linear forms τ_k on $\bigoplus_{0\leq l\leq k} \mathcal{B}^l$ which vanishes on $\bigoplus_{0\leq l\leq k-1} \mathcal{B}^l$ and such that if A is in $\bigoplus_{0\leq l\leq k} \mathcal{B}^l$ and if B is in $\bigoplus_{0\leq l\leq m} \mathcal{B}^l$ then $\tau_{k+m}(AB) = \tau_{k+m}(BA)$.

Under the assumption of the uniqueness of a trace τ_0 on $\mathcal{A}_{C\ell}$ we show that there exists a unique graded trace $(\overline{\tau_0}^k)_{k\in\mathbb{N}}$ on the whole algebra $\mathcal{A}$ extending τ_0 (see Theorem **??**

Paragraph 3). We also prove that if there exists a trace on $\mathcal{A}$ extending τ_0 then this extension is unique (see Theorem **??** Paragraph 3).

Our first result applies to the Wodzicki-Guillemin residue (also called the non commutative residue) res on the algebra $C\ell$ provided M is of dimension $n \geq 2$. It is well known that the Wodzicki-Guillemin residue res is the unique trace on $C\ell$. Setting $\mathrm{res}_0 = \mathrm{res}$, the extension $\overline{\mathrm{res}_0}^k$ coincides with the higher non commutative residue res_k introduced by M. Lesch up to a multiplicative factor.

The k-th residue of an operator L in $\mathcal{L}$ of log-degree k with local symbol $\sigma(L) = a_k(x,\xi)\log^k|\xi| + a_{k-1}(x,\xi)\log^{k-1}|\xi| + \cdots + a_0(x,\xi)$ is defined as

$$\mathrm{res}_k(L) = (k+1)! \int_M \int_{S^*M} tr((a_k)_{-n}(x,\xi))d\xi dx.$$

On $\mathcal{L}$ seen as a graded algebra (the grading is given by log-degrees), M. Lesch ([7]) has shown that the sequence $(\mathrm{res}_k)_{k\in\mathbb{N}}$ is the unique graded trace. We recover the same result by an alternative approach.

Our second result applies to the canonical trace on the algebra of odd-class log-polyhomogeneous PDOs when the manifold M is odd dimensional.

According to M. Kontsevitch and S. Vishik [5], a classical operator A of order $m \in \mathbb{Z}$ is of *odd-class* if locally the positively homogeneous components of its symbol $\{a_{m-j} : j \in \mathbb{Z}\}$ are simply homogeneous, i.e. they have the property

$$a_{m-j}(x,-\xi) = (-1)^{m-j} a_{m-j}(x,\xi). \tag{3}$$

The odd-class classical PDOs form an algebra. Following [10] we say that a log-polyhomogeneous PDO L is of odd-class if locally, all the classical symbols $a_0, \cdots, a_k$ arising as coefficients of powers of $\log|\xi|$ in its symbol (see (1)) have the above property (3). Similarly to odd-class classical PDOs, one can easely check that odd-class log-polyhomogeneous PDOs form an algebra.

When M is odd dimensional, the canonical trace TR of [5] is well defined on the algebra of odd-class classical PDOs. Recently, L. Maniccia, E. Schrohe and J. Seiler ([8]) have proved that TR is the unique trace on this algebra. The canonical trace has been first extended

by M. Lesch ([7]) to log-polyhomogeneous PDOs of non integer orders (this means that the orders of $a_0, \cdots, a_k$ are not integers) and then by S. Paycha and S. Scott to odd-class log-polyhomogeneous PDOs when M is odd dimensional ([10]). We refer the reader to Paragraph 4 for more details. To our knowledge, the proof of the uniqueness of TR on odd-class log-polyhomogeneous PDOs is new.

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