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▶ **To cite this version:**

Vincent Laurain, Marion Gilson, Roland Toth, Hugues Garnier. Direct identification of continuous-time LPV models. American Control Conference, ACC 2011, Jun 2011, San Francisco, United States. pp.CDROM. hal-00603629

HAL Id: hal-00603629

https://hal.science/hal-00603629

Submitted on 26 Jun 2011

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Direct identification of continuous-time LPV models

V. Laurain*, M. Gilson*, R. Tóth**, H. Garnier*

***Abstract*— Controllers in the linear parameter-varying (LPV) framework are commonly designed in continuous-time (CT) requiring accurate and low-order CT models of the system. Nonetheless, most of the methods dedicated to the identification of LPV systems are addressed in discrete-time (DT) settings. In practice when discretizing models which are naturally expressed in CT, the dependency on the scheduling variables becomes non-trivial and over-parameterized. Consequently, direct identification of CT LPV systems in an input-output setting is investigated. To provide consistent model parameter estimates in this setting, a refined instrumental variable (IV) approach is proposed. The statistical properties of this approach is illustrated through a relevant Monte Carlo simulation example.**

I. Introduction

The framework of *linear parameter-varying* (LPV) systems was introduced in the 1990s with the purpose to handle in a simple but efficient way the often nonlinear or time-varying nature of systems encountered in practice. The LPV system class forms an intermediate step between *linear time-invariant* (LTI) systems and nonlinear/time-varying plants as the signal relations in LPV systems are considered to be linear just as in the LTI case, but the parameters are assumed to be functions of a measurable time-varying signal, the so-called *scheduling variable* $p : \mathbb{Z} \rightarrow \mathbb{P}$. Here the compact set $\mathbb{P} \subset \mathbb{R}^{n_\mathbb{P}}$ denotes the *scheduling space*. This LPV modeling concept allows for a wide representation capability of physical processes, but the real practical significance of the LPV framework lays in its well worked out and industrially reputed control synthesis approaches, *e.g.* [1], [18], [24], that have led to many successful applications of LPV control in practice [3], [13], [14], [23].

However a major drawback of the LPV framework today is that, despite the advances of the LPV control field, identification of such systems is not well developed as the current methods are unable to support practical control design. Commonly LPV controllers are synthesized in *continuous time* (CT) as stability and performance requirements of the closed loop behavior can be more conveniently expressed in CT, like in a mixed-sensitivity setting [28].

However, LPV identification methods are almost exclusively developed for *discrete-time* (DT) (for a recent survey see [20]), as in this setting it is much easier to handle the estimation of parameter-varying dynamics. Nonetheless, the absence of CT methods represents a gap between the available identification approaches and the needs of LPV control synthesis.

There is therefore a growing need of the LPV framework for efficient identification methods that directly deliver reliable CT models.

In practice, CT systems can only be identified based on sampled measured data records. Thus in general, for delivering a CT model estimate, the available approaches in system identification can be categorized as follows:

- **Indirect approaches:** These methods involve the identification of a DT model in a completely DT setting which is followed by the transformation of the DT model estimate into a CT form.
- **Direct approaches:** The methods formulate the identification of the CT model directly based on samples of the measured CT signals.

Unfortunately, transformation of DT-LPV models to CT-LPV models is more complicated than in the LTI case and despite recent advances in LPV discretization theory (see [21], [22]) the theory of CT realization of DT models is still in an immature state. The discretization of a CT LPV model results in a system order increase and more importantly in complicated dynamic dependency on p (dependency of the model coefficients on time-shifted versions of p with non-trivial rational functions) for which the available LPV identification methods are not well suited.

Even for a very simple CT-LPV model, estimation of a DT model with the purpose of obtaining afterwards a CT realization is a tedious task with many underlaying problems for which there are no general theoretical solutions available.

Unlike an indirect approach, a direct solution offers a way to efficiently overcome these problems but presents intrinsic difficulties mostly linked to the inaccessibility to signal time-derivatives from the acquired sampled data. The offered solutions in the LTI case often require the use of signals prefiltering [4]. These filters depend on hyperparameters input by the user and their efficiency is strongly linked to the adequacy with the considered system. In the CT-LTI case, one of the methods for relaxing the need of prefiltering is the *Refined Instrumental Variable for Continuous-time* (RIVC) method. This method is attractive in the sense that it provides consistent estimates under the realistic assumption of an unknown noise model and it achieves similar performance as *prediction-error-minimization* (PEM) methods [17].

Another problem related to CT identification based on sampled data is the mathematical complexity of the CT random process used to describe the noise added to the system. An efficient way to overcome this problem is to consider a discrete-time Box-Jenkins noise model leading to

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hybrid models (see [15], [7]). In order to avoid the different issues linked to the simultaneous use of CT and DT filters, this paper focuses on the case where the noise added onto the output is a white gaussian noise: the Output Error (OE) case.

Recently, an LPV identification approach has been introduced, which uniquely among other approaches, addresses the minimization of the prediction error for LPV-OE models by using a *Multiple Input Single Ouput* (MISO)-LTI reformulation of the data-generating LPV system [9]. This paper aims at providing the very first step towards bridging the existing gap between LPV control and identification via the introduction of a direct CT identification approach that benefits from the properties of RIV methods in the continuous-time case and uses the recent advances of the prediction error minimization framework [9], [20].

The paper is organized as follows: in Section II, the general class of CT-LPV systems in an IO representation form is introduced. In Section III, the proposed CT LPV-IV method is described and analyzed, while its performance is illustrated in Section IV through a representative simulation example.

II. Problem description

A. System description

Consider the data generating CT LPV system described by the following equations

$$\mathcal{S}_\text{o}\begin{cases} A_\text{o}(p_t, d)\chi_\text{o}(t) = B_\text{o}(p_t, d)u(t) \\ y(t) = \chi_\text{o}(t) + e_\text{o}(t), \end{cases} \tag{1}$$

where d denotes the differentiation operator w.r.t. time, i.e. $d = \frac{\mathrm{d}}{\mathrm{d}t}$, $p : \mathbb{R} \to \mathbb{P}$ is the scheduling variable with $p_t = p(t)$, χ_o is the noise-free output and e_o is a white gaussian noise process with variance $\sigma^2_{e_\text{o}}$. A_o, B_o are polynomials in d with coefficients a^o_i and b^o_i that are meromorphic functions[1] of p with no singularity on $\mathbb{P}$:

$$A_\text{o}(p_t, d) = d^{n_a} + \sum_{i=1}^{n_\text{a}} a^\text{o}_i(p_t) d^{n_a - i}, \tag{2}$$

$$B_\text{o}(p_t, d) = \sum_{j=0}^{n_\text{b}} b^\text{o}_j(p_t) d^{n_b - j}. \tag{3}$$

Note that a^o_i and b^o_j are functions of p at time t, which is called *static dependence*. In LPV system theory, a more general p-dependence of coefficients than static is required to establish equivalence of representations. In particular, it can be required that the coefficients a^o_i and b^o_j depend also on time derivatives of p, which is called dynamic dependence [20]. In order to simplify the upcoming discussion, we restrict our attention to static dependence. Nevertheless, the established results hold also in the case of *dynamic dependence* of (1) and of the proposed model structure.

In terms of identification we can assume that sampled measurements of (y, p, u) are available with a sampling period $\mathtt{T}_\text{s} > 0$. Hence, we will denote the discrete-time samples of these signals as $u(t_k) = u(k\mathtt{T}_\text{s})$, where $k \in \mathbb{Z}$.

[1] A function f is called meromorphic if $f = \frac{g}{h}$ where g, h are holomorphic (analytic) functions and h is not the zero function.

B. Model structure considered

The process model is denoted by $\mathcal{G}_\rho$ and defined in a form of an LPV-IO representation with a static scheduling dependence:

$$\mathcal{G}_\rho : (A(p_t, d, \rho), B(p_t, d, \rho)) \tag{4}$$

where the p-dependent polynomials A and B given as

$$\begin{aligned} A(p_t, d, \rho) &= d^{n_a} + \sum_{i=1}^{n_\text{a}} a_i(p_t) d^{n_a - i}, \\ B(p_t, d, \rho) &= \sum_{j=0}^{n_\text{b}} b_j(p_t) d^{n_b - j}, \end{aligned}$$

are parameterized as

$$\begin{aligned} a_i(p_t) &= a_{i,0} + \sum_{l=1}^{n_\alpha} a_{i,l} f_l(p_t) \quad i = 1, \dots, n_\text{a} \\ b_j(p_t) &= b_{j,0} + \sum_{l=1}^{n_\beta} b_{j,l} g_l(p_t) \quad j = 0, \dots, n_\text{b} \end{aligned}$$

In this parametrization, $\{f_l\}_{l=1}^{n_\alpha}$ and $\{g_l\}_{l=1}^{n_\beta}$ are meromorphic functions of p, with static dependence, allowing the identifiability of the model (they can be chosen for example as linearly independent functions on $\mathbb{P}$). The associated model parameters are stacked columnwise:

$$\rho = [\ \mathtt{a}_1 \ \dots \ \mathtt{a}_{n_\text{a}} \ \mathtt{b}_0 \ \dots \ \mathtt{b}_{n_\text{b}}\ ]^\top \in \mathbb{R}^{n_\rho}, \tag{5}$$

where

$$\begin{aligned} \mathtt{a}_i &= [\ a_{i,0} \ a_{i,1} \ \dots \ a_{i,n_\alpha}\ ] \in \mathbb{R}^{n_\alpha + 1} \\ \mathtt{b}_j &= [\ b_{j,0} \ b_{j,1} \ \dots \ b_{j,n_\beta}\ ] \in \mathbb{R}^{n_\beta + 1} \end{aligned}$$

and $n_\rho = n_\text{a}(n_\alpha + 1) + (n_\text{b} + 1)(n_\beta + 1)$. Introduce also $\mathcal{G} = \{\mathcal{G}_\rho \mid \rho \in \mathbb{R}^{n_\rho}\}$, as the collection of all process models in the form of (4).

With respect to the considered OE structure, the signal relations of the LPV-BJ model, denoted in the sequel as $\mathcal{M}_\rho$, are defined as:

$$\mathcal{M}_\rho \begin{cases} A(p_k, d, \rho)\chi(t) = B(p_k, d, \rho)u(t) \\ y(t_k) = \chi(t_k) + e(t_k) \end{cases} \tag{6}$$

Based on this model structure, the model set, denoted as $\mathcal{M} = \{\mathcal{M}\rho \mid \rho \in \mathbb{R}^{n_\rho}\} = \mathcal{G}$, corresponds to the set of candidate models in which we seek the model that explains data gathered from $\mathcal{S}_\text{o}$ the best, under a given identification criterion (cost function).

C. Predictors and prediction error

Similar to the LTI case, in the LPV prediction error framework, one is concerned about finding a model in a given LPV model structure $\mathcal{M}$, which minimizes the statistical mean of the squared prediction error based on past samples of (y, u, p). However in the LPV case, no transfer function representation of systems is available.

Furthermore, multiplication with $\mathscr{d}$ is not commutative over the p-dependent coefficients [20], meaning that $\mathscr{d}(B(p,\mathscr{d})u(t)) = B(\mathscr{d}p,\mathscr{d})\mathscr{d}u(t)$ which is not equal to $B(p,\mathscr{d})\mathscr{d}u(t)$.

1) System reformulation and prediction error: Following the same idea developed in [9] and if the system belongs to the model set defined with a deterministic p signal, it is possible to express the CT LPV system as a CT MISO LTI system by rewriting the signal relations of (1) as

$$\underbrace{\chi_{\mathrm{o}}^{(n_a)}(t) + \sum_{i=1}^{n_{\mathrm{a}}} a_{i,0}^{\mathrm{o}} \chi_{\mathrm{o}}^{(n_a-i)}(t)}_{F_{\mathrm{o}}(\mathscr{d})\chi_{\mathrm{o}}(t)} + \sum_{i=1}^{n_{\mathrm{a}}} \sum_{l=1}^{n_\alpha} a_{i,l}^{\mathrm{o}} \underbrace{f_l(p(t))\chi_{\mathrm{o}}^{(n_a-i)}(t)}_{\chi_{i,l}^{\mathrm{o}}(t)}$$
$$= \sum_{j=0}^{n_{\mathrm{b}}} \sum_{l=0}^{n_\beta} b_{j,l}^{\mathrm{o}} \underbrace{g_l(p(t))u^{(n_b-j)}(t)}_{u_{j,l}(t)} \qquad (7)$$

where $g_0(t) = 1$ and the superscript (n) for a signal, like $u^{(n)}$, denotes the n^{th} time-derivative of the signal, *e.g.* $u^{(n)}(t) = \mathscr{d}^n u(t)$. Furthermore, $F_{\mathrm{o}}(\mathscr{d}) = \mathscr{d}^{n_a} + \sum_{i=1}^{n_{\mathrm{a}}} a_{i,0}\mathscr{d}^{n_a-i}$ while $u^{(n)}(t_k)$ represents the value of the signal $u^{(n)}(t)$ sampled at time instance t_k.

Note that in this way, the time variation of the coefficients is transposed onto the signals $\chi_{i,l}^{\mathrm{o}}(t)$ and $u_{j,l}(t)$:

$$\chi_{i,l}^{\mathrm{o}}(t) = f_l(p(t))\chi_{\mathrm{o}}^{(n_a-i)}(t) \quad \{i,l\} \in \{1\dots n_{\mathrm{a}}, 1\dots n_\alpha\},$$
$$u_{j,l}(t) = g_l(p(t))u^{(n_b-j)}(t) \quad \{j,l\} \in \{1\dots n_{\mathrm{b}}, 1\dots n_\alpha\}.$$

Therefore, the process part of the LPV-BJ model is rewritten as a *Multiple-Input Single-Output* (MISO) system with $(n_{\mathrm{b}}+1)(n_\beta+1)+n_{\mathrm{a}}n_\alpha$ inputs $\{\chi_{i,l}^{\mathrm{o}}\}_{i=1,l=1}^{n_{\mathrm{a}},n_\alpha}$ and $\{u_{j,l}\}_{j=0,l=0}^{n_{\mathrm{b}},n_\beta}$. By using (7), (6) can be rewritten in terms of the sampled output signal $y(t_k)$ as

$$y(t_k) = -\left(\sum_{i=1}^{n_{\mathrm{a}}}\sum_{l=1}^{n_\alpha} \frac{a_{i,l}^{\mathrm{o}}}{F_{\mathrm{o}}(\mathscr{d})}\chi_{i,l}^{\mathrm{o}}\right)(t_k)$$
$$+ \left(\sum_{j=0}^{n_{\mathrm{b}}}\sum_{l=0}^{n_\beta} \frac{b_{j,l}^{\mathrm{o}}}{F_{\mathrm{o}}(\mathscr{d})}u_{k,j}\right)(t_k) + e_{\mathrm{o}}(t_k), \qquad (9)$$

which is a sampled LTI representation of the system (1).

2) Prediction Error Model: Similarly to the LTI case, the *one-step-ahead prediction error* can be expressed and defined as [10]:

$$\varepsilon_\rho(t_k) = y(t_k) - \hat{y}_\rho(t_k), \qquad (10)$$

where $\hat{y}_\rho(t_k)$ is the *one step ahead predictor* based on the model (6) written as a MISO LTI form (9) and defined as:

$$\hat{y}_\rho(t_k) = -\left(\sum_{i=1}^{n_{\mathrm{a}}}\sum_{l=1}^{n_\alpha} \frac{a_{i,l}}{F(\mathscr{d},\rho)}\chi_{i,l}\right)(t_k)$$
$$+ \left(\sum_{j=0}^{n_{\mathrm{b}}}\sum_{l=0}^{n_\beta} \frac{b_{j,l}}{F(\mathscr{d},\rho)}u_{k,j}\right)(t_k). \qquad (11)$$

3) Prediction error minimization: Denote $\mathcal{D}_N = \{y(t_k), u(t_k), p(t_k)\}_{k=1}^N$ a data sequence of $\mathcal{S}_{\mathrm{o}}$. Then to provide an estimate of ρ based on the minimization of ε_ρ, an identification criterion $W(\mathcal{D}_N,\rho)$ can be introduced, like the *least square* criterion

$$W(\mathcal{D}_N,\rho) = \frac{1}{N}\sum_{k=1}^{N}\varepsilon_\rho^2(t_k), \qquad (12)$$

such that the parameter estimate is

$$\hat{\rho}_N = \arg \min_{\rho\in\mathbb{R}^{n_\rho+n_\eta}} W(\mathcal{D}_N,\rho). \qquad (13)$$

4) CT filtering and sampled data: The CT representation of the model (6) consists in a CT filtering operation. In this paper, we considered the practically feasible situation such that only sampled measurements of the CT signals (y,p,u) are available. In order to apply a CT filter on sampled data one can either interpolate the samples to obtain a continuous-time signal and apply the CT filter on this reconstructed signal or use a numerical approximation, *i.e.* DT approximation of the considered system. This is a common problem for simulation of CT systems. For simulation purposes, DT approximation of the system can efficiently be dealt with, by using powerful numerical algorithms available [2].

Note that to derive an accurate DT approximation of the system, it is often sufficient in terms of the classical discretization theory to assume that the sampled free CT signals of the system are restricted to be constant in the sampling period [5]. This has been also shown in case of LPV systems with static dependence [20]. This provides the hypothesis, also used in [15], [7], that if CT (p,u) are piecewise constant between two samples, then the trajectory of y is completely determined by its observations at the sample period $\mathtt{T}_{\mathrm{s}}k$. Therefore, under these inter-sampling conditions, DT filtering and numerical approximation of CT filtering operations commute [6]. Nevertheless, it is important to notice that the numerical approximation method used for the evaluation of a CT filter does not have any impact on the coefficients to be estimated which remain, in terms of (11), the coefficients of the parsimonious CT model.

D. Identification problem statement

Based on the previous considerations, the identification problem addressed in the sequel can now be defined.

Problem 1: Given a CT-LPV data generating system $\mathcal{S}_{\mathrm{o}}$ defined as in (1) and a data set $\mathcal{D}_N$ collected from $\mathcal{S}_{\mathrm{o}}$. Based on the CT LPV model structure $\mathcal{M}_\rho$ defined by (6), estimate the parameter vector ρ using $\mathcal{D}_N$ under the following assumptions:

- A1 $\mathcal{S}_{\mathrm{o}} \in \mathcal{M}$.
- A2 In the parametrization $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, $\{f_l\}_{l=1}^{n_\alpha}$ and $\{g_l\}_{l=1}^{n_\beta}$ are chosen such that $(\mathcal{G}_{\mathrm{o}})$ is identifiable for any trajectory of p.
- A3 $u(t_k)$ is not correlated to $e_{\mathrm{o}}(t_k)$.
- A4 $\mathcal{D}_N$ is informative with respect to $\mathcal{M}$.
- A5 $\mathcal{S}_{\mathrm{o}}$ is globally BIBO stable, i.e. for any trajectory of $p:\mathbb{R}\to\mathbb{P}$ and any bounded input signal u, the output of $\mathcal{S}_{\mathrm{o}}$ is bounded [20].

III. Refined Instrumental variable for LPV systems

Based on the MISO-LTI formulation (11), it becomes possible in theory to achieve optimal PEM using linear regression [9]. This allows to extend the *Refined Instrumental Variable* (RIV) approach of the LTI identification framework to provide an efficient way of identifying CT LPV models.

A. Linear Regression for CT LPV-BJ models

Using the LTI model (6) reformulated as in (11), $y(t_k)$ can be written in the regression form:

$$y^{(n_a)}(t_k) = \varphi^\top(t_k)\rho + \tilde{v}(t_k) \tag{14}$$

where,

$$\varphi(t_k) = [-y^{(n_a-1)}(t_k)\ \ldots\ -y(t_k)\ -\chi_{1,1}(t_k)\ \ldots \\ \ldots -\chi_{n_\mathrm{a},n_\alpha}(t_k)\ u_{0,0}(t_k)\ \ldots\ u_{n_\mathrm{b},n_\beta}(t_k)]^\top$$

$$\rho = [a_{1,0}\ \ldots\ a_{n_\mathrm{a},0}\ a_{1,1}\ \ldots\ a_{n_\mathrm{a},n_\alpha}\ b_{0,0}\ \ldots\ b_{n_\mathrm{b},n_\beta}]^\top$$

$$\tilde{v}(t_k) = F(d, \rho)e(t_k).$$

The extended regressor in (14) contains the noise-free output terms $\{\chi_{i,k}\}$. Therefore, by momentary assuming that $\{\chi_{i,l}(t_k)\}_{i=1,l=0}^{n_a,n_\alpha}$ are known *a priori*, the prediction error $\varepsilon_\rho(t_k)$ for (14) is given in terms of (10) as:

$$\varepsilon_\rho(t_k) = (F(d, \rho)y_\mathrm{f})(t_k) - \sum_{i=1}^{n_\mathrm{a}}\sum_{l=1}^{n_\alpha} a_{i,l}\chi_{i,l}^\mathrm{f}(t_k) \\ + \sum_{j=0}^{n_\mathrm{b}}\sum_{l=0}^{n_\beta} b_{j,l}u_{k,j}^\mathrm{f}(t_k) \tag{15}$$

where $y_\mathrm{f}(t_k)$, $u_{j,l}^\mathrm{f}(t_k)$ and $\chi_{i,l}^\mathrm{f}(t_k)$ represent the outputs of an hybrid prefiltering operation, involving the continuous-time filter (see [27]):

$$Q_\mathrm{c}(d, \rho) = \frac{1}{F(d, \rho)}, \tag{16}$$

Based on (15), the associated linear-in-the-parameters model takes the form [27]:

$$y_\mathrm{f}^{(n_a)}(t_k) = \varphi_\mathrm{f}^\top(t_k)\rho + \tilde{v}_\mathrm{f}(t_k), \tag{17}$$

where

$$\varphi_\mathrm{f}(t_k) = [-y_\mathrm{f}^{(n_a-1)}(t_k)\ \ldots\ -y_\mathrm{f}(t_k)\ -\chi_{1,1}^\mathrm{f}(t_k)\ \ldots \\ \ldots\ -\chi_{n_\mathrm{a},n_\alpha}^\mathrm{f}(t_k)\ u_{0,0}^\mathrm{f}(t_k)\ \ldots\ u_{n_\mathrm{b},n_\beta}^\mathrm{f}(t_k)]^\top$$

$$\tilde{v}_\mathrm{f}(t_k) = Q_\mathrm{c}(d, \rho)\tilde{v}(t_k) = e(t_k).$$

B. The refined instrumental variable approach

Under the assumption that the CT filter $Q_\mathrm{c}(d, \rho)$ and $\{\chi_{i,l}(t_k)\}_{i=1,l=0}^{n_a,n_\alpha}$ are known *a priori*, traditional parametric estimation methods from the LTI framework could provide efficient estimates of ρ. However, in a practical situation, $Q_\mathrm{c}(d, \rho)$ is unknown and only some estimates will be available.

Furthermore, it is important to notice here that the regressors in (17) and (14) contain some time-derivatives of y and u which, in the assumed framework considering sampled data, can only be approximated. It is well-known that the approximation of derivatives requires a low pass filtering on the input and output. The most commonly used filters for this purpose are Poisson's filters, or state-variable filters [4]. The drawback of these filters is that they require the choice of a design variable. Therefore, it is a particular strength of the presented method that the estimated filter $F(d, \rho)$ is not only used for the minimisation of the prediction error but it also provides the filtering for the approximation of the time derivatives. In other words, the regressor φ_f in (17) can be well-approximated numerically whereas the regressor φ from (14) cannot as it requires prefiltering of the data which must be chosen by the user. In order to estimate the parameter vector in (17) without the prior knowledge of $Q_\mathrm{c}(d, \rho)$, the RIV method is chosen for the following reasons:

- RIV methods lead to optimal estimates in the LTI case if $\mathcal{S}_\mathrm{o} \in \mathcal{M}$ (see [19], [26], [27]).
- In a practical situation of identification, $\mathcal{G}_\mathrm{o} \in \mathcal{G}$ might be fulfilled due to first principle or expert's knowledge. However, it is commonly fair to assume that the model is not OE. In such case, RIV methods have the advantage of providing consistent estimates whereas methods such as extended LS are biased and more advanced PEM methods need robust initialisation [12].
- The RIV algorithm has been successfully used for LTI model with similar CT structure, in the case of linear models [16], [26] and nonlinear ones [8].

Aiming at the extension of the RIV approach for the estimation of CT LPV models, consider the relationship between the process input and output signals as in (14). Based on this form, the extended-IV estimate is given as [26]:

$$\hat{\rho}_\mathtt{XIV}(N) = \arg\min_{\rho\in\mathbb{R}^{n_\rho}} \left\| \left[\frac{1}{N}\sum_{k=1}^{N}\zeta_\mathrm{f}(t_k)\varphi_\mathrm{f}^\top(t_k)\right]\rho \right. \\ \left. - \left[\frac{1}{N}\sum_{t=1}^{N}\zeta_\mathrm{f}(t_k)y_\mathrm{f}^{(n_a)}(t_k)\right]\right\|_W^2, \tag{18}$$

where $\zeta(t_k)$ is the instrument, $\|x\|_W^2 = x^TWx$, with W a positive definite weighting matrix and the filtered variables ζ_f, φ_f and y_f are filtered using a stable prefilter. If $G_\mathrm{o} \in \mathcal{G}$, the extended-IV estimate is consistent under the following two conditions[2]:

C1 $\bar{\mathbb{E}}\{\zeta_\mathrm{f}(t_k)\varphi_\mathrm{f}^\top(t_k)\}$ is full column rank.

C2 $\bar{\mathbb{E}}\{\zeta_\mathrm{f}(t_k)\tilde{v}_\mathrm{f}(t_k)\} = 0$.

Moreover it has been shown in [19], [25] and [26] that the minimum variance estimator can be achieved if:

C3 $W = I$.

C4 ζ is chosen as the noise-free version of the extended regressor in (14) and is therefore defined in the present LPV case as:

$$\zeta(t_k) = \left[-\chi^{(n_\mathrm{a}-1)}(t_k)\ \ldots\ -\chi(t_k)\ -\chi_{1,1}(t_k)\ \ldots \right. \\ \left. \ldots\chi_{n_\mathrm{a},n_\alpha}(t_k)\ u_{0,0}(t_k)\ \ldots\ u_{n_\mathrm{b},n_\beta}(t_k)\right]^\top$$

[2]The notation $\bar{\mathbb{E}}\{.\} = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}\{.\}$ is adopted from the prediction error framework of [10].

C5 $\mathcal{G}_\mathrm{o} \in \mathcal{G}$ and n_ρ is equal to the minimal number of parameters required to represent $\mathcal{G}_\mathrm{o}$ with the considered model structure.

C6 The CT filter used is chosen as the filter (16).

While conditions C1, C2, C3 and C5 are quite straightforward to fulfill (see [19], [25]), the obtention of the suitable instrument that fulfills C4 and of the optimal filter fulfilling C6 is not trivial in practical case. The RIV algorithm involves an iterative (or relaxation) algorithm in which, at each iteration, an 'auxiliary model' is used to generate the instrumental variables (which guarantees C2), as well as the associated prefilters. This auxiliary model is based on the parameter estimates obtained at the previous iteration. Consequently, if convergence occurs, C4 and C6 are fulfilled. Thus, the RIV is the most suitable method to simultaneously i) efficiently estimate the parameter vector ρ in (17) in the case $S_\mathrm{o} \in \mathcal{M}$ and ii) consistently estimate ρ in the practical assumption where the noise model is misspecified.

C. The `LPV-RIVC` Algorithm

Step 1 The usual initialisation for `CT-RIV` algorithm is a DT model estimate issued from an `LS` method or a `DT-RIV` algorithm. In the LPV case however, the transformation of a DT model into a CT model is not trivial. Consequently, the initial estimate proposed for the `LPV-RIVC` algorithm is an `LTI-RIVC` estimate of $\mathcal{M}_\rho$, i.e. $\hat{\rho}^{(0)}$ is given. Set $\tau = 0$.

Step 2 Compute an estimate of $\chi(t_k)$ via numerical approximation of

$$A(p_t, d, \hat{\rho}^{(\tau)})\hat{\chi}(t) = B(p_t, d, \hat{\rho}^{(\tau)})u(t),$$

where $\hat{\rho}^{(\tau)}$ is estimated in the previous iteration. Based on $\mathcal{M}_{\hat{\rho}^{(\tau)}}$, deduce $\hat{\chi}(t_k)$ which is bounded according to Assumption A5.

Step 3 Compute the estimated continuous-time filter $\hat{Q}_\mathrm{c}(d, \hat{\rho}^{(\tau)}) = \frac{1}{F(d, \hat{\rho}^{(\tau)})}$, where $F(d, \hat{\rho}^{(\tau)})$ is as given in (7).

Step 4 Use the CT filter $\hat{Q}_\mathrm{c}(d, \hat{\rho}^{(\tau)})$ as well as $\hat{\chi}(t_k)$ in order to generate the estimates of the derivatives which are needed:

Step 5 Build the filtered estimated regressor $\hat{\varphi}_\mathrm{f}(t_k)$ and, in terms of C4, the filtered instrument $\hat{\zeta}_\mathrm{f}(t_k)$.

Step 6 The solution of the IV optimization problem is then

$$\hat{\rho}^{(\tau+1)}(N) = \left[\sum_{k=1}^{N} \hat{\zeta}_\mathrm{f}(t_k)\hat{\varphi}_\mathrm{f}^\top(t_k)\right]^{-1} \sum_{k=1}^{N} \hat{\zeta}_\mathrm{f}(t_k) y_\mathrm{f}^{(n_a)}(t_k) \quad (19)$$

Step 7 If $\rho^{(\tau+1)}$ has converged or the maximum number of iterations is reached, then stop, else increase τ by 1 and go to Step 2.

IV. Simulation Example

In order to show the relevance of the presented algorithm, the following data-generating system is considered:

$$\mathcal{S}_\mathrm{o} \begin{cases} A_\mathrm{o}(d, p) = d^2 + a_1^\mathrm{o}(p)d + a_2^\mathrm{o}(p) \\ B_\mathrm{o}(d, p) = b_0^\mathrm{o}(p)d + b_1^\mathrm{o}(p) \end{cases}$$

where

$$a_1^\mathrm{o}(p) = 1 - 0.5p, \quad (20a)$$
$$a_2^\mathrm{o}(p) = 5 + 3p \quad (20b)$$
$$b_0^\mathrm{o}(p) = 2 + p, \quad (20c)$$
$$b_1^\mathrm{o}(p) = 5 - p. \quad (20d)$$

3000 samples are collected from a $15\mathrm{sec}$ simulation ($\mathtt{T}_\mathrm{s} = 0,005\mathrm{sec}$). The input signal u is chosen as a uniformly distributed sequence $\mathcal{U}(-1, 1)$ while the scheduling variable is chosen as $p(t) = \sin(\frac{2}{3}\pi t)$.

The following model structure in terms of (4) is considered to capture the dynamics of $\mathcal{S}_\mathrm{o}$:

$$\mathcal{M} \begin{cases} A(d, p) = d^2 + a_1(p)d + a_2(p) \\ B(d, p) = b_0(p)d + b_1(p) \end{cases}$$

where

$$a_1(p) = a_{1,0} + a_{1,1}p, \quad (21a)$$
$$a_2(p) = a_{2,0} + a_{2,1}p, \quad (21b)$$
$$b_0(p) = b_{0,0} + b_{0,1}p, \quad (21c)$$
$$b_1(p) = b_{1,0} + b_{1,1}p. \quad (21d)$$

As previously pointed out, the efficiency of the `LPV-RIVC` estimator can not be yet proven. Nonetheless, in order to analyse its statistical properties on this example the model is estimated using both the `LPV-RIVC` algorithm and the MATLAB `LSQNONLIN` method. The `LSQNONLIN` method is a nonlinear statistically optimal optimization method but such nonlinear method is also sensitive to initialisation [11]. Therefore, in order to put this latter method at its best, it is initialized on the true parameters.

Monte Carlo simulation results obtained using the different methods are presented in Table I. The statistical properties of each method is evaluated using the estimated parameters mean and standard deviation. These results are based on $N_\mathrm{run} = 100$ random realizations under a *Signal-to-Noise Ratio* (SNR) of 10dB with:

$$\mathrm{SNR} = 10 \log \frac{P_{\chi_\mathrm{o}}}{P_{e_\mathrm{o}}}, \quad (22)$$

where P_x is the power of signal x.

It can be seen from Table I that according to the theory, the estimated parameters using the `LPV-RIVC` algorithm are unbiased. Moreover, in case of correct parametrization and on this example, the `LPV-RIVC` method performs equivalently to the optimal `LSQNONLIN` method which is theoretically optimal but is also known to be sensitive to initialisation in comparison to `RIV` based algorithms [11].

Consequently, the presented algorithm constitutes the first direct continuous-time method aiming at minimizing the error prediction. It looks from this example that the empirically accepted properties of RIV based method might apply to the LPV case even though this cannot be yet proven. Moreover, this method does not requires any hyperparameters from the user for the signals time-derivative approximation.

TABLE I

MONTE CARLO SIMULATION FOR $SNR = 10$ dB

	Method	LSQNONLIN		LPV-RIVC	
Name	True Value	mean	st. dev.	mean	st. dev.
$a_{1,0}$	1	1.0026	0.0408	1.0040	0.0421
$a_{1,1}$	-0.5	-0.5054	0.0707	-0.5089	0.0745
$a_{2,0}$	5	5.0017	0.0698	5.0016	0.0731
$a_{2,1}$	3	2.9996	0.1278	2.9973	0.1308
$b_{0,0}$	2	2.0004	0.0298	1.9999	0.0311
$b_{0,1}$	1	0.9988	0.0550	0.9981	0.0578
$b_{1,0}$	5	5.0008	0.1469	5.0021	0.1559
$b_{1,1}$	-1	-1.0274	0.2670	-1.0355	0.2732

V. CONCLUSION

The proposed approach provides on of the very first direct global LPV identification method that is able to give consistent estimates of LPV-IO models in continuous-time and has a low computational load. The proposed algorithm has been tested on a representative numerical simulation example and it has been shown that the proposed procedure is robust to noise and can compete with the optimal nonlinear optimization method even in the case where the latter is initialized knowing the true parameters. Furthermore, based on previous work on CT-LTI systems operating in closed loop, this methods opens the possibility for closed-loop CT-LPV identification.

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