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Existence of weak solutions for compressible Navier-Stokes equations in a moving domain

B. DI MARTINO

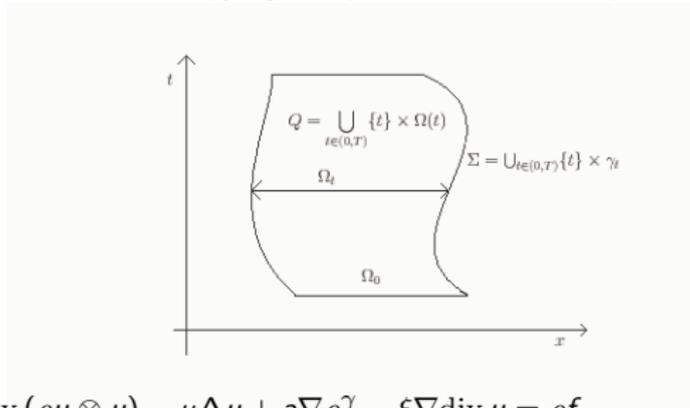
(IN COLLABORATION WITH P. ORENGA AND M. PEYBERNES)

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Physical description of the problem

We consider a compressible fluid occupying an open bounded set depending on time.



$$\frac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + a \nabla \rho^\gamma - \xi \nabla \operatorname{div} u = \rho f, \quad \text{on } Q \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \quad \text{on } Q \quad (2)$$

where $\mu > 0$, $\mu + \xi > 0$, $a > 0$, $T > 0$, $\gamma \in (1, \infty)$.

Boundary condition :

$$u(X + d(X, t), t) = \frac{\partial d}{\partial t} \quad \text{on } \Sigma$$

where d is a given smooth function describing the Σ evolution.

References :

- For a fixed domain, main results have been presented by P.L. Lions (*Mathematical topics in fluid mechanics, volume 2, Compressible models*, Oxford science Publication, 1996)
- For time depending domain, some works have been realized by B. Desjardins and M.J. Esteban, and by M. Boulakia when a fluid is contained in a fixed bounded set and some elastic structures are immersed in this fluid.
- The problem of interaction between a compressible fluid bounded by an elastic plate is considered F. Flori and P. Orenca with the *quasi Navier Stokes* equations.
- An existence result for a free boundary shallow water model is proposed by F. Flori, P. Orenca and M. Peybernes using a Lagrangian scheme.

Some preliminary remarks :

- On boundary conditions

Usually, for non homogeneous problem, it is necessary to impose value of density when $u \cdot n < 0$. But, in our problem, even if we have non homogeneous boundary condition on the velocity, no fluid go across the boundary and it is no necessary to impose the value of density when $u \cdot n < 0$.

- On Eulerian or Lagrangian description

The more natural method to treat the difficulty linked to a moving domain is to use a lagrangian description of the equations. But we are not able to obtain energy estimate using this Lagrangian form.

The homogeneous problem

As usual for a non homogeneous Dirichlet problem, we assume that there exists a sufficiently smooth function w on Q such that $w = \frac{\partial d}{\partial t}$ on γ_t and verifying

$$w \in L^2(0, T, H^1(\Omega_t)) \cap L^\infty(0, T; L^{2\gamma/(\gamma-1)}(\Omega_t))$$

$$\operatorname{div} w \in L^2(0, T; L^\infty(\Omega_t))$$

$$\frac{\partial w}{\partial t} \text{ in a good space.}$$

By tacking $v = u - w$, we can prove the following *a priori* result :

Lemma

Let (v, ρ) a classical solution of the problem, then under some assumption on initial data and for a finite time T , v and ρ verify :

$$v \in L^2(0, T; H_0^1(\Omega_t)), \quad \rho \in L^\infty(0, T; L^\gamma(\Omega_t)), \quad \sqrt{\rho}v \in L^\infty(0, T; L^2(\Omega_t)).$$

Proof

A large part of the estimate is usual and we give only a sketch of proof to treat the pressure term $a\nabla\rho^\gamma$. To handle this one, we multiply the mass equation by $a\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$:

$$\frac{a}{\gamma-1} \frac{\partial \rho^\gamma}{\partial t} + \frac{\gamma a}{\gamma-1} \rho^{\gamma-1} \operatorname{div}(\rho u) = 0,$$

with

$$\operatorname{div}(\rho \rho^{\gamma-1} u) = \rho^{\gamma-1} \operatorname{div}(\rho u) + \frac{\gamma-1}{\gamma} u \nabla \rho^\gamma.$$

After some straight-forward manipulations and an integration on Ω_t , we obtain

$$\int_{\Omega_t} a \nabla \rho^\gamma dx = a \int_{\Omega_t} \rho^\gamma \operatorname{div} w dx + \frac{a}{\gamma-1} \int_{\Omega_t} \frac{\partial \rho^\gamma}{\partial t} dx + \frac{a}{\gamma-1} \int_{\Gamma_t} \rho^\gamma w \cdot n dS.$$

Finally, by using the Leibniz formula, the boundary terms vanish, and we obtain the following result :

$$\int_{\Omega_t} a \nabla \rho^\gamma dx = a \int_{\Omega_t} \rho^\gamma \operatorname{div} w dx + \frac{a}{\gamma-1} \frac{d}{dt} \int_{\Omega_t} \rho^\gamma dx.$$

Energy estimate is the following

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} \rho \frac{|v|^2}{2} + \frac{a}{\gamma-1} \rho^\gamma dx + \int_{\Omega_t} \mu |Dv^2| dx \\ = \int_{\Omega_t} \rho \left(f - \frac{\partial w}{\partial t} \right) v + (\rho u \cdot \nabla) w \cdot v - a \rho^\gamma \operatorname{div} w + \mu \operatorname{div} (v \nabla w) - \mu Dv Dw \, dx. \end{aligned}$$

The first two terms of the right-hand side are easily estimated by the Cauchy Schwarz inequality. For the two remaining terms, we use the bound on $\operatorname{div} w$ in $L^1(0, T; L^\infty(\Omega_t))$ and we can write

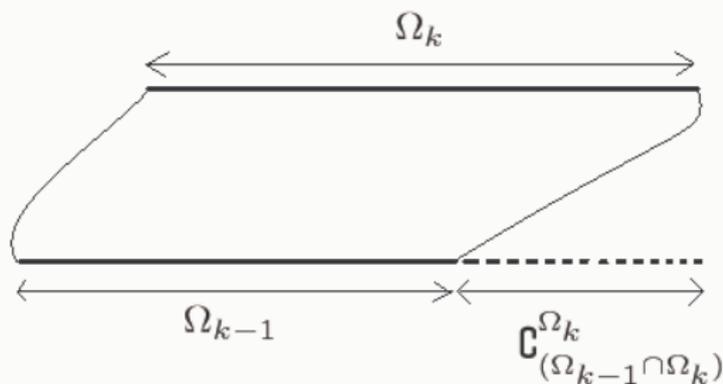
$$\begin{aligned} \int_{\Omega_t} \rho^\gamma \operatorname{div} w \, dx &\leq \| \operatorname{div} w \|_{L^\infty(\Omega_t)} \int_{\Omega_t} \rho^\gamma \, dx, \\ \int_{\Omega_t} (\rho u \cdot \nabla) w \cdot v \, dx &\leq C \| \operatorname{div} w \|_{L^\infty(\Omega_t)} \left[\int_{\Omega_t} \rho v^2 \, dx + \| \rho \|_{L^\gamma(\Omega_t)} \| w \|_{L^{2\gamma/(\gamma-1)}(\Omega_t)}^{\frac{1}{2}} \right]. \end{aligned}$$

Then we deduce from Grönwall's lemma the desired bounds.

Time discretization

Proof of the existence is obtained by building a sequence of approached solutions based on the time discretisation of the continuous equations. The main difficulty is linked to the time depending domain. A classical Eulerian discretization requires to evaluate the value at a given position x_i at time t_k depending of the value at the same position at time t_{k-1} . But this point can be out of the domain Q .

$$\frac{\partial \rho}{\partial t}(x_i, t_k) \simeq \frac{\rho(x_i, t_k) - \rho(x_i, t_{k-1})}{\Delta t}$$



For $M \geq 1$, we denote $\Delta t = \frac{T}{M}$ and we define for $k \geq 0$ a sequence of $(m+1)$ domains Ω_k by taking $\Omega_k = Q \cap k\Delta t$.

Let ρ_k defined on Ω_k , we define $\bar{\rho}_{k-1}$ a function also defined on Ω_k such that $\bar{\rho}_{k-1} = \rho_{k-1}$ on $(\Omega_{k-1} \cap \Omega_k)$ and $\bar{\rho}_{k-1} = 0$ on $(\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_k})$.

Similarly, we define \bar{u}_{k-1} a function such that $\bar{u}_{k-1} = u_{k-1}$ on $(\Omega_{k-1} \cap \Omega_k)$, but \bar{u}_{k-1} are non imposed on $(\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_k})$. Since this term is systematically multiplied par $\bar{\rho}_{k-1}$, the value of the product is zero in $(\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_k})$ for all chosen value of \bar{u}_{k-1}

Discretized problem

Let $w_k = w(k\Delta t)$ where w is the function defined previously, by induction for $k \geq 1$, we define (ρ_k, v_k) to be the solution of our discretized problem on Ω_k :

$$\rho_k - \bar{\rho}_{k-1} + \Delta t \operatorname{div}(\rho_k u_k) - C_{k-1} = 0$$

$$\begin{aligned} \rho_k v_k - \bar{\rho}_{k-1} \bar{v}_{k-1} + \bar{\rho}_{k-1}(w_k - \bar{w}_{k-1}) + w_k(\rho_k - \bar{\rho}_{k-1}) + \Delta t \operatorname{div}(\rho_k u_k \otimes u_k) \\ - \mu \Delta t \Delta u_k + \Delta t a \nabla \rho_k^\gamma = 0 \end{aligned}$$

where $C_{k-1} = \frac{1}{\operatorname{meas} \Omega_k} \int_{\mathbb{C}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_{k-1}}} \rho_{k-1} dx$ and $u_k = v_k + w_k$.

At this time, we have to consider existence of solution for this stationary problem. This one can be obtained by a fixed point argument after obtention of some compactness results.

Compactness results

Lemma

Let (ρ_k, v_k) ($0 \leq k \leq M$) a sequence of weak solutions of our problem with the homogeneous boundary conditions on v_k and some initial conditions $\rho_0 \geq 0$, v_0 a.e. in Ω_0 . Let $\rho_k = 0$ on $\{x \in \partial\Omega_k, w(x) \cdot n < 0\}$

$$\sup_{0 \leq l \leq M} \|\rho_l\|_{L^\gamma(\Omega_l)} + \|\rho_l |v_l|^2\|_{L^1(\Omega_l)} \leq C$$

$$\sum_{k=1}^M \Delta t \|\nabla v_l\|_{L^2(\Omega_l)}^2 \leq C$$

$$\sum_{k=1}^M \|\sqrt{\bar{\rho}_{k-1}}(v_k - \bar{v}_{k-1})\|_{L^2(\Omega_k)}^2 \leq C$$

$$\sum_{k=1}^M \int_{\Omega_k} [(\gamma - 1)\rho_k^\gamma + \bar{\rho}_{k-1}^\gamma - \gamma\rho_k^{\gamma-1}\bar{\rho}_{k-1}] dx \leq C$$

$$\sum_{k=1}^M \int_{\Omega_k} |\rho_k - \bar{\rho}_{k-1}|^\gamma dx \leq C$$

Where, here and below, C denotes various positive constants independent of k , Δt and thus M .

Proof

We give only one step of the demonstration to treat the pressure term.

To do this, we multiply the discretized mass equation by $a \frac{\rho_k^{\gamma-1}}{\gamma-1}$ and after some manipulations, we deduce :

$$a\Delta t v_k \nabla \rho_k^\gamma = \frac{\gamma a}{\gamma-1} \rho_k^\gamma - \frac{\gamma a}{\gamma-1} \rho_k^{\gamma-1} \bar{\rho}_{k-1} + \frac{a\Delta T \gamma}{\gamma-1} \operatorname{div}(u_k \rho_k^\gamma) - a\Delta t w_k \nabla \rho_k^\gamma - \frac{\gamma C_{k-1}}{\gamma-1} \rho_k^{\gamma-1}.$$

Then, after integration over Ω_k , we obtain

$$\begin{aligned} a\Delta t \int_{\Omega_k} v_k \nabla \rho_k^\gamma dx &= \frac{a}{\gamma-1} \int_{\Omega_k} \rho_k^\gamma dx - \frac{a}{\gamma-1} \int_{\Omega_{k-1}} \rho_{k-1}^\gamma dx + \frac{a}{\gamma-1} \int_{\mathbb{C}_{(\Omega_{k-1} \cap \Omega_k)}} \rho_{k-1}^\gamma dx \\ &+ \frac{a\gamma\Delta T}{\gamma-1} \int_{\Omega_k} \operatorname{div}(u_k \rho_k^\gamma) dx - a\Delta T \int_{\Omega_k} w_k \nabla \rho_k^\gamma dx - \frac{\gamma C_{k-1}}{\gamma-1} \int_{\Omega_k} \rho_k^{\gamma-1} dx \\ &+ \frac{a}{\gamma-1} \int_{\Omega_k} (\gamma-1)\rho_k^\gamma + \bar{\rho}_{k-1}^\gamma - \gamma \rho_k^{\gamma-1} \bar{\rho}_{k-1} dx \end{aligned}$$

By adding from $k = 1$ to M we obtain

$$\begin{aligned}
& \frac{1}{2} \|\sqrt{\rho_M} v_M\|_{L^2(\Omega_M)}^2 + \frac{1}{2} \sum_{k=1}^M \int_{\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_{k-1}}} \rho_{k-1} v_{k-1}^2 + \frac{3\mu}{2} \sum_{k=1}^M \Delta t \|\nabla v_k\|_{L^2(\Omega_k)}^2 + \frac{a}{\gamma-1} \|\rho_M\|_{L^\gamma(\Omega_M)}^\gamma \\
& + \left(\frac{a}{\gamma-1} - \varepsilon \right) \sum_{k=1}^M \|\rho_{k-1}\|_{L^\gamma(\mathbb{G}_{\Omega_{k-1} \cap \Omega_k}^{\Omega_{k-1}})}^\gamma + \sum_{k=1}^M \int_{\Omega_k} [(\gamma-1)\rho_k^\gamma + \bar{\rho}_{k-1}^\gamma - \gamma\rho_k^{\gamma-1}\bar{\rho}_{k-1}] \\
& + \frac{1}{2} \sum_{k=1}^M \|(1-\Delta t)\sqrt{\bar{\rho}_{k-1}}(v_k - \bar{v}_{k-1})\|_{L^2(\Omega_k)}^2 \\
\leq & \frac{1}{2} \|\sqrt{\rho_0} v_0\|_{L^2(\Omega_0)}^2 + \frac{a}{\gamma-1} \|\rho_0\|_{L^\gamma(\Omega_0)}^\gamma + \sum_{k=1}^M \Delta t \left[\frac{3}{2} \|Dw_k\|_{L^\infty(\Omega_k)} + \|f_k\|_{L^{2\gamma/(\gamma-1)}(\Omega_k)}^2 \right] \|\sqrt{\rho_k} v_k\|_{L^2(\Omega_k)}^2 \\
& + \sum_{k=1}^M \Delta t \left[\|Dw_k\|_{L^\infty(\Omega_k)} \frac{1}{2} \|w_k\|_{L^{2\gamma/(\gamma-1)}(\Omega_k)}^{\frac{1}{2}} - aC \|Dw_k\|_{L^\infty(\Omega_k)} \right. \\
& \left. + \frac{3}{2} \|f_k\|_{L^{2\gamma/(\gamma-1)}(\Omega_k)}^2 + C_\varepsilon \right] \|\rho_k\|_{L^\gamma(\Omega_k)} \\
& - \mu \sum_{k=1}^M \Delta t \frac{1}{2} \|\nabla w_k\|_{L^2(\Omega_k)}^2 + \frac{3}{2} \|f_0\|_{L^{2\gamma/(\gamma-1)}(\Omega_0)}^2 \|\rho_0\|_{L^\gamma(\Omega_0)}
\end{aligned}$$

Using the discrete Gronwall lemma, we deduce the bound on $\rho_k v_k^2$, ρ_k and ∇v_k .

To ensure convergence of the solution of our equations to this of the continuous problem, we need to obtain an other bound for ρ to treat the pressure term. We now can prove the following lemma

Lemma

If $\gamma > 2$ for $N = 2$ or $\gamma \geq 3$ for $N = 3$, we have $\forall \varphi \in C^\infty([0, T]; C_0^\infty(\Omega_t))$

$$\sum_{k=1}^M \Delta t \|\varphi \rho_k\|_{L^{\gamma+1}}^{\gamma+1} \leq C$$

Proof

The proof follow these proposed by P.L. Lions by applying the $\Delta^{-1} \operatorname{div}$ operator at the discrete momentum equation. To apply this operator in case of Dirichlet boundary condition, we need to localize the equations by applying a cut-off function. In our case of moving domain, we define for any fixed compact set $K \subset Q$, a cut-off function $\varphi \in C^\infty([0, T]; C_0^\infty(\Omega_t))$ such that $\varphi \equiv 1$ on K and $\varphi \geq 0$ on Q .

The end of the proof corresponds to this given by P.L. Lions.

Passage to the limit

Introducing for any function $(\varphi_0, \varphi_1, \dots, \varphi_M)$ the piecewise constant (respectively piecewise affine) function $\hat{\varphi}$ and $\bar{\varphi}$ (respectively $\tilde{\varphi}$) defined by

$$\hat{\varphi}(x, t) = \varphi_{k+1}(x), \quad \text{if } k\Delta t \leq t < (k+1)\Delta t$$

$$\bar{\varphi}(x, t) = \bar{\varphi}_k(x), \quad \text{if } k\Delta T \leq t < (k+1)\Delta T$$

$$\text{(respectively } \tilde{\varphi}(x, t) = \varphi_k(x) + (t - k\Delta t) \left(\frac{\varphi_{k+1} - \bar{\varphi}_k}{\Delta t} \right) (x), \quad \text{if } k\Delta t \leq t < (k+1)\Delta t)$$

We have

$$\int_0^T \hat{\varphi}(t) dt = \sum_{k=1}^M \Delta t \varphi_k, \quad \int_0^T \tilde{\varphi}(t) dt = \sum_{k=1}^M \Delta t \frac{\varphi_{k-1} + \varphi_k}{2}$$

and

$$\frac{\partial \tilde{\varphi}}{\partial t} = \frac{\varphi_{k+1} - \bar{\varphi}_k}{\Delta t} \quad \text{if } k\Delta t \leq t < (k+1)\Delta t$$

With this notation, we can observe that our discretized equations on $[(k-1)\Delta t, k\Delta t[$, ($k \geq 1$)

$$\rho_k - \bar{\rho}_{k-1} + \Delta t \operatorname{div}(\rho_k u_k) - C_{k-1} = 0$$

$$\begin{aligned} \rho_k v_k - \bar{\rho}_{k-1} \bar{v}_{k-1} + \bar{\rho}_{k-1}(w_k - \bar{w}_{k-1}) + w_k(\rho_k - \bar{\rho}_{k-1}) + \Delta t \operatorname{div}(\rho_k u_k \otimes u_k) \\ - \mu \Delta t \Delta u_k + \Delta t a \nabla \rho_k^\gamma = 0 \end{aligned}$$

can be written as

$$\begin{aligned} \frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\hat{\rho} \hat{u}) &= \frac{\bar{C}}{\Delta t} \\ \frac{\partial \tilde{\rho} \tilde{u}}{\partial t} + \operatorname{div}(\hat{\rho} \hat{u} \otimes \hat{u}) - \mu \Delta \hat{u} + a \nabla \hat{\rho}^\gamma &= 0 \end{aligned}$$

where $\hat{u} = \hat{v} + \hat{w}$ and $\tilde{\rho} \tilde{u} = \tilde{\rho} \tilde{v} + \tilde{\rho} \tilde{w}$.

It is easy to see that the term $\frac{\bar{C}}{\Delta t}$ goes to zero when Δt goes to zero. We recall that

$$C_{k-1} = \frac{1}{\text{meas } \Omega_k} \int_{\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_{k-1}}} \rho_{k-1} dx.$$

$$\int_{\mathbb{G}_{(\Omega_k \cap \Omega_{k+1})}^{\Omega_{k-1}}} \rho_{k-1} dx \leq \|\rho_{k-1}\|_{L^\gamma(\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_{k-1}})} \text{meas}(\mathbb{G}_{(\Omega_k \cap \Omega_{k+1})}^{\Omega_{k-1}})^{\frac{\gamma}{\gamma-1}}$$

We introduce then the d_k sequence such that $d_0 = 0$ and for all k and all point X belonging on the boundary of Ω_k , $d_k(X) = d_{k-1}(X) + \int_{(k-1)\Delta t}^{k\Delta t} w(X, t) dt$. Then

$$\text{meas}(\mathbb{G}_{(\Omega_k \cap \Omega_{k-1})}^{\Omega_{k-1}}) \leq C \int_{\gamma_0} |d_k - d_{k-1}| dS \leq C' \Delta T$$

where C and C' depend of the regularity of the domain. Then

$$\frac{\bar{C}}{\Delta t} \leq (C' \Delta T)^{\frac{1}{\gamma-1}} \|\rho_{k-1}\|_{L^\gamma(\mathbb{G}_{(\Omega_{k-1} \cap \Omega_k)}^{\Omega_{k-1}})} \leq C'' \Delta T^{\frac{1}{\gamma-1}}$$

From the estimate $\sup_{0 \leq l \leq M} \|\rho_l\|_{L^\gamma(\Omega_l)} \leq C$ and $\sum_{k=0}^M \Delta t \|\rho_k\|_{L^{\gamma+1}}^{\gamma+1} \leq C$, we may assume that, extracting subsequences if necessary, as Δt goes to 0, $\hat{\rho}$ converge weakly in $L^\infty(0, T; L^\gamma(\Omega_t))$ (weak-*) and in $L^{\gamma+1}(L_{loc}^{\gamma+1}(\Omega_t))$ for some $\rho \geq 0$.

In addition, from $\sum_{k=1}^M \int_{\Omega_k} |\rho_k - \bar{\rho}_{k-1}|^\gamma dx \leq C$ we deduce that

$$\int_0^T \int_{\Omega_t} |\hat{\rho}(t) - \bar{\rho}(t)|^\gamma dx dt = \Delta t \sum_{k=1}^M \int_{\Omega_k} |\rho_k - \bar{\rho}_{k-1}|^\gamma dx \leq C \Delta t$$

and we see easily that $\hat{\rho} - \bar{\rho}$ converge to 0 as Δt goes to 0_+ in $L^q(0, T; L^\gamma)$ for all $1 \leq q < \infty$. Moreover,

$$\tilde{\rho} - \bar{\rho} = \frac{\rho_k - \bar{\rho}_{k-1}}{\Delta t} \underbrace{(t - (k-1)\Delta t)}_{\leq \Delta t} \quad \text{on } [(k-1)\Delta t; k\Delta t[,$$

then,

$$\int_0^T \int_{\Omega_t} |\tilde{\rho} - \bar{\rho}|^\gamma dx dt \leq \int_0^T \int_{\Omega_t} |\rho_k - \bar{\rho}_{k-1}|^\gamma dx dt \leq \|\hat{\rho} - \bar{\rho}\|_{L^\gamma(Q)}^\gamma$$

and we deduce that $\tilde{\rho} - \bar{\rho}$ converge to 0 as Δt goes to 0_+ in $L^q(0, T; L^\gamma)$ for all $1 \leq q < \infty$.

Passage to the limit in non linear terms

To ensure convergence of non linear terms, we recall the following general lemma (P.L. Lions)

Lemma

Let g_n, h_n converge weakly to g, h respectively in $L^{p_1}(O, T; L^{p_2}(\Omega)), L^{q_1}(O, T; L^{q_2}(\Omega))$ where $1 \leq p_1, p_2 \leq \infty$,

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

We assume in addition that

$\frac{\partial g_n}{\partial t}$ is bounded in $L^1(0, T; W^{-m,1}(\Omega))$ for some $m \geq 0$ independent of n .

$\|h_n - h_n(\cdot + \xi, t)\|_{L^{q_1}(O, T; L^{q_2}(\Omega))} \rightarrow 0$ as $|\xi| \rightarrow 0$, uniformly in n .

Then, $g_n h_n$ converge to gh (in the sens of distribution in $\Omega \times (0, T)$).

The estimate $\sup_{0 \leq l \leq M} \|\rho_l\|_{L^\gamma(\Omega_l)} + \|\rho_l |u_l|^2\|_{L^1(\Omega_l)} \leq C$ imply that $\sqrt{\hat{\rho}}\hat{u}, \sqrt{\tilde{\rho}}\tilde{u}$ are bounded in $L^\infty(0, T; L^1(\Omega_t))$ and $\sqrt{\hat{\rho}}, \sqrt{\tilde{\rho}}$ are bounded in $L^\infty(0, T; L^{2\gamma}(\Omega_t))$.

$\Rightarrow \hat{\rho}\hat{u}, \tilde{\rho}\tilde{u}$ are bounded in $L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega_t))$.

We can also claim that $\hat{\rho}\hat{u} - \bar{\rho}\bar{u}$ and $\hat{\rho}\hat{u} - \tilde{\rho}\tilde{u}$ converge to 0 as Δt goes to 0_+ in $L^q(0, T; L^r(\Omega_t))$ with $2 \leq q < \infty$, $1 \leq r < \frac{2\gamma}{\gamma+1}$. This convergence is obvious using previous bounds and observing that $\rho_k u_k - \bar{\rho}_{k-1} \bar{u}_{k-1} = (\rho_k - \bar{\rho}_k) u_k + \bar{\rho}_{k-1} (u_k - u_{k-1})$

From $\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\hat{\rho}\hat{u}) = \frac{\bar{C}}{\Delta t}$ and using the bound on $\hat{\rho}\hat{u}$, we deduce that $\frac{\partial \tilde{\rho}}{\partial t}$ is bounded in $L^\infty(0, T; W^{-1, 2\gamma/(\gamma-1)}(\Omega_t)) \subset L^1(0, T; W^{-1, 1})$. Then, applying previous lemma we deduce that $\tilde{\rho}\hat{u}$ converge to ρu (in the sense of distributions on Q).

This equation can then be written as follow :

$$\frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\tilde{\rho}\hat{u}) - \operatorname{div}((\tilde{\rho} - \hat{\rho})\hat{u}) = \frac{\bar{C}}{\Delta t}$$

and, since $(\tilde{\rho} - \hat{\rho})$ goes to 0 we can pass to the limit in this equation in the sense of distribution on Q to obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0$$

To pass to the limit in momentum equation, we first observe that

$$|\hat{\rho}\hat{u} \otimes \hat{u} - \widetilde{\rho u} \otimes \hat{u}| \leq |\hat{u}| |\hat{\rho} - \bar{\rho u}|$$

We deduce then that $\hat{\rho}\hat{u} \otimes \hat{u} - \widetilde{\rho u} \otimes \hat{u}$ converge to 0 as Δt goes to 0_+ in $L^1(0, T; L^r(\Omega_t)) \cap L^q(0, T; L^1(\Omega_t))$ for all $1 \leq r \leq r_1$ if $N \geq 3$, $1 \leq r < \gamma$ if $N = 2$ and for all $q \in [1, +\infty$ with $\frac{1}{r_1} = \frac{1}{\gamma} + \frac{N-2}{N}$

In view of bounds show above, we may prove that $\frac{\partial \widetilde{\rho u}}{\partial t}$ is bounded in $L^\infty(0, T; W^{-1,1}(\Omega_t)) + L^2(0, T; H^{-1}(\Omega_t)) \subset L^1(0, T; W^{-1,1}(\Omega_t))$. Using the Lemma 5.1 (PL2), we deduce that $\widetilde{\rho u} \otimes \hat{u}$, and thus $\hat{\rho}(\hat{u} \otimes \hat{u})$ converge weakly in $L^1(0, T; L^r(\Omega_t)) \cap L^q(0, T; L^1(\Omega_t))$ to $\rho u \otimes u$.

To finish the proof, we need then to obtain a strong convergence of $\hat{\rho}$ in $L^\gamma(Q)$. We follow for thus the proof proposed by P.L. Lions in the case where $\hat{\rho}$ is bounded in $L^{\gamma+1}(Q) \cap L^\infty(0, T, L^s(\Omega_t))$ for some $s > N$, which is precisely the case here.

Then, when Δt goes to 0_+ , our discretized system converge to the continuous system.

Perspectives

1) This work is restricted at the case of given moving domain (Q thus d are given). We propose to extend this result at a problem where d is unknown, for instance, d can be a solution of a PDE describing movement of a plate or a shell.

The main difficulty is then that we need to construct the domain at each time step and the solution can be obtained by a fixed point method. Moreover, the equation verified by d need to give sufficient regularity to obtain a sufficient regularity on boundary Σ that permit to apply all previous result.

2) Developments of a numerical code based on the constructive method developed here.

- 1 How to calculate W ?
- 2 Since we do not pass numerically to the limit, how many manage ρ on the boundary.