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Absence of traveling wave solutions of conductivity type for the Novikov-Veselov equation at zero energy

A.V. Kazeykina ¹

Abstract. We prove that the Novikov-Veselov equation (an analog of KdV in dimension $2 + 1$) at zero energy does not have sufficiently localized soliton solutions of conductivity type.

1 Introduction

In this note we are concerned with the Novikov-Veselov equation at zero energy

$$\begin{aligned} \partial_t v &= 4\operatorname{Re}(4\partial_z^3 v + \partial_z(vw)), \\ \partial_{\bar{z}} w &= -3\partial_z v, \quad v = \bar{v}, \\ v &= v(x, t), \quad w = w(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).$$

Definition 1. A pair (v, w) is a sufficiently localized solution of equation (1) if

- $v, w \in C(\mathbb{R}^2 \times \mathbb{R})$, $v(\cdot, t) \in C^3(\mathbb{R}^3)$,
- $|\partial_x^j v(x, t)| \leq \frac{q(t)}{(1 + |x|)^{2+\varepsilon}}$, $|j| \leq 3$, for some $\varepsilon > 0$, $w(x, t) \rightarrow 0$, $|x| \rightarrow \infty$,
- (v, w) satisfies (1).

Definition 2. A solution (v, w) of (1) is a soliton (a traveling wave) if $v(x, t) = V(x - ct)$, $c \in \mathbb{R}^2$.

Equation (1) is an analog of the classic KdV equation. When $v = v(x_1, t)$, $w = w(x_1, t)$, then equation (1) is reduced to KdV. Besides, equation (1) is integrable via the scattering transform for the 2-dimensional Schrödinger equation

$$\begin{aligned} L\psi &= 0, \\ L &= -\Delta + v(x, t), \quad \Delta = 4\partial_z\partial_{\bar{z}}, \quad x \in \mathbb{R}^2. \end{aligned} \tag{2}$$

Equation (1) is contained implicitly in [M] as an equation possessing the following representation

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{3}$$

where L is defined in (2), A and B are suitable differential operators of the third and zero order respectively and $[\cdot, \cdot]$ denotes the commutator. In the explicit form equation (1) was written in [NV1], [NV2], where it was also studied in the periodic setting. For the rapidly decaying potentials the studies of equation (1) and the scattering problem for (2) were carried out in [BLMP], [GN] [T], [LMS]. In [LMS] the relation with the Calderón conductivity problem was discussed in detail.

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Definition 3. A potential $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, is of conductivity type if $v = \gamma^{-1/2} \Delta \gamma^{1/2}$ for some real-valued positive $\gamma \in L^\infty(\mathbb{R}^2)$, such that $\gamma \geq \delta_0 > 0$ and $\nabla \gamma^{1/2} \in L^p(\mathbb{R}^2)$.

The potentials of conductivity type arise naturally when the Calderón conductivity problem is studied in the setting of the boundary value problem for the 2-dimensional Schrödinger equation at zero energy (see [Nov1], [N], [LMS]); in addition, in [N] it was shown that for this type of potentials the scattering data for (2) are well-defined everywhere.

The main result of the present note consists in the following: there are no solitons of conductivity type for equation (1). The proof is based on the ideas proposed in [Nov2].

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

2 Scattering data for the 2-dimensional Schrödinger equation at zero energy with a potential of conductivity type

Consider the Schrödinger equation (2) on the plane with the potential $v(z)$, $z = x_1 + ix_2$, satisfying

$$\begin{aligned} v(z) &= \overline{v(\bar{z})}, \quad v(z) \in L^\infty(\mathbb{C}), \\ |v(z)| &< q(1 + |z|)^{-2-\varepsilon} \text{ for some } q > 0, \varepsilon > 0. \end{aligned} \quad (4)$$

For $k \in \mathbb{C}$ we consider solutions $\psi(z, k)$ of (2) having the following asymptotics

$$\psi(z, k) = e^{ikz} \mu(z, k), \quad \mu(z, k) = 1 + o(1), \text{ as } |z| \rightarrow \infty, \quad (5)$$

i.e. Faddeev's exponentially growing solutions for the two-dimensional Schrödinger equation (2) at zero energy, see [F], [GN], [Nov1].

It was shown that if v satisfies (4) and is of conductivity type, then $\forall k \in \mathbb{C} \setminus 0$ there exists a unique continuous solution of (2) satisfying (5) (see [N]). Thus the scattering data b for the potential v of conductivity type are well-defined and continuous:

$$b(k) = \iint_{\mathbb{C}} e^{i(ky + \bar{k}\bar{y})} v(y) \mu(y, k) d\text{Re}y d\text{Im}y, \quad k \in \mathbb{C} \setminus 0. \quad (6)$$

In addition (see [N]), the function $\mu(z, k)$ from (5) satisfies the following $\bar{\partial}$ -equation

$$\frac{\partial \mu(z, k)}{\partial \bar{k}} = \frac{1}{4\pi \bar{k}} e^{-i(kz + \bar{k}\bar{z})} b(k) \overline{\mu(z, k)}, \quad z \in \mathbb{C}, \quad k \in \mathbb{C} \setminus 0 \quad (7)$$

and the following limit properties:

$$\mu(z, k) \rightarrow 1, \text{ as } |k| \rightarrow \infty, \quad (8)$$

$$\mu(z, k) \text{ is bounded in the neighborhood of } k = 0. \quad (9)$$

The following lemma describes the scattering data corresponding to a shifted potential.

Lemma 1. Let $v(z)$ be a potential satisfying (4) with the scattering data $b(k)$. The scattering data $b_y(k)$ for the potential $v_y(z) = v(z - y)$ are related to $b(k)$ by the following formula

$$b_y(k) = e^{i(ky + \bar{k}\bar{y})} b(k), \quad k \in \mathbb{C} \setminus 0, \quad y \in \mathbb{C}. \quad (10)$$

Proof. We note that $\psi(z - y, k)$ satisfies (2) with $v_y(z)$ and has the asymptotics $\psi(z - y, k) = e^{ik(z-y)}(1 + o(1))$ as $|z| \rightarrow \infty$. Thus $\psi_y(z, k) = e^{iky}\psi(z - y, k)$ and $\mu_y(z, k) = \mu(z - y, k)$. Finally, we have

$$\begin{aligned} b_y(k) &= \iint_{\mathbb{C}} e^{i(k\zeta + \bar{k}\bar{\zeta})} v_y(\zeta) \mu_y(\zeta, k) d\operatorname{Re}\zeta d\operatorname{Im}\zeta = \\ &= \iint_{\mathbb{C}} e^{i(k\zeta + \bar{k}\bar{\zeta})} v(\zeta - y) \mu(\zeta - y, k) d\operatorname{Re}\zeta d\operatorname{Im}\zeta = e^{i(ky + \bar{k}\bar{y})} b(k). \end{aligned}$$

□

As for the time dynamics of the scattering data, in [BLMP], [GN] it was shown that if the solution (v, w) of (1) exists and the scattering data for this solution are well-defined, then the time evolution of these scattering data is described as follows:

$$b(k, t) = e^{i(k^3 + \bar{k}^3)t} b(k, 0), \quad k \in \mathbb{C} \setminus 0, \quad t \in \mathbb{R}. \quad (11)$$

3 Absence of solitons of conductivity type

Theorem 1. *Let (v, w) be a sufficiently localized traveling wave solution of (1) of conductivity type. Then $v \equiv 0$, $w \equiv 0$.*

Scheme of proof. From (10), (11), continuity of $b(k)$ on $\mathbb{C} \setminus 0$ and the fact that the functions k , \bar{k} , k^3 , \bar{k}^3 , 1 are linearly independent in the neighborhood of any point, it follows that $b \equiv 0$. Equation (7) implies that in this case the function $\mu(z, k)$ is holomorphic on k , $k \in \mathbb{C} \setminus 0$. Using properties (8) and (9) we apply Liouville theorem to obtain that $\mu \equiv 1$. Then $\psi(z, k) = e^{ikz}$ and from (2) it follows that $v \equiv 0$. □

References

- [BLMP] Boiti M., Leon J.J.P., Manna M., Pempinelli F.: On a spectral transform of a KdV-like equation related to the Schrodinger operator in the plane. *Inverse Problems*. 3, 25–36 (1987)
- [F] Faddeev L.D. Growing solutions of the Schrödinger equation. *Dokl. Akad. Nauk SSSR*. 165, 514-517 (1965); translation in *Sov. Phys. Dokl.* 10, 1033-1035 (1966)
- [GN] Grinevich P.G., Novikov S.P.: Two-dimensional “inverse scattering problem” for negative energies and generalized-analytic functions. I. Energies below the ground state. *Funct. Anal. Appl.* 22(1), 19–27 (1988)
- [LMS] Lassas M., Mueller J.L., Siltanen S.: Mapping properties of the nonlinear Fourier transform in dimension two. *Communications in Partial Differential Equations*. 32, 591–610 (2007)
- [M] Manakov S.V.: The inverse scattering method and two-dimensional evolution equations. *Uspekhi Mat. Nauk.* 31(5), 245–246 (1976) (in Russian)
- [N] Nachman A.I.: Global uniqueness for a two-dimensional inverse boundary value problem. *Annals of Mathematics*. 143, 71–96 (1995)

- [Nov1] Novikov R.G.: Multidimensional inverse spectral problem for the equation $-\Delta\psi+(v(x)-Eu(x))\psi = 0$. *Funkt. Anal. i Pril.* 22(4), 11-22 (1988); translation in *Funct. Anal. Appl.* 22, 263-272 (1988)
- [Nov2] Novikov R.G.: Absence of exponentially localized solitons for the Novikov–Veselov equation at positive energy. *Physics Letters A.* 375, 1233–1235 (2011)
- [NV1] Novikov S.P., Veselov A.P.: Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formula and evolutions equations. *Dokl. Akad. Nauk SSSR.* 279, 20–24 (1984), translation in *Sov. Math. Dokl.* 30, 588-591 (1984)
- [NV2] Novikov S.P., Veselov A.P.: Finite-zone, two-dimensional Schrödinger operators. Potential operators. *Dokl. Akad. Nauk SSSR.* 279, 784–788 (1984), translation in *Sov. Math. Dokl.* 30, 705–708 (1984)
- [T] Tsai T.-Y. The Schrödinger operator in the plane. *Inverse Problems.* 9, 763–787 (1993)