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**UNIQUENESS OF SOLUTIONS
FOR THE NON-CUTOFF BOLTZMANN EQUATION
WITH SOFT POTENTIAL**

R. ALEXANDRE, Y. MORIMOTO, S. UKAI, C.-J. XU, AND T. YANG

ABSTRACT. In this paper, we consider the Cauchy problem for the non-cutoff Boltzmann equation in the soft potential case. By using a singular change of velocity variables before and after collision, we prove the uniqueness of weak solutions to the Cauchy problem in the space of functions with polynomial decay in the velocity variable.

1. INTRODUCTION

Consider the Cauchy problem for the spatially inhomogeneous Boltzmann equation,

$$(1.1) \quad \partial_t f + v \cdot \nabla_x f = Q(f, f), \quad f(0, x, v) = f_0(x, v),$$

where $f = f(t, x, v)$ is the density distribution function of particles with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t . The right hand side of (1.1) is given by the Boltzmann bilinear collision operator

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g(v'_*)f(v') - g(v_*)f(v)\} d\sigma dv_*,$$

which is well-defined for suitable functions f and g specified later. Notice that the collision operator $Q(\cdot, \cdot)$ acts only on the velocity variable $v \in \mathbb{R}^3$. In the following discussion, we will use the σ -representation, that is, for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

which give the relations between pre- and post- collisional velocities. The non-negative cross section $B(z, \sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|} \cdot \sigma$. As in our previous works, we assume that it takes the form

$$B(|v - v_*|, \cos \theta) = \Phi(|v - v_*|)b(\cos \theta), \quad \cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where

$$(1.2) \quad \Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \quad b(\cos \theta)\theta^{2+2s} \rightarrow K \quad \text{when } \theta \rightarrow 0+,$$

for some $\gamma > -3, 0 < s < 1$ and $K > 0$. The angle θ is the deviation angle, i.e., the angle between pre- and post- collisional velocities. The range of θ is a full

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interval $[0, \pi]$, but it is customary [20] to restrict it to $[0, \pi/2]$, replacing $b(\cos \theta)$ by its ‘‘symmetrized’’ version

$$[b(\cos \theta) + b(\cos(\pi - \theta))] \mathbf{1}_{0 \leq \theta \leq \pi/2},$$

which is possible due to the invariance of the product $f(v')f(v'_*)$ in the collision operator $Q(f, f)$ under the change of variables $\sigma \rightarrow -\sigma$.

We will use the following weighted function spaces: For $p \geq 1$ and $\beta \in \mathbb{R}$, we set

$$\|f\|_{L_\beta^p} = \left(\int_{\mathbb{R}^3} |\langle v \rangle^\beta f(v)|^p dv \right)^{1/p},$$

and for $m \in \mathbb{R}$

$$\|f\|_{H_\beta^m(\mathbb{R}_v^3)} = \left(\int_{\mathbb{R}^3} |\langle D_v \rangle^m (\langle v \rangle^\beta f(v))|^2 dv \right)^{1/2},$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$ and $\langle D_v \rangle = (1 - \Delta_v)^{1/2}$.

$f(t, x, v)$ is called a weak solution of the Cauchy problem (1.1), if

$$f \in C(\mathbb{R}^+; \mathcal{S}'(\mathbb{R}_{x,v}^6)) \cap L^1([0, T]; L^\infty(\mathbb{R}_x^3, L_{2+\gamma^+}^1(\mathbb{R}_v^3))),$$

and it satisfies (1.1) in the following weak sense:

$$\begin{aligned} (1.3) \quad & \int_{\mathbb{R}^6} f(t, x, v) \eta(t, x, v) dx dv - \int_{\mathbb{R}^6} f_0(x, v) \eta(0, x, v) dx dv \\ & - \int_0^t d\tau \int_{\mathbb{R}^6} f(\tau, x, v) (\partial_\tau + v \cdot \nabla_x) \eta(\tau, x, v) dx dv \\ & = \int_0^t d\tau \int_{\mathbb{R}^6} Q(f, f)(\tau, x, v) \eta(\tau, x, v) dx dv, \end{aligned}$$

where $\eta \in C^1(\mathbb{R}; C_0^\infty(\mathbb{R}^6))$. Here, the right hand side of the last integral can be defined by

$$\int_{\mathbb{R}^3} Q(f, g)(v) \eta(v) dv = \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} B f(v_*) g(v) (\eta(v') - \eta(v)) dv dv_* d\sigma.$$

For the uniqueness of weak solutions, we consider the function space with polynomial decay in the velocity variable. More precisely, for $m \in \mathbb{R}, \ell \geq 0$ and $T > 0$, set

$$\begin{aligned} \mathcal{P}_\ell^m([0, T] \times \mathbb{R}_{x,v}^6) &= \left\{ f \in C^0([0, T]; \mathcal{S}'(\mathbb{R}_{x,v}^6)); \right. \\ &\quad \left. s.t. f \in L^\infty([0, T] \times \mathbb{R}_x^3; H_\ell^m(\mathbb{R}_v^3)) \right\}. \end{aligned}$$

Our theorem is concerned with the uniqueness of solutions for the case when $\gamma \leq 0$ in the cross-section that includes the soft potential and Maxwell molecule for the inverse power law.

Theorem 1.1. *For $0 < s < 1$ and $\max\{-3, -3/2 - 2s\} < \gamma \leq 0$, suppose that the Cauchy problem (1.1) admits two weak solutions $f_1(t), f_2(t) \in \mathcal{P}_{\ell_0}^{2s}([0, T] \times \mathbb{R}_{x,v}^6)$ with $0 < T < +\infty$ and $\ell_0 \geq 14$ having the same initial datum $f_0 \in L^\infty(\mathbb{R}_x^3; H_{\ell_0}^{2s}(\mathbb{R}_v^3))$. If one solution is non-negative, then $f_1(t) \equiv f_2(t)$.*

Remark 1.2. *The above result holds true for the spatially homogeneous Boltzmann equation. Moreover, according to the proof of the above theorem, the uniqueness holds also true for the cutoff Boltzmann equation in the function space $\mathcal{P}_{\ell_0}^0([0, T] \times \mathbb{R}_{x,v}^6)$.*

Let us now review the previous results on the cutoff spatially inhomogeneous Boltzmann equation. First of all, there is an extensive literature on the existence of classical and weak solutions, which is verified basically in two settings, that is, as a small perturbation of a profile or a global Maxwellian and as a large perturbation of vacuum. For the small perturbation problem, the uniqueness usually follows from the construction of the solutions, cf. [12, 15, 18] and references therein. Here, we would like to mention that the weak perturbation solution in $L^\infty_\beta \cap L^2$ around a global Maxwellian was proved to be unique by the fixed point theorem, [19]. However, for large perturbation solutions, even though the uniqueness of classical solution can be well justified, the uniqueness for weak solutions, such as the renormalized solutions introduced by [11], remains unsolved as a challenging open problem in this area. A preliminary result is found in [14, 16] that if the Cauchy problem (1.1) has one renormalized solution and one classical solution, then they should coincide.

On the other hand, for the Boltzmann equation without angular cutoff, the uniqueness problem was studied in our joint works [3, 4, 7] for solutions with exponential decay in the velocity variable. Therefore, the uniqueness result proved in this paper for solutions with polynomial decay in the velocity variable can be viewed as one step forward in the study on the uniqueness for the weak solutions. Finally, we would like to mention that there are also some interesting results on the uniqueness for the spatially homogeneous Boltzmann equation, for example, for the Maxwellian case in [17] for entropy solution; and for the mild singularity, that is, $0 < s < \frac{1}{2}$, in [10] in the function space $W_\ell^{1,1}$.

Throughout this paper, we will use the following notation: $f \lesssim g$ means that there exists a generic positive constant C such that $f \leq C g$.

The rest of the paper will be arranged as follows. In the next section, we will give the strategy in the proof. Some basic properties of the weight function in (x, v) will be given in Section 3. The two main estimates, one on the commutator of the weight function between the collision operator and another one on the upper bound of the collision operator with weight, will be given in the last section. These two main estimates lead to the completion of the proof of our uniqueness Theorem 1.1.

2. OUTLINE OF THE PROOF OF THEOREM 1.1

Set $F = f_1 - f_2$. Then it follows from (1.1), in the weak sense of (1.3), that

$$(2.1) \quad \begin{cases} F_t + v \cdot \nabla_x F = Q(f_1, F) + Q(F, f_2), \\ F|_{t=0} = 0, \end{cases}$$

which is equivalent to, for any $t, t' \in [0, T]$,

$$(2.2) \quad \begin{aligned} & \int_{\mathbb{R}^6} F(t, x, v) \eta(x, v) dx dv - \int_{\mathbb{R}^6} F(t', x, v) \eta(x, v) dx dv \\ & - \int_{t'}^t d\tau \int_{\mathbb{R}^6} F(\tau, x, v) (v \cdot \nabla_x) \eta(x, v) dx dv \\ & = \int_{t'}^t d\tau \int_{\mathbb{R}^6} \left(Q(f_1, F) + Q(F, f_2) \right) (\tau, x, v) \eta(x, v) dx dv, \end{aligned}$$

where the test function η is chosen to be independent of t .

Now we choose a mollification of the function F and take it as a test function. Let $S(\tau) \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$

Then, for any $N \in \mathbb{N}$ and any $m \in \mathbb{R}$, we have

$$S_N(D_x) = S(2^{-2N}|D_x|^2) : \quad H^m(\mathbb{R}^3) \rightarrow H^\infty(\mathbb{R}^3),$$

and for any $f \in H^m$,

$$\lim_{N \rightarrow \infty} \|S_N(D_x)f - f\|_{H^m} = 0.$$

For $\ell \in \mathbb{R}$, we set also $\varphi(v, x) = 1 + |v|^2 + |x|^2$ and

$$W_\ell(v) = \langle v \rangle^\ell, \quad W_{\varphi, \ell} = \frac{W_\ell(v)}{\varphi(v, x)} = \frac{(1 + |v|^2)^{\ell/2}}{1 + |v|^2 + |x|^2}.$$

Then for $F = f_1 - f_2$, with f_1 and f_2 given as in the statement of Theorem 1.1, we have

$$\eta(t, x, v) = W_{\varphi, \ell} S_N(D_x)^2 W_{\varphi, \ell} F \in L^\infty([0, T]; H^\infty(\mathbb{R}_x^3; H_{\ell_0 - 2\ell}^{2s}(\mathbb{R}_v^3))).$$

Similarly to Lemma 4.3 of [9], by taking $\eta(\bar{t}, x, v)$, for a fixed \bar{t} , as a test function in (2.2), we can prove that

$$S_N(D_x) W_{\varphi, \ell} F \in Lip([0, T]; H^\infty(\mathbb{R}_x^3; L_{\ell_0 - 2\ell}^2(\mathbb{R}_v^3))).$$

Hence, for any $0 < t < T$, we have

$$\begin{aligned} & \|S_N(D_x) W_{\varphi, \ell} F(t)\|_{L^2(\mathbb{R}^6)}^2 \\ &= 2 \int_0^t \left\{ (v \cdot \nabla_x(\varphi^{-1}) W_\ell F(\tau), S_N^2(D_x) W_{\varphi, \ell} F(\tau))_{L^2(\mathbb{R}^6)} \right. \\ & \quad \left. + (W_{\varphi, \ell} Q(f_1(\tau), F(\tau)) + W_{\varphi, \ell} Q(F(\tau), f_2(\tau)), S_N^2(D_x) W_{\varphi, \ell} F(\tau))_{L^2(\mathbb{R}^6)} \right\} d\tau, \end{aligned}$$

because

$$(v \cdot \nabla_x (S_N(D_x) W_{\varphi, \ell} F(\tau)), S_N(D_x) W_{\varphi, \ell} F(\tau))_{L^2(\mathbb{R}^6)} = 0.$$

Taking the limit $N \rightarrow \infty$, we get that, for any $0 < t < T$,

$$(2.3) \quad \|W_{\varphi, \ell} F(t)\|_{L^2(\mathbb{R}^6)}^2 = 2 \int_0^t \left\{ (v \cdot \nabla_x(\varphi^{-1}) W_\ell F(\tau), W_{\varphi, \ell} F(\tau))_{L^2(\mathbb{R}^6)} \right. \\ \left. + (W_{\varphi, \ell} Q(f_1(\tau), F(\tau)) + W_{\varphi, \ell} Q(F(\tau), f_2(\tau)), W_{\varphi, \ell} F(\tau))_{L^2(\mathbb{R}^6)} \right\} d\tau.$$

The first term on the right hand side is estimated by $\|W_{\varphi, \ell} F\|_{L^2(\mathbb{R}^6)}^2$ because

$$|v \cdot \nabla_x(\varphi^{-1})| \lesssim \varphi^{-1}.$$

If we admit the following two estimates

$$(2.4) \quad (W_{\varphi, \ell} Q(f_1, F), W_{\varphi, \ell} F)_{L^2(\mathbb{R}^6)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}_x^3, H_{2\ell+3/2+\varepsilon}^{2s}(\mathbb{R}_v^3))} \|W_{\varphi, \ell} F\|_{L^2(\mathbb{R}^6)}^2,$$

and

$$(2.5) \quad (W_{\varphi, \ell} Q(F, f_2), W_{\varphi, \ell} F)_{L^2(\mathbb{R}^6)} \lesssim \|f_2\|_{L^\infty(\mathbb{R}_x^3, H_{\ell+2s}^{2s}(\mathbb{R}_v^3))} \|W_{\varphi, \ell} F(t)\|_{L^2(\mathbb{R}^6)}^2,$$

we can obtain

$$(2.6) \quad \|W_{\varphi, \ell} F(t)\|_{L^2(\mathbb{R}^6)}^2 \leq B \int_0^t \|W_{\varphi, \ell} F(\tau)\|_{L^2(\mathbb{R}^6)}^2 d\tau,$$

with

$$B = C \left(\|f_1\|_{L^\infty([0,T] \times \mathbb{R}_x^3, H_{2\ell+3/2+\varepsilon}^{2s}(\mathbb{R}_v^3))} + \|f_2\|_{L^\infty([0,T] \times \mathbb{R}_x^3, H_{\ell+2s}^{2s}(\mathbb{R}_v^3))} \right),$$

which concludes the proof of Theorem 1.1.

It remains to prove the two estimates (2.4) and (2.5). Set

$$\mathcal{D}(f, h) = \iiint B(v - v_*, \sigma) f_*(h - h')^2 dv dv_* d\sigma.$$

From here and now on, we will use the notations $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$ and $f'_* = f(v'_*)$. The estimate (2.4) is a consequence of the following two propositions.

Proposition 2.1. *Let $0 < s < 1$ and $0 \geq \gamma > \max\{-3, -2s - 3/2\}$. Then we have*

$$\left(Q(f, h), h \right)_{L^2(\mathbb{R}^6)} \leq -\frac{1}{2} \int \mathcal{D}(f, h) dx + C \|f\|_{L^\infty(\mathbb{R}_x^3, H_{3/2+\varepsilon}^{2s'}(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}^2,$$

where $s' \geq 0$ satisfies $\gamma + 2s' > -3/2$ and $s' < \min\{s, \frac{3}{4}\}$.

Proof. Regarding x as a parameter we have

$$\begin{aligned} \left(Q(f, h), h \right)_{L^2(\mathbb{R}_v^3)} &= \frac{1}{2} \left(-\mathcal{D}(f, h) + \iiint B f_* (h'^2 - h^2) dv dv_* d\sigma \right) \\ &= \frac{1}{2} \left(-\mathcal{D}(f, h) + R(f, h) \right). \end{aligned}$$

It follows from the cancellation lemma [2] that

$$\begin{aligned} |R(f, h)| &\lesssim \iiint |v - v_*|^\gamma |f_*| |h|^2 dv dv_* \\ &\lesssim \iiint_{\{|v-v_*|>1\}} |f_*| |h|^2 dv dv_* + \iiint_{\{|v-v_*|\leq 1\}} |v - v_*|^\gamma |f_*| |h|^2 dv dv_* \\ &\lesssim \|f\|_{L^1(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2 \\ &\quad + \int \left(\int_{\{|v-v_*|\leq 1\}} |v - v_*|^{2(\gamma+2s')} dv_* \right)^{1/2} \left(\int \frac{|f_*|^2}{|v - v_*|^{4s'}} dv_* \right)^{1/2} |h|^2 dv \\ &\lesssim \|f\|_{L_{3/2+\varepsilon}^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2 + \|f\|_{H^{2s'}(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2, \end{aligned}$$

where we have used Hardy inequality. \square

Remark that if $0 \geq \gamma > -3/2$, then we can get

$$(2.7) \quad \left(Q(f, h), h \right)_{L^2(\mathbb{R}^6)} \leq -\frac{1}{2} \int \mathcal{D}(f, h) dx + C \|f\|_{L^\infty(\mathbb{R}_x^3, L_{3/2+\varepsilon}^2(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)}^2.$$

The next result takes care of commutator's estimates.

Proposition 2.2. *Let $\ell \geq 6$. If $\max\{-3, -2s - 3/2\} < \gamma \leq 0$ and $0 < s < 1$, then*

$$(2.8) \quad \begin{aligned} &\left| \left((W_{\varphi, \ell} Q(f, g) - Q(f, W_{\varphi, \ell} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \\ &\lesssim \|f\|_{L^\infty(\mathbb{R}_x^3, H_{3/2+\varepsilon}^{(2s-1)^+}(\mathbb{R}_v^3))} \|h\|_{L^2(\mathbb{R}^6)} \|W_{\varphi, \ell} g\|_{L^2(\mathbb{R}^6)} \\ &\quad + \left(\int \mathcal{D}(|f|, h) dx \right)^{1/2} \|f\|_{L^\infty(\mathbb{R}_x^3, L_{2\ell+3/2+\varepsilon}^2(\mathbb{R}_v^3))}^{1/2} \|W_{\varphi, \ell} g\|_{L^2(\mathbb{R}^6)}. \end{aligned}$$

Remark 2.3. *If $0 < s < 1/2$, we have*

$$\begin{aligned} & \left| \left((W_{\varphi,\ell} Q(f, g) - Q(f, W_{\varphi,\ell} g)), h \right)_{L^2(\mathbb{R}^6)} \right| \\ & \lesssim \left(\|f\|_{L^\infty(\mathbb{R}_x^3; L^2_{3+3/2+\varepsilon}(\mathbb{R}_v^3))} \|W_{\varphi,\ell} g\|_{L^2(\mathbb{R}^6)} \right. \\ & \quad \left. + \|f\|_{L^\infty(\mathbb{R}_x^3; L^2_\ell(\mathbb{R}_v^3))} \|W_{\varphi,3/2+\varepsilon} g\|_{L^2(\mathbb{R}^6)} \right) \|h\|_{L^2(\mathbb{R}^6)}. \end{aligned}$$

When f_1 is non-negative, the combination of (2.7) and (2.8) gives (2.4) by using the Cauchy-Schwarz inequality.

Together with the last result, the estimate (2.5) will be a consequence of the following proposition.

Proposition 2.4. *If $\max\{-3, -2s - 3/2\} < \gamma \leq 0$ and $0 < s < 1$, then for $\ell \geq 6$, we have*

$$\left(W_{\varphi,\ell} Q(f, g), h \right)_{L^2(\mathbb{R}_v^3)} \lesssim \|g\|_{H^{2s}_{\ell+2s}(\mathbb{R}_v^3)} \|W_{\varphi,\ell} f\|_{L^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}.$$

Thus we obtain (2.6) with $\ell = 6$ if

$$f_1, f_2 \in L^\infty([0, T] \times \mathbb{R}_x^3; H^{2s}_{14}(\mathbb{R}_v^3)).$$

The rest of this paper is devoted to the proof of the above two Propositions 2.2 and 2.4.

3. PRELIMINARY LEMMAS

For the estimation on the commutator between the collision operator and the weight function $W_{\varphi,\ell}$, we prepare some technical lemmas.

Lemma 3.1. *For $\ell \geq 4$, we have*

$$(3.1) \quad |W_{\varphi,\ell}(v) - W_{\varphi,\ell}(v')| \lesssim \sin\left(\frac{\theta}{2}\right) \frac{W_\ell(v) + W_\ell(v_*)}{\varphi(v, x)} \lesssim \theta W_{\varphi,\ell}(v) W_\ell(v_*),$$

and

$$(3.2) \quad \begin{aligned} |W_{\varphi,\ell}(v) - W_{\varphi,\ell}(v')| & \leq C \sin(\theta/2) \left(\frac{W_\ell(v) + W_{\ell-3}(v) W_3(v_*)}{\varphi(v_*, x)} + \sin^{\ell-3}(\theta/2) W_{\varphi,\ell}(v_*) \right) \\ & \lesssim (\theta W_\ell(v) W_{\varphi,3}(v_*) + \theta^{\ell-2} W_{\varphi,\ell}(v_*)). \end{aligned}$$

Remark 3.2. *Remark that we can improve (3.2) to*

$$(3.3) \quad |W_{\varphi,\ell}(v) - W_{\varphi,\ell}(v')| \lesssim \frac{\theta W_\ell(v) W_3(v_*) + \theta^{\ell-2} W_\ell(v_*)}{1 + |v|^2 + |v_*|^2 + |x|^2}.$$

Proof. For $k \geq 0, a \geq 0$, set

$$F_k(\lambda) = \frac{\lambda^k}{\lambda + a}, \quad \lambda \in [1, \infty[.$$

Then, for $j = 1, 2$, we have $\frac{d^j}{d\lambda^j} F_k(\lambda) \geq 0$ if $k \geq j$. Thus if $k \geq 2$, it follows from the mean value theorem that for $\lambda, \lambda' \geq 1$

$$|F_k(\lambda) - F_k(\lambda')| \leq \left(\frac{d}{d\lambda} F_k \right) (\lambda + |\lambda - \lambda'|) |\lambda - \lambda'|,$$

because $\frac{d}{d\lambda}F_k(\lambda)$ is positive and increasing on $[1, \infty[$ for $k \geq 2$. Setting $\lambda = \langle v \rangle^2, \lambda' = \langle v' \rangle^2$, then

$$|\langle v \rangle^2 - \langle v' \rangle^2| \leq 2|v - v'| |v| + |v - v'|^2 \leq |v|^2 + 2|v - v'|^2.$$

So that we have

$$\begin{aligned} |F_k(\langle v \rangle^2) - F_k(\langle v' \rangle^2)| &\leq \left(\frac{d}{d\lambda} F_k \right) \left(2(\langle v \rangle^2 + |v - v'|^2) \right) (2|v| + |v - v'|) |v - v'| \\ &\leq 2kF_{k-1/2}(2(\langle v \rangle^2 + |v - v'|^2)) |v - v'|, \end{aligned}$$

because $\sqrt{\lambda} \frac{d}{d\lambda} F_k(\lambda) \leq kF_{k-1/2}(\lambda)$. Therefore, choosing $a = |x|^2$ and $k = \frac{\ell}{2} \geq 2$, we get

$$\begin{aligned} |W_{\varphi, \ell}(v) - W_{\varphi, \ell}(v')| &\lesssim \left(\frac{\langle v \rangle^{\ell-1} |v - v'|}{\langle v \rangle^2 + |v - v'|^2 + |x|^2} + \frac{|v - v'|^\ell}{\langle v \rangle^2 + |v - v'|^2 + |x|^2} \right) \\ (3.4) \quad &\lesssim |v - v'| \langle v \rangle^{\ell-3} F_1(\langle v \rangle^2) + \frac{|v - v'|^\ell}{\langle v \rangle^2 + |v - v'|^2 + |x|^2} \\ &= \mathcal{B}_1 + \mathcal{B}_2. \end{aligned}$$

Note that $\langle v \rangle^2 \leq 2\langle v_* \rangle^2 + 2|v - v_*|^2$. Then the increasing property of F_1 implies

$$\begin{aligned} \mathcal{B}_1 &\lesssim |v - v'| \langle v \rangle^{\ell-3} \frac{\langle v_* \rangle^2 + |v - v_*|^2}{\langle v_* \rangle^2 + |v - v_*|^2 + |x|^2} \\ &\lesssim \sin\left(\frac{\theta}{2}\right) \frac{W_\ell(v) + W_{\ell-3}(v)W_3(v_*)}{|v|^2 + \varphi(v_*, x)}, \end{aligned}$$

where we have used $|v - v_*|^2 \geq \frac{1}{2}|v|^2 - |v_*|^2$ and

$$|v - v'|^2 = \sin^2\left(\frac{\theta}{2}\right) |v - v_*|^2.$$

This implies also

$$\begin{aligned} \mathcal{B}_2 &\lesssim \frac{|v - v_*|^\ell \sin^\ell\left(\frac{\theta}{2}\right)}{1 + (1 - \sin^2\left(\frac{\theta}{2}\right))|v|^2 + \frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)|v_*|^2 + |x|^2} \\ &\lesssim \sin^{\ell-2}\left(\frac{\theta}{2}\right) \frac{W_\ell + W_{\ell,*}}{|v|^2 + \varphi(v_*, x)}. \end{aligned}$$

Hence, we get the desired estimate (3.1), (3.2) and (3.3). \square

When the change of variables is singular (see below), we need also a high order moment estimate.

Lemma 3.3. *For $l \geq 6$, we have*

$$\begin{aligned} &\left| W_{\varphi, \ell}(v) - W_{\varphi, \ell}(v') - \left(\nabla_v W_{\varphi, \ell} \right)(v') \cdot (v - v') \right| \\ &= \left| \int_0^1 (1 - \tau) \nabla^2 W_{\varphi, \ell}(v' + \tau(v - v')) d\tau (v - v')^2 \right| \\ (3.5) \quad &\lesssim \sin^2\left(\frac{\theta}{2}\right) \frac{W_\ell(v) + W_{\ell-4}(v)W_4(v_*)}{\varphi(v_*, x)} + \sin^{\ell-2}\left(\frac{\theta}{2}\right) W_{\varphi, \ell}(v_*) \\ &\lesssim \theta^2 W_\ell(v) W_{\varphi, 4}(v_*) + \theta^{\ell-2} W_{\varphi, \ell}(v_*), \end{aligned}$$

and

$$\begin{aligned}
(3.6) \quad & \left| \left\{ \left(\nabla_v W_{\varphi, \ell} \right) (v) - \left(\nabla_v W_{\varphi, \ell} \right) (v') \right\} \cdot (v - v') \right| \\
& \lesssim \sin^2 \left(\frac{\theta}{2} \right) \frac{W_{\ell}(v) + W_{\ell-4}(v)W_4(v_*)}{\varphi(v_*, x)} + \sin^{\ell-2} \left(\frac{\theta}{2} \right) W_{\varphi, \ell}(v_*) \\
& \lesssim \theta^2 W_{\ell}(v) W_{\varphi, 4}(v_*) + \theta^{\ell-2} W_{\varphi, \ell}(v_*).
\end{aligned}$$

We also have

$$\begin{aligned}
(3.7) \quad & \left| W_{\varphi, \ell}(v') - W_{\varphi, \ell}(v) - \left(\nabla_v W_{\varphi, \ell} \right) (v) \cdot (v' - v) \right| \\
& \lesssim \sin^2 \left(\frac{\theta}{2} \right) \varphi(v, x)^{-1} \left\{ (W_{\ell}(v) + W_{\ell}(v_*)) \mathbf{1}_{\{|v-v_*| \geq 1\}} \right. \\
& \quad \left. + (W_{\ell-2}(v) + W_{\ell-2}(v_*)) |v - v_*|^2 \mathbf{1}_{\{|v-v_*| < 1\}} \right\} \\
& \lesssim \theta^2 W_{\ell}(v_*) W_{\varphi, \ell}(v) \left(\mathbf{1}_{\{|v-v_*| \geq 1\}} + |v - v_*|^2 \mathbf{1}_{\{|v-v_*| < 1\}} \right).
\end{aligned}$$

Proof. As for (3.5), we use the Taylor expansion of second order

$$\begin{aligned}
& W_{\varphi, \ell}(v) - W_{\varphi, \ell}(v') - \left(\nabla W_{\varphi, \ell} \right) (v') \cdot (v - v') \\
& = \int_0^1 (1 - \tau) \left(\nabla^2 W_{\varphi, \ell} \right) (v' + \tau(v - v')) d\tau (v - v')^2 \\
& = I_2(v, v').
\end{aligned}$$

We have with $v_{\tau} = v' + \tau(v - v')$

$$\begin{aligned}
(3.8) \quad & |I_2(v, v')| \lesssim |v - v'|^2 \int_0^1 \left(F'_{\ell/2}(\langle v_{\tau} \rangle^2) + \langle v_{\tau} \rangle^2 F''_{\ell/2}(\langle v_{\tau} \rangle^2) \right) d\tau \\
& \lesssim |v - v'|^2 F_{\ell/2-1}(2(\langle v \rangle^2 + |v' - v|^2)) \\
& \lesssim \frac{\langle v \rangle^{\ell-2} |v' - v|^2 + |v' - v|^{\ell}}{\langle v \rangle^2 + |v' - v|^2 + |x|^2},
\end{aligned}$$

because $F'_k(\lambda), F''_k(\lambda)$ are positive in $[1, \infty)$, and

$$F'_k \leq CF_{k-1}, \quad F''_k \leq CF_{k-2}$$

and F_{k-1} is increasing there, if $k = \ell/2 \geq 3$. Here we have used the fact that $\langle v_{\tau} \rangle^2 \leq 2\langle v \rangle + 2|v' - v|^2$. Noticing again that $\langle v \rangle^2 \leq 2\langle v_* \rangle^2 + 2|v - v_*|^2$ and F_1 is increasing, we have

$$\begin{aligned}
& |I_2(v, v')| \lesssim |v - v_*|^2 \sin^2(\theta/2) \langle v \rangle^{\ell-4} F_1(\langle v \rangle^2) + \mathcal{B}_2 \\
& \lesssim \sin^2(\theta/2) \frac{W_{\ell}(v) + W_{\ell-4}(v)W_4(v_*)}{\varphi(v_*, x)} + \sin^{\ell-2}(\theta/2) W_{\varphi, \ell}(v_*),
\end{aligned}$$

which yields (3.5). The proof of (3.6) is similar. The last inequality (3.7) follows easily from (3.8). \square

4. PROOFS OF PROPOSITIONS 2.2 AND 2.4

In this section, we regard (t, x) as a parameter.

Proof of Proposition 2.2 :

First of all, we have

$$\begin{aligned}
& \left((W_{\varphi,\ell} Q(f, g) - Q(f, W_{\varphi,\ell} g)), h \right)_{L^2(\mathbb{R}_v^3)} \\
&= \int B (W'_{\varphi,\ell} - W_{\varphi,\ell}) f_* g h' dv dv_* d\sigma \\
&= \int B (W'_{\varphi,\ell} - W_{\varphi,\ell}) f_* g (h' - h) dv dv_* d\sigma \\
&\quad + \int B (\nabla_v W_{\varphi,\ell})(v) \cdot (v' - v) f_* g h dv dv_* d\sigma \\
&\quad + \int B (W'_{\varphi,\ell} - W_{\varphi,\ell} - (\nabla_v W_{\varphi,\ell})(v) \cdot (v' - v)) f_* g h dv dv_* d\sigma \\
&= D_1 + D_2 + D_3.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we get in view of (3.1),

$$\begin{aligned}
|D_1| &\lesssim \left(\mathcal{D}(|f|, h) \right)^{1/2} \left(\iiint B |W'_{\varphi,\ell} - W_{\varphi,\ell}|^2 |f_*| g^2 dv dv_* d\sigma \right)^{1/2} \\
&\lesssim \left(\mathcal{D}(|f|, h) \right)^{1/2} \|f\|_{L_{2\ell}^{1/2}}^{1/2} \|W_{\varphi,\ell} g\|_{L^2(\mathbb{R}_v^3)} \\
&\lesssim \left(\mathcal{D}(|f|, h) \right)^{1/2} \|f\|_{L_{2\ell+3/2+\varepsilon}^{1/2}}^{1/2} \|W_{\varphi,\ell} g\|_{L^2(\mathbb{R}_v^3)}.
\end{aligned}$$

Using (3.7) gives

$$\begin{aligned}
|D_3| &\lesssim \iiint B \theta^2 W_{\ell,*} W_{\varphi,\ell} (\mathbf{1}_{|v-v_*| \geq 1} + |v-v_*|^2 \mathbf{1}_{|v-v_*| < 1}) |f_* g h| dv dv_* d\sigma \\
&\lesssim \iint |(W_{\ell} f)_*| (W_{\varphi,\ell} g) h| dv dv_* + \iint \mathbf{1}_{|v-v_*| < 1} |v-v_*|^{\gamma+2} |(W_{\ell} f)_*| (W_{\varphi,\ell} g) h| dv dv_* \\
&\lesssim \|f\|_{L_t^1(\mathbb{R}_v^3)} \|W_{\varphi,\ell} g\|_{L^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)} \\
&\quad + \int \left(\int_{|v-v_*| < 1} |v-v_*|^{2(\gamma+2)} dv_* \right)^{1/2} \|W_{\ell} f\|_{L_t^2(\mathbb{R}_v^3)} (W_{\varphi,\ell} g) h| dv \\
&\lesssim \|f\|_{L_{\ell+3/2+\varepsilon}^2(\mathbb{R}_v^3)} \|W_{\varphi,\ell} g\|_{L^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}.
\end{aligned}$$

We now consider the estimate of the term D_2 by first noticing that

$$v' - v = \frac{1}{2}(v_* - v)(1 - \mathbf{k} \cdot \sigma) + \frac{1}{2}|v_* - v|(\sigma - (\mathbf{k} \cdot \sigma)\sigma),$$

where $\mathbf{k} = (v - v_*)/|v - v_*|$. It follows from the symmetry on σ that

$$\int_{\mathbb{S}^2} b(\mathbf{k} \cdot \sigma)(\sigma - (\mathbf{k} \cdot \sigma)\sigma) d\sigma = 0.$$

Hence, $(1 - \mathbf{k} \cdot \sigma) = 2 \sin^2(\theta/2)$ implies

$$\begin{aligned}
|D_2| &= \frac{1}{2} \left| \int B (\nabla_v W_{\varphi,\ell})(v) \cdot (v_* - v)(1 - \mathbf{k} \cdot \sigma) f_* g h dv dv_* d\sigma \right| \\
&\lesssim \int b \theta^2 \left| \nabla_v W_{\varphi,\ell} \right| (v) |v_* - v|^{\gamma+1} |f_* g h| dv dv_* d\sigma.
\end{aligned}$$

Since

$$\left| \nabla_v W_{\varphi,\ell} \right| |v_* - v|^{\gamma+1} \lesssim W_{\varphi,\ell} W_{1,*} \mathbf{1}_{\{|v-v_*| \geq 1\}} + W_{\varphi,\ell-1} |v_* - v|^{\gamma+1} \mathbf{1}_{\{|v-v_*| < 1\}},$$

we get

$$\begin{aligned}
|D_2| &\lesssim \iint |(W_\ell f)_*| (W_{\varphi, \ell} g) h| \, dv dv_* \\
&\quad + \iint \mathbf{1}_{\{|v-v_*|<1\}} |v-v_*|^{\gamma+1} |(W_\ell f)_*| (W_{\varphi, \ell} g) h| \, dv dv_* \\
&\lesssim \|f\|_{L^1_v(\mathbb{R}^3)} \|W_{\varphi, \ell} g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} \\
&\quad + \int \left(\int_{|v-v_*|<1} |v-v_*|^{2(\gamma+2s)} dv_* \right)^{1/2} \left(\int \frac{(W_\ell f)_*^2}{|v-v_*|^{2(2s-1)}} dv_* \right)^{1/2} |(W_{\varphi, \ell} g) h| \, dv \\
&\lesssim \|f\|_{L^2_{\ell+3/2+\varepsilon}(\mathbb{R}^3)} \|W_{\varphi, \ell} g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} \\
&\quad + \|f\|_{H_\ell^{(2s-1)^+}(\mathbb{R}^3)} \|W_{\varphi, \ell} g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

By summing up the above estimates and integrating with respect to x , we finish the proof of Proposition 2.2.

We now turn to

Proof of Proposition 2.4 :

Here we need to use the mollification of the function. In the estimate stated in Proposition 2.4, we put the weight on the first function in the collision operator. To estimate this, we need a singular change of the variables between the pre and post collision velocities as follows:

$$v_* \mapsto v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2} \sigma,$$

where the Jacobian is

$$\left| \frac{\partial v_*}{\partial v'} \right| = \frac{8}{|I - \mathbf{k} \otimes \sigma|} = \frac{8}{|1 - \mathbf{k} \cdot \sigma|} = \frac{4}{\sin^2(\theta/2)}, \quad \theta \in [0, \pi/2],$$

where again $\mathbf{k} = (v-v_*)/|v-v_*|$. Note that this change of variables is singular when $\theta = 0$. After this change of variables, $\mathbf{k} = (v-v_*)/|v-v_*|$ is a function of v, v', σ , so that θ no longer plays the role as the polar angle. In fact, ‘‘pole \mathbf{k} ’’ moves with σ and hence the measure $d\sigma$ is no longer given by $\sin \theta d\theta d\phi$. Hence, we need to choose a new pole which is independent of σ . Choose $\mathbf{k}'' = (v'-v)/|v'-v|$, then the polar angle ψ defined by $\cos \psi = \mathbf{k}'' \cdot \sigma$ satisfies,

$$\psi = \frac{\pi}{2} - \frac{\theta}{2}, \quad d\sigma = \sin \psi d\psi d\phi, \quad \psi \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right].$$

Note that now the angular singularity in $b(\cos \theta) d\sigma$ becomes $\theta^{-2-2-2s}$, which is stronger than (1.2) where it is of order θ^{-1-2s} .

On the other hand, there is another singularity in the kinetic factor of the cross section for soft potential. To study this, we decompose the kinetic factor of collision operator $\Phi_\gamma(v-v_*) = |v-v_*|^\gamma$ in two part by using a cutoff function. Let $0 \leq \phi(z) \leq 1$ be a smooth radial function with value 1 for z close to 0, and 0 for large values of z . Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z) + \Phi_\gamma(z)(1-\phi(z)) = \Phi_{sing}(z) + \Phi_{reg}(z).$$

Then we write

$$Q(f, g) = Q_{sing}(f, g) + Q_{reg}(f, g),$$

where the kinetic factor in the collision operator is defined according to the decomposition respectively. We consider firstly the regular part.

Since $\Phi_{reg}(z)$ is smooth, and $\Phi_{reg}(z) \leq \tilde{\Phi}_\gamma(z) = (1 + |z|^2)^{\gamma/2}$, $Q_{reg}(f, g)$ has similar upper bound and commutator estimates as for $Q_{\tilde{\Phi}_\gamma}(f, g)$.

Let us recall several propositions obtained in [3]. For $0 < s < 1$, $\gamma \in \mathbb{R}$, we proved the following upper bound estimate (Theorem 2.1 of [3])

$$(4.1) \quad |(Q_{\tilde{\Phi}_\gamma}(f, g), h)| \lesssim \|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}} \|g\|_{H^{m+s}_{(\alpha+\gamma+2s)^+}} \|h\|_{H^{s-m}_{-\alpha}},$$

for any $m, \alpha \in \mathbb{R}$, and the estimate of commutators with weight (Lemma 2.4 of [3])

$$(4.2) \quad \left| \left(W_\ell Q_{\tilde{\Phi}_\gamma}(f, g) - Q_{\tilde{\Phi}_\gamma}(f, W_\ell g), h \right) \right| \lesssim \|f\|_{L^1_{\ell+(2s-1)^+ + \gamma^+}} \|g\|_{H^{(2s-1+\epsilon)^+}_{\ell+(2s-1)^+ + \gamma^+}} \|h\|_{L^2}.$$

It also follows from Lemma 2.8 of [4] that

$$(4.3) \quad \iiint b |f_*| (g' - g)^2 d\sigma dv dv_* \lesssim \|f\|_{L^1_{2s}} \|g\|_{H^s}^2.$$

We now study the estimate of Proposition 2.4 for the regular part.

Proposition 4.1. *Let $\ell \geq 6$. If $\max\{-3, -2s - 3/2\} < \gamma \leq 0$ and $0 < s < 1$, then*

$$\left(W_{\varphi, \ell} Q_{reg}(f, g), h \right)_{L^2(\mathbb{R}_v^3)} \lesssim \|g\|_{H^{2s}_{\ell+2s}(\mathbb{R}_v^3)} \|W_{\varphi, \ell} f\|_{L^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}.$$

Proof. Write

$$\begin{aligned} \left(W_{\varphi, \ell} Q_{reg}(f, g), h \right)_{L^2(\mathbb{R}^3)} &= \left(Q_{reg}(f, W_{\varphi, \ell} g), h \right)_{L^2(\mathbb{R}^3)} \\ &+ \left((W_{\varphi, \ell} Q_{reg}(f, g) - Q_{reg}(f, W_{\varphi, \ell} g)), h \right)_{L^2(\mathbb{R}^3)} = A + B. \end{aligned}$$

By using the upper bound estimate (4.1) with $m = s, \alpha = 0$, we have

$$\begin{aligned} |A| &\lesssim \|f\|_{L^1_{(\gamma+2s)^+}(\mathbb{R}_v^3)} \|W_{\varphi, \ell} g\|_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)} \\ &\lesssim \left\| \frac{\varphi}{\langle x \rangle^2} \frac{f}{\varphi} \right\|_{L^2_{2s+3/2+\epsilon}(\mathbb{R}_v^3)} \left\| \frac{\langle x \rangle^2}{\varphi} W_\ell g \right\|_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)} \\ &\lesssim \|W_{\varphi, 2s+7/2+\epsilon} f\|_{L^2(\mathbb{R}_v^3)} \|g\|_{H^{2s}_{\ell+2s}(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}. \end{aligned}$$

Here we have used $\frac{\varphi}{\langle x \rangle^2} \lesssim \langle v \rangle^2$.

For the term B , we have

$$\begin{aligned} B &= \left(W_{\varphi, \ell} Q_{reg}(f, g) - Q_{reg}(f, W_{\varphi, \ell} g), h \right)_{L^2(\mathbb{R}_v^3)} \\ &= \iiint b \Phi_{reg} \left((W_{\varphi, \ell})' - (W_{\varphi, \ell}) \right) f_* g h' dv dv_* d\sigma \\ &= \iiint b \Phi_{reg} \left((\nabla_v W_{\varphi, \ell})(v') \cdot (v' - v) \right) f_* g h' dv dv_* d\sigma \\ &+ \iiint b \Phi_{reg} \left((W_{\varphi, \ell})' - (W_{\varphi, \ell}) - (\nabla_v W_{\varphi, \ell})(v') \cdot (v' - v) \right) f_* g h' dv dv_* d\sigma \\ &= B_1 + B_2. \end{aligned}$$

By using (3.5), we have

$$\begin{aligned}
|B_2| &\lesssim \iiint b \theta^2 |(W_{\varphi,4f})_*| |(W_{\ell}g)| |h'| dv dv_* d\sigma \\
(4.4) \quad &+ \iiint b \theta^{\ell-2} |(W_{\varphi,\ell f})_*| |gh'| dv dv_* d\sigma \\
&= M_1 + M_2,
\end{aligned}$$

where we have used the fact that $\gamma \leq 0$. By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
M_1^2 &\lesssim \left(\iiint b \theta^2 |(W_{\varphi,4f})_*| |(W_{\ell}g)|^2 dv dv_* d\sigma \right) \\
&\quad \times \left(\iiint b \theta^2 |(W_{\varphi,4f})_*| |(h')|^2 dv dv_* d\sigma \right) \\
&\lesssim \|W_{\varphi,4f}\|_{L^1(\mathbb{R}_v^3)}^2 \|g\|_{L^2(\mathbb{R}_v^3)}^2 \|h\|_{L^2(\mathbb{R}_v^3)}^2.
\end{aligned}$$

Here we have used the regular change of variables

$$v \rightarrow v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma,$$

whose Jacobian is given by

$$\left| \frac{\partial v}{\partial v'} \right| = \frac{8}{|I + \mathbf{k} \otimes \sigma|} = \frac{8}{|1 + \mathbf{k} \cdot \sigma|} = 4 / \cos^2(\theta/2) \leq 8.$$

On the other hand, by the Cauchy-Schwarz inequality again, we have

$$\begin{aligned}
M_2^2 &\lesssim \left(\iiint b \theta^{\ell-7/2} |g| |W_{\varphi,\ell f}|_*^2 dv_* dv d\sigma \right) \\
&\quad \times \left(\iiint b \theta^{\ell-1/2} |g| |h'|^2 dv_* dv d\sigma \right),
\end{aligned}$$

if we choose ℓ so that

$$\ell - 7/2 - (1 + 2s) = \ell - 1/2 - (2 + 2s + 2) > -1.$$

Then a direct calculation reduces the first integral to

$$\iiint b \theta^{\ell-2-3/2} |g| |W_{\varphi,\ell f}|_*^2 dv_* dv d\sigma \lesssim \|g\|_{L^1(\mathbb{R}_v^3)} \|W_{\varphi,\ell f}\|_{L^2(\mathbb{R}_v^3)}^2.$$

For the second integral, we now use the singular change of variables $v_* \rightarrow v'$ whose Jacobian is

$$\left| \frac{\partial v_*}{\partial v'} \right| = \frac{4}{\sin^2(\theta/2)}.$$

Then, we have

$$\begin{aligned}
&\iiint_{\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times \mathbb{S}^2} b(\cos \theta) \theta^{\ell-1/2} |g| |h'|^2 dv_* dv d\sigma \\
&\lesssim \iint_{\mathbb{R}_v^3 \times \mathbb{R}_{v'}^3} \int_0^{\pi/2} \theta^{\ell-1/2-(2+2s+2)} |g| |h'|^2 dv' dv d\theta \\
&\lesssim \|g\|_{L^1(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2.
\end{aligned}$$

Thus,

$$M_2 \lesssim \|g\|_{L^1(\mathbb{R}_v^3)} \|W_{\varphi,\ell f}\|_{L^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}.$$

For the term B_1 , we decompose it further into

$$\begin{aligned}
B_1 &= \iiint \Phi_{reg} b \left\{ \left(\nabla_v W_{\varphi, \ell} \right) (v') - \left(\nabla_v W_{\varphi, \ell} \right) (v) \right\} \cdot (v - v') f_* g h' dv dv_* d\sigma \\
&\quad + \iiint \Phi_{reg} b \left\{ \left(\nabla_v W_{\varphi, \ell} \right) (v) g(v) - \left(\nabla_v W_{\varphi, \ell} \right) (v') g(v') \right\} \cdot (v - v') f_* h' dv dv_* d\sigma \\
&\quad + \iiint \Phi_{reg} b f_* \left(g h \nabla_v W_{\varphi, \ell} \right) (v') \cdot (v - v') dv dv_* d\sigma \\
&= B_1^{(1)} + B_1^{(2)} + B_1^{(3)}.
\end{aligned}$$

It follows from the symmetry of σ variable that $B_1^{(3)}$ vanishes, see the Figure 1 below.

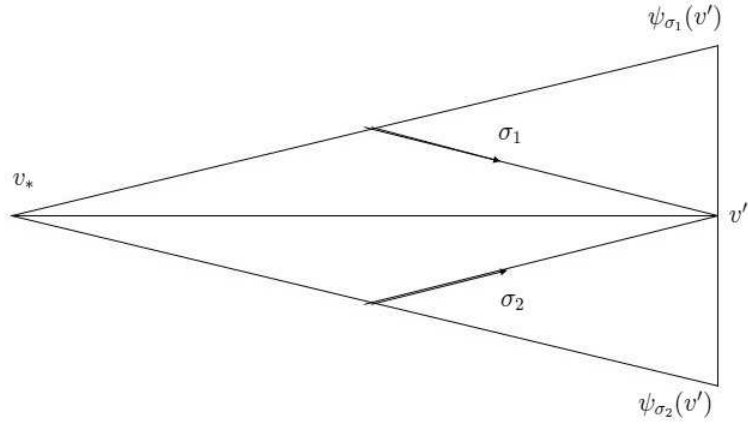


FIGURE 1. Symmetry of σ_1 and σ_2 , $v' \rightarrow v = \psi_\sigma(v')$

Since $|v - v'| = |v - v_*| \sin(\theta/2)$ and $|\Phi_{reg}| \lesssim 1$ for $\gamma \leq 0$, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|B_1^{(2)}|^2 &\lesssim \iiint b |f_*| |v - v_*|^2 \left| \left(\nabla_v W_{\varphi, \ell} \right) (v) g(v) - \left(\nabla_v W_{\varphi, \ell} \right) (v') g(v') \right|^2 dv dv_* d\sigma \\
&\quad \times \iiint b \sin^2(\theta/2) |f_*| |h'|^2 dv dv_* d\sigma = B_1^{(2,1)} \times B_1^{(2,2)}.
\end{aligned}$$

Using the regular change of variables $v \rightarrow v'$, we get

$$B_1^{(2,2)} \lesssim \|f\|_{L^1(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2 \lesssim \|f\|_{L_{3/2+\varepsilon}^2(\mathbb{R}_v^3)} \|h\|_{L^2(\mathbb{R}_v^3)}^2.$$

Putting $G = \left(\nabla_v W_{\varphi, \ell} \right) g$, in view of $|v - v_*| \lesssim |v' - v_*| \leq \langle v_* \rangle + \langle v' \rangle$ we have

$$\begin{aligned}
B_1^{(2,1)} &\lesssim \iiint b |\langle v_* \rangle^2 f_*| \left| G(v) - G(v') \right|^2 dv dv_* d\sigma \\
&\quad + \iiint b |f_*| \left| \langle v \rangle G(v) - \langle v' \rangle G(v') \right|^2 dv dv_* d\sigma \\
&\quad + \iiint b |f_*| \left| (\langle v \rangle - \langle v' \rangle) G(v) \right|^2 dv dv_* d\sigma.
\end{aligned}$$

We apply (4.3) to the first two terms. Note that

$$|\langle v \rangle - \langle v' \rangle| \lesssim |v - v'| \lesssim \langle v_* \rangle \langle v \rangle \sin(\theta/2),$$

in the last term. Then we get

$$B_1^{(2,1)} \lesssim \|f\|_{L^1_2} \|W_{\varphi, \ell} g\|_{H^s_s(\mathbb{R}^3_v)}^2.$$

On account of (3.6), we have

$$\begin{aligned} |B_1^{(1)}| &\lesssim \iiint b \theta^2 |(W_{\varphi, 4} f)_*| |(W_{\ell} g)| |h'| dv dv_* d\sigma \\ &\quad + \iiint b \theta^{\ell-2} |(W_{\varphi, \ell} f)_*| |gh'| dv dv_* d\sigma, \end{aligned}$$

which is the same as for B_2 in (4.4). We have proved Proposition 4.1. \square

We finally turn to the singular part $Q_{sing}(f, g)$. As shown in [7], the singular part Q_{sing} requires fairly long computations. For our use, we now recall some estimates in [7]. The following upper bound estimate is a consequence of Proposition 2.1 from [7]: for $0 < s < 1, \gamma > \max\{-3, -2s - 3/2\}$ and $m \in [s - 1, s]$,

$$(4.5) \quad |(Q_{sing}(f, g), h)_{L^2(\mathbb{R}^3_v)}| \lesssim \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{H^{s+m}(\mathbb{R}^3)} \|h\|_{H^{s-m}(\mathbb{R}^3)}.$$

And the following commutator estimate is implied by Proposition 2.5 of [7]: letting $0 < s < 1, \gamma > \max\{-3, -2s - 3/2\}$, for any $\ell, \beta, \delta \in \mathbb{R}$,

$$(4.6) \quad \left| (W_{\ell} Q_{sing}(f, g) - Q_{sing}(f, W_{\ell} g), h) \right| \lesssim \|f\|_{L^2_{\ell-1-\beta-\delta}} \|g\|_{H^{(2s-1+\epsilon)+}_{\beta}} \|h\|_{L^2_{\delta}}.$$

For the estimate of singular part with weight introduced in this paper, we now want to prove

Proposition 4.2. *Let $\ell \geq 2$. If $\max\{-3, -2s - 3/2\} < \gamma \leq 0$ and $0 < s < 1$, then*

$$(4.7) \quad \left| (W_{\varphi, \ell} Q_{sing}(f, g), h)_{L^2(\mathbb{R}^3)} \right| \lesssim \|g\|_{H^{\gamma}_{\ell}(\mathbb{R}^3)} \|W_{\varphi, \ell} f\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}.$$

Proof. Write

$$\begin{aligned} & \left(W_{\varphi, \ell} Q_{sing}(f, g), h \right)_{L^2(\mathbb{R}^3)} \\ &= \left(Q_{sing}\left(\frac{f}{\langle x \rangle^2}, W_{\ell} g\right), \frac{\langle x \rangle^2}{\varphi(v, x)} h \right)_{L^2(\mathbb{R}^3_v)} \\ &+ \left(W_{\ell} Q_{sing}\left(\frac{f}{\langle x \rangle^2}, g\right) - Q_{sing}\left(\frac{f}{\langle x \rangle^2}, W_{\ell} g\right), \frac{\langle x \rangle^2}{\varphi(v, x)} h \right)_{L^2(\mathbb{R}^3_v)} \\ &= A_1 + A_2. \end{aligned}$$

It follows from (4.5) that with $m = s$,

$$\begin{aligned} |A_1| &\lesssim \left\| \frac{\varphi(v, x)}{\langle x \rangle^2} \frac{f}{\varphi(v, x)} \right\|_{L^2(\mathbb{R}^3_v)} \|W_{\ell} g\|_{H^{2s}(\mathbb{R}^3_v)} \left\| \frac{\langle x \rangle^2}{\varphi(v, x)} h \right\|_{L^2(\mathbb{R}^3_v)} \\ &\lesssim \|g\|_{H^{\gamma}_{\ell}(\mathbb{R}^3)} \|W_{\varphi, 2} f\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

because $\langle x \rangle^2 \leq \varphi(v, x) \leq \langle x \rangle^2 \langle v \rangle^2$.

Using (4.6) with $\beta = \ell - 1, \delta = 0$, we have

$$\begin{aligned} |A_2| &\lesssim \left\| \frac{\varphi(v, x)}{\langle x \rangle^2} \frac{f}{\varphi(v, x)} \right\|_{L^2(\mathbb{R}^3_v)} \|g\|_{H^{(2s-1+\epsilon)+}_{\ell-1}(\mathbb{R}^3_v)} \left\| \frac{\langle x \rangle^2}{\varphi(v, x)} h \right\|_{L^2(\mathbb{R}^3_v)} \\ &\lesssim \|g\|_{H^{\gamma}_{\ell}(\mathbb{R}^3)} \|W_{\varphi, 2} f\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

The above two estimates complete the proof of Proposition 2.4. \square

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