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# Absence of traveling wave solutions of conductivity type for the Novikov-Veselov equation at zero energy

A.V. Kazeykina <sup>1</sup>

**Abstract.** We prove that the Novikov-Veselov equation (an analog of KdV in dimension  $2 + 1$ ) at zero energy does not have sufficiently localized soliton solutions of conductivity type.

## 1 Introduction

In this note we are concerned with the Novikov-Veselov equation at zero energy

$$\begin{aligned} \partial_t v &= 4\operatorname{Re}(4\partial_z^3 v + \partial_z(vw)), \\ \partial_{\bar{z}} w &= -3\partial_z v, \quad v = \bar{v}, \\ v &= v(x, t), \quad w = w(x, t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).$$

**Definition 1.** A pair  $(v, w)$  is a sufficiently localized solution of equation (1) if

- $v, w \in C(\mathbb{R}^2 \times \mathbb{R})$ ,  $v(\cdot, t) \in C^3(\mathbb{R}^3)$ ,
- $|\partial_x^j v(x, t)| \leq \frac{q(t)}{(1 + |x|)^{2+\varepsilon}}$ ,  $|j| \leq 3$ , for some  $\varepsilon > 0$ ,  $w(x, t) \rightarrow 0$ ,  $|x| \rightarrow \infty$ ,
- $(v, w)$  satisfies (1).

**Definition 2.** A solution  $(v, w)$  of (1) is a soliton (a traveling wave) if  $v(x, t) = V(x - ct)$ ,  $c \in \mathbb{R}^2$ .

Equation (1) is an analog of the classic KdV equation. When  $v = v(x_1, t)$ ,  $w = w(x_1, t)$ , then equation (1) is reduced to KdV. Besides, equation (1) is integrable via the scattering transform for the 2-dimensional Schrödinger equation

$$\begin{aligned} L\psi &= 0, \\ L &= -\Delta + v(x, t), \quad \Delta = 4\partial_z\partial_{\bar{z}}, \quad x \in \mathbb{R}^2. \end{aligned} \tag{2}$$

Equation (1) is contained implicitly in [M] as an equation possessing the following representation

$$\frac{\partial(L - E)}{\partial t} = [L - E, A] + B(L - E), \tag{3}$$

where  $L$  is defined in (2),  $A$  and  $B$  are suitable differential operators of the third and zero order respectively and  $[\cdot, \cdot]$  denotes the commutator. In the explicit form equation (1) was written in [NV1], [NV2], where it was also studied in the periodic setting. For the rapidly decaying potentials the studies of equation (1) and the scattering problem for (2) were carried out in [BLMP], [GN] [T], [LMS]. In [LMS] the relation with the Calderón conductivity problem was discussed in detail.

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**Definition 3.** A potential  $v \in L^p(\mathbb{R}^2)$ ,  $1 < p < 2$ , is of conductivity type if  $v = \gamma^{-1/2} \Delta \gamma^{1/2}$  for some real-valued positive  $\gamma \in L^\infty(\mathbb{R}^2)$ , such that  $\gamma \geq \delta_0 > 0$  and  $\nabla \gamma^{1/2} \in L^p(\mathbb{R}^2)$ .

The potentials of conductivity type arise naturally when the Calderón conductivity problem is studied in the setting of the boundary value problem for the 2-dimensional Schrödinger equation at zero energy (see [Nov1], [N], [LMS]); in addition, in [N] it was shown that for this type of potentials the scattering data are well-defined everywhere.

The main result of the present note consists in the following: there are no solitons of conductivity type for equation (1). The proof is based on the ideas proposed in [Nov2].

This work was fulfilled in the framework of research carried out under the supervision of R.G. Novikov.

## 2 Scattering data for the 2-dimensional Schrödinger equation at zero energy with a potential of conductivity type

Consider the Schrödinger equation (2) on the plane with the potential  $v(z)$ ,  $z = x_1 + ix_2$ , satisfying

$$\begin{aligned} v(z) &= \overline{v(\bar{z})}, \quad v(z) \in L^\infty(\mathbb{C}), \\ |v(z)| &< q(1 + |z|)^{-2-\varepsilon} \text{ for some } q > 0, \varepsilon > 0. \end{aligned} \quad (4)$$

For  $k \in \mathbb{C}$  we consider solutions  $\psi(z, k)$  of (2) having the following asymptotics

$$\psi(z, k) = e^{ikz} \mu(z, k), \quad \mu(z, k) = 1 + o(1), \text{ as } |z| \rightarrow \infty, \quad (5)$$

i.e. Faddeev's exponentially growing solutions for the two-dimensional Schrödinger equation (2) at zero energy, see [F], [GN], [Nov1].

It was shown that if  $v$  satisfies (4) and is of conductivity type, then  $\forall k \in \mathbb{C} \setminus 0$  there exists a unique continuous solution of (1) satisfying (5) (see [N]). Thus the scattering data  $b$  for the potential  $v$  of conductivity type are well-defined and continuous:

$$b(k) = \iint_{\mathbb{C}} e^{i(ky + \bar{k}\bar{y})} v(y) \mu(y, k) d\text{Re}y d\text{Im}y, \quad k \in \mathbb{C} \setminus 0. \quad (6)$$

In addition, the function  $\mu(z, k)$  from (5) satisfies the following  $\bar{\partial}$ -equation

$$\frac{\partial \mu(z, k)}{\partial \bar{k}} = \frac{1}{4\pi \bar{k}} e^{-i(kz + \bar{k}\bar{z})} b(k) \overline{\mu(z, k)}, \quad z \in \mathbb{C}, \quad k \in \mathbb{C} \setminus 0 \quad (7)$$

and the following limit properties:

$$\mu(z, k) \rightarrow 1, \text{ as } |k| \rightarrow \infty, \quad (8)$$

$$\mu(z, k) \text{ is bounded in the neighborhood of } k = 0. \quad (9)$$

The following lemma describes the scattering data corresponding to a shifted potential.

**Lemma 1.** Let  $v(z)$  be a potential satisfying (4) with the scattering data  $b(k)$ . The scattering data  $b_y(k)$  for the potential  $v_y(z) = v(z - y)$  are related to  $b(k)$  by the following formula

$$b_y(k) = e^{i(ky + \bar{k}\bar{y})} b(k), \quad k \in \mathbb{C} \setminus 0, \quad y \in \mathbb{C}. \quad (10)$$

*Proof.* We note that  $\psi(z - y, k)$  satisfies (1) with  $v_y(z)$  and has the asymptotics  $\psi(z - y, k) = e^{ik(z-y)}(1 + o(1))$  as  $|z| \rightarrow \infty$ . Thus  $\psi_y(z, k) = e^{iky}\psi(z - y, k)$  and  $\mu_y(z, k) = \mu(z - y, k)$ . Finally, we have

$$\begin{aligned} b_y(k) &= \iint_{\mathbb{C}} e^{i(k\zeta + \bar{k}\bar{\zeta})} v_y(\zeta) \mu_y(\zeta, k) d\operatorname{Re}\zeta d\operatorname{Im}\zeta = \\ &= \iint_{\mathbb{C}} e^{i(k\zeta + \bar{k}\bar{\zeta})} v(\zeta - y) \mu(\zeta - y, k) d\operatorname{Re}\zeta d\operatorname{Im}\zeta = e^{i(ky + \bar{k}\bar{y})} b(k). \end{aligned}$$

□

As for the time dynamics of the scattering data, in [BLMP], [GN] it was shown that if the solution  $(v, w)$  of (1) exists and the scattering data for this solution are well-defined, then the time evolution of these scattering data is described as follows:

$$b(k, t) = e^{i(k^3 + \bar{k}^3)t} b(k, 0), \quad k \in \mathbb{C} \setminus 0, \quad t \in \mathbb{R}. \quad (11)$$

### 3 Absence of solitons of conductivity type

**Theorem 1.** *Let  $(v, w)$  be a sufficiently localized traveling wave solution of (1) of conductivity type. Then  $v \equiv 0$ ,  $w \equiv 0$ .*

*Proof.* From (10), (11), continuity of  $b(k)$  on  $\mathbb{C} \setminus 0$  and the fact that the functions  $k, \bar{k}, k^3, \bar{k}^3, 1$  are linearly independent in the neighborhood of any point, it follows that  $b \equiv 0$ . Equation (7) implies that in this case the function  $\mu(z, k)$  is holomorphic on  $k, k \in \mathbb{C} \setminus 0$ . Using properties (8) and (9) we apply Liouville theorem to obtain that  $\mu \equiv 1$ . Then  $\psi(z, k) = e^{ikz}$  and from (2) it follows that  $v \equiv 0$ . □

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