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# A PSEUDO-RIP FOR MULTIVARIATE REGRESSION

CHRISTOPHE GIRAUD

ABSTRACT. We give a suitable RI-Property under which recent results for trace regression translate into strong risk bounds for multivariate regression. This pseudo-RIP is compatible with the setting  $n < p$ .

## 1. INTRODUCTION

1.1. **Statistical framework.** Multivariate regression deals with  $n$  observations of a  $T$ -dimensional vector

$$y_i = A_0^T x_i + \varepsilon_i, \quad i = 1, \dots, n$$

where  $A_0^T$  is the transpose of a  $p \times T$  matrix  $A_0$ . We have in mind that  $A_0$  has a small (unknown) rank and the design  $x_i$  is non-random. Writing  $Y$ ,  $X$  and  $E$  for the matrices with respective rows  $y_i^T$ ,  $x_i^T$  and  $\varepsilon_i^T$ , the above equation translate into

$$Y = XA_0 + E.$$

Anderson [1] and Izenman [5] have introduced reduced-rank estimators

$$\hat{A}_r \in \underset{A : \text{rank}(A) \leq r}{\text{argmin}} \|Y - XA\|^2, \quad r = 0, \dots, \min(p, T),$$

where  $\|\cdot\|$  is the Hilbert-Schmidt norm associated to the scalar product  $\langle \cdot, \cdot \rangle$ . The problem of selecting among the family of estimators  $\{\hat{A}_r, r = 0, \dots, \min(p, T)\}$  by minimizing the criterion

$$\text{Crit}(r) = \|Y - X\hat{A}_r\|^2 + \text{pen}(r)\sigma^2 \quad \text{and} \quad \text{Crit}'(r) = \log\left(\|Y - X\hat{A}_r\|^2\right) + \text{pen}'(r)$$

has been investigated recently from a non-asymptotic point of view by Bunea *et al.* [3] and Giraud [4]. Both papers provide oracle bounds for the predictive risk

$$R(\hat{A}) = \mathbb{E} \left[ \|X\hat{A} - XA\|^2 \right]$$

with *no assumption* on the design  $X$ .

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Multivariate regression corresponds to a special case of the trace regression model

$$\mathcal{Y}_j = \langle Z_j, A_0 \rangle + \xi_j, \quad j = 1, \dots, N,$$

where  $\langle Z_j, A_0 \rangle = \text{tr}(Z_j^T A_0)$ . Indeed, we have for all  $i \in \{1, \dots, n\}$  and  $t \in \{1, \dots, T\}$

$$\begin{aligned} Y_{it} &= \langle A_0^T x_i, e_t \rangle + E_{it} \\ &= \underbrace{\langle x_i e_t^T, A_0 \rangle}_{=: Z_{it}} + E_{it}, \end{aligned}$$

where  $\{e_1, \dots, e_T\}$  is the canonical basis of  $\mathbb{R}^T$ . Many recent works [2, 7, 8, 6] have investigated trace regression with nuclear norm penalization. Translated in terms of multivariate regression, Nuclear-Norm-Penalized regression estimators are defined by

$$(1) \quad \widehat{A}_\lambda \in \underset{A \in \mathbb{R}^{p \times T}}{\text{argmin}} \left\{ \|Y - XA\|^2 + \lambda \sum_k \sigma_k(A) \right\},$$

where  $\sigma_1(A) \geq \sigma_2(A) \geq \dots$  are the singular values of  $A$ . Several risk bounds have been obtained for the predictive risk of  $\widehat{A}_\lambda$  and they all require the assumption (semi-RI Property)

$$(2) \quad \|A\| \leq \mu \|XA\|, \quad \text{for all } A \in \mathbb{R}^{p \times T}$$

for some positive  $\mu$ . In other words the smallest eigenvalue of  $X^T X$  must be larger than  $1/\mu^2 > 0$ . This enforces the sample size  $n$  to be larger than the number  $p$  of parameters. This assumption on the design needed for  $\widehat{A}_\lambda$  is thus very strong, in contrast with the reduced-rank estimator  $\widehat{A}_{\hat{r}}$  which requires no assumption on the design.

**1.2. Object of this note.** In this note, we emphasize that the Assumption (2) coming from the general trace regression framework can be weakened for the multivariate regression framework. Under this (much) weaker assumption, we show that the analysis of Kolchinskii *et al.* [6] gives an oracle bound with leading constant 1 for the estimators  $\widehat{A}_\lambda$ .

## 2. SEMI-RIP FOR MULTIVARIATE REGRESSION

The Condition (2) requires the sample size  $n$  to be larger than the number  $p$  of covariates. Is-it still possible to get an oracle bound on  $\|X\widehat{A}_\lambda - XA\|^2$  when  $n$  is smaller than  $p$  ?

The analysis of Theorem 12 in Bunea *et al.* [3] suggests that the Condition (2) only need to hold true for matrices  $A$  of rank at most twice the rank of  $A_0$ . Unfortunately, when the rank of  $A_0$  is positive this condition is still equivalent to require that the smallest eigenvalue of  $X^T X$  is larger than  $1/\mu^2 > 0$ .

In the analysis of in Kolchinskii *et al.* [6], the Condition (2) is needed for comparing  $\|\widehat{A}_\lambda - A\|$  to  $\|X\widehat{A}_\lambda - XA\|$ , see for example the Display (2.17) of [6]. We point out below, that this inequality needs not to hold for all matrices  $\widehat{A}_\lambda$  and  $A$ , so that Condition (2) can be relaxed to handle cases where  $p > n$ .

**Assumption 1.**

$$\sigma_{\text{rank}(X)}(X) \geq \frac{1}{\mu} > 0$$

where  $\sigma_1(X) \geq \sigma_2(X) \geq \dots$  are the singular values of  $X$ .

The singular value  $\sigma_{\text{rank}(X)}(X)$  is always positive but can be arbitrary small. Assumption 1 requires a positive lower bound on this singular value.

**2.1. Risk bound under Assumption 1.** Write  $\text{rg}(X^T)$  for the range of the linear operator  $X^T$  and  $\Pi_{\text{rg}(X^T)}$  for the orthogonal projection onto the range of  $X^T$  in  $\mathbb{R}^p$ . Since we have the orthogonal decomposition  $\mathbb{R}^p = \ker(X) + \text{rg}(X^T)$ , we have  $X\Pi_{\text{rg}(X^T)}A = XA$  for any matrix  $A$ . In addition,  $\sigma_k(\Pi_{\text{rg}(X^T)}A) \leq \sigma_k(A)$  for any  $k$  and matrix  $A$ , so

$$\sum_k \sigma_k(\Pi_{\text{rg}(X^T)}A) \leq \sum_k \sigma_k(A),$$

with strict inequality if  $\Pi_{\text{rg}(X^T)}A \neq A$ . As a consequence, we have  $\Pi_{\text{rg}(X^T)}\widehat{A}_\lambda = \widehat{A}_\lambda$ , so  $\widehat{A}_\lambda$  is also a minimizer of

$$(3) \quad \min_{A \in \mathbb{A}} \left\{ \|Y - XA\|^2 + \lambda \sum_k \sigma_k(A) \right\}.$$

where  $\mathbb{A} := \{A \in \mathbb{R}^{p \times T} : \text{rg}(A) \subset \text{rg}(X^T)\}$ .

Under Assumption 1, we have

$$\|A\| \leq \mu \|XA\|, \quad \text{for all } A \in \mathbb{A}.$$

Theorem 1 of Kolchinskii *et al.* [6] then gives the upper bound

$$\|X\widehat{A}_\lambda - XA_0\|^2 \leq \inf_{A \in \mathbb{A}} \left\{ \|XA - XA_0\|^2 + \left( \frac{1 + \sqrt{2}}{2} \right)^2 \mu^2 \lambda^2 \text{rank}(A) \right\}$$

for  $\lambda \geq 2\sigma_1(X^T E)$ . Again, since  $X\Pi_{\text{rg}(X^T)}A = XA$  and  $\text{rank}(\Pi_{\text{rg}(X^T)}A) \leq \text{rank}(A)$ , the infimum on the right hand side coincides with the infimum on the whole space  $\mathbb{R}^{p \times T}$ . We then have the following result.

**Theorem 1.** *Let  $\widehat{A}_\lambda$  be defined by (1). Then, under Assumption 1, for  $\lambda \geq 2\sigma_1(X^T E)$  we have*

$$\begin{aligned} \|X\widehat{A}_\lambda - XA_0\|^2 &\leq \inf_{A \in \mathbb{R}^{p \times T}} \left\{ \|XA - XA_0\|^2 + \frac{3}{2} \mu^2 \lambda^2 \text{rank}(A) \right\} \\ &= \inf_r \left\{ \sum_{k \geq r+1} \sigma_k(XA_0)^2 + \frac{3}{2} \mu^2 \lambda^2 r \right\}. \end{aligned}$$

**2.2. Case of Gaussian errors.** The above statement is purely deterministic. In the case of Gaussian errors we have the following corollary.

**Corollary 1.** *Assume that the entries of  $E$  are i.i.d. with Gaussian  $\mathcal{N}(0, \sigma^2)$  distribution. Let  $K > 1$  and set*

$$\lambda = 2K\sigma_1(X) \left( \sqrt{T} + \sqrt{q} \right) \sigma, \quad \text{with } q = \text{rank}(X).$$

*Then, with probability larger than  $1 - e^{-(K-1)^2(T+q)/2}$  we have*

$$\begin{aligned} \|X\hat{A}_\lambda - XA_0\|^2 &\leq \inf_{A \in \mathbb{R}^{p \times T}} \left\{ \|XA - XA_0\|^2 + 6K^2 \frac{\sigma_1(X)^2}{\sigma_q(X)^2} \left( \sqrt{T} + \sqrt{q} \right)^2 \sigma^2 \text{rank}(A) \right\} \\ (4) \quad &= \inf_r \left\{ \sum_{k \geq r+1} \sigma_k(XA_0)^2 + 6K^2 \frac{\sigma_1(X)^2}{\sigma_q(X)^2} \left( \sqrt{T} + \sqrt{q} \right)^2 \sigma^2 r \right\} \end{aligned}$$

### 3. DISCUSSION

The Assumption 1, which requires that the smallest *positive* singular value of  $X$  is lower bounded, is much weaker than the Assumption (2). In particular, this condition is fully compatible with the setting where the sample size  $n$  is smaller than the number  $p$  of covariables.

The inequality (4) for the Nuclear Norm Penalized estimator suggests that the suitable "RI-Property" for prediction in multivariate regression is

**RI-Property :** *There exists  $\eta \in [1, +\infty[$  such that*

$$1 \leq \frac{\sigma_1(X)}{\sigma_q(X)} \leq \eta, \quad \text{with } q = \text{rank}(X).$$

When this condition is met with  $\eta$  of reasonable size, the NNP-estimator achieves under the assumptions of Corollary 1, the oracle inequality

$$\|X\hat{A}_\lambda - XA_0\|^2 \leq \inf_{A \in \mathbb{R}^{p \times T}} \left\{ \|XA - XA_0\|^2 + 6K^2 \eta^2 \left( \sqrt{T} + \sqrt{q} \right)^2 \sigma^2 \text{rank}(A) \right\}$$

with probability larger than  $1 - e^{-(K-1)^2(T+q)/2}$ . This inequality ensures that the NNP-estimator is adaptive rate-minimax.

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