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Modified Leland's Strategy for a Constant Transaction Costs Rate

Emmanuel Denis

Dedicated to Agnes and my daughter Louise.

Abstract In 1985 Leland suggested an approach to price contingent claims under proportional transaction costs. Its main idea is to use the classical Black–Scholes formula with a suitably adjusted volatility for a periodical revision of the portfolio whose terminal value approximates the pay-off. Unfortunately, if the transaction costs rate does not depend on the number of revisions, the approximation error does not converge to zero as the frequency of revisions tends to infinity. In the present paper, we suggest a modification of Leland's strategy ensuring that the approximation error vanishes in the limit.

Key words Black–Scholes formula · Transaction costs · Leland's strategy · Approximate hedging.

Mathematics Subject Classification (2000) 60G44

JEL Classification G11 · G13

1 Introduction

In his seminal paper [7] Leland suggested a modification of the Black–Scholes approach to contingent claim pricing in the framework of a two-asset financial market model with proportional transaction costs. The idea is the following: one may use the Black–Scholes formula with an artificially enlarged volatility $\hat{\sigma}$ instead

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of σ , the true one. The intuition behind is to compensate for transaction costs by increasing volatility. A theoretical justification for this approach is based on the replication principle. The terminal value of a self-financing portfolio, revised at a sufficiently large number of dates $\{t_k : k = 1, \dots, n\}$, should approximate the terminal pay-off. Leland gave an explicit formula for the adjusted volatility $\hat{\sigma}$ which may depend on n , the number of revisions. His pricing methodology is of practical importance: it is easy to implement.

However, a mathematical justification of this “approximate replication principle” turns out to be quite difficult to obtain.

The first rigorous result conjectured by Leland was obtained by Lott [9]: for the call option the approximation error tends to zero in probability if the transaction costs coefficients $k = k_n$ decrease to zero with the rate proportional to $n^{-1/2}$ (in this case $\hat{\sigma}$ does not depend on n). On the other hand, the replication principle fails to be true if $k = k_0$ is a constant and the Leland strategy leads to a systematic limiting error. This was observed by Kabanov and Safarian [6] who proved also that the replication error tends to zero when $k = k_n$ decreases to zero with the rate $n^{-\alpha}$, $\alpha \in (0, 1/2)$.

There are a number of studies treating the case $\alpha \in (0, 1/2)$ and, especially, $\alpha = 1/2$: for more general pay-off functions, non-uniform revision intervals, on the asymptotics of the L^2 -norm of the approximation error, see papers [1], [2], [3], [4], and the monograph [6].

The practically interesting case $\alpha = 0$ (i.e., k_0 is constant), where there is a systematic error also attracted a lot of attention. Limit theorems were obtained by Granditz and Schachinger [5] and Pergamenshchikov [10]. Zhao and Ziemba [12], [13] provides a numerical study of the limiting error for practical values of parameters. Sekine and Yano, [11] suggested some scheme to reduce it. In the paper [10] it was suggested a modification of the Leland strategy for the call option eliminating the limiting error. Unfortunately, the approach is based on the explicit formulae and, seemingly, cannot be generalized for more general pay-off functions.

Our modification of the Leland strategy has the following features:

- 1) we use the same enlarged volatility;
- 2) the initial value of the portfolio V_0^n is exactly the same;
- 3) the only difference is at the revision dates t_i ; We apply not the modified “delta” of the Black–Scholes formula with enlarged volatility, but correct it on the basis of previous revisions, see the formula (2.3).

We show that the terminal values of portfolios for the proposed strategy converge to the terminal pay-off. We believe that practitioners will benefit from the suggested modification. Our strategy outperforms the conventional one even for a small number of revision dates. The simulations presented in our paper show this quite clearly.

2 Main Result

We consider the standard two asset model with time horizon $T = 1$ assuming that it is specified under the martingale measure. The non risky asset is the *numéraire*

and the price of the risky asset is given by the formula

$$S_t = S_0 \exp \left\{ -\frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s \right\},$$

that is $dS_t = \sigma_t S_t dW_t$ where W is a Wiener process. We assume that $\sigma(t)$ is a strictly positive and continuous function on $[0, 1]$ verifying

$$|\sigma(t) - \sigma(u)| \leq \mu |t - u|$$

where $\mu > 0$ is a constant. In particular, we have $\sigma(t) \in [\sigma_1, \sigma_2]$ where $\sigma_1 > 0$. Note that

$$S_t \sim S_0 \exp\{\alpha_t \xi - \alpha_t^2/2\}$$

where $\alpha_t^2 = \int_0^t \sigma_s^2 ds$ and $\xi \sim \mathcal{N}(0, 1)$.

By virtue of Black and Scholes, the price of the contingent claim $G(S_1)$ is the initial value of the replicating portfolio

$$V_t = V_0 + \int_0^t H_r dS_r = E(G(S_1)|\mathcal{F}_t) = C(t, S_t),$$

where

$$C(t, x) = E G(x \exp\{\bar{\rho}_t \xi - \bar{\rho}_t^2/2\}),$$

$$\bar{\rho}_t = \int_t^1 \sigma_s^2 ds$$

and the replication strategy is $H_r = C_x(r, S_r)$.

In the model with proportional transaction costs and a finite number of revisions the current value of the portfolio process at time t is described as

$$V_t^n = V_0^n + \int_0^t D_u^n dS_u - \sum_{t_i \leq t} k_0 S_{t_i} |D_{t_{i+1}}^n - D_{t_i}^n| \quad (2.1)$$

where D^n is a piecewise constant process with $D^n = D_{t_i}^n$ on the interval $]t_{i-1}, t_i]$. Here $t_i = t_i^n$, $i \leq n$, are the revision dates and $D_{t_i}^n$ are $\mathcal{F}_{t_{i-1}}$ -measurable random variables. We assume that the transaction costs coefficient is a constant $k_0 > 0$ and the dates t_i are defined by a strictly increasing function $g \in C^2[0, 1]$ with $g(0) = 0$, $g(1) = 1$, so that $t_i = g(\frac{i}{n})$, $i = 0, \dots, n$. Let us denote by f the inverse of g . The adjusted volatility depending on n is given by the formula

$$\hat{\sigma}_t^2 = \sigma_t^2 + \sigma_t k_0 n^{\frac{1}{2}} \sqrt{8/\pi} \sqrt{f'(t)} = \sigma_t^2 + \sigma_t \gamma_n(t). \quad (2.2)$$

We modify the usual Leland strategy (see [1], [2], [3]) by considering the process D^n with $D_t^n = D_{t_i}^n$ on $t \in]t_{i-1}, t_i]$ and

$$D_{t_i}^n = \hat{C}_x(t_{i-1}, S_{t_{i-1}}) - \int_0^{t_{i-1}} \hat{C}_{xt}(t_u, S_u) du \quad (2.3)$$

where the function $\hat{C}(t, x)$ is the solution of the Cauchy problem:

$$\hat{C}_t(t, x) + \frac{1}{2} \hat{\sigma}_t^2 x^2 \hat{C}_{xx}(t, x) = 0, \quad \hat{C}(1, x) = G(x). \quad (2.4)$$

Its solution can be written as

$$\widehat{C}(t, x) = \int_{-\infty}^{\infty} G(xe^{\rho y - \rho^2/2})\varphi(y)dy \quad (2.5)$$

where $\rho^2 = (\rho_t^n)^2 = \int_t^1 \widehat{\sigma}_s^2 ds$ and φ is the standard Gaussian density.

Note that $\widehat{\sigma}_s^2 \geq \sigma_1^2 + cn^{\frac{1}{2}}$ for a constant $c > 0$ and, therefore

$$\rho^2 \geq (\sigma_1^2 + cn^{\frac{1}{2}})(1-t).$$

We shall use the following hypothesis on the cadence of revisions:

Assumption (R): $g' > 0$, $g \in C^2[0, 1]$.

The basic example is $g(t) = t$ but it could be interesting to concentrate revision dates near the horizon date (see for example [3]) in order to increase the convergence rate.

Note that there exists a constant $c > 0$ such that $\Delta t_i = t_i - t_{i-1} \leq cn^{-1}$. We shall repeatedly make use of results from [1], [2] since Assumption (R) is stronger than the similar one (G1) of these papers.

We use the abbreviations $\widehat{H}_t = \widehat{C}_x(t, S_t)$, $\widehat{h}_t = \widehat{C}_{xx}(t, S_t)$ and

$$K_t^n = \sum_{t_i^n < t} \Delta K_{t_i^n}^n$$

where $\Delta K_{t_0^n}^n = 0$ and for $i \geq 1$,

$$\Delta K_{t_i^n}^n = - \int_{t_{i-1}}^{t_i} \widehat{C}_{xt}(t_u, S_u) du.$$

Our hypothesis on the payoff function is as follows (see [1], [2]):

Assumption (G): G is a continuous convex function on $[0, \infty)$, two-times differentiable except at the points $K_1 < \dots < K_p$ where G' and G'' admit right and left limits; $|G''(x)| \leq Mx^{-\beta}$ for all $x \geq K_p$ where $\beta \geq 3/2$.

Let $K_0 = 0$ and $K_{p+1} = \infty$. Then G' , G'' exist on each interval $[K_i, K_{i+1})$ and are bounded while G verifies the inequality $|G(x)| \leq M(1+x)$ for some constant M . The function $\widehat{C}(t, x)$ is continuous on $[0, 1] \times \mathbf{R}$. Standard examples of such functions G satisfying **Assumption (G)** are the continuous convex functions which are piecewise affine.

The main result of this paper is the following:

Theorem 2.1 *Let $k_0 > 0$. Suppose that assumption (R) and (G) hold. Then*

$$P\text{-}\lim_n V_1^n = G(S_1). \quad (2.6)$$

Remark 2.1 The proof is given in Appendix. If G is not convex, an approximation error appears which is explicitly formulated in [1] and [2].

Remark 2.2 From a practical viewpoint, it is more convenient to consider the modified hedging strategy defined by:

$$D_{t_i}^n := \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) - \sum_{j=0}^{i-2} \widehat{C}_{xt}(t_j, S_{t_j})(t_{j+1} - t_j).$$

Theorem 2.1 still holds with this latter. Indeed, using Taylor approximations and using estimations from [1], [2] and [3], we can replace $\Delta K_{t_i}^n$ by

$$\Delta K_{t_i}^n = -\widehat{C}_{xt}(t_i, S_{t_i})\Delta t_i, \quad i \geq 1.$$

To prove this, it suffices to study the additional residual terms appearing by such a modification in the decomposition of the deviation $V_1^n - G(S_1)$ we use in the proof above. They involve the analysis of the differences

$$\int_{t_{i-1}}^{t_i} \widehat{C}_{xt}(t_u, S_u) du - \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}})\Delta t_i.$$

We conclude by using a Taylor approximation and the bounds of the successive derivatives we get from [1], [2] and [3].

3 Monte Carlo Simulations

This section presents an empirical study of the performance of Leland's method and the modified one. We consider the price process $(S_t)_{t \geq 0}$ with $\sigma = 0.2$ and $S_0 = 100$. The transaction costs coefficient is $k_0 = 0.01$, the strike price is $K = 100$. The simulation of the price $(S_t)_{t \geq 0}$ is obtained from N uniform subdivisions of the interval $[0, 1]$. For convenience, we choose uniform revision dates. The sample size is 5000. For different values of $N = n$ and different convex payoff functions, we compare the simulation of the gain $V_1^n - G(S_1)$ with $n = N$, using the initial Leland's method and the modified one, by computing the empirical mean \overline{gain}_n and the empirical standard deviation Sd_n of $V_1^n - G(S_1)$. Moreover, we provide the corresponding 95% confidence intervals I_n . We use the abbreviations \overline{gain}_n^{Mod} or \overline{gain}_n^{Lel} to denote whether we use the modified Leland strategy or not. Recall that the goal of Leland's method is to replicate approximatively $G(S_1)$ by the terminal wealth V_1^n . Hence our goal is to obtain, if possible, a gain $\overline{gain}_n \geq 0$.

3.1 The European Call

The payoff function is $G(x) = (x - K)^+$. The solution of the Cauchy problem (2.4) is given by

$$C(t, x) = x\phi(d(t, x)) - K\phi(d(t, x) - \widehat{\sigma}\sqrt{1-t}),$$

$$d(t, x) = \frac{\ln(x/K)}{\widehat{\sigma}\sqrt{1-t}} + \frac{\widehat{\sigma}\sqrt{1-t}}{2}$$

where

$$\phi(x) := \int_{-\infty}^x \varphi(y) dy.$$

Leland's strategy is defined by

$$C_x(t, x) = \phi(d(t, x)).$$

The modified Leland strategy is

$$D_{t_i}^n := C_x(t_{i-1}, S_{t_{i-1}}) - \sum_{j=0}^{i-2} C_{xt}(t_j, S_{t_j})(t_{j+1} - t_j)$$

where

$$C_{xt}(t, x) = \frac{1}{\hat{\sigma}\sqrt{1-t}} \left(\frac{1}{2} \frac{1}{1-t} \ln(x/K) - \frac{1}{4} \hat{\sigma}^2 \right) \varphi(d(t, x)).$$

We obtain the following results:

n	$\hat{\sigma}$	\overline{gain}_n^{Mod}	\overline{gain}_n^{Lel}	I_n^{Mod}	I_n^{Lel}	Sd_n^{Mod}	Sd_n^{Lel}
10	0.2238	0.1669	0.1063	[0.0155, 0.3182]	[-0.0422, 0.2548]	5.46	5.36
100	0.2681	0.083	0.031	[0.0170, 0.1489]	[-0.0169, 0.0789]	2.38	1.73
200	0.2917	0.046	0.003	[-0.0116, 0.1036]	[-0.032, 0.0387]	2.08	1.29
500	0.3337	0.039	-0.016	[-0.012, 0.09]	[-0.0387, 0.0067]	1.84	0.82
1000	0.3754	0.054	-0.006	[0.0032, 0.1047]	[-0.0229, 0.0109]	1.83	0.6121

As we can notice, a trader using the suggested modified strategy can increase in average his gain by 57% versus Leland's strategy when $n = 10$.

3.2 The European Put

The payoff function is $G(x) = (K - x)^+$. The solution of the Cauchy problem (2.4) is given by

$$P(t, x) = x\phi(d(t, x)) - K\phi(d(t, x) - \hat{\sigma}\sqrt{1-t}) - x + K,$$

$$d(t, x) = \frac{\ln(x/K)}{\hat{\sigma}\sqrt{1-t}} + \frac{\hat{\sigma}\sqrt{1-t}}{2}.$$

Leland's strategy is defined by

$$P_x(t, x) = \phi(d(t, x)) - 1.$$

The modified Leland strategy is

$$D_{t_i}^n := P_x(t_{i-1}, S_{t_{i-1}}) - \sum_{j=0}^{i-2} P_{xt}(t_j, S_{t_j})(t_{j+1} - t_j)$$

where

$$P_{xt}(t, x) = \frac{1}{\hat{\sigma}\sqrt{1-t}} \left(\frac{1}{2} \frac{1}{1-t} \ln(x/K) - \frac{1}{4} \hat{\sigma}^2 \right) \varphi(d(t, x)).$$

We obtain the following results:

n	$\hat{\sigma}$	\overline{gain}_n^{Mod}	\overline{gain}_n^{Lel}	I_n^{Mod}	I_n^{Lel}	Sd_n^{Mod}	Sd_n^{Lel}
10	0.2238	0.15	0.08	[0.0277, 0.2722]	[-0.0378, 0.1978]	4.41	4.25
100	0.2681	0.069	0.05	[0.0091, 0.1288]	[0.0092, 0.0907]	2.16	1.47
200	0.2917	0.01	-0.004	[-0.0432, 0.0632]	[-0.0325, 0.0245]	1.92	1.03
500	0.3337	0.01	-0.031	[-0.0396, 0.0596]	[-0.0509, -0.011]	1.79	0.72
1000	0.3754	0.01	-0.005	[-0.0387, 0.0587]	[-0.0191, 0.0091]	1.76	0.51

These simulations show that the adjusted Leland portfolio not only approximately super-replicates the payoff but greatly outperforms the initial Leland one. This is particularly significant for a small number of a revision dates.

4 Appendix

4.1 Proof of Theorem 2.1

By the Ito formula we get

$$\widehat{C}_x(t, S_t) = \widehat{C}_x(0, S_0) + M_t^n + A_t^n \quad (4.1)$$

where

$$M_t^n := \int_0^t \sigma_u S_u \widehat{C}_{xx}(u, S_u) dW_u,$$

$$A_t^n := \int_0^t \left[\widehat{C}_{xt}(u, S_u) + \frac{1}{2} \sigma_u^2 S_u^2 \widehat{C}_{xxx}(u, S_u) \right] du.$$

The process M^n is a square integrable martingale on $[0, 1]$ by virtue of [1] (see also [2]).

We note $\Delta H_{t_i}^n = H_{t_i+}^n - H_{t_i-}^n$, $\Delta K_{t_i}^n = K_{t_i+}^n - K_{t_i-}^n$ and we set the difference $V_1^n - G(S_1)$ in a convenient way.

Lemma 4.1 *We have $V_1^n - G(S_1) = F_1^n + F_2^n + F_3^n$ where*

$$F_1^n := \int_0^1 (H_t^n - \widehat{H}_t) dS_t - k_0 |\Delta H_{t_n}^n + \Delta K_{t_n}^n| S_{t_n}, \quad (4.2)$$

$$F_2^n := \frac{1}{2} \int_0^1 \sigma_t \gamma_n(t) S_t^2 |\widehat{C}_{xx}(t, S_t)| dt - k_0 \sum_{i=1}^{n-1} |\Delta H_{t_i}^n + \Delta K_{t_i}^n| S_{t_i}, \quad (4.3)$$

$$F_3^n := \int_0^1 K_t^n dS_t. \quad (4.4)$$

We recall the next result deduced from Lemma 4.8 [2] (see also [1]):

Lemma 4.2 *We have the following equality*

$$\int_s^t \widehat{C}_{xt}(u, S_u) du = \int_{\rho_u^2}^{\rho_s^2} \widehat{C}_{xt}(u, S_u) \widehat{\sigma}_u^{-2} dx,$$

where $u = u(x, n)$ is defined by $x = \rho_u^2$ and verifies $\lim_{n \rightarrow \infty} u(x, n) = 1$. Moreover,

$$C_{xt}(u, S_u) \widehat{\sigma}_u^{-2} = \frac{1}{2x} \int_{-\infty}^{\infty} G'(S_u e^{\sqrt{xy} + x/2}) (-y^2 - \sqrt{xy} + 1) \varphi(y) dy$$

satisfies the following inequality

$$|\widehat{C}_{xt}(u, S_u) \widehat{\sigma}_u^{-2}| du \leq c E_1(x, S_u)$$

where

$$E_1(x, y) = \frac{1}{x} e^{-x/8} \left(\sum_{j=1}^p \frac{|\ln(y/K_j)|}{\sqrt{x}} \exp \left\{ -\frac{\ln^2(y/K_j)}{2x} \right\} + \sqrt{x} + x \right).$$

Corollary 4.3 *Assume that we have two sequences $\{t_{k^n} : k = 0, \dots, n\}_{n \in \mathbf{N}}$ and $\{s_{k^n} : k = 0, \dots, n\}_{n \in \mathbf{N}}$ in $[0, 1]$ such that $\rho_{t_{k^n}}$ and $\rho_{s_{k^n}}$ respectively converge to $a \in [0, \infty]$ and $b \in [0, \infty]$. Then,*

$$\lim_{n \rightarrow \infty} \int_{s_{k^n}}^{t_{k^n}} \widehat{C}_{xt}(u, S_u) du = \int_a^b E_1^\infty(x, S_1) dx < \infty, \text{ a.s.}$$

where

$$E_1^\infty(x, S) := \frac{1}{2x} \int_{-\infty}^{\infty} G'(S e^{\sqrt{xy} + x/2}) (-y^2 - \sqrt{xy} + 1) \varphi(y) dy.$$

Proof. We apply Lemma 4.2 with the change of variable $x = \rho_u^2$. Recall that $0 \leq 1 - u \leq c x n^{-1/2}$ so that $u \rightarrow 1$ as $n \rightarrow \infty$ for a given $x \geq 0$. We can apply the Lebesgue theorem by dominating the function $E_1(x, S_u)$ whether $x \leq 1$ or not because $x \leq 1$ implies that u is sufficiently near from 1 independently of x for $n \geq n_0$. Indeed, outside of the null-set $\cup_i \{S_1 = K_i\}$, we have $0 < a \leq |\ln(S_u/K_j)| \leq b$ for some constants a, b (depending on ω) provided that u is sufficiently near one. \square

Lemma 4.4 *We have*

$$P\text{-}\lim_n F_1^n = 0. \tag{4.5}$$

Proof. The first term in (4.2) converges to 0; the proof is given in [1], [2]. We split the second term into two terms:

$$k_0 |\Delta H_{t_n}^n + \Delta K_{t_n}^n| S_{t_n} \leq k_0 |\Delta H_{t_n}^n| S_{t_n} + k_0 |\Delta K_{t_n}^n| S_{t_n}$$

where $k_0 |\Delta H_{t_n}^n| S_{t_n}$ converges to 0 by virtue of [1], [2] whereas the second term converges to 0 because of Corollary 4.3. \square

We write $F_2^n = \sum_{i=1}^5 L_i^n$ with the summands

$$\begin{aligned} L_1^n &= \frac{1}{2} \int_0^1 \sigma_t \gamma_n(t) S_t^2 |\widehat{h}_t| dt - \frac{1}{2} \int_0^1 \sum_{i=1}^{n-1} \sigma_{t_{i-1}} \gamma_n(t_{i-1}) S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}| I_{[t_{i-1}, t_i]}(t) dt \\ L_2^n &= \sum_{i=1}^{n-1} |\widehat{h}_{t_{i-1}}| S_{t_{i-1}}^2 \left(\frac{1}{2} \sigma_{t_{i-1}} \gamma_n(t_{i-1}) \Delta t_i - k_0 \sigma_{t_{i-1}} n^{1/2} \sqrt{\Delta t_i} f'(t_{i-1}) |\Delta W_{t_i}| \right), \\ L_3^n &= k_0 \sum_{i=1}^{n-1} \sigma_{t_{i-1}} S_{t_{i-1}}^2 |\widehat{h}_{t_{i-1}}| n^{1/2} \sqrt{\Delta t_i} f'(t_{i-1}) |\Delta W_{t_i}| - k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta M_{t_i}|, \\ L_4^n &= k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta M_{t_i}| - k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta H_{t_i}^n + \Delta K_{t_i}^n|, \\ L_5^n &= -k_0 \sum_{i=1}^{n-1} \Delta S_{t_i} |\Delta H_{t_i}^n + \Delta K_{t_i}^n| \end{aligned}$$

where we use the abbreviations $\Delta W_{t_i} = W_{t_i} - W_{t_{i-1}}$, etc.

We recall from [1], [2] the following lemmas.

Set

$$\theta_1(x, S_1) := \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} G'(S_1 e^{\sqrt{x}y + x/2}) y \varphi(y) dy.$$

Lemma 4.5 *Both terms whose difference defines L_1^n converge almost surely, as $n \rightarrow \infty$, to J_0 defined by the formula*

$$J_0 = \frac{1}{2} S_1 \int_0^{\infty} |\theta_1(x, S_1)| dx \quad (4.6)$$

Therefore, $L_1^n \rightarrow 0$ a.s.

Lemma 4.6 *We have $P\text{-}\lim_n L_2^n = 0$.*

Lemma 4.7 *We have $P\text{-}\lim_n L_3^n = 0$.*

The only main difference between the analyze of F_2^n in this paper and [2] (or [1]) is due to the sequence L_4^n .

Lemma 4.8 *We have $P\text{-}\lim_n L_4^n = 0$.*

Proof. Using the inequality $||a_1| - |a_2|| \leq |a_1 - a_2|$ we obtain that

$$\begin{aligned} |L_4^n| &\leq k_0 \sum_{i=1}^{n-1} S_{t_{i-1}} |\Delta A_{t_i} + \Delta \widehat{K}_{t_i}| \\ &\leq k_0 \int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u, S_u)| du. \end{aligned}$$

Using Lemma 4.13 and the change of variable $x = \rho_u^2$, we deduce:

$$\int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u, S_u)| du \leq cn^{-1/2} \int_0^{\rho_0^2} E_2(x) dx$$

where

$$E_2(x) = \frac{1}{x^{3/2}} \sum_{j=1}^p \exp \left\{ -\frac{\ln^2(S_u/K_j)}{2x} \right\} + \frac{1}{x^2} e^{-x/8}.$$

Thus,

$$k_0 \int_0^1 \sigma_u^2 S_u^2 |\widehat{C}_{xxx}(u, S_u)| du \rightarrow 0.$$

Indeed, the reasoning is similar to that of Corollary 4.3. We have $0 \leq 1 - u \leq cxn^{-1/2}$ so that $u \rightarrow 1$ as $n \rightarrow \infty$. One can apply the Lebesgue theorem by dominating the function E_2 independently of n whether $x \leq 1$ or not since $x \leq 1$ implies that u is sufficiently near from 1 independently of x for $n \geq n_0$. \square

Lemma 4.9 *We have $P\text{-}\lim_n L_5^n = 0$.*

Proof. Since $\max_i |\Delta S_{t_i}| \rightarrow 0$ as $n \rightarrow \infty$, it suffices to verify that the sequence $k_0 \sum_{i=1}^n |\Delta \widehat{H}_{t_i}|$ is bounded in probability. But this follows from the preceding lemmas. \square

Inspecting the formulations of above lemmas, we observe that all terms $L_j^n \rightarrow 0$ in probability and, hence, F_2^n converges to 0 in probability.

Lemma 4.10 *We have $P\text{-}\lim_n F_3^n = 0$.*

Proof. We rewrite F_3^n as

$$\begin{aligned} F_3^n &= \sum_{i=1}^n K_{t_i}^n \Delta S_{t_i}, \\ &= - \sum_{i=1}^{n-1} S_{t_i} (K_{t_{i+1}}^n - K_{t_i}^n) - K_{t_1}^n S_0 + K_1^n S_1. \end{aligned} \quad (4.7)$$

Recall that $K_{t_1}^n = 0$ and

$$K_1^n = - \int_0^{t_{n-1}} \widehat{C}_{xt}(t_u, S_u) du$$

which converges a.s. to $-\int_0^\infty E_1(x, S_1) dx$ by virtue of Corollary 4.3. The first term on the right hand side of (4.7) can be seen as

$$P^n = \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \widehat{C}_{xt}(t_u, S_u) S_{t_i} du = \int_0^{t_{n-1}} \widehat{C}_{xt}(t_u, S_u) S_u du + \widetilde{P}^n$$

where

$$\lim_{n \rightarrow \infty} \int_0^{t_{n-1}} \widehat{C}_{xt}(t_u, S_u) S_u du = S_1 \int_0^\infty E_1^\infty(x, S_1) dx$$

by virtue of the same reasoning used in Corollary 4.3. Moreover, \tilde{P}^n is defined as follows

$$\tilde{P}^n := \sum_{i=2}^{n-1} \int_{t_{i-1}}^{t_i} \hat{C}_{xt}(t_u, S_u) (S_{t_i} - S_u) du.$$

Since $\sup_u S_u < \infty$ a.s., $\tilde{P}^n \leq c(\omega) \bar{P}^n$ a.s. (ω) where

$$\bar{P}^n := \sum_{i=2}^{n-1} \int_{t_{i-1}}^{t_i} |\hat{C}_{xt}(t_u, S_u)| |S_{t_i}/S_u - 1| du.$$

Recall that $E(S_{t_i}/S_u - 1)^2 \leq c\Delta t_i$ and $S_{t_i}/S_u - 1$ is independent of S_u because of the increments of the Wiener process. Using Lemma 4.13, it follows the inequality

$$\|\bar{P}^n\|_1 \leq \frac{c}{\sqrt{n}} \int_0^{t_{n-1}} \frac{du}{1-u} \leq \frac{c \ln n}{\sqrt{n}}$$

proving that $\tilde{P}^n \rightarrow 0$ a.s. \square

From above, we can conclude about Theorem 2.1.

4.2 Auxiliary Results

In the following, we only recall some properties satisfied by the solution of the Cauchy problem (2.4) we need in the proof of Theorem 2.1. For more details, see [1], [2].

Lemma 4.11 *Let $\hat{C}(t, x)$ is given by (2.5). Then*

$$\frac{\partial^{k+1} \hat{C}(t, x)}{\partial x^{k+1}} = \frac{1}{\rho^k x^k} \int_{-\infty}^{\infty} G'(xe^{\rho y + \rho^2/2}) P_k(y) \varphi(y) dy, \quad k \geq 0,$$

where $P_k(y) = y^k + a_k(\rho)y^k + \dots + a_0(\rho)$ is a polynomial of degree k whose coefficients $a_i(\rho)$ are polynomials in ρ of degree $k-1$.

In particular, we deduce that $|\hat{C}_x(t, x)| \leq \|G'\|_{\infty}$ and we have the following expression:

Lemma 4.12 *Let $\hat{C}(t, x)$ is given by (2.5). Then*

$$\hat{C}_{tx}(t, x) = \frac{\hat{\sigma}_t^2}{2\rho^2} \int_{-\infty}^{\infty} G'(xe^{\rho y + \rho^2/2}) (-y^2 - \rho y + 1) \varphi(y) dy \quad (4.1)$$

Lemma 4.13 *There exists a constant c such that*

$$|\hat{C}_{tx}(t, x)| \leq \frac{c\hat{\sigma}_t^2 e^{-\frac{\rho^2}{8}}}{x^{1/2}\rho^2} (L(x, \rho) + \rho + \rho^2)$$

where

$$L(x, \rho) = \sum_{j=1}^p \frac{|\ln(x/K_j)|}{\rho} \exp\left\{-\frac{\ln^2(x/K_j)}{2\rho^2}\right\}.$$

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¹ <https://sites.google.com/site/emmanueldenis6/introduction>