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# UNIQUENESS OF TRACES ON LOG-POLYHOMOGENEOUS PDOs

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(June 28, 2011)

*Dedicated to Alan Carey, on the occasion of his 60<sup>th</sup> birthday*

## Abstract

We show how to derive the uniqueness of graded or ordinary traces on some algebras of log-polyhomogeneous pseudodifferential operators (PDOs) from the uniqueness of their restriction to classical pseudodifferential ones.

*Keywords and phrases:* log-polyhomogeneous pseudodifferential operators, traces.

## 1. Introduction

We consider a closed connected riemannian manifold  $M$  of finite dimension  $n$  and a finite rank hermitian vector bundle  $E$  over  $M$ . A pseudodifferential operator acting on smooth sections of  $E$  is called classical (or polyhomogeneous) ([12]) if locally its symbol is classical, i.e. it admits an asymptotic expansion in positively homogeneous components.

A pseudodifferential operator  $L$  acting on smooth sections of  $E$  is called log-polyhomogeneous if locally its symbol has the form

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$$a_k(x, \xi) \log^k |\xi| + a_{k-1}(x, \xi) \log^{k-1} |\xi| + \cdots + a_0(x, \xi), \quad (1)$$

where  $k \in \mathbb{N}$  and  $a_0, \dots, a_k$  are classical symbols. We call the integer  $k$  the *log-degree* of  $L$ . We denote by

- $\mathcal{L}$  the algebra of log-polyhomogeneous PDOs acting on smooth sections of  $E$
- $\mathcal{C}\ell$  the subalgebra of classical PDOs of  $\mathcal{L}$
- $Q$  an admissible classical PDO of positive order such that  $\text{Log}Q$  exists
- $\mathcal{A}_{\mathcal{C}\ell}$  a subalgebra of  $\mathcal{C}\ell$  such that  $[\mathcal{A}_{\mathcal{C}\ell}, \text{Log}Q] \subset \mathcal{A}_{\mathcal{C}\ell}$  ( in general we only have that the commutator of a classical PDO and a logarithm is a classical PDO, hence we only have  $[\mathcal{A}_{\mathcal{C}\ell}, \text{Log}Q] \subset \mathcal{C}\ell$ ).
- $\mathcal{A}$  the subalgebra of  $\mathcal{L}$  generated by  $\text{Log}Q$  and  $\mathcal{A}_{\mathcal{C}\ell}$ .

The assumption on  $\mathcal{A}_{\mathcal{C}\ell}$  implies the following fundamental decomposition of  $\mathcal{A}$  (see Lemma ?? Paragraph 2)

$$\mathcal{A} = \bigoplus_{k=0}^{+\infty} \mathcal{A}_{\mathcal{C}\ell} \text{Log}^k Q. \quad (2)$$

We assume that  $\mathcal{A}_{\mathcal{C}\ell}$  does not only contain smoothing operators. Otherwise,  $\mathcal{A}_{\mathcal{C}\ell}$  and  $\mathcal{A}$  are algebras of smoothing operators and V. Guillemin ([4]) has shown that the  $L^2$  trace is the unique trace on such algebras.

On an algebra, we say that a linear form  $\tau$  is a trace if for any operators  $A$  and  $B$  in the algebra, we have  $\tau(AB) = \tau(BA)$  i.e.  $\tau$  vanishes on commutators. On a graded algebra  $\mathcal{B} = \bigoplus_{k \geq 0} \mathcal{B}^k$ , a graded trace is a sequence  $(\tau_k)_{k \in \mathbb{N}}$  of linear forms  $\tau_k$  on  $\bigoplus_{0 \leq l \leq k} \mathcal{B}^l$  which vanishes on  $\bigoplus_{0 \leq l \leq k-1} \mathcal{B}^l$  and such that if  $A$  is in  $\bigoplus_{0 \leq l \leq k} \mathcal{B}^l$  and if  $B$  is in  $\bigoplus_{0 \leq l \leq m} \mathcal{B}^l$  then  $\tau_{k+m}(AB) = \tau_{k+m}(BA)$ .

Under the assumption of the uniqueness of a trace  $\tau_0$  on  $\mathcal{A}_{\mathcal{C}\ell}$  we show that there exists a unique graded trace  $(\overline{\tau_0}^k)_{k \in \mathbb{N}}$  on the whole algebra  $\mathcal{A}$  extending  $\tau_0$  (see Theorem ??

Paragraph 3). We also prove that if there exists a trace on  $\mathcal{A}$  extending  $\tau_0$  then this extension is unique (see Theorem ?? Paragraph 3).

Our first result applies to the Wodzicki-Guillemin residue (also called the non commutative residue)  $\text{res}$  on the algebra  $C\ell$  provided  $M$  is of dimension  $n \geq 2$ . It is well known that the Wodzicki-Guillemin residue  $\text{res}$  is the unique trace on  $C\ell$ . Setting  $\text{res}_0 = \text{res}$ , the extension  $\overline{\text{res}_0}^k$  coincides with the higher non commutative residue  $\text{res}_k$  introduced by M. Lesch up to a multiplicative factor.

The  $k$ -th residue of an operator  $L$  in  $\mathcal{L}$  of log-degree  $k$  with local symbol  $\sigma(L) = a_k(x, \xi) \log^k |\xi| + a_{k-1}(x, \xi) \log^{k-1} |\xi| + \dots + a_0(x, \xi)$  is defined as

$$\text{res}_k(L) = (k+1)! \int_M \int_{S^*M} \text{tr}((a_k)_{-n}(x, \xi)) d\xi dx.$$

On  $\mathcal{L}$  seen as a graded algebra (the grading is given by log-degrees), M. Lesch ([7]) has shown that the sequence  $(\text{res}_k)_{k \in \mathbb{N}}$  is the unique graded trace. We recover the same result by an alternative approach.

Our second result applies to the canonical trace on the algebra of odd-class log-polyhomogeneous PDOs when the manifold  $M$  is odd dimensional.

According to M. Kontsevitch and S. Vishik [5], a classical operator  $A$  of order  $m \in \mathbb{Z}$  is of *odd-class* if locally the positively homogeneous components of its symbol  $\{a_{m-j} : j \in \mathbb{Z}\}$  are simply homogeneous, i.e. they have the property

$$a_{m-j}(x, -\xi) = (-1)^{m-j} a_{m-j}(x, \xi). \quad (3)$$

The odd-class classical PDOs form an algebra. Following [10] we say that a log-polyhomogeneous PDO  $L$  is of odd-class if locally, all the classical symbols  $a_0, \dots, a_k$  arising as coefficients of powers of  $\log |\xi|$  in its symbol (see (1)) have the above property (3). Similarly to odd-class classical PDOs, one can easily check that odd-class log-polyhomogeneous PDOs form an algebra.

When  $M$  is odd dimensional, the canonical trace  $\text{TR}$  of [5] is well defined on the algebra of odd-class classical PDOs. Recently, L. Maniccia, E. Schrohe and J. Seiler ([8]) have proved that  $\text{TR}$  is the unique trace on this algebra. The canonical trace has been first extended

by M. Lesch ([7]) to log-polyhomogeneous PDOs of non integer orders (this means that the orders of  $a_0, \dots, a_k$  are not integers) and then by S. Paycha and S. Scott to odd-class log-polyhomogeneous PDOs when  $M$  is odd dimensional ([10]). We refer the reader to Paragraph 4 for more details. To our knowledge, the proof of the uniqueness of TR on odd-class log-polyhomogeneous PDOs is new.

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