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EXISTENCE OF RADIAL STATIONARY SOLUTIONS FOR A SYSTEM IN COMBUSTION THEORY

JÉRÔME COVILLE AND JUAN DÁVILA

ABSTRACT. In this paper, we construct radially symmetric solutions of a nonlinear noncooperative elliptic system derived from a model for flame balls with radiation losses. This model is based on a one step kinetic reaction and our system is obtained by approximating the standard Arrhenius law by an ignition nonlinearity, and by simplifying the term that models radiation. We prove the existence of 2 solutions using degree theory.

1. INTRODUCTION

This paper deals with radial solutions of the system of equations

$$(1.1) \quad \begin{cases} \Delta u - \varepsilon u = -vf(u) & \text{in } \mathbb{R}^3 \\ \Delta v = vf(u) & \text{in } \mathbb{R}^3 \\ u \geq 0, \quad v \geq 0 & \text{in } \mathbb{R}^3 \\ \lim_{|x| \rightarrow \infty} u(x) = 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 1, \end{cases}$$

where $\varepsilon \geq 0$ and f is an ignition type nonlinearity, that is, there exists $0 < \theta < 1$ such that

$$(1.2) \quad f(t) = 0 \quad \forall t \leq \theta, \quad f(t) > 0 \quad \forall t > \theta.$$

System (1.1) arises as a model problem for some reaction diffusion systems in combustion theory. We will describe the connection with these models at the end of the section.

We call $(u, v) = (0, 1)$ the trivial solution of (1.1). Note that for any solution (u, v) of (1.1) one has

$$0 \leq u \leq 1 \quad \text{and} \quad 0 \leq v \leq 1,$$

as can be deduced from the maximum principle.

We are mainly interested in existence and multiplicity of nontrivial radial solutions of (1.1). The first observation in this direction is that for $\varepsilon > 0$ large (1.1) has no nontrivial solutions. Indeed, by the maximum principle $u + v \leq 1$ and then

$$-\Delta u \leq (1 - u)f(u) - \varepsilon u \quad \text{in } \mathbb{R}^3.$$

If $\varepsilon > 0$ is large then $(1 - u)f(u) - \varepsilon u \leq 0$ for all $u \in [0, 1]$ and we conclude that $u \equiv 0$. Better estimates for the quantity

$$\varepsilon^* = \sup\{\varepsilon > 0 : (1.1) \text{ has a nontrivial radial solution}\}$$

are given in Section 6.

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Suppose that u, v is a nontrivial radial solution. Then for some $r > 0$ we must have $u(r) > \theta$. Otherwise $f(u) \equiv 0$ and then $\Delta v = 0$, and by the Liouville theorem $v \equiv 1$. Since $0 \leq u + v \leq 1$ this would imply $u \equiv 0$ and then u, v is trivial. So for a nontrivial solution u, v , since $\lim_{r \rightarrow \infty} u(r) = 0$, there is $\beta_u > 0$ such that

$$u(\beta_u) = \theta \quad \text{and} \quad u(r) < \theta \quad \forall r > \beta_u.$$

The parameter β_u , which we will write simply as β , then serves to distinguish different solutions.

Theorem 1.1. *Assume $f : [0, 1] \rightarrow [0, +\infty)$ is continuous, satisfies (1.2) and for some $C > 0$*

$$(1.3) \quad f(t) \leq Cf(u) \quad \forall t \leq u, \quad t, u \in [\theta, 1].$$

Then $\varepsilon^ > 0$ and there exist $0 < \varepsilon_1 < \varepsilon^*$ such that for $0 < \varepsilon < \varepsilon_1$ there are at least 2 solutions of (1.1). One of them has bounded β as $\varepsilon \rightarrow 0$ and the other has β in the range $\delta/\sqrt{\varepsilon} \leq \beta \leq 1/(\theta\sqrt{\varepsilon})$ as $\varepsilon \rightarrow 0$, where $\delta > 0$ is fixed.*

A very natural and interesting question is the stability of the solutions constructed in Theorem 1.1. Based on the works [10, 11] we conjecture that the solution with bounded β is unstable and that in part of branch of solutions with large β the solution is stable, at least with respect to radial perturbations.

In some cases one may want to consider a discontinuous nonlinearity, such as the Heavisde function $f(u) = \chi_{[u > \theta]}$. With this example in mind we introduce the following hypothesis

$$(1.4) \quad f \text{ is continuous in } (\theta, 1] \text{ and } \lim_{u \rightarrow \theta^+} f(u) \text{ exists.}$$

Theorem 1.2. *Assume $f : [0, 1] \rightarrow [0, +\infty)$ satisfies (1.2), (1.3), and (1.4). Then $\varepsilon^* > 0$ and there exist $0 < \varepsilon_1 < \varepsilon^*$ such that for $0 < \varepsilon < \varepsilon_1$ there are at least 2 solutions of (1.1). One of them has bounded β as $\varepsilon \rightarrow 0$ and the other has β in the range $\delta/\sqrt{\varepsilon} \leq \beta \leq 1/(\theta\sqrt{\varepsilon})$ as $\varepsilon \rightarrow 0$, where $\delta > 0$ is fixed.*

The solution constructed in Theorem 1.2 is such that the set $\{r \in [0, \infty) : u(r) = \theta\}$ is finite, and hence the equation (1.1) holds a.e. in \mathbb{R}^3 .

The motivation to consider a discontinuous nonlinearity is only mathematical. However the example $f(u) = \chi_{[u > \theta]}$ is interesting since it provides a situation where explicit calculations are possible. Theorem 1.2 shows that in part the conclusions obtained for $f(u) = \chi_{[u > \theta]}$ remain valid for more general non-linearities.

For $f(u) = \chi_{[u > \theta]}$ explicit calculations lead to an equation for β and ε in order for a radial solution to exist. In Figure 1 we show the numerical solution for this relation when $\theta = 0.5$, with β in the vertical axis and $\sqrt{\varepsilon}$ in the horizontal axis. It shows that for $0 < \varepsilon < \varepsilon^*$ there are 2 solutions. Solutions in the lower branch satisfy $u > \theta$ in $[0, \beta)$ that is, the reaction takes place in the ball of radius β . The same happens for points in the upper branch which are to the right of the special point marked in the graph. To the left of that point the solution satisfies $u > \theta$ in an annulus of the form $r \in (\beta - L_\beta, \beta)$. Thanks to the explicit form of the relation between β and ε we can compute the asymptotic behavior of the curve as $\beta \rightarrow \infty$, and we find that

$$\lim_{\beta \rightarrow \infty} \sqrt{\varepsilon} \beta = a_0 > 0 \quad \text{and} \quad \lim_{\beta \rightarrow \infty} L_\beta = L_0 > 0,$$

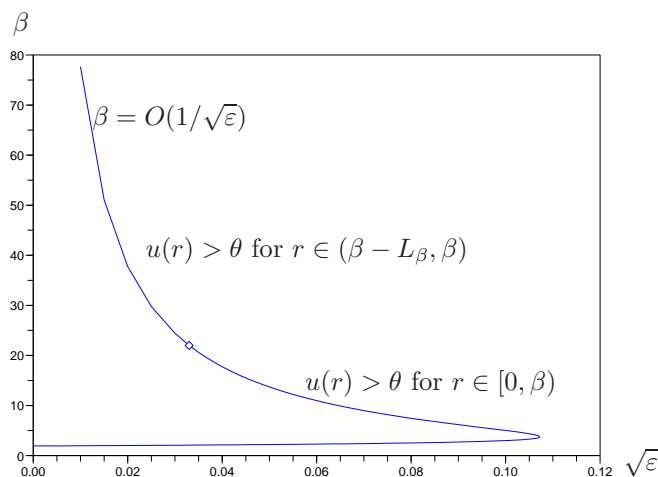


FIGURE 1. Bifurcation diagram for $f = \chi_{[u>\theta]}$ with $\theta = 0.5$; β is in the vertical axis and $\sqrt{\varepsilon}$ in the horizontal axis

where a_0, L_0 is the unique solution of the system of equations

$$\theta \left(\frac{a}{\tanh(a)} - 1 \right) + \theta(1 + a) = 1$$

$$1 - \frac{1}{L \tanh(L)} + \frac{1}{L \sinh(L)} = \theta(1 + a).$$

Because of the information on the Heaviside nonlinearity one can conjecture that for general f there should be a similar relation for β and ε as $\beta \rightarrow +\infty$. We present in Section 2 nonexistence results for general ignition nonlinearities satisfying (1.2) and (1.3), that capture this relation, and roughly speaking say that no solution can exist if $\sqrt{\varepsilon}\beta$ is either too large or too small, provided β is taken large enough. Using these nonexistence results and degree theory we can give a proof of Theorem 1.1. This is done in Section 3. In Section 4 we give the proof of Theorem 1.2 by approximating the discontinuous nonlinearity by continuous ones. Section 5 is devoted to the explicit computations for the Heaviside function. Finally Section 6 contains a finer estimate of ε^* .

As mentioned before, system (1.1) arises in connection with some models in combustion theory, more precisely, in the flame ball problem for a weakly premixed gas sensitive to radiative heat losses. In such a mixture, it is known that apparently stationary spherical structures appear, which are called flame balls [27, 30, 10, 11].

In [27, 30, 10, 11, 21, 22, 29] the following reaction diffusion system has been used to model a combustion process where flame balls can appear:

$$(1.5) \quad \begin{cases} \rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T) + \frac{QBY\rho}{m} e^{-\frac{E}{RT}} - q(T) \\ \rho \frac{\partial Y}{\partial t} = \nabla \cdot (\mu \nabla Y) - BY\rho e^{-\frac{E}{RT}} \\ Y \rightarrow Y_\infty, \quad T \rightarrow T_\infty \quad \text{as } |x| \rightarrow \infty \end{cases}$$

where T is the temperature, Y the reactant concentration, $Y_\infty > 0$, $T_\infty > 0$ are the reactant concentration and temperature at infinity, and C_p , R , Q and m are respectively the specific heat capacity at constant pressure, the perfect gas constant, the chemical heat release and the molecular mass of the reactant. The term $q(T)$ represents radiative losses. The reaction is characterized by the one-step Arrhenius kinetics $\sim Be^{-\frac{E}{RT}}$ where B is a constant. Furthermore, the hydrodynamics effects are neglected, i.e the density ρ , the thermal conductivity λ and the diffusion coefficient μ are constant. See also numerical simulations in [7, 21, 22].

After the seminal work [34], the traveling front problem for systems like (1.5) has been investigated by several authors, for example, [16, 17, 31].

In absence of radiation, i.e. when $q = 0$, there are many works dealing with (1.5), see for instance [3, 6, 25, 28, 32] and the references therein. Also we remark that the stationary version of (1.5) without radiation, leads to system (1.1) with $\varepsilon = 0$ which reduces to a scalar equation, since $v = 1 - u$. There is a huge amount of literature concerning existence of radial ground states for semilinear equations, so we mention here only some classical references [5, 24] on the problem in entire space. When the problem is treated in a bounded domain see the book [1] and [15, 23] for multiplicity results in the case of Arrhenius non-linearity. The paper [9] contains interesting numerical computations of the bifurcation diagram in the case of the full coupled system in a interior of a sphere.

A common simplification of (1.5) under the assumption of large activation energy, that is $E \gg 1$, is to assume that the source term for the reaction is concentrated on a very thin layer, typically a sphere. This approach is taken for example in [10, 11, 26, 29] and leads to the free boundary problem

$$(1.6) \quad \begin{cases} \frac{\partial T}{\partial t} = \Delta T + Be^{-\frac{E}{2T^*}} \delta(r - R(t)) - q(T) & \text{in } \mathbb{R}^3 \\ \frac{\partial Y}{\partial t} = \frac{1}{Le} \Delta Y - Be^{-\frac{E}{2T^*}} \delta(r - R(t)) & \text{in } \mathbb{R}^3 \\ Y \equiv 0 & \text{in } B(0, R(t)) \\ q \equiv 0 & \text{in } \mathbb{R}^3 \setminus B(0, R(t)) \\ Y \rightarrow Y_\infty, \quad T \rightarrow T_\infty & \text{as } |x| \rightarrow \infty \end{cases}$$

where $R(t)$ is the radius of the front where the reaction takes place, δ is the Dirac measure and T^* is the front temperature. In [10, 11, 12] the authors analyze the stability of stationary solutions of (1.6). In a similar framework, existence and stability of flame balls and travelling flame balls have also been studied in [29, 18, 19, 33, 2].

We arrive at (1.1) by introducing the following simplifications:

- a) assume the radiative loss to be linear, i.e., $q(T) = a(T - T_\infty)$ where $a > 0$,

- b) approximate $e^{-\frac{E}{RT}}$ by $\eta(T - T_0)BYe^{-\frac{E}{RT}}$ where T_0 is an activation temperature and η is a cut-off function satisfying $\eta > 0$ in \mathbb{R}^+ and $\eta \equiv 0$ in \mathbb{R}^- .

As in [22, 33] one can model radiative heat losses using Stefan's law $q(T) = \varepsilon(T^4 - T_\infty^4)$ for some constant $\varepsilon > 0$. When T is close to T_∞ we can write $q(T) \approx 4\varepsilon T_\infty^3(T - T_\infty)$. As a step towards understanding more general situations, we assume that this linear relation holds for all T , that is, we assume a). Other linear or piecewise linear approximations have been used before, for instance in [8, 29, 33]. Assumption b) corresponds to a standard approximation in combustion theory to avoid the *cold boundary difficulty*, see [4].

After introducing dimensionless variables u, v , corresponding to temperature and reactant concentration, the stationary version of (1.5) becomes

$$(1.7) \quad \begin{cases} \Delta u + vg(u) - cu = 0 & \text{in } \mathbb{R}^3 \\ \frac{1}{Le}\Delta v - vg(u) = 0 & \text{in } \mathbb{R}^3 \\ u \rightarrow 0, \quad v \rightarrow v_\infty & \text{as } |x| \rightarrow +\infty \end{cases}$$

where $Le > 0$ is the Lewis number, $c > 0$, $v_\infty > 0$ and g is an ignition type function, that is, there is $\theta > 0$ such that $g(u) = 0$ if $u \leq \theta$.

We stress that our results are valid for any value of $Le > 0$. Indeed, since we are considering stationary solutions, the following change of variables will allow us to assume that $Le = 1$. Letting $v(x) = v_\infty \tilde{v}(\sqrt{Le}x)$, $u(x) = \frac{v_\infty}{Le} \tilde{u}(\sqrt{Le}x)$ transforms system (1.7) into (1.1), where $\varepsilon = \frac{c}{Le} > 0$ and $f(u) = g(\frac{v_\infty}{Le}u)$. We observe that f is still an ignition type nonlinearity.

2. APRIORI ESTIMATES

The purpose in this section is to establish nonexistence results in some ranges of the parameters.

Given a nontrivial solution u, v of (1.1) let $\beta > 0$ be such that

$$u(\beta) = \theta \quad \text{and} \quad u(r) < \theta \quad \forall r > \beta.$$

Setting $\tilde{u}(r) = u(\beta r)$, $\tilde{v}(r) = v(\beta r)$ these new functions satisfy

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) - \varepsilon r^2 \beta^2 u = -r^2 \beta^2 v f(u) & \text{in } \mathbb{R}^+ \\ \frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) = r^2 \beta^2 v f(u) & \text{in } \mathbb{R}^+ \\ \lim_{r \rightarrow \infty} u(r) = 0, \quad \lim_{r \rightarrow \infty} v(r) = 1 \end{cases}$$

In the sequel we will study (2.1) in the following set of functions

$$\mathcal{S} = \{(u, v) : u, v \text{ are } C^1([0, \infty)) \text{ and } u(1) = \theta \text{ and } u(r) < \theta \text{ for } r > 1\}.$$

Let $h_0 : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$(2.2) \quad h_0(t) = 0 \quad \forall t \leq \theta \quad \text{and} \quad h_0(t) > 0 \quad \forall t > \theta$$

$$(2.3) \quad h_0 \text{ is nondecreasing.}$$

We consider now functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(2.4) \quad f(t) = 0 \quad \forall t \leq \theta,$$

$$(2.5) \quad f \geq h_0 \quad \text{in } \mathbb{R}$$

$$(2.6) \quad f(t) \leq C_0 f(u) \quad \forall \theta \leq t \leq u \leq 1.$$

Lemma 2.1. *There is no solution in \mathcal{S} to (2.1) if*

$$\beta\sqrt{\varepsilon} > \frac{1}{\theta}.$$

Proof. Since

$$\Delta u - \varepsilon\beta^2 u = 0 \quad \text{for } r > 1, \quad u(1) = \theta, \quad \lim_{r \rightarrow +\infty} u(r) = 0,$$

we have an explicit formula

$$u(r) = \theta \frac{e^{-(r-1)\beta\sqrt{\varepsilon}}}{r} \quad \text{for all } r \geq 1.$$

From this we find

$$(2.7) \quad u'(1) = -\theta(1 + \beta\sqrt{\varepsilon}).$$

Similarly, since $\Delta v = 0$ for $r \geq 1$ and $\lim_{r \rightarrow \infty} v(r) = 1$ we have

$$v(r) = 1 - \frac{\gamma}{r} \quad \forall r \geq 1$$

where $0 < \gamma < 1$. This yields

$$v'(1) = \gamma \in (0, 1).$$

Integrating the equation for v in $(0, 1)$ implies

$$(2.8) \quad \beta^2 \int_0^1 s^2 v f(u) ds = v'(1) = \gamma.$$

Let $r_0 \in [0, 1)$ be such that $u'(r_0) = 0$. Then integrating the equation for u in $(r_0, 1)$ yields

$$u'(1) = \varepsilon\beta^2 \int_{r_0}^1 s^2 u ds - \beta^2 \int_{r_0}^1 s^2 v f(u) ds.$$

This formula together with (2.7) and (2.8) gives

$$v'(1) \geq \beta^2 \int_{r_0}^1 s^2 v f(u) ds = \varepsilon\beta^2 \int_{r_0}^1 s^2 u ds + \theta(1 + \beta\sqrt{\varepsilon})$$

and it follows that

$$\theta\beta\sqrt{\varepsilon} \leq 1.$$

□

Lemma 2.2. *There is $\beta_0 > 0$, $\delta > 0$ depending only on h_0 , θ , C_0 such that for all $\beta \geq \beta_0$, $\beta\sqrt{\varepsilon} \leq \delta$ and all f satisfying (2.4), (2.5), (2.6) there is no solution in \mathcal{S} to the system (2.1).*

Proof. We treat the case $\varepsilon > 0$ since the situation $\varepsilon = 0$ is similar.

As before

$$u(r) = \theta \frac{e^{-(r-1)\beta\sqrt{\varepsilon}}}{r} \quad \text{for all } r \geq 1.$$

From this we find

$$(2.9) \quad u'(1) = -\theta(1 + \beta\sqrt{\varepsilon}).$$

Similarly

$$v(r) = 1 - \frac{\gamma}{r} \quad \forall r \geq 1$$

where $0 < \gamma < 1$. This yields

$$v'(1) = \gamma \in (0, 1).$$

Integrating the equation for u we see that

$$(2.10) \quad r^2 u'(r) = \varepsilon \beta^2 \int_0^r s^2 u \, ds - \beta^2 \int_0^r s^2 v f(u) \, ds.$$

From this and (2.9) it follows that

$$(2.11) \quad \beta^2 \int_0^1 s^2 v f(u) \, ds = \theta(1 + \beta\sqrt{\varepsilon}) + \varepsilon \beta^2 \int_0^1 s^2 u \, ds \leq \theta + \delta(\theta + \delta).$$

Integrating the equation for v in $(0, 1)$ implies

$$(2.12) \quad \beta^2 \int_0^1 s^2 v f(u) \, ds = v'(1) = \gamma$$

and combined with (2.11) yields

$$(2.13) \quad \gamma \leq \theta + \delta(\theta + \delta).$$

Integrating the equation for v in $(0, r)$ we obtain

$$(2.14) \quad v'(r) = \beta^2 \int_0^r (s/r)^2 v(s) f(u(s)) \, ds.$$

Integrating this on $(0, R)$ with $0 \leq R \leq 1$ yields

$$\begin{aligned} v(R) - v(0) &= \beta^2 \int_0^R \int_0^r (s/r)^2 v(s) f(u(s)) \, ds \, dr \\ &= \beta^2 \int_0^R \int_s^R r^{-2} \, dr s^2 v(s) f(u(s)) \, ds \\ &= \beta^2 \int_0^R (s^{-1} - R^{-1}) s^2 v(s) f(u(s)) \, ds. \end{aligned}$$

In particular, with $R = 1$

$$\beta^2 \int_0^1 (s - s^2) v(s) f(u(s)) \, ds = v(1) - v(0) \leq 1$$

This and (2.12) give

$$\beta^2 \int_0^1 s v(s) f(u(s)) \, ds \leq 2.$$

Now going back to (2.14) we obtain

$$(2.15) \quad v'(r) = \beta^2 \int_0^r (s/r)^2 v(s) f(u(s)) ds \leq \beta^2 \int_0^r (s/r) v(s) f(u(s)) ds \leq \frac{2}{r} \quad 0 < r \leq 1.$$

Now consider the function $z = u + v$ which satisfies $\Delta z = \varepsilon \beta^2 u$ in \mathbb{R}^3 . Integrating the relation $(s^2 z')' = \varepsilon \beta^2 s^2 u$ in $(0, r)$

$$z'(r) = \varepsilon \beta^2 \int_0^r (s/r)^2 u(s) ds$$

so that

$$(2.16) \quad 0 \leq z'(r) \leq \delta^2 r \quad r \geq 0.$$

It follows that

$$(2.17) \quad -\frac{2}{r} \leq u'(r) \leq \delta^2 r \quad 0 < r \leq 1.$$

Observe that integrating (2.16) on $(r, 1)$ we find

$$0 \leq z(1) - z(r) \leq \delta^2$$

and since $z(1) = \theta + 1 - \gamma$

$$1 + \theta - \gamma - \delta^2 \leq z(r) \leq 1 + \theta - \gamma \quad 0 \leq r \leq 1.$$

In particular, using (2.13),

$$(2.18) \quad v = z - u \geq 1 + \theta - \gamma - \delta^2 - u \geq 1 - \delta(\theta + 2\delta) - u \quad 0 \leq r \leq 1.$$

Step 1. For any $r_0 > 0$ there exists $\beta_1(r_0)$ depending on r_0 , h_0 and θ only such that

$$(2.19) \quad \max_{[r_0, 1]} u < \frac{1 + \theta}{2} \quad \forall \beta \geq \beta_1.$$

To prove this, suppose that $\max_{[r_0, 1]} u \geq \frac{1 + \theta}{2}$ and let $r \in [r_0, 1]$ be such that $u(r) = \frac{1 + \theta}{2}$. Let $M = 2/r_0$ so that from (2.17)

$$|u'(s)| \leq M \quad r_0 \leq s \leq 1.$$

Then

$$(2.20) \quad \frac{1 + \theta}{2} - \frac{1 - \theta}{4} \leq u(s) \leq \frac{1 + \theta}{2} + \frac{1 - \theta}{4} \quad \forall s \in [r_0, 1] \quad \text{such that} \quad |s - r| \leq \frac{1 - \theta}{4M}.$$

Using (2.12) and since $f \geq h_0$

$$1 \geq \beta^2 \int_{r_0}^1 s^2 v(s) f(u(s)) ds \geq \beta^2 \int_{r_0}^1 s^2 v(s) h_0(u(s)) ds.$$

If $r \leq \frac{1 + r_0}{2}$ let $I = [r - \frac{1 - \theta}{4M}, r]$ and if $r \geq \frac{1 + r_0}{2}$ let $I = [r, r + \frac{1 - \theta}{4M}]$. Using (2.18) and (2.20):

$$\begin{aligned} 1 &\geq \beta^2 \int_I s^2 v(s) f(u(s)) ds \\ &\geq \beta^2 r_0^2 \frac{1 - \theta}{4M} h_0 \left(\frac{1 + \theta}{2} - \frac{1 - \theta}{4} \right) \left(1 - \delta(\theta + 2\delta) - \frac{3 + \theta}{4} \right). \end{aligned}$$

We take $\delta > 0$ small only depending on θ such that

$$(2.21) \quad 1 - \delta(\theta + 2\delta) - \frac{3 + \theta}{4} \geq \frac{1 - \theta}{8}.$$

Then

$$1 \geq \beta^2 r_0^3 \frac{(1 - \theta)^2}{64} h_0 \left(\frac{1 + \theta}{2} - \frac{1 - \theta}{4} \right)$$

and the claim follows.

Step 2. For any $r_0 > 0$ there exists $C > 0$ such that

$$\left(\max_{[r_0, 1]} u - \theta \right) h_0 \left(\frac{\max_{[r_0, 1]} u - \theta}{2} + \theta \right) \leq \frac{C}{\beta^2}.$$

The constant C depends only on r_0 and θ . The conclusion from this is that

$$(2.22) \quad \limsup_{\beta \rightarrow \infty} \max_{[r_0, 1]} (u - \theta) \leq 0,$$

and this is uniform with respect to u and f .

As before, setting $M = 2/r_0$ we have $|u'(s)| \leq M$ for $r_0 \leq s \leq 1$ by (2.17). Let $r_m \in [r_0, 1]$ be a point such that $u(r_m) = \max_{[r_0, 1]} u$ and let us write $u_m = u(r_m) = \max_{[r_0, 1]} u$. Then

$$u(r) \geq \frac{u_m - \theta}{2} + \theta \quad \forall r \in [r_0, 1] \quad \text{such that} \quad |r - r_m| \leq \frac{u_m - \theta}{2M}.$$

Since $f \geq h_0$, using (2.12), (2.18), (2.19), (2.21) we obtain

$$\begin{aligned} 1 &\geq \beta^2 \int_0^1 s^2 v(s) f(u(s)) ds \geq \beta^2 \int_0^1 s^2 v(s) h_0(u(s)) ds \\ &\geq \beta^2 r_0^2 \frac{u_m - \theta}{2M} h_0 \left(\frac{u_m - \theta}{2} + \theta \right) \left(1 - \delta(\theta + 2\delta) - \frac{1 + \theta}{2} \right) \\ &\geq \beta^2 r_0^2 \frac{u_m - \theta}{2M} h_0 \left(\frac{u_m - \theta}{2} + \theta \right) \frac{1 - \theta}{8}. \end{aligned}$$

Thus there exists C depending only on r_0, θ such that for $\beta \geq \beta_1$

$$(u_m - \theta) h_0 \left(\frac{u_m - \theta}{2} + \theta \right) \leq \frac{C}{\beta^2}.$$

This proves the claim.

Step 3. For $0 \leq a \leq 1$

$$(2.23) \quad \int_a^1 r^2 (u')^2 dr \leq 2a + \delta^2 + \max_{[a, 1]} (u - \theta).$$

Indeed multiplying the equation

$$(2.24) \quad -(r^2 u')' = -\varepsilon \beta^2 r^2 u + \beta^2 r^2 v f(u)$$

by $(u - \theta)$ and integrating over $(a, 1)$ we get

$$\int_a^1 r^2 (u')^2 dr = r^2 u' (u - \theta) \Big|_a^1 - \varepsilon \beta^2 \int_a^1 r^2 u (u - \theta) dr + \beta^2 \int_a^1 r^2 v f(u) (u - \theta) dr.$$

By (2.17)

$$r^2 u' (u - \theta) \Big|_a^1 = -a^2 u'(a) (u(a) - \theta) \leq 2a.$$

Since $u \leq 1$ we have

$$\varepsilon\beta^2 \int_a^1 r^2 u(u - \theta) dr \leq \delta^2$$

Finally using (2.12)

$$\beta^2 \int_a^1 r^2 v f(u)(u - \theta) dr \leq \max_{[a,1]}(u - \theta) \beta^2 \int_a^1 r^2 v f(u) dr \leq \max_{[a,1]}(u - \theta).$$

Step 4. We finish the proof of the Lemma using a modification of Pohozaev's identity. Let $0 < a < 1$ be a fixed number. Multiplying (2.24) by $(r - a)u'$ and integrating over $(a, 1)$

(2.25)

$$- \int_a^1 (r^2 u')'(r - a)u' dr = \varepsilon\beta^2 \int_a^1 r^2(r - a)uu' dr + \beta^2 \int_a^1 r^2(r - a)v f(u)u' dr.$$

A computation shows that the left hand side is given by

$$\begin{aligned} \int_a^1 (r^2 u')'(r - a)u' dr &= \frac{1}{2} r^2(r - a)(u')^2 \Big|_a^1 + \frac{1}{2} \int_a^1 r^2 (u')^2 dr - a \int_a^1 r (u')^2 dr \\ (2.26) \qquad \qquad \qquad &= \frac{1}{2}(1 - a)u'(1)^2 + \frac{1}{2} \int_a^1 r^2 (u')^2 dr - a \int_a^1 r (u')^2 dr. \end{aligned}$$

Thanks to (2.23) we find

$$a \int_a^1 r (u')^2 dr \leq \int_a^1 r^2 (u')^2 dr \leq 2a + \delta^2 + \max_{[a,1]}(u - \theta).$$

Since $u'(1) = -\theta(1 + \sqrt{\varepsilon}\beta)$ we obtain from (2.26), (2.23) and the previous estimate

(2.27)

$$\left| \int_a^1 (r^2 u')'(r - a)u' dr - \frac{1}{2}(1 - a)\theta^2(1 + \sqrt{\varepsilon}\beta)^2 \right| \leq 4a + 2\delta^2 + 2 \max_{[a,1]}(u - \theta).$$

Let us compute

$$\varepsilon\beta^2 \int_a^1 r^2(r - a)uu' dr = \frac{1}{2}\varepsilon\beta^2 r^2(r - a)u^2 \Big|_a^1 - \frac{1}{2}\varepsilon\beta^2 \int_a^1 (3r^2 - 2ar)u^2 dr.$$

Recalling that $\sqrt{\varepsilon}\beta \leq \delta$ this shows that

$$(2.28) \qquad \qquad \qquad \left| \varepsilon\beta^2 \int_a^1 r^2(r - a)uu' dr \right| \leq 3\delta^2.$$

Define

$$F(u) = \int_\theta^u f(t) dt.$$

Then

$$\begin{aligned} \beta^2 \int_a^1 r^2(r - a)v f(u)u' dr &= \beta^2 r^2(r - a)v F(u) \Big|_a^1 - \beta^2 \int_a^1 (3r^2 - 2ar)v F(u) dr \\ &\quad - \beta^2 \int_a^1 (r^3 - ar^2)v' F(u) dr \\ &= \beta^2 \int_a^1 (3r^2 - 2ar)v F(u) dr - \beta^2 \int_a^1 (r^3 - ar^2)v' F(u) dr. \end{aligned}$$

These terms can be estimated using (2.6):

$$F(u) \leq C_0 f(u)(u - \theta) \quad \forall \theta \leq u \leq 1$$

and hence by (2.12)

$$\beta^2 \int_a^1 r^2 v F(u) dr \leq C_0 \int_a^1 r^2 v f(u)(u - \theta) dr \leq C_0 \max_{[a,1]}(u - \theta)$$

Similarly

$$a\beta^2 \int_a^1 r v F(u) dr \leq \beta^2 \int_a^1 r^2 v F(u) dr \leq C_0 \max_{[a,1]}(u - \theta).$$

To estimate the remaining terms we use (2.15)

$$\beta^2 \int_a^1 r^3 v' F(u) dr \leq 2\beta^2 \int_a^1 r^2 F(u) dr \leq \frac{16}{1-\theta} \int_a^1 r^2 v F(u) dr$$

where the last inequality is consequence of (2.18), (2.19) and (2.21) provided $\beta \geq \beta_1(a)$ ($a \in (0, 1)$ is fixed). Hence

$$\beta^2 \int_a^1 r^3 v' F(u) dr \leq \frac{16C_0}{1-\theta} \max_{[a,1]}(u - \theta),$$

and this implies also

$$\beta^2 \int_a^1 ar^2 v' F(u) dr \leq \beta^2 \int_a^1 r^3 v' F(u) dr \leq \frac{16C_0}{1-\theta} \max_{[a,1]}(u - \theta).$$

Therefore

$$(2.29) \quad \left| \beta^2 \int_a^1 r^2 (r - a) v f(u) u' dr \right| \leq K_1 \max_{[a,1]}(u - \theta)$$

where $K_1 = 5C_0 + \frac{32C_0}{1-\theta}$.

From (2.25), (2.27), (2.28), (2.29) we arrive at

$$(2.30) \quad \frac{1}{2}(1-a)\theta^2(1 + \sqrt{\varepsilon}\beta)^2 \leq 4a + 5\delta^2 + (K_1 + 2) \max_{[a,1]}(u - \theta) \quad \forall \beta \geq \beta_1(a).$$

We fix $\delta > 0$ even smaller if necessary so that

$$5\delta^2 \leq \frac{1}{3} \frac{\theta^2}{4}.$$

Then fix $0 < a < \frac{1}{2}$ such that

$$4a \leq \frac{1}{3} \frac{\theta^2}{4}.$$

Then for (2.30) yields

$$\frac{1}{3} \frac{\theta^2}{4} \leq (K_1 + 2) \max_{[a,1]}(u - \theta) \quad \forall \beta \geq \beta_1(a)$$

which is not possible for β large enough by (2.22). \square

3. EXISTENCE OF SOLUTIONS

In this section we prove Theorem 1.1, and throughout it we assume that f is continuous and satisfies (1.2) and (1.3).

We work in the Banach space

$$X = \{(u, \beta) : u \in C^{1,\alpha}([0, 1]), u(1) = 0, \beta \in \mathbb{R}\}$$

endowed with its natural norm

$$\|(u, \beta)\|_X = \|u\|_{C^{1,\alpha}([0,1])} + |\beta|,$$

where $0 < \alpha < 1$ is fixed.

Define for $t \in [0, 1]$

$$f_t(u) = tf(u) + (1-t)(u-\theta)^+ \quad t \in [0, 1],$$

where $s^+ = \max(s, 0)$. Let us consider (2.1) with nonlinearity f_t that is

$$(3.1) \quad \begin{cases} \Delta u - \varepsilon\beta^2 u = -\beta^2 v f_t(u) & \text{in } \mathbb{R}^3 \\ \Delta v = \beta^2 v f_t(u) & \text{in } \mathbb{R}^3 \\ \lim_{|x| \rightarrow \infty} u(x) = 0, & \lim_{|x| \rightarrow \infty} v(x) = 1 \end{cases}$$

To apply the non-existence results of the previous section we need to exhibit a function h_0 satisfying (2.2) and (2.3). For this purpose define

$$g(t) = \inf \{r \in [0, 1] : f \geq t \text{ in } [r, 1]\} \quad \text{for all } t \geq 0,$$

and

$$h(r) = \inf \{t \geq 0 : g(t) \geq r\} \quad \text{for all } r \in [0, 1].$$

The following properties then follow from these definitions:

- a) g is strictly increasing, continuous from the left and satisfies $\lim_{t \rightarrow 0^+} g(t) = \theta$.
- b) $h(r) = \theta$ for $r \in [0, \theta]$, $h(r) > 0$ for $r \in (\theta, 1]$, h is nondecreasing and continuous in $[0, 1]$.
- c) $f \geq h$ in $[0, 1]$.

The function

$$h_0 = \min(h, (u-\theta)^+)$$

satisfies (2.2) and (2.3) and the nonlinearity f_t satisfies (2.4), (2.5), (2.6) for all $t \in [0, 1]$.

Since any solution to (3.1) is bounded above by 1 we may redefine $f_t(u)$ as a constant for $u \geq 1$. Thus we may assume that for some constant $M > 0$ we have

$$|f_t(u)| \leq M \quad \forall u \in \mathbb{R}, \quad \forall t \in [0, 1].$$

We define a nonlinear map $T_{\varepsilon, t} : X \rightarrow X$ as follows. Let $(\tilde{u}, \tilde{\beta}) \in X$. Then solve

$$(3.2) \quad \Delta v = \tilde{\beta}^2 v f_t(\tilde{u} + \theta) \quad \text{in } B_1, \quad v(1) + v'(1) = 1 \quad \text{on } \partial B_1.$$

This problem has a unique solution which can be found for example by minimizing

$$\frac{1}{2} \int_{B_1} (|\nabla v|^2 + \tilde{\beta}^2 f_t(\tilde{u} + \theta) v^2) + \frac{1}{2} \int_{\partial B_1} v^2 - \int_{\partial B_1} v$$

with $v \in H^1(B_1)$. Since $f_t(\tilde{u} + \theta) \in L^\infty(B_1)$ by standard elliptic regularity $v \in C^{1,\alpha}([0, 1])$. Then find $\beta > 0$ and u such that

$$(3.3) \quad \begin{cases} -\Delta u + \varepsilon \tilde{\beta}^2 u = \beta^2 v f_t(\tilde{u} + \theta) - \varepsilon \tilde{\beta}^2 \theta + \beta^2 - \tilde{\beta}^2 & \text{in } B_1 \\ u(1) = 0, \quad u'(1) = -\theta(1 + \tilde{\beta} \sqrt{\varepsilon}) \end{cases}$$

admits a solution. This problem has a unique solution (u, β) which furthermore is $C^{1,\alpha}([0, 1])$ (this assertion is verified in the proof of Lemma 3.1 below). We define

$$T_{\varepsilon,t}(\tilde{u}, \tilde{\beta}) = (u, \beta).$$

Observe that (u, β) is a fixed point of $T_{\varepsilon,t}$ if and only if $(u + \theta, v)$ is a solution of (3.1).

Lemma 3.1. *Write $T(u, \beta, \varepsilon, t) = T_{\varepsilon,t}(u, \beta)$. Then*

$$T : X \times [0, \infty) \times [0, 1] \rightarrow X$$

is compact. Moreover

$$(3.4) \quad \|T(u, \beta, \varepsilon, t)\|_X \leq C(1 + \sinh(\sqrt{\varepsilon}\beta) + \beta^2) \quad \forall (u, \beta) \in X, \forall \varepsilon \geq 0, \forall t \in [0, 1].$$

Proof. First we remark that the solution v to (3.2) satisfies

$$0 \leq v \leq 1 \quad \text{in } B_1$$

because 0 is a subsolution and 1 is a supersolution. Given $a \geq 0$ let Z_a be the solution to

$$-\Delta Z_a + a Z_a = 0 \quad \text{in } B_1, \quad Z_a(1) = 1.$$

Note that Z_a is explicit:

$$Z_a(r) = \frac{\sinh(ar)}{r \sinh(a)} \quad \forall 0 < r \leq 1$$

and hence

$$(3.5) \quad \frac{a}{\sinh(a)} \leq Z_a \leq 1.$$

Let $(u, \beta) = T(\tilde{u}, \tilde{\beta}, \varepsilon, t)$. Then choosing $a = \varepsilon \tilde{\beta}^2$, multiplying (3.3) by Z_a and integrating in B_1 we find

$$-u'(1) = \beta^2 \int_{B_1} (v f(\tilde{u} + \theta) + 1) - \tilde{\beta}^2 \int_{B_1} (\varepsilon \theta + 1) Z_a.$$

From the boundary condition in (3.3) it follows that β is explicitly given by

$$(3.6) \quad \beta^2 = \frac{\theta(1 + \sqrt{\varepsilon}\tilde{\beta}) + \tilde{\beta}^2 \int_{B_1} (\varepsilon \theta + 1) Z_a}{\int_{B_1} (v f(\tilde{u} + \theta) + 1) Z_a},$$

where $a = \varepsilon \tilde{\beta}^2$. By (3.5)

$$(3.7) \quad \int_{B_1} (v f(\tilde{u} + \theta) + 1) Z_a \geq \frac{\sqrt{\varepsilon}\tilde{\beta}}{\sinh(\sqrt{\varepsilon}\tilde{\beta})}$$

From formulas (3.6), (3.7) we see that

$$\beta^2 \leq C(1 + \sinh(\sqrt{\varepsilon}\tilde{\beta}) + \tilde{\beta}^2).$$

Using this inequality, standard elliptic estimates and the facts that $v \leq 1$ and f_t is uniformly bounded we obtain (3.4). From here we deduce that T is continuous and compact. Indeed, for the latter assertion, note that if \mathcal{B} be a bounded set of

$X \times [0, \infty) \times [0, 1]$ then $T(\mathcal{B})$ is a bounded set in $C^{1,\mu}([0, 1]) \times \mathbb{R}$ for any $\mu \in (0, 1)$ and taking $\mu > \alpha$ it follows that this set is precompact in $C^{1,\alpha}([0, 1]) \times \mathbb{R}$. \square

Lemma 3.2. *The operator $T_{0,0}$ has a unique fixed point (u_0, β_0) in X .*

Proof. In this situation $u + v \equiv 1$ and hence the system reduces to

$$-\Delta u = \beta^2(1 - u)(u - \theta)^+ \quad \text{in } B_1, \quad u|_{\partial B_1} = \theta$$

with the additional requirement that

$$-\frac{\partial u}{\partial \nu} = \theta.$$

Let $w = u - \theta$. Then the equation for w becomes

$$(3.8) \quad -\Delta w = \beta^2(1 - \theta - w)w^+ \quad \text{in } B_1, \quad w|_{\partial B_1} = 0.$$

This equation is of logistic type and many properties are well known (see [13, 20]):

- a) Any solution w to (3.8) satisfies $0 \leq w < 1 - \theta$, and either $w \equiv 0$ or $w > 0$ in B_1 .
- b) (3.8) has a nontrivial solution if and only if $\beta > \beta^*$ where β^* is such that $(\beta^*)^2(1 - \theta) = \lambda_1$ and λ_1 denotes the first Dirichlet eigenvalue of $-\Delta$ in the unit ball.
- c) For $\beta > \beta^*$ there is a unique non-trivial solution, which we write as w_β .
- d) w_β is monotone increasing with respect to β and

$$\lim_{\beta \rightarrow \infty} w_\beta = 1 - \theta$$

uniformly on compact sets of B_1 and

$$\lim_{\beta \rightarrow \beta^*} w_\beta = 0.$$

From the above properties it follows that

$$\lim_{\beta \rightarrow \infty} -\frac{\partial w_\beta}{\partial \nu} = \infty \quad \text{and} \quad \lim_{\beta \rightarrow 0} -\frac{\partial w_\beta}{\partial \nu} = 0.$$

Moreover

$$\beta \mapsto -\frac{\partial w_\beta}{\partial \nu} \quad \text{is strictly increasing.}$$

It follows that there is a unique $\beta > \beta^*$ such that

$$-\frac{\partial w_\beta}{\partial \nu} = \theta.$$

We call this value β_0 and let $u_0 = w_{\beta_0}$. Then $(u_0, \beta_0) \in X$ is the fixed point of $T_{0,0}$. \square

Lemma 3.3. *Let (u_0, β_0) denote the unique fixed point of $T_{0,0}$ found in Lemma 3.2. Then (u_0, β_0) is nondegenerate.*

Proof. Let us write $(u_0, \beta_0) \in X$ the solution of $I - T_{0,0} = 0$ of Lemma 3.2. We have to verify that the linearization of $I - T_{0,0}$ around this solution is an invertible operator. Note that the operator $T_{0,0}$ involves only the nonlinearity $f_0(s) = (s - \theta)^+$ and that the fixed point (u_0, β_0) from Lemma 3.2 satisfies $u_0 > 0$ in $[0, 1]$ and $u_0'(1) < 0$. Therefore for φ in a neighborhood of u_0 in the topology of $C^{1,\alpha}([0, 1])$ we have $f_0(u_0 + \varphi + \theta) = (u_0 + \varphi)^+ = u_0 + \varphi$. Hence $T_{0,0}$ is differentiable at (u_0, β_0) .

We next compute the derivative of $T_{0,0}$ at (u_0, β_0) in the direction of $(\varphi, \sigma) \in X$, which we write as

$$(\psi, \gamma) = DT_{0,0}(u_0, \beta_0)(\varphi, \sigma).$$

Then

$$(3.9) \quad \psi = \psi_2 + \sigma\psi_1, \quad \gamma = \gamma_2 + \sigma\gamma_1$$

where (ψ_2, γ_2) and (ψ_1, γ_1) are computed as follows.

To compute ψ_2 we have linearize (3.3) with respect to \tilde{u} and then set $\tilde{u} = u = u_0$, $\tilde{\beta} = \beta = \beta_0$. Since for $t = 0$, we have $f_0(u_0 + \theta) = u_0$ and $f'_0(u_0 + \theta) = 1$, the function ψ_2 satisfies

$$(3.10) \quad \begin{cases} -\Delta\psi_2 = 2\beta_0\gamma_2v_0u_0 + \beta_0^2\frac{\partial v}{\partial \tilde{u}}u_0 + \beta_0^2v_0\varphi + 2\beta_0\gamma_2 & \text{in } B_1 \\ \psi_2(1) = 0, \quad \psi'_2(1) = 0, \end{cases}$$

where γ_2 is adjusted so that ψ_2 satisfies both boundary conditions, $\frac{\partial v}{\partial \tilde{u}}$ is found solving

$$(3.11) \quad \begin{cases} \Delta\frac{\partial v}{\partial \tilde{u}} = \beta_0^2\frac{\partial v}{\partial \tilde{u}}u_0 + \beta_0^2v_0\varphi & \text{in } B_1 \\ \frac{\partial v}{\partial \tilde{u}}(1) + \frac{\partial}{\partial r}\frac{\partial v}{\partial \tilde{u}}(1) = 0, \end{cases}$$

and v_0 is the solution of (3.2) with $\tilde{\beta} = \beta_0$, $\tilde{u} = u_0$ and $t = 0$. As explained in the proof of Lemma 3.2 then

$$v_0 = 1 - \theta - u_0.$$

To compute ψ_1 we linearize (3.3) with respect to $\tilde{\beta}$ and then set $\tilde{u} = u = u_0$, $\tilde{\beta} = \beta = \beta_0$. Therefore ψ_1 satisfies

$$(3.12) \quad \begin{cases} -\Delta\psi_1 = 2\beta_0\gamma_1v_0u_0 + \beta_0^2\frac{\partial v}{\partial \tilde{\beta}}u_0 + 2\beta_0\gamma_1 - 2\beta_2 & \text{in } B_1 \\ \psi_1(1) = 0, \quad \psi'_1(1) = 0, \end{cases}$$

where γ_1 is chosen so that both boundary conditions are satisfied and $\frac{\partial v}{\partial \tilde{\beta}}$ is computed from the equation

$$(3.13) \quad \begin{cases} \Delta\frac{\partial v}{\partial \tilde{\beta}} = 2\beta_0v_0u_0 + \beta_0^2\frac{\partial v}{\partial \tilde{\beta}}u_0 & \text{in } B_1 \\ \frac{\partial v}{\partial \tilde{\beta}}(1) + \frac{\partial}{\partial r}\frac{\partial v}{\partial \tilde{\beta}}(1) = 0. \end{cases}$$

We need to find the kernel of $I - D_{\tilde{u}, \tilde{\beta}}T_{0,0}(u_0, \beta_0)$, that is solutions $(\varphi, \sigma) \in X$ of:

$$(3.14) \quad D_{\tilde{u}, \tilde{\beta}}T_{0,0}(u_0, \beta_0)(\varphi, \sigma) = (\varphi, \sigma)$$

Using the notation above we may write $\varphi = \psi_2 + \sigma\psi_1$, $\sigma = \gamma_2 + \sigma\gamma_1$. We claim that

$$(3.15) \quad \varphi = -\left(\frac{\partial v}{\partial \tilde{u}} + \sigma\frac{\partial v}{\partial \tilde{\beta}}\right).$$

To prove this, let $z = \varphi + \frac{\partial v}{\partial \tilde{u}} + \sigma\frac{\partial v}{\partial \tilde{\beta}}$. Then a calculation shows that

$$\Delta z = 0 \quad \text{in } B_1$$

and $z(1) + z'(1) = 0$. This implies $z = 0$ and (3.15) follows. From the formulas (3.9)–(3.13) we get

$$-\Delta\varphi = \beta_0^2(1 - \theta - 2u_0)\varphi + 2\sigma\beta_0(1 - \theta - u_0)u_0$$

But also φ satisfies

$$\varphi(1) = 0, \quad \varphi'(1) = 0.$$

The linear elliptic operator $-\Delta - \beta_0^2(1 - \theta - 2u_0)$ with Dirichlet boundary condition arises as the linearization of (3.8) around the solution u_0 , and it is well known that it has a positive first eigenvalue. Since $\beta_0(1 - \theta - u_0)u_0 > 0$, for positive σ we would have $\varphi > 0$ and $\varphi'(1) < 0$ and for negative σ we would have $\varphi < 0$ and $\varphi'(1) > 0$, which are both impossible since $\varphi'(1) = 0$. Hence $\sigma = 0$ and also $\varphi = 0$. This shows that the linear equation $D_{\tilde{u}, \tilde{\beta}}T_{0,0}(u_0, \beta_0)(\varphi, \sigma) = (\varphi, \sigma)$ admits only the trivial solution. Since $D_{\tilde{u}, \tilde{\beta}}T_{0,0}(u_0, \beta_0)$ is compact we see that $I - D_{\tilde{u}, \tilde{\beta}}T_{0,0}(u_0, \beta_0)$ is an isomorphism. \square

Proof of Theorem 1.1. By Lemma 2.2 there exists $\delta > 0$ and $\beta_1 > 0$ such that for all $\beta \geq \beta_1$, $\beta\sqrt{\varepsilon} \leq \delta$ and all $t \in [0, 1]$ there is no solution in \mathcal{S} to the system (2.1). This means that $T_{\varepsilon, t}$ has no fixed point (u, β) if $\beta \geq \beta_1$ and $\beta\sqrt{\varepsilon} \leq \delta$. Let

$$R_1 = C(1 + \sinh(\delta) + \beta_1^2)$$

and define

$$\Omega_1 = \{(u, \beta) \in X : \|u\|_{C^{1,\alpha}([0,1])} < R_1, 0 < \beta < \beta_1\}.$$

Let $\varepsilon_1 = (\delta/\beta_1)^2$. Then Ω_1 is a bounded open set of X and for $0 \leq \varepsilon \leq \varepsilon_1$ and $0 \leq t \leq 1$ the operator $T_{\varepsilon, t}$ has no fixed point in $\partial\Omega_1$. Indeed, suppose $(u, \beta) \in \partial\Omega_1$ is a fixed point of $T_{\varepsilon, t}$. It is not possible that $\beta = \beta_1$ by Lemma 2.2. The case $\beta = 0$ is also impossible. This means that $0 < \beta < \beta_1$ and $\|u\|_{C^{1,\alpha}([0,1])} = R_1$. But by inequality (3.4) we would have

$$R_1 = \|u\|_{C^{1,\alpha}([0,1])} \leq C(1 + \sinh(\sqrt{\varepsilon}\beta) + \beta^2) < C(1 + \sinh(\delta) + \beta_1^2) = R_1$$

which is impossible. Thus the Leray-Schauder degree $\deg(I - T_{\varepsilon, t}, \Omega_1, 0)$ is well defined for $0 \leq \varepsilon \leq \varepsilon_1$, $t \in [0, 1]$. By Lemmas 3.2 and 3.3 we have

$$\deg(I - T_{0,0}, \Omega_1, 0) = \pm 1$$

and by homotopy invariance

$$\deg(I - T_{\varepsilon, t}, \Omega_1, 0) = \pm 1 \quad \forall 0 \leq \varepsilon \leq \varepsilon_1, \quad \forall t \in [0, 1].$$

This shows $T_{\varepsilon, t}$ has at least one fixed point in Ω_1 for $0 \leq \varepsilon \leq \varepsilon_1$ and $t \in [0, 1]$. In particular the system (2.1) has a solution with bounded β for any $0 \leq \varepsilon \leq \varepsilon_1$.

We know there exists $\varepsilon^* > 0$ such that for $\varepsilon > \varepsilon^*$ the system (2.1) has no solution, and hence $T_{\varepsilon, 1}$ has no fixed points in X for such ε . Let $0 < \varepsilon_0 < \varepsilon_1$ and define

$$\Omega_2 = \{(u, \beta) \in X : \|u\|_{C^{1,\alpha}([0,1])} < R_2, 0 < \beta < \beta_2\}$$

where

$$R_2 = C(1 + \sinh(\sqrt{2\varepsilon^*}\beta_2) + \beta_2^2) \quad \text{and} \quad \beta_2 = \theta/\sqrt{\varepsilon_0}.$$

Then $T_{\varepsilon, t}$ has no fixed points on $\partial\Omega_2$ for $\varepsilon_0 \leq \varepsilon \leq 2\varepsilon^*$ and $t \in [0, 1]$. Then using a homotopy along $\varepsilon \in [\varepsilon_0, 2\varepsilon^*]$ we find

$$\deg(I - T_{\varepsilon, 1}, \Omega_2, 0) = 0 \quad \forall \varepsilon_0 \leq \varepsilon \leq 2\varepsilon^*.$$

This implies that for $\varepsilon_0 \leq \varepsilon \leq \varepsilon_1$ the system has at least another solution and that this solution has β in the range $\delta/\sqrt{\varepsilon} \leq \beta \leq \theta/\sqrt{\varepsilon}$. Since ε_0 is arbitrary we obtain the same conclusion for $0 < \varepsilon \leq \varepsilon_1$. \square

4. EXISTENCE WHEN f IS DISCONTINUOUS

In this section we prove Theorem 1.2. We assume that f satisfies (1.2), (1.4), (1.3) and that

$$\eta = \lim_{u \rightarrow \theta^+} f(u) > 0.$$

For $n \geq 1$ define

$$f_n(t) = \min(f(t), n(t - \theta)^+).$$

Then f_n is continuous and satisfies (1.2) and (1.3) with a fixed constant. Moreover there is h_0 satisfying (2.2) and (2.3) and such that $f_n \geq h_0$ for all n . Such h_0 can be taken for instance as

$$h_0(t) = \min((t - \theta)^+, \sigma)$$

for some $\sigma > 0$ small enough.

By Theorem 1.1 there is $\varepsilon_1 > 0$ such that the system (2.1) admits 2 solutions $(u_n^1, v_n^1), (u_n^2, v_n^2)$ for all $\varepsilon \in (0, \varepsilon_1)$. Moreover there is a fixed number β_1 such that $\beta_n^1 \leq \beta_1$ and $\beta_n^2 \in (\delta/\sqrt{\varepsilon}, 1/(\theta\sqrt{\delta}))$ for all $n \geq 1$ and $\varepsilon \in (0, \varepsilon_1)$. From now on we fix $\varepsilon \in (0, \varepsilon_1)$ and study the limit as $n \rightarrow \infty$ of any of the 2 solutions which we call just (u_n, v_n) with parameter β_n . Since β_n is bounded and by standard elliptic estimates we may assume that $\beta_n \rightarrow \beta$, $u_n \rightarrow u$ and $v_n \rightarrow v$ in $C^{1,\alpha}([0, 1])$.

Since $f_n(u_n)$ is uniformly bounded we may extract a further subsequence such that $f_n(u_n) \rightarrow \varphi$ weakly-* in $L^\infty(0, 1)$. The function φ then satisfies $0 \leq \varphi \leq \max_{[0,1]} f$ in $[0, 1]$. Since u_n satisfies a linear equation for $r > 1$ and vanishes at infinity it has an explicit formula

$$u_n(r) = \theta \frac{e^{-(r-1)\beta_n\sqrt{\varepsilon}}}{r} \quad \text{for all } r \geq 1.$$

Setting $\varphi(r) = 0$ for $r > 1$ we see that u, v satisfy

$$\begin{cases} \Delta u - \varepsilon\beta^2 u = -\beta^2 v\varphi & \text{in } \mathbb{R}^3 \\ \Delta v = \beta^2 v\varphi & \text{in } \mathbb{R}^3 \\ \lim_{|x| \rightarrow \infty} u(x) = 0, & \lim_{|x| \rightarrow \infty} v(x) = 1 \end{cases}$$

The set $D = \{r \in [0, \infty) : u(r) \neq \theta\}$ is open (relative to $[0, \infty)$) and for any $r \in D$ we have $f_n(u_n(r)) \rightarrow f(u(r))$ as $n \rightarrow \infty$. It follows that $\varphi = f(u)$ in D . Outside D we have $\varphi \leq \eta$ a.e. Indeed, let $\psi \in C_0^\infty((0, 1))$, $\psi \geq 0$. Then

$$\int_0^1 f_n(u_n)\psi = \int_{D \cap [0,1]} f_n(u_n)\psi + \int_{D^c \cap [0,1]} f_n(u_n)\psi.$$

Since $f_n(u_n) \rightarrow f(u)$ in D by dominated convergence we have

$$\int_{D \cap [0,1]} f_n(u_n)\psi \rightarrow \int_{D \cap [0,1]} f(u)\psi.$$

On the other hand, if $r \in D^c$ $\limsup_{n \rightarrow +\infty} f_n(u_n(r)) \leq \eta$ so that $(f_n(u_n) - \eta)^+ \rightarrow 0$ on D^c . Then

$$\int_{D^c \cap [0,1]} (f_n(u_n) - \eta)\psi = \int_{D^c \cap [0,1]} (f_n(u_n) - \eta)^+ \psi - \int_{D^c \cap [0,1]} (f_n(u_n) - \eta)^- \psi \leq o(1)$$

where $o(1)$ denotes a sequence converging to 0 as $n \rightarrow +\infty$. It follows that

$$\int_{D^c \cap [0,1]} f_n(u_n) \psi \leq \int_{D^c \cap [0,1]} \eta \psi + o(1)$$

and hence

$$\int_0^1 \varphi \psi = \lim_{n \rightarrow +\infty} \int_0^1 f_n(u_n) \psi \leq \int_{D \cap [0,1]} f(u) \psi + \int_{D^c \cap [0,1]} \eta \psi.$$

This shows that $\varphi \leq \eta$ a.e. in D^c .

Our main objective is to show that the complement of D is finite. If $u'(r) \neq 0$ whenever $u(r) = \theta$ then D^c is discrete and since it is contained in $[0, 1]$ it is finite.

Let us analyze the case where for some $r \in [0, 1]$ we have $u(r) = \theta$ and $u'(r) = 0$. Let

$$r_0 = \sup \{r > 0 : u(r) = \theta, u'(r) = 0\}.$$

Then $r_0 < 1$ and $u(r_0) = \theta$, $u'(r_0) = 0$. We assert that there is a small interval $(r_0, r_0 + \sigma)$, $\sigma > 0$ such that $u > \theta$ in $(r_0, r_0 + \sigma)$. To prove this we start ruling out the possibility that $u(r_n) = \theta$ for some infinite sequence $r_n \searrow r_0$. We actually may assume that if n is even then $u > \theta$ on (r_{n+1}, r_n) and if n is odd $u < \theta$ on (r_{n+1}, r_n) . Let us see that the following holds

$$(4.1) \quad u'(r_{n+1}) = -u'(r_n) + O(\ell_n^2)$$

where

$$\ell_n = r_{n+1} - r_n$$

and $O(\ell_n^2)$ denotes a sequence bounded by $C\ell_n^2$ with C independent of n as $n \rightarrow \infty$. Suppose first that $u < \theta$ on (r_{n+1}, r_n) and define \tilde{u}_n by

$$u(r) = \theta + \ell_n^2 \tilde{u}_n((r - r_{n+1})/\ell_n) \quad r \in [r_{n+1}, r_n].$$

Then

$$\begin{aligned} \tilde{u}_n'' + \frac{2\ell_n}{r\ell_n + r_{n+1}} \tilde{u}_n' &= \varepsilon\beta^2(\ell_n^2 \tilde{u}_n + \theta) \quad \text{for } r \in [0, 1]. \\ \tilde{u}_n(0) &= \tilde{u}_n(1) = 0. \end{aligned}$$

This equation implies that \tilde{u}_n and \tilde{u}_n' are uniformly bounded on $[0, 1]$. Therefore

$$\tilde{u}_n'' = \varepsilon\beta^2\theta + O(\ell_n) \quad \text{for } r \in [0, 1]$$

hence integrating

$$\tilde{u}_n(r) = \frac{1}{2}\varepsilon\beta^2(1 + O(\ell_n))r(r-1) \quad \text{for } r \in [0, 1]$$

and

$$(4.2) \quad \tilde{u}_n'(r) = \frac{1}{2}\varepsilon\beta^2(1 + O(\ell_n))(2r-1) \quad \text{for } r \in [0, 1].$$

This proves (4.1) in this case.

Now suppose that $u > \theta$ on (r_{n+1}, r_n) . Defining \tilde{u}_n as before we now obtain the following equation

$$\begin{aligned} \tilde{u}_n'' + \frac{2\ell_n}{r\ell_n + r_{n+1}} \tilde{u}_n' &= \varepsilon\beta^2(\ell_n^2 \tilde{u}_n + \theta) - \beta^2 \tilde{v}_n f(\theta + \ell_n^2 \tilde{u}_n) \quad \text{for } r \in [0, 1]. \\ \tilde{u}_n(0) &= \tilde{u}_n(1) = 0 \end{aligned}$$

where

$$\tilde{v}_n(r) = v(r_{n+1} + \ell_n r) \quad r \in [0, 1].$$

This equation implies that $\tilde{u}_n, \tilde{u}'_n$ are uniformly bounded on $[0, 1]$. Thus

$$\tilde{u}''_n = \varepsilon\beta^2\theta - \beta^2\tilde{v}_n(0)f(\theta + \ell_n^2\tilde{u}_n) + O(\ell_n) \quad \text{for } r \in [0, 1].$$

Multiplying this equation by \tilde{u}_n and integrating in $[0, 1]$ yields

$$\frac{1}{2}(\tilde{u}_n(1)')^2 - \frac{1}{2}(\tilde{u}_n(0)')^2 = O(\ell_n)$$

which shows that

$$\tilde{u}_n(1)' = -\tilde{u}_n(0)' + O(\ell_n)$$

and proves (4.1) in this case.

Using (4.1) inductively we find

$$u'(r_n) = (-1)^k u'(r_{n+k}) + \sum_{j=n}^{n+k-1} O(\ell_j^2) \quad \forall n \geq 1, k \geq 1.$$

Letting $k \rightarrow +\infty$, and using that $u'(r_0) = 0$ yields

$$u'(r_n) = \sum_{j=n}^{\infty} O(\ell_j^2) \quad \forall n \geq 1.$$

Choose a subsequence $n_i \rightarrow \infty$ as $i \rightarrow \infty$ such that $\ell_{n_i} \geq \ell_j$ for all $j \geq n_i$. Then

$$|u'(r_{n_i})| \leq C\ell_{n_i} \sum_{j=n_i}^{\infty} \ell_j.$$

But (4.2) implies that $|u'(r_n)| = \ell_n \varepsilon \beta^2 (1 + O(\ell_n))/2$. It follows that

$$\ell_{n_i} \varepsilon \beta^2 (1 + O(\ell_{n_i}))/2 \leq C\ell_{n_i} \sum_{j=n_i}^{\infty} \ell_j$$

which is impossible as $i \rightarrow \infty$.

This establishes the assertion that u cannot oscillate infinitely many times around θ to the right of r_0 . It follows that for some $\sigma > 0$ either $u > \theta$ in $(r_0, r_0 + \sigma)$ or $u < \theta$ in $(r_0, r_0 + \sigma)$. The latter can in fact not occur by the Hopf lemma.

Since u satisfies the ODE $u'' + \frac{2}{r}u' = \varepsilon\beta^2u - \beta^2vf(u)$ in $(r_0, r_0 + \sigma)$ and in this interval $u > \theta$ we see that u is C^2 in $[r_0, r_0 + \sigma)$. Since r_0 is a minimum of u restricted to $[r_0, r_0 + \sigma)$ it follows that $\lim_{r \rightarrow r_0^+} u'' \geq 0$. Which yields the following inequality

$$(4.3) \quad \varepsilon\beta^2\theta - \beta^2v(r_0)\eta \geq 0.$$

Now, if v is constant in $[0, r_0]$ this means that $\varphi = 0$ a.e. in $[0, r_0]$ and $u \leq \theta$ in $[0, r_0]$. Actually $u < \theta$ in $[0, r_0)$ by the strong maximum principle and this finishes the proof in this case. If v is not constant, then $v(r) < v(r_0)$ for all $r \in [0, r_0)$. Let us verify that D^c is discrete. Suppose that for some $r_1 \in [0, r_0)$ we have $u(r_1) = \theta$. If $u'(r_1) \neq 0$ then r_1 is isolated. If $u'(r_1) = 0$ then integrating the equation for u we have

$$r^2 u'(r) = \int_{r_1}^r t^2 (\varepsilon\beta^2 u - \beta^2 v \varphi) dt.$$

For r near r_1 we have

$$\begin{aligned} & \text{if } u(r) < \theta \text{ then } \varepsilon\beta^2u(r) - \beta^2v(r)\varphi(r) > 0 \\ & \text{if } u(r) = \theta \text{ then } \varepsilon\beta^2u(r) - \beta^2v(r)\varphi(r) > \varepsilon\beta^2\theta - \beta^2v(r_0)\eta \geq 0 \\ & \text{if } u(r) > \theta \text{ then } \varepsilon\beta^2u(r) - \beta^2v(r)\varphi(r) = \varepsilon\beta^2u(r) - \beta^2v(r)f(u(r)). \end{aligned}$$

In the second line above we have used that $v(r) < v(r_0)$, $\varphi \leq \eta$ and (4.3). In the third line above we may say that $\varepsilon\beta^2u(r) - \beta^2v(r)f(u(r)) > 0$ if r is sufficiently close to r_1 by continuity, $v(r) < v(r_0)$ and (4.3) (we may regard f as continuous here since we are working with values above θ). This shows that u is strictly convex in a neighborhood of r_1 and hence there are no other points in D^c close to r_1 . This shows that D^c is discrete, hence finite, and finishes the proof of the theorem.

5. THE HEAVISIDE IGNITION

In this section we perform explicit calculations for the ignition nonlinearity $f(u) = \chi_{[u>\theta]}$, where $0 < \theta < 1$. We first reduce the differential equations to a finite number of equations in some parameters. In a second part we obtain rigorously the bifurcation diagram of Figure 1 for small ε .

We rewrite (2.1) in the form

$$(5.1) \quad \begin{cases} \Delta u - \beta^2\varepsilon u = 0 & \text{in } (1, +\infty) \\ \Delta v = 0 & \text{in } (1, +\infty) \\ u(1) = \theta, \quad v(1) = \gamma \\ u(+\infty) = 0 \quad v(+\infty) = 1 \end{cases}$$

and

$$(5.2) \quad \begin{cases} \Delta u - \beta^2\varepsilon u = -\beta^2vf(u) & \text{in } (0, 1) \\ \Delta v = \beta^2vf(u) & \text{in } (0, 1) \\ u(1) = \theta \quad v(1) = \gamma \\ u'(0) = 0 \quad v'(0) = 0 \\ u'(1^-) = u'(1^+) \quad v'(1^-) = v'(1^+) \end{cases}$$

where γ is a parameter to be adjusted.

Observe that problem (5.1) can explicitly be solved. Indeed, the following functions solve (5.1)

$$u = \frac{\theta e^{-\beta\sqrt{\varepsilon}(r-1)}}{r} \quad v = 1 - \frac{1-\gamma}{r}.$$

Hence the last condition in (5.2) becomes

$$u'(1^+) = -\theta(1 + \beta\sqrt{\varepsilon}) \quad v'(1^+) = 1 - \gamma.$$

5.1. Solving (5.2) with the assumption $u > \theta$ in $B(0, 1)$. Assuming that $u > \theta$, in $B(0, 1)$ the subsystem (5.2) is reduced to the following linear problem

$$(5.3) \quad \Delta u - \beta^2\varepsilon u = -\beta^2v \quad \text{in } B(0, 1)$$

$$(5.4) \quad \Delta v = \beta^2v \quad \text{in } B(0, 1)$$

$$(5.5) \quad u(1) = \theta \quad v(1) = \gamma$$

$$(5.6) \quad v'(1^-) = 1 - \gamma.$$

$$(5.7) \quad u'(1^-) = -\theta(1 + \beta\sqrt{\varepsilon})$$

For $\varepsilon \neq 1$, observe that the two following functions

$$(5.8) \quad v := \frac{1}{\beta \operatorname{ch}(\beta)} \frac{\operatorname{sh}(\beta r)}{r} \quad r \in (0, 1)$$

$$(5.9) \quad u := -\frac{v}{1-\varepsilon} + \frac{\left(\theta + \frac{th(\beta)}{\beta(1-\varepsilon)}\right) \operatorname{sh}(\beta\sqrt{\varepsilon}r)}{\operatorname{sh}(\beta\sqrt{\varepsilon})} \frac{1}{r} \quad r \in (0, 1)$$

solve equations (5.3)-(5.6). From the formula for v we can see that

$$\gamma = \frac{th(\beta)}{\beta}.$$

Now to obtain a solution of subsystem (5.2), it then remains to adjust β and ε to get (5.7). This equation is

$$-\theta(1 + \beta\sqrt{\varepsilon}) = u'(1^-) = \frac{v'(1)}{1-\varepsilon} + \frac{\left(\theta - \frac{th(\beta)}{\beta(1-\varepsilon)}\right)}{\operatorname{sh}(\beta\sqrt{\varepsilon})} (\beta\sqrt{\varepsilon} \operatorname{ch}(\beta\sqrt{\varepsilon}) - \operatorname{sh}(\beta\sqrt{\varepsilon}))$$

which after simplification yields

$$(5.10) \quad \theta(1-\varepsilon)(1 + th(\beta\sqrt{\varepsilon})) + \frac{th(\beta)}{\beta} = \frac{th(\beta\sqrt{\varepsilon})}{\beta\sqrt{\varepsilon}}.$$

Let us define

$$(5.11) \quad g(\varepsilon, \beta) := \theta(1-\varepsilon)(1 + th(\beta\sqrt{\varepsilon})) + \frac{th(\beta)}{\beta} - \frac{th(\beta\sqrt{\varepsilon})}{\beta\sqrt{\varepsilon}}.$$

It can be shown that there is $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*)$, $g(\varepsilon, \cdot)$ has 2 zeros, $\beta_\varepsilon^- < \beta_\varepsilon^+$. The function $g(0, \cdot)$ has a unique zero $\beta_0 = \lim_{\varepsilon \rightarrow 0} \beta_\varepsilon^-$. A calculation (see Lemma 5.1) shows that for $\varepsilon > 0$ small and $\beta = \beta_\varepsilon^-$ the solution u obtained by the formula (5.9) satisfies $u > \theta$ in $[0, 1)$. Along the lower branch β_ε^- , $0 \leq \varepsilon \leq \varepsilon^*$ we observe numerically that $u > \theta$ in $[0, 1)$. After reaching ε^* we continue the curve along the upper branch β_ε^+ now decreasing ε from ε^* to 0 until we reach a critical value $\varepsilon = \varepsilon_0$ where the corresponding solution given by (5.9) satisfies $u > \theta$ in $(0, 1)$ but $u(0) = \theta$.

5.2. Solving the subsystem (5.2) with the assumption $u > \theta$ in $B(0, 1) \setminus B(0, \eta)$ and $u \leq \theta$ in $B(0, \eta)$ for some $\eta > 0$.

Under this assumption the problem (5.2) decouples again into two subsystems

$$(5.12) \quad \begin{cases} \Delta u - \beta^2 \varepsilon u = -\beta^2 v & \text{in } B(0, 1) \setminus B(0, \eta) \\ \Delta v = \beta^2 v & \text{in } B(0, 1) \setminus B(0, \eta) \\ u(1) = \theta = u(\eta) & v(1) = \gamma \quad v(\eta) = \delta \\ u'(1^-) = u'(1^+) & v'(1^-) = v'(1^+) \\ u'(\eta^-) = u'(\eta^+) & v'(\eta^-) = v'(\eta^+) \end{cases}$$

and

$$(5.13) \quad \begin{cases} \Delta u - \beta^2 \varepsilon u = 0 & \text{in } B(0, \eta) \\ \Delta v = 0 & \text{in } B(0, \eta) \\ u(\eta) = \theta \quad v(\eta) = \delta \end{cases}$$

where δ, γ, η are parameters to be found. Using the conditions at $r = \eta$ and $r = 1$ we will be able to reduce the parameters to only β and η which will be implicitly defined as functions of ε by 2 equations.

Observe that since v is harmonic in $B(0, \eta)$ with the Dirichlet condition $v(\eta) = \delta$, we deduce that $v \equiv \delta$ and therefore $v'(\eta^-) = 0$.

Let us now solve (5.12) explicitly. First, observe that the function $v = A \frac{sh(\beta r)}{r} + B \frac{ch(\beta r)}{r}$ satisfies

$$\Delta v - \beta^2 v = 0 \quad \text{in} \quad B(0, 1) \setminus B(0, \eta).$$

Therefore, choosing A and B such that

$$\begin{aligned} \gamma &= A sh(\beta) + B ch(\beta) \\ 1 - \gamma &= A[\beta ch(\beta) - sh(\beta)] + B[\beta sh(\beta) - ch(\beta)], \end{aligned}$$

it follows that v satisfies the right boundary conditions on $\partial B(0, 1)$. Solving for A and B we find

$$A = \frac{ch(\beta) - \gamma \beta sh(\beta)}{\beta} \quad \text{and} \quad B = \frac{sh(\beta) - \gamma \beta ch(\beta)}{\beta}.$$

To obtain a full solution it remains to adjust γ in such way that $v'(\eta) = 0$. But

$$v' = A \left[\frac{\beta ch(\beta r)}{r} - \frac{sh(\beta r)}{r^2} \right] + B \left[\frac{\beta sh(\beta r)}{r} - \frac{ch(\beta r)}{r^2} \right]$$

and setting $v'(\eta) = 0$ in the above equation yields

$$\begin{aligned} A \left[\frac{\beta ch(\beta \eta)}{\eta} - \frac{sh(\beta \eta)}{\eta^2} \right] + B \left[\frac{\beta sh(\beta \eta)}{\eta} - \frac{ch(\beta \eta)}{\eta^2} \right] &= 0 \\ A \left(\eta \beta ch(\beta \eta) - sh(\beta \eta) \right) + B \left(\eta \beta sh(\beta \eta) - ch(\beta \eta) \right) &= 0 \end{aligned}$$

Substituting now the values of A and B in the above equation, it follows

$$\left(ch(\beta) - \gamma \beta sh(\beta) \right) C - \left(sh(\beta) - \gamma \beta ch(\beta) \right) D = 0,$$

where

$$\begin{aligned} C &:= \left(\eta \beta ch(\beta \eta) - sh(\beta \eta) \right) \\ D &:= \left(\eta \beta sh(\beta \eta) - ch(\beta \eta) \right) \end{aligned}$$

Therefore, we have

$$(5.14) \quad \gamma = \frac{C - D th(\beta)}{\beta (C th(\beta) - D)}$$

which after simplification reads

$$\gamma = \gamma(\beta, \eta) = \frac{\eta \beta - th(\eta \beta) - th(\beta) [\eta \beta th(\beta \eta) - 1]}{\beta \left[(\eta \beta - th(\eta \beta)) th(\beta) + 1 - \eta \beta th(\eta \beta) \right]}.$$

Thus we may compute now $\delta = v(\eta)$:

$$\delta = \left(\frac{ch(\beta) - \gamma \beta sh(\beta)}{\beta} \right) \frac{sh(\beta \eta)}{\eta} - \left(\frac{sh(\beta) - \gamma \beta ch(\beta)}{\beta} \right) \frac{ch(\beta \eta)}{\eta}.$$

From (5.13) we know that in $B(0, \eta)$, we have the following

$$u = \frac{\theta \eta}{sh(\eta \beta \sqrt{\varepsilon})} \frac{sh(r \beta \sqrt{\varepsilon})}{r}.$$

Therefore,

$$(5.15) \quad u'(\eta) = \theta \left[\frac{\beta\sqrt{\varepsilon}ch(\eta\beta\sqrt{\varepsilon})}{sh(\eta\beta\sqrt{\varepsilon})} - \frac{1}{\eta} \right] = \frac{\theta ch(\eta\beta\sqrt{\varepsilon})}{\eta sh(\eta\beta\sqrt{\varepsilon})} [\eta\beta\sqrt{\varepsilon} - th(\eta\beta\sqrt{\varepsilon})].$$

Let us now find the solution u in $B(0, 1) \setminus B(0, \eta)$. As for the construction of v , let us observe that for $\varepsilon \neq 1$,

$$(5.16) \quad u := \frac{1}{\varepsilon - 1}v + E \frac{sh(r\beta\sqrt{\varepsilon})}{r} + F \frac{ch(r\beta\sqrt{\varepsilon})}{r}$$

solves the following equation

$$\Delta u - \beta^2 \varepsilon u = -\beta^2 v$$

By taking

$$(5.17) \quad E(\beta, \eta, \gamma) := \frac{ch(\beta\sqrt{\varepsilon}) - \gamma\beta\sqrt{\varepsilon}sh(\beta\sqrt{\varepsilon})}{(1 - \varepsilon)\beta\sqrt{\varepsilon}} - \theta \left(ch(\beta\sqrt{\varepsilon}) + sh(\beta\sqrt{\varepsilon}) \right)$$

$$(5.18) \quad F(\beta, \eta, \gamma) := -\frac{sh(\beta\sqrt{\varepsilon}) - \gamma\beta\sqrt{\varepsilon}ch(\beta\sqrt{\varepsilon})}{(1 - \varepsilon)\beta\sqrt{\varepsilon}} + \theta \left(ch(\beta\sqrt{\varepsilon}) + sh(\beta\sqrt{\varepsilon}) \right)$$

we easily verify that $u(1) = \theta$ and $u'(1^-) = -\theta(1 + \sqrt{\varepsilon}\beta)$. It remains now to impose the boundary condition on $\partial B(0, \eta)$. From the formula for u (5.16) we have

$$(5.19) \quad \begin{aligned} u(\eta) &= \frac{\delta}{\varepsilon - 1} + E \frac{sh(\eta\beta\sqrt{\varepsilon})}{\eta} + F \frac{ch(\eta\beta\sqrt{\varepsilon})}{\eta} \\ u'(\eta) &= E\beta\sqrt{\varepsilon} \frac{ch(\beta\sqrt{\varepsilon}\eta)}{\eta} + F\beta\sqrt{\varepsilon} \frac{sh(\beta\sqrt{\varepsilon}\eta)}{\eta} - E \frac{sh(\beta\sqrt{\varepsilon}\eta)}{\eta^2} - F \frac{ch(\beta\sqrt{\varepsilon}\eta)}{\eta^2} \end{aligned}$$

since $v'(\eta) = 0$. Thus the equation $u(\eta) = \theta$ becomes

$$(5.20) \quad \frac{\delta}{\varepsilon - 1} + E(\beta, \eta, \gamma(\beta, \eta)) \frac{sh(\eta\beta\sqrt{\varepsilon})}{\eta} + F(\beta, \eta, \gamma(\beta, \eta)) \frac{ch(\eta\beta\sqrt{\varepsilon})}{\eta} = \theta$$

and (5.15) combined with (5.19) becomes

$$(5.21) \quad \begin{aligned} &E(\beta, \eta, \gamma(\beta, \eta))\eta\beta\sqrt{\varepsilon}ch(\beta\sqrt{\varepsilon}\eta) + F(\beta, \eta, \gamma(\beta, \eta))\eta\beta\sqrt{\varepsilon}sh(\beta\sqrt{\varepsilon}\eta) \\ &\quad - E(\beta, \eta, \gamma(\beta, \eta))sh(\beta\sqrt{\varepsilon}\eta) - F(\beta, \eta, \gamma(\beta, \eta))ch(\beta\sqrt{\varepsilon}\eta) \\ &= \theta \left[\beta\sqrt{\varepsilon}coth(\eta\beta\sqrt{\varepsilon}) - \frac{1}{\eta} \right]. \end{aligned}$$

We solve numerically equations (5.20) and (5.21) with β and η as unknowns that depend on ε , for $0 < \varepsilon < \varepsilon_0$, where ε_0 is the critical value of ε described in Subsection 5.1. The result from this numerical computation is shown in Figure 1.

5.3. Results for small ε .

Lemma 5.1. *For $\varepsilon > 0$ suitably small β_ε^- is a smooth function of ε and the solution given by (5.9) corresponding to $\beta = \beta_\varepsilon^-$ satisfies $u > \theta$ in $[0, 1)$.*

Proof. Let β_1 the unique solution of the equation

$$\frac{th(\beta)}{\beta} = \frac{(1 - \theta)}{2}.$$

Then

$$\begin{aligned}
g(\varepsilon, \beta_1, \theta) &= \theta(1 - \varepsilon)(1 + th(\beta_1\sqrt{\varepsilon})) + \frac{1 - \theta}{2} - \frac{th(\beta_1\sqrt{\varepsilon})}{\beta_1\sqrt{\varepsilon}} \\
&= \theta(1 - \varepsilon)th(\beta_1\sqrt{\varepsilon}) + \frac{1 + \theta - 2\theta\varepsilon}{2} - \frac{th(\beta_1\sqrt{\varepsilon})}{\beta_1\sqrt{\varepsilon}} \\
&= \theta(1 - \varepsilon)th(\beta_1\sqrt{\varepsilon}) - \frac{1 - \theta}{2} - \theta\varepsilon + \left[1 - \frac{th(\beta_1\sqrt{\varepsilon})}{\beta_1\sqrt{\varepsilon}}\right]
\end{aligned}$$

For $\beta_1\sqrt{\varepsilon} < 1$, using that $\beta_1\sqrt{\varepsilon} - \frac{(\beta_1\sqrt{\varepsilon})^3}{3} \leq th(\beta_1\sqrt{\varepsilon}) \leq \beta_1\sqrt{\varepsilon}$, we end up with

$$\begin{aligned}
g(\varepsilon, \beta_1, \theta) &\leq \theta(1 - \varepsilon)\beta_1\sqrt{\varepsilon} - \frac{1 - \theta}{2} - \theta\varepsilon + \frac{(\beta_1\sqrt{\varepsilon})^2}{3} \\
&\leq \frac{(\beta_1\sqrt{\varepsilon})^2}{3} + \theta\beta_1\sqrt{\varepsilon} - \frac{1 - \theta}{2} - \theta\varepsilon.
\end{aligned}$$

Computing now the positive roots of $\frac{X^2}{3} + \theta X - \frac{1 - \theta}{2}$, yields $g(\varepsilon, \beta_1, \theta) < 0$ for

$$\sqrt{\varepsilon}\beta_1 \leq \frac{3\theta\left(\sqrt{1 + \frac{2(1-\theta)}{3\theta^2}} - 1\right)}{2}.$$

Therefore, we achieve $g(\varepsilon, \beta_1, \theta) < 0$, for ε in $(0, \min\left\{\left(\frac{3\theta\left(\sqrt{1 + \frac{2(1-\theta)}{3\theta^2}} - 1\right)}{2\beta_1}\right)^2, \frac{1}{\beta_1^2}\right\})$.

Let us denote

$$\varepsilon_1 := \left(\frac{3\theta\left(\sqrt{1 + \frac{2(1-\theta)}{3\theta^2}} - 1\right)}{2\beta_1}\right)^2.$$

Hence, for any (ε, θ) in $(0, \min\{\varepsilon_1, \frac{1}{\beta_1^2}\}) \times (0, 1)$ fixed there exists two possible solution to $g(\varepsilon, \beta, \theta) = 0$. Moreover, $\beta^- < \beta_1 < \beta^+$.

To obtain a solution, to problem (5.2), we still need to show that the constructed solutions effectively satisfies the conditions $u > \theta$ in $B(0, 1)$. Let us observe that $u - \theta$ satisfies:

$$\Delta(u - \theta) - \varepsilon\beta^2(u - \theta) = -\beta^2(v - \varepsilon\theta).$$

If $v(0) - \varepsilon\theta > 0$ then using the maximum principle, it follows that $(u - \theta) > 0$.

Since $v(r) = \left(\frac{1}{ch(\beta)\beta}\right) \frac{sh(\beta r)}{r}$, it follows that $v(0) = \frac{2}{ch(\beta)}$. Hence, we end up with the condition

$$\frac{2}{ch(\beta)} \geq \varepsilon\theta.$$

We can conclude with a final estimate on ε , namely since $\beta^- < \beta_1$, and $\frac{1}{ch(\beta)}$ is a decreasing function, we have the following uniform estimates on ε

$$\varepsilon \leq \frac{(1 - \theta)\beta_1}{\theta sh(\beta_1)}.$$

We therefore have construct a solution to the problem (1), when (ε, θ) belongs to $(0, \varepsilon_0(\theta)) \times (0, 1)$ where

$$\varepsilon_0(\theta) := \min\left\{\frac{1}{\beta_1^2}; \frac{(1 - \theta)\beta_1}{\theta sh(\beta_1)}; \varepsilon_1; \frac{(1 - \theta)^2}{2}\right\}$$

□

Lemma 5.2. For $\varepsilon > 0$ small the system (5.20), (5.21) has a unique solution $\beta_\varepsilon^+, \eta$ with $\eta > 0$, β_ε^+ is a smooth function of ε and u defined by (5.16) satisfies $u > \theta$ in $(\eta, 1)$.

Proof. The 2 equations (5.20) and (5.21) can be written in the form

$$E \sinh(a\eta) + F \cosh(a\eta) = \eta\theta + \frac{\eta\delta}{1-\varepsilon}$$

$$E (a\eta \cosh(a\eta) - \sinh(a\eta)) + F (a\eta \sinh(a\eta) - \cosh(a\eta)) = \eta\theta \left(\frac{a\eta}{\tanh(a\eta)} - 1 \right)$$

where

$$a = \beta\sqrt{\varepsilon}.$$

Instead of ε, β, η consider the variables t, a, x defined by the following relations

$$t = \sqrt{\varepsilon}, \quad a = \beta\sqrt{\varepsilon}, \quad \eta = 1 - \frac{x}{\beta}.$$

Consider also the function $G(t, a, x) = (G_1(t, a, x), G_2(t, a, x))$ where

$$G_1(t, a, x) = \beta \left[E \sinh(a\eta) + F \cosh(a\eta) - \eta\theta - \frac{\eta\delta}{1-\varepsilon} \right]$$

$$G_2(t, a, x) = E (a\eta \cosh(a\eta) - \sinh(a\eta)) + F (a\eta \sinh(a\eta) - \cosh(a\eta))$$

$$- \eta\theta \left(\frac{a\eta}{\tanh(a\eta)} - 1 \right)$$

where δ, γ, E, F are as before. We see that we have a solution to the system if and only if $G(t, x, a) = 0$.

Using the fact that

$$\lim_{t \rightarrow 0} \beta\gamma = \frac{1}{\tanh(x)} \quad \text{and} \quad \lim_{t \rightarrow 0} \beta\delta = \frac{1}{\sinh(x)}$$

it is not difficult to verify that G maybe extended in a C^1 manner for $t = 0$, $a, x \in (0, \infty)$ with the values

$$G_1(0, a, x) = -x + \theta ax + \frac{1}{\tanh(x)} + x\theta - \frac{1}{\sinh(x)}$$

$$G_2(0, a, x) = 1 - \theta(1+a) - \theta \left(\frac{a}{\tanh(a)} - 1 \right)$$

Note that the equation $G_2(0, a, x) = 0$ is equivalent to

$$\frac{\frac{a}{\tanh(a)} - 1}{1 - \theta(1+a)} = \frac{1}{\theta}$$

which is seen to have a unique solution $a_0 \in (0, 1/\theta - 1)$ since the left hand side defines a strictly increasing function of a which goes to zero as $a \rightarrow 0$ and goes to $+\infty$ as $a \nearrow 1/\theta - 1$.

Now the equation $G_1(0, a_0, x) = 0$ can be written as

$$(5.22) \quad 1 - \frac{1}{x \tanh(x)} + \frac{1}{x \sinh(x)} = \theta(1+a_0) \in (0, 1).$$

The above left hand side is strictly increasing with limits $1/2$ as $x \rightarrow 0$ and 1 as $x \rightarrow \infty$. Using the inequality

$$\frac{a}{\tanh(a)} - 1 \leq a \quad \forall a \geq 0$$

we see that $1 - \theta - \theta a_0 \leq \theta a_0$ which yields $a_0 \geq \frac{1-\theta}{2\theta}$. Hence

$$\theta(1 + a_0) \geq \frac{1}{2} + \frac{\theta}{2} > \frac{1}{2}.$$

Thus (5.22) has a unique solution $x_0 \in (0, \infty)$. \square

6. ESTIMATE ON ε^*

As mentioned before the largest value of ε such that (1.1) has a nontrivial solution satisfies

$$\varepsilon^* \leq \varepsilon^0$$

where

$$\varepsilon^0 = \inf\{\varepsilon > 0 : g_\varepsilon(s) \leq 0 \quad \forall s \in [0, 1]\},$$

and

$$g_\varepsilon(s) := (1 - s)f(s) - \varepsilon s.$$

In the particular case of $f(u) = \chi_{[u > \theta]}$, $0 < \theta < 1$ we obtain the value $\varepsilon^0 = \frac{1}{\theta}$. This estimate can be sharpened.

For this, given $\varepsilon > 0$ let us introduce

$$G_\varepsilon(b) = \int_0^b g_\varepsilon(s) ds.$$

For $0 < \varepsilon < \varepsilon_0$ define also $a(\varepsilon)$ as the smallest zero of g_ε in the interval $(0, 1)$. We $b(\varepsilon)$ be the largest b in $[0, 1]$ such that g_ε is positive on $(a(\varepsilon), b(\varepsilon))$.

Proposition 6.1. *Assume that f satisfies (1.2) and is continuous. Then (1.1) has no nontrivial solution if $\varepsilon > \varepsilon^1$ where*

$$(6.1) \quad \varepsilon^1 = \sup\{\varepsilon > 0 \mid G_\varepsilon(b(\varepsilon)) > 0\}.$$

Before proving the above proposition, let us recall a result on the scalar problem on a bounded domain $\Omega \subset \mathbb{R}^N$:

$$(6.2) \quad \begin{cases} -\Delta w = h(w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

There is a vast literature concerning the existence of positive solutions of the above equation. We will just mention a necessary and sufficient condition for the existence of positive solutions.

Theorem 6.2. *(Clément and Sweers [14]) Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $h \in C^1(\mathbb{R})$. Assume there exists $b_1 < b_2$ with $b_2 > 0$ such that $h(b_1) = h(b_2) = 0$ and $h > 0$ on (b_1, b_2) . Then there exists positive solution w to (6.2) with $\max w \in (b_1, b_2)$ if and only if $\int_b^{b_2} h(s) ds > 0$ for all $b \in [0, b_2)$.*

Proof of Proposition 6.1. Fix $\varepsilon > \varepsilon^1$. From our assumption, we have $G(b(\varepsilon), \varepsilon) < 0$. Let \tilde{g} a smooth function such that $\tilde{g}_\varepsilon \geq g_\varepsilon$ and $\int_0^{\tilde{b}(\varepsilon)} \tilde{g}_\varepsilon(s) ds < 0$. This is always possible since $G(b(\varepsilon), \varepsilon) < 0$. Observe now that from the necessary and sufficient condition given in theorem 6.2, the following scalar problem

$$(6.3) \quad \begin{cases} \Delta w = -\tilde{g}_\varepsilon(w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

does not have any positive solution for any bounded domain Ω .

Now, we argue by contradiction, assume there exists a non trivial couple (u, v) solution of the system (1.1). An easy computation shows that u is a sub-solution of the problem (6.3) with $\Omega = \mathbb{R}^n$. Take $\delta > 0$ small, and define

$$g_{\varepsilon, \delta} = \tilde{g}_\varepsilon(s + \delta).$$

So for δ small enough, (i.e $\delta < \delta_0$), we have:

$$G_\delta(\tilde{b}(\varepsilon) - \delta) = \int_0^{\tilde{b}(\varepsilon) - \delta} g_{\varepsilon, \delta}(s) ds = \int_\delta^{\tilde{b}(\varepsilon)} \tilde{g}_\varepsilon(s) ds = G(\tilde{b}(\varepsilon), \varepsilon) - \int_0^\delta \tilde{g}_\varepsilon(s) ds \leq 0$$

Therefore $g_{\varepsilon, \delta}$ does not satisfy the necessary existence condition of Theorem 6.2. Now, take $\Omega = B(0, R)$. Since u goes uniformly to 0 when $|x|$ goes to infinity, we have $\sup_{\partial\Omega} u \rightarrow 0$ as $R \rightarrow +\infty$. Thus, with $\delta = \sup_{\partial\Omega} u < \delta_0$, $u_\delta = u - \delta$ is a sub-solution of the following scalar problem:

$$(6.4) \quad \begin{cases} \Delta w = -g_\delta(w) & \text{in } B(0, R) \\ w = 0 & \text{on } \partial B(0, R) \end{cases}$$

Observe that the constant $\tilde{b}(\varepsilon) - \delta$ is a super-solution of (6.4) and $u_\delta < \tilde{b}(\varepsilon) - \delta$, then we can apply the monotone iterative scheme to obtain at least one positive solution which contradicts theorem 6.2. \square

In the case $f(u) = \chi_{[u > \theta]}$ where $0 < \theta < 1$ with a similar argument we can show that for $\varepsilon > \frac{(1-\theta)^2}{2}$ the system (1.1) has no nontrivial solution. Indeed, given any $\sigma > 0$ choose a smooth function $\tilde{f} \geq f$ such that $\tilde{f}(u) = 1$ for $u \geq \theta$ and $\tilde{f}(u) = 0$ for $u \leq \theta - \sigma$. With the same argument as before, (1.1) has no nontrivial solution for $\varepsilon > \tilde{\varepsilon}^1$ where $\tilde{\varepsilon}^1$ is given by (6.1) with f replaced by \tilde{f} . A computation then shows that as $\sigma \rightarrow 0$, $\tilde{\varepsilon}^1 \rightarrow \frac{(1-\theta)^2}{2}$.

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REFERENCES

- [1] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford, 1975.
- [2] J. B. van den Berg, V. Guyonne, J. Hulshof, *Flame balls for a free boundary combustion model with radiative transfer*. *SIAM J. Appl. Math.* **67** (2006), no. 1, 116–137.

- [3] H. Berestycki, B. Larrouturou, *Quelques aspects mathématiques de la propagation des flammes prmlanges*. Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. X (Paris, 1987–1988), 65–129, Pitman Res. Notes Math. Ser., 220, Longman Sci. Tech., Harlow, 1991.
- [4] H. Berestycki, B. Larrouturou, J.-M. Roquejoffre, *Mathematical investigation of the cold boundary difficulty in flame propagation theory*. Dynamical issues in combustion theory (Minneapolis, MN, 1989), 37–61, IMA Vol. Math. Appl., 35, Springer, New York, 1991.
- [5] H. Berestycki, P. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*. Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.
- [6] H. Berestycki, B. Nicolaenko, B. Scheurer, *Traveling wave solutions to combustion models and their singular limits*. SIAM J. Math. Anal. **16** (1985), no. 6, 1207–1242.
- [7] H. Bockhorn, J. Frhlich, K. Schneider, *An adaptive two-dimensional wavelet-vaguelette algorithm for the computation of flame balls*. Combustion Theory and Modelling **3** (1999) , no. 1, 177 – 198.
- [8] I. Brailovsky, G.I. Sivashinsky, *On stationay and travelling flameballs*. Combustion and Flame **110** (1997), 524-529
- [9] J. Brindley, N. A. Jivraj; J. H. Merkin; S. K. Scott, *Stationary-state solutions for coupled reaction-diffusion and temperature-conduction equations. II. Spherical geometry with Dirichlet boundary conditions*. Proc. Roy. Soc. London Ser. A **430** (1990), no. 1880, 479–488.
- [10] J. Buckmaster, G. Joulin, P. Ronney, *The structure and stability of nonadiabatic flame balls*. Combustion and Flame **79** (1990), no. 3-4, 381–392.
- [11] J.D. Buckmaster, G. Joulin, P.D. Ronney, *The structure and stability of nonadiabatic flame balls: II. Effects of far-field losses*. Combustion and Flame **84** (1991), no. 3-4, 411–422.
- [12] C.J. Lee, J. Buckmaster, *The structure and stability of flame balls: a near-equidiffusional flame analysis*. SIAM J. Appl. Math. **51** (1991), no. 5, 1315–1326.
- [13] R. S. Cantrell, C. Cosner, *Spatial ecology via reaction-diffusion equations*, Wiley Series in Mathematical and Computational Biology, John Wiley & Sons Ltd., Chichester, 2003.
- [14] P. Clément, G. Sweers, *Existence and multiplicity results for a semilinear elliptic eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 1, 97–121.
- [15] Y. Du, Y. Lou, *Proof of a conjecture for the perturbed Gelfand equation from combustion theory*. J. Differential Equations **173** (2001), no. 2, 213–230.
- [16] V. Giovangigli, *Nonadiabatic plane laminar flames and their singular limits*. SIAM J. Math. Anal. **21** (1990), no. 5, 1305–1325.
- [17] L. Glangetas, J.-M. Roquejoffre, *Rigorous derivation of the dispersion relation in a combustion model with heat losses*. Preprint Publications du Laboratoire d’Analyse Numérique R95032 (1995).
- [18] V. Guyonne, L. Lorenzi, *Instability in a flame ball problem*. Discrete Contin. Dyn. Syst. Ser. B **7** (2007), no. 2, 315–350.
- [19] V. Guyonne, P. Noble, *On a model of flame ball with radiative transfer*. SIAM J. Appl. Math. **67** (2007), no. 3, 854–868.
- [20] V. Hutson, J. López-Gómez, K. Mischaikow, and G. Vickers, *Limit behaviour for a competing species problem with diffusion*, Dynamical systems and applications, World Sci. Ser. Appl. Anal., vol. 4, World Sci. Publ., River Edge, NJ, 1995, pp. 343–358.
- [21] Kagan L., and Sivashinsky G., *Self-fragmentation of nonadiabatic cellular flames*, Combust. Flame **108** (1997), 220 – 226.
- [22] Kagan, L. and Minaev, S. and Sivashinsky, G., *On self-drifting flame balls*, Math. Comput. Simulation, **65** (2004), no. 4-5, 511–520.
- [23] A. K. Kapila, B. J. Matkowsky, J. Vega, *Reactive-diffusive system with Arrhenius kinetics: peculiarities of the spherical geometry*. SIAM J. Appl. Math. **38** (1980), no. 3, 382–401.
- [24] M.K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n* . Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266.
- [25] C. Lederman, J.-M. Roquejoffre, N. Wolanski, *Mathematical justification of a nonlinear integrodifferential equation for the propagation of spherical flames*. Ann. Mat. Pura Appl. (4) **183** (2004), no. 2, 173–239.
- [26] C. J. Lee, J. Buckmaster, *The structure and stability of flame balls: a near-equidiffusional flame analysis*, SIAM J. Appl. Math. **51** (1991), no. 5, 1315–1326.
- [27] B.Lewis and von Elbe, *Combustion, Flames and Explosion of Gases*, 3erd ed. Academic Press, Orlando 1987.

- [28] M. Marion, *Qualitative properties of a nonlinear system for laminar flames without ignition temperature*. Nonlinear Anal. 9 (1985), no. 11, 1269–1292.
- [29] Minaev, S., Kagan, L., Joulin G., Sivashinsky, G., *On self-drifting flame balls*, Combust. Theory Model. **5** (2001), no. 4, 609–622.
- [30] P. Ronney *Near-Limit Flame Structures at Low Lewis Number*. Combustion and Flame **82** (1990), 1–14.
- [31] L. Roques, *Study of the premixed flame model with heat losses. The existence of two solutions*. European J. Appl. Math. 16 (2005), no. 6, 741–765.
- [32] G. Sagon, *Steady fronts solutions of semilinear elliptic equations in exterior domains arising in flame propagation*. Ann. Mat. Pura Appl. (4) 185 (2006), no. 2, 273–291.
- [33] A.A. Shah, R.W. Thatcher, J. W. Dold, *Stability of a spherical flame ball in a porous medium*, Combustion Theory and Modelling **4** (2000), no. 4, 511 – 534.
- [34] Y.B. Zeldovich, *A theory of the limit of slow flame propagation*. Zh. Prikl. Mekh. i Tekhn. Fiz. (11)1 (1941), 159–169.

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