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# ON THE GENERAL ADDITIVE DIVISOR PROBLEM

ALEKSANDAR IVIĆ AND JIE WU

ABSTRACT. We obtain a new upper bound for  $\sum_{h \leq H} \Delta_k(N, h)$  for  $1 \leq H \leq N$ ,  $k \in \mathbb{N}$ ,  $k \geq 3$ , where  $\Delta_k(N, h)$  is the (expected) error term in the asymptotic formula for  $\sum_{N < n \leq 2N} d_k(n)d_k(n+h)$ , and  $d_k(n)$  is the divisor function generated by  $\zeta(s)^k$ . When  $k = 3$  the result improves, for  $H \geq N^{1/2}$ , the bound given in the recent work [1] of Baier, Browning, Marasingha and Zhao, who dealt with the case  $k = 3$ .

*In honor of Professor A.A. Karatsuba's 75th birthday*

## 1. INTRODUCTION

Let  $d_k(n)$  denote that (generalized) divisor function, which represents the number of ways  $n$  can be written as a product of  $k$  ( $\in \mathbb{N}$ ) factors. Thus

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta(s)^k \quad (\Re s > 1),$$

where  $\zeta(s)$  is the familiar zeta-function of Riemann. In particular  $d_1(n) \equiv 1$  and  $d_2(n) = \sum_{\delta|n} 1$  is the number of positive divisors of  $n$ . The function  $d_k(n)$  is a multiplicative function of  $n$ , and

$$d_k(p^\nu) = (-1)^\nu \binom{-k}{\nu} = \frac{k(k+1) \cdots (k+\nu-1)}{\nu!}$$

for primes  $p$  and  $\nu \in \mathbb{N}$ . The general divisor problem deals with the estimation of  $\Delta_k(x)$ , the error term in the asymptotic formula (see Chapter 13 of Ivić [3] and Chapter 12 of Titchmarsh [15] for an extensive discussion)

$$(1.1) \quad D_k(x) := \sum_{n \leq x} d_k(n) = xp_{k-1}(\log x) + \Delta_k(x),$$

where

$$(1.2) \quad p_{k-1}(\log x) = \operatorname{Res}_{s=1} \left( \zeta(s)^k \frac{x^{s-1}}{s} \right).$$

Since  $\zeta(s)$  is regular in  $\mathbb{C}$  except at  $s = 1$  where it has a simple pole with residue 1, it transpires that  $p_{k-1}(y)$  is a polynomial of degree  $k - 1$ , whose coefficients may

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be explicitly evaluated, and in particular  $p_1(y) = y + 2\gamma - 1$ , where  $\gamma$  is Euler's constant. The important constants  $\alpha_k, \beta_k$  are defined as

$$(1.3) \quad \begin{aligned} \alpha_k &:= \inf \{ a_k : \Delta_k(x) \ll x^{a_k} \}, \\ \beta_k &:= \inf \left\{ b_k : \int_1^X |\Delta_k(x)|^2 dx \ll X^{1+2b_k} \right\}. \end{aligned}$$

It is known that  $\alpha_k \geq \beta_k \geq (k-1)/(2k)$  for all  $k \in \mathbb{N}$ , and the conjecture that  $\alpha_k = \beta_k = (k-1)/(2k)$  for all  $k \in \mathbb{N}$  is equivalent to the Lindelöf hypothesis that  $\zeta(\frac{1}{2} + it) \ll_\varepsilon (|t|+1)^\varepsilon$ . Here and later  $\varepsilon (> 0)$  denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while  $\ll_{a,b,\dots}$  means that the implied constant in the  $\ll$ -symbol depends on  $a, b, \dots$ .

The general additive divisor problem is another important problem involving the divisor function  $d_k(n)$ . It consists of the estimation of the quantity  $\Delta_k(x, h)$ , given by the formula

$$(1.4) \quad \sum_{n \leq x} d_k(n) d_k(n+h) = x P_{2k-2}(\log x; h) + \Delta_k(x, h).$$

In (1.4) it is assumed that  $k \geq 2$  is a fixed integer, and  $P_{2k-2}(\log x; h)$  is a suitable polynomial of degree  $2k-2$  in  $\log x$ , whose coefficients depend on  $k$  and  $h$ , while  $\Delta_k(x, h)$  is supposed to be the error term. This means that we should have

$$(1.5) \quad \Delta_k(x, h) = o(x) \quad \text{as } x \rightarrow \infty,$$

but unfortunately (1.5) is not yet known to hold for any  $k \geq 3$ , even for fixed  $h$ . However, when we consider the sum

$$(1.6) \quad \sum_{h \leq H} \Delta_k(x, h),$$

we may reasonably hope that a certain cancelation will occur among the individual summands  $\Delta_k(x, h)$ , since there are no absolute value signs in (1.6). It turns out that it is precisely the estimation of the sum in (1.6) which is relevant for bounding the integral

$$(1.7) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt,$$

which is of great importance in the theory of the Riemann zeta-function (see the monographs [3, 4, 15]).

For  $k = 1$  the sum in (1.6) is trivial, while for  $k = 2$  it was extensively studied by many authors, including Kuznetsov [10], Motohashi [13], Ivić & Motohashi [8] and Meurman [12]. The natural next step in (1.6) is to deal with the case  $k = 3$ , but the works of A.I. Vinogradov and Takhtadžjan [19, 20] and A.I. Vinogradov [16, 17, 18] show that the analytic problems connected with the Dirichlet series generated by  $d_3(n)d_3(n+h)$  are overwhelmingly hard. The ensuing problems are connected with the group  $SL(3, \mathbb{Z})$ , and they are much more difficult than the corresponding problems connected with the group  $SL(2, \mathbb{Z})$  which appear in the case  $k = 2$ . The latter involve the spectral theory of the non-Euclidean Laplacian, which was extensively developed in recent times by Kuznetsov (see e.g., [11]), Iwaniec and others (see Motohashi's monograph [14] for applications of spectral theory to the

theory of  $\zeta(s)$ . Thus at present in the case  $k = 2$  we have sharp explicit formulas, while in the case  $k > 2$  we have none.

A.I. Vinogradov [18] conjectured that  $\Delta_k(x, h) \ll x^{1-1/k}$ , without stating for which range of  $h$  this sharp bound should hold. Very likely this bound is too strong, and (even for fixed  $h$ ) it seems probable that a power of a logarithm should be included on the right-hand side. More importantly, one hopes that the bound

$$(1.8) \quad \sum_{h \leq H} \Delta_k(x, h) \ll_{k, \varepsilon} H x^{1-1/k+\varepsilon} \quad \text{for } 1 \leq H \leq x^{(k-2)/k+\delta_k}$$

holds uniformly in  $H$  for fixed  $k \geq 3$  and some  $\delta_k > 0$ , which was stated in [5]. Note that Vinogradov's conjecture in the form  $\Delta_k(x, h) \ll_{k, \varepsilon} x^{1-1/k+\varepsilon}$  trivially implies (1.8), but the important point is that there are no absolute value signs in the sum in (1.8). One can also assume (1.8) to hold in the case  $k = 2$  for  $1 \leq H \leq \sqrt{x}$ , say. Then it would follow that the inequality

$$(1.9) \quad \int_{T-G}^{T+G} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \ll_{\varepsilon} GT^{\varepsilon}$$

holds with  $G = T^{5/6}$ , whereas it is known (see e.g., [7]) that  $G = T^{2/3}$  is unconditionally permissible. It was conjectured in [5] that for any  $k \geq 2$  and  $h \geq 1$  one has

$$(1.10) \quad \Delta_k(x, h) = \Omega(x^{1-1/k}).$$

For  $k = 2$  and fixed  $h$  this conjecture was proved by Motohashi [13]. As usual,  $f(x) = \Omega(g(x))$  means that  $\lim_{x \rightarrow \infty} f(x)/g(x) \neq 0$ .

The general additive divisor problems is connected to the power moments of  $|\zeta(\frac{1}{2} + it)|$  (see e.g., [3] and [4] for an extensive account). In 1996 the first author [5] proved that

$$(1.11) \quad \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 dt \ll_{\varepsilon} T^{1+\varepsilon} + T^{(\alpha+3\beta-1)/2+\varepsilon}$$

provided that

$$\sum_{h \leq H} \Delta_3(x, h) \ll_{\varepsilon} H^{\alpha} x^{\beta+\varepsilon}$$

holds for  $1 \leq H \leq x^{1/3+\delta_3}$  for some constant  $\delta_3 > 0$ ,  $0 \leq \alpha, \beta \leq 1$ ,  $\alpha + \beta \geq 1$ . The conjecture (1.8) with  $k = 3$  means that we can take  $\alpha = 1, \beta = 2/3$  in (1.11) so that the sixth moment in the form

$$(1.12) \quad \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 dt \ll_{\varepsilon} T^{1+\varepsilon}$$

follows. Note that the best known exponent of  $T$  for the right-hand side of the above integral is  $5/4$  (see [3, Chapter 8]).

In [6] the research begun in [5] was continued, and a plausible heuristic evaluation of the polynomial  $P_{2k-2}(x; h)$  in (1.4) was made. Yet another (heuristic) evaluation

of the sum in (1.5) was made later by Conrey and Gonek [2] in 2001. Moreover, it was shown in [6] that, for a fixed integer  $k \geq 3$  and any fixed  $\varepsilon > 0$ , we have

$$(1.13) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \ll_{k,\varepsilon} T^{1+\varepsilon} \left( 1 + \sup_{T^{1+\varepsilon} < M \ll T^{k/2}} \frac{G_k(M; T)}{M} \right),$$

if, for  $T^{1+\varepsilon} \leq M \ll T^{k/2}$  and  $M < M' \leq 2M$ ,

$$(1.14) \quad G_k(M; T) := \sup_{\substack{M \leq x \leq M' \\ 1 \leq t \leq M^{1+\varepsilon}/T}} \left| \sum_{h \leq t} \mathbb{D}_k(x, h) \right|.$$

This result, which generalizes (1.11), provides a direct link between upper bounds for the  $2k$ -th moment of  $|\zeta(\frac{1}{2} + it)|$  and sums of  $\mathbb{D}_k(x, h)$  over the shift parameter  $h$ . The result also gives an insight as to the limitations of the attack on the  $2k$ -th moment of  $|\zeta(\frac{1}{2} + it)|$  via the use of estimates for  $\Delta_k(x, h)$ . Of course the problem greatly increases in complexity as  $k$  increases, and this is one of the reasons why in [5] only the case  $k = 3$  was considered. The case  $k = 2$  was not treated, since for the fourth moment of  $|\zeta(\frac{1}{2} + it)|$  we have an asymptotic formula with precise results for the corresponding error term (see e.g., [7] and [14]). Note that (1.13)–(1.14) again lead to the sixth moment bound (1.12) if the conjecture (1.8) holds with  $k = 3$ .

## 2. THE GENERAL ADDITIVE DIVISOR PROBLEM

The main objective of this note is to study the averaged sum (1.6), when  $k \geq 3$  is a fixed integer. To this end we introduce more notation, defining

$$(2.1) \quad D_k(N, h) := \sum_{N < n \leq 2N} d_k(n) d_k(n + h),$$

and letting henceforth

$$(2.2) \quad \Delta_k(N; h) := D_k(N, h) - \int_N^{2N} \mathfrak{S}_k(x, h) dx,$$

so that  $\Delta_k(N; h)$  in (2.2) differs slightly from (1.4); in fact it equals  $\Delta_k(2N, h) - \Delta_k(N, h)$  in the notation of (1.4). Here we follow the notation of [1], based on the approach of Conrey and Gonek [2], who made conjectures on the high moments of  $|\zeta(\frac{1}{2} + it)|$ . Let us also define

$$\mathfrak{S}_k(x, h) := \sum_{q=1}^{\infty} \frac{c_q(h)}{q^2} Q_k(x, q)^2,$$

where  $\mu(n)$  is the Möbius function,  $c_q(h) := \sum_{d|(h,q)} d\mu(q/d)$  is the Ramanujan sum and  $Q_k(x, q)$  is defined as follows. If  $\varphi(n)$  is the Euler totient function, set

$$\Psi_{d,e}(s, q, k) := \frac{d\mu(d)\mu(e)}{\varphi(d)e} \prod_{p|(eq/d)} \left\{ \left( 1 - \frac{1}{p^s} \right)^k \sum_{\nu=0}^{\infty} \frac{d_k(p^{\nu+\nu_p(eq/d)})}{p^{\nu s}} \right\},$$

where here and later  $\nu_p(m)$  is the  $p$ -adic valuation of  $m$ . Then we define

$$\begin{aligned} Q_k(x, q) &:= \frac{1}{2\pi i} \int_{|s-1|=1/8} \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q, k) \left(\frac{ex}{dq}\right)^{s-1} ds \\ &= \operatorname{Res}_{s=1} \left\{ \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q, k) \left(\frac{ex}{dq}\right)^{s-1} \right\}, \end{aligned}$$

by the residue theorem. Thus  $Q_k(x, q)$  is a polynomial of degree  $2k - 2$  whose coefficients depend on  $q$ , and may be explicitly evaluated. The work of Conrey-Gonek (op. cit.) predicts, as stated in (2.2), that  $D_k(N, h)$  is well approximated by  $\int_N^{2N} \mathfrak{S}_k(x, h) dx$ , which equals  $N$  times a polynomial in  $\log N$  of degree  $2k - 2$ , all of whose coefficients depend on  $h$  and  $k$ . This is in agreement with [5] (when  $k = 3$  and [6] (in the general case), although the shape of the polynomial in question is somewhat different. Conrey and Gonek even predict that uniformly

$$\Delta_k(N; h) \ll_{\varepsilon} N^{1/2+\varepsilon} \quad \text{for } 1 \leq h \leq N^{1/2}.$$

This conjecture is probably too strong, and one feels that more likely the bound  $\Delta_k(N; h) \ll_{\varepsilon} N^{1-1/k+\varepsilon}$  is closer to the truth (see (1.10)).

In a recent work [1], Baier, Browning, Marasingha and Zhao obtain new results involving averages of  $\Delta_3(N; h)$  (they employ the terminology “shifted convolutions of  $d_3(n)$ ”, which seems appropriate). They proved that

$$(2.3) \quad \sum_{h \leq H} \Delta_3(N; h) \ll_{\varepsilon} N^{\varepsilon} (H^2 + H^{1/2} N^{13/12}) \quad (1 \leq H \leq N),$$

and if  $N^{1/3+\varepsilon} \leq H \leq N^{1-\varepsilon}$ , then there exists  $\delta(\varepsilon) > 0$  such that

$$(2.4) \quad \sum_{h \leq H} |\Delta_3(N; h)|^2 \ll_{\varepsilon} H N^{2-\delta(\varepsilon)}.$$

These results can be used, in conjunction with the bounds (1.13)–(1.14) when  $k = 3$ , to bound the integral in (1.11), but they will produce only the exponent  $11/8$  on the right-hand side of (1.11), hence no improvement on the known result for the sixth moment of  $|\zeta(\frac{1}{2} + it)|$ .

**Remark 1.** Note that (2.3), in the range  $N^{1/6+\varepsilon} \leq H \leq N^{1-\varepsilon}$ , provides an asymptotic formula for the averaged sum  $\sum_{h \leq H} D_3(N, h)$  (see (2.1)). However, it should be noted that no asymptotic formula for the individual  $D_3(N, h)$  has been found yet, and in general for  $\Delta_k(N; h)$  when  $k \geq 3$ . In fact, it is worth pointing out that when  $1 \leq H \leq N^{1/6}$ , the bound in (2.3) is worse than the trivial bound  $HN^{1+\varepsilon}$ . Namely we have

$$\sum_{h \leq H} D_k(N, h) \ll_{\varepsilon} \sum_{h \leq H} \sum_{N < n \leq 2N} (n+h)^{\varepsilon/2} \ll_{\varepsilon} (HN)^{1+\varepsilon/2} \ll_{\varepsilon} HN^{1+\varepsilon}.$$

On the other hand we have

$$\sum_{h \leq H} \int_N^{2N} \mathfrak{S}_k(x, h) dx \ll_{\varepsilon} HN^{1+\varepsilon},$$

which is obvious from (3.8). Hence by (2.2) it follows that

$$(2.5) \quad \sum_{h \leq H} \Delta_k(N; h) \ll_{\varepsilon} H N^{1+\varepsilon} \quad (1 \leq H \leq N),$$

and clearly (2.5) for  $k = 3$  improves (2.3) for  $1 \leq H \leq N^{1/6}$ . The aim of this note is to give a bound for the sum in (1.6), or equivalently for the average of (2.2), which for  $k = 3$  improves (2.3) for a certain range of  $H$ . The result is contained in the following

**Theorem 1.** *For fixed  $k \geq 3$  we have*

$$(2.6) \quad \sum_{h \leq H} \Delta_k(N; h) \ll_{\varepsilon} N^{\varepsilon} (H^2 + N^{1+\beta_k}) \quad (1 \leq H \leq N),$$

where  $\beta_k$  is defined by (1.3).

Note that we have  $\beta_3 = 1/3, \beta_4 = 3/8$  (see Chapter 13 of [3]),  $\beta_5 \leq 9/20$  (see Zhang [21]),  $\beta_6 \leq 1/2$ , etc. For a discussion of the values of  $\alpha_k$  and  $\beta_k$ , see also the paper by Ouellet and Ivić [9].

**Corollary 1.** *We have, for  $1 \leq H \leq N$ ,*

$$(2.7) \quad \begin{aligned} \sum_{h \leq H} \Delta_3(N; h) &\ll_{\varepsilon} N^{\varepsilon} (H^2 + N^{4/3}), \\ \sum_{h \leq H} \Delta_4(N; h) &\ll_{\varepsilon} N^{\varepsilon} (H^2 + N^{11/8}), \\ \sum_{h \leq H} \Delta_5(N; h) &\ll_{\varepsilon} N^{\varepsilon} (H^2 + N^{29/20}), \\ \sum_{h \leq H} \Delta_6(N; h) &\ll_{\varepsilon} N^{\varepsilon} (H^2 + N^{3/2}). \end{aligned}$$

**Remark 2.** Since it is known that  $\beta_k < 1$  for any  $k$ , this means that the bound in (2.6) improves on the trivial bound  $H N^{1+\varepsilon}$  in the range  $N^{\beta_k+\varepsilon} \leq H \leq N^{1-\varepsilon}$ . Our result thus supports the assertion that  $\Delta_k(N; h)$  is really the error term in the asymptotic formula for  $D_k(N, h)$ , as given by (3.1) and (3.2). In the case when  $k = 3$ , we have by (2.7) an improvement of (2.3) when  $H \geq N^{1/2}$ .

### 3. PROOF OF THEOREM 1

We begin by noting that obviously

$$\sum_{h \leq H} d_k(n+h) = \sum_{m \leq n+H} d_k(m) - \sum_{m \leq n} d_k(m).$$

Therefore by (1.1)–(1.2) and (2.1)–(2.2) we can write

$$(3.1) \quad \begin{aligned} \sum_{h \leq H} \Delta_k(N, h) &= \sum_{N < n \leq 2N} d_k(n) \sum_{h \leq H} d_k(n+h) - \sum_{h \leq H} \int_N^{2N} \mathfrak{S}_k(x, h) dx \\ &= M_k(N, H) + R_k(N, H) - \sum_{h \leq H} \int_N^{2N} \mathfrak{S}_k(x, h) dx, \end{aligned}$$

say, where

$$M_k(N, H) := \sum_{N < n \leq 2N} d_k(n) \operatorname{Res}_{s=1} \left( \zeta(s)^k \frac{(n+H)^s - n^s}{s} \right),$$

$$R(N, H) := \sum_{N < n \leq 2N} d_k(n) (\Delta_k(n+H) - \Delta_k(n)),$$

where  $\Delta_k(x)$  is defined by (1.1). It is rather easy to estimate  $R_k(N, H)$ . Namely since  $d_k(n) \ll_\varepsilon n^\varepsilon$ , we have trivially

$$R_k(N, H) \ll_\varepsilon N^\varepsilon \sum_{n \leq 3N} |\Delta_k(n)|.$$

For  $n < t < n+1$ , we obviously have

$$\Delta_k(n) - \Delta_k(t) = tp_{k-1}(\log t) - np_{k-1}(\log n) \ll (\log n)^{k-1}.$$

Thus

$$(3.2) \quad \begin{aligned} R_k(N, H) &\ll_\varepsilon N^\varepsilon \sum_{n \leq 3N} \int_n^{n+1} |\Delta_k(n)| dt \\ &\ll_\varepsilon N^\varepsilon \sum_{n \leq 3N} \int_n^{n+1} |\Delta_k(t)| dt + N^{1+\varepsilon} \\ &\ll_\varepsilon N^\varepsilon \int_1^{4N} |\Delta_k(t)| dt + N^{1+\varepsilon} \\ &\ll_\varepsilon N^\varepsilon \left( N \int_1^{4N} |\Delta_k(t)|^2 dt \right)^{1/2} + N^{1+\varepsilon} \\ &\ll_\varepsilon N^{1+\beta_k+\varepsilon}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality for integrals and the mean square bound (1.3) in the last step.

To estimate  $M_k(N, H)$ , set

$$u_k(x) := \operatorname{Res}_{s=1} \left( \zeta(s)^k \frac{(x+H)^s - x^s}{s} \right).$$

Then we can write

$$M_k(N, H) = \int_N^{2N+0} u_k(x) dD_k(x).$$

But we have, since

$$D_k(x) = \operatorname{Res}_{s=1} \left( \zeta(s)^k \frac{x^s}{s} \right) + \Delta_k(x)$$

in view of (1.1) and (1.2),

$$(3.3) \quad M_k(N, H) = \int_N^{2N} u_k(x) \operatorname{Res}_{s=1} (\zeta(s)^k x^{s-1}) dx + \int_N^{2N} u_k(x) d\Delta_k(x).$$



Further note that

$$(3.4) \quad \begin{aligned} u_k(x) &= yp_{k-1}(\log y) \Big|_x^{x+H} \ll H(\log x)^{k-1}, \\ u'_k(x) &= \operatorname{Res}_{s=1} \zeta(s)^k \{(x+H)^{s-1} - x^{s-1}\} \ll_\varepsilon x^\varepsilon. \end{aligned}$$

On integrating by parts and using (1.3) and (3.4) we obtain, similarly to (3.2),

$$(3.5) \quad \begin{aligned} \int_N^{2N} u_k(x) d\Delta_k(x) &= u_k(x)\Delta_k(x) \Big|_N^{2N} - \int_N^{2N} \Delta_k(x)u'_k(x) dx \\ &\ll_\varepsilon HN^{\alpha_k+\varepsilon} + N^{1+\beta_k+\varepsilon}. \end{aligned}$$

As for the other integral in (3.3), note that

$$\frac{(x+H)^s - x^s}{s} = \frac{x^s}{s} \left\{ 1 + \frac{sH}{x} + \binom{s}{2} \frac{H^2}{x^2} + \cdots - 1 \right\}.$$

This gives

$$(3.6) \quad \int_N^{2N} u_k(x) \operatorname{Res}_{s=1} (\zeta(s)^k x^{s-1}) dx = H \int_N^{2N} (\operatorname{Res}_{s=1} \zeta(s)^k x^{s-1})^2 dx + O_\varepsilon(H^2 N^\varepsilon).$$

Therefore from (3.3), (3.5) and (3.6) we obtain

$$(3.7) \quad \begin{aligned} M_k(N, H) &= H \int_N^{2N} (\operatorname{Res}_{s=1} \zeta(s)^k x^{s-1})^2 dx \\ &\quad + O_\varepsilon(H^2 N^\varepsilon + NH^{\alpha_k+\varepsilon} + N^{1+\beta_k+\varepsilon}). \end{aligned}$$

Next we shall prove that

$$(3.8) \quad \sum_{h \leq H} \int_N^{2N} \mathfrak{S}_k(x, h) dx = H \int_N^{2N} (\operatorname{Res}_{s=1} \zeta(s)^k x^{s-1})^2 dx + O_\varepsilon(N^{1+\varepsilon}).$$

The case of  $k = 3$  has been treated in [1]. Here we repeat the same argument with some simplification in the general case, obtaining (3.8).

First write

$$x^{s-1} = \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} (s-1)^n.$$

Since  $\Psi_{d,e}(s, q)$  and  $(s-1)^n \zeta(s)^k$  with  $n \geq k$  are holomorphic for  $\Re s > 0$ , Cauchy's theorem allows us to deduce that

$$Q_k(x, q) = \frac{1}{2\pi i} \sum_{n=0}^{k-1} \int_{|s-1|=1/8} \zeta(s)^k \sum_{d|q} \sum_{e|d} \Psi_{d,e}(s, q) \frac{(\log(dx/eq))^n}{n!} (s-1)^n ds.$$

Clearly for  $\Re s > \frac{1}{2}$ , we have

$$\begin{aligned} \Psi_{d,e}(s, q) &\ll \frac{d}{\varphi(d)e} \prod_{p|(eq/d)} \left\{ \left(1 + \frac{1}{p^{1/2}}\right)^k \sum_{\nu=0}^{\infty} \frac{d_k(p^{\nu+\nu_p(eq/d)})}{p^{\nu s}} \right\} \\ &\ll_{\varepsilon} \frac{d}{\varphi(d)e} \prod_{p|(eq/d)} \left\{ \left(1 + \frac{1}{p^{1/2}}\right)^k p^{\nu_p(eq/d)\varepsilon/4} \sum_{\nu \geq 0} \frac{p^{\nu\varepsilon/4}}{p^{\nu/2}} \right\} \\ &\ll_{\varepsilon} q^{\varepsilon/2}. \end{aligned}$$

Thus

$$(3.9) \quad Q_k(x, q) \ll_{\varepsilon, k} q^{\varepsilon} (\log x)^{k-1},$$

where the implied constant depends only on  $\varepsilon$  and  $k$ .

In view of (3.9) and the bound  $|c_q(h)| \leq (h, q)$ , we have

$$(3.10) \quad \begin{aligned} \sum_{h \leq H} \sum_{q > H} \frac{c_q(h)}{q^2} Q_k(x, q)^2 &\ll (\log x)^{k-1} \sum_{h \leq H} \sum_{q > H} \frac{(h, q)}{q^{2-\varepsilon}} \\ &\ll_{\varepsilon, k} H^{\varepsilon} (\log x)^{k-1}. \end{aligned}$$

On the other hand, it is well known that  $\sum_{h \leq q} c_q(h) = 0$  if  $q > 1$ . From this it is easy to deduce that

$$\sum_{h \leq H} c_q(h) = \begin{cases} H + O(1) & \text{if } q = 1, \\ O_{\varepsilon}(q^{1+\varepsilon}) & \text{if } q > 1. \end{cases}$$

With the help of this relation and (3.9), we can write

$$(3.11) \quad \begin{aligned} &\sum_{h \leq H} \sum_{q \leq H} \frac{c_q(h)}{q^2} Q_k(x, q)^2 \\ &= \{H + O(1)\} Q_k(x, 1)^2 + O\left((\log x)^{k-1} \sum_{1 < q \leq H} \frac{1}{q^{1-\varepsilon}}\right) \\ &= H \left(\operatorname{Res}_{s=1} \zeta(s)^k x^{s-1}\right)^2 + O((\log x)^{k-1} H^{\varepsilon}), \end{aligned}$$

where we have used the fact that

$$Q_k(x, 1) = \operatorname{Res}_{s=1} (\zeta(s)^k x^{s-1}) \ll_k (\log x)^{k-1}.$$

By combining (3.10) and (3.11), we obtain (3.8).

From (3.1), (3.2), (3.7) and (3.8) we obtain

$$(3.12) \quad \sum_{h \leq H} \Delta_k(N, H) \ll_{\varepsilon} N^{\varepsilon} (H^2 + HN^{\alpha_k} + N^{1+\beta_k}) \quad (1 \leq H \leq N).$$

But we always have

$$(3.13) \quad \alpha_k \leq \frac{1}{2} + \frac{1}{2}\beta_k.$$

To see this note that, for  $1 \leq H \leq x$ , the defining relation (1.1) and  $d_k(n) \ll_\varepsilon n^\varepsilon$  give

$$\begin{aligned} \Delta_k(x) - \frac{1}{H} \int_x^{x+H} \Delta_k(y) \, dy &= \frac{1}{H} \int_x^{x+H} (\Delta_k(x) - \Delta_k(y)) \, dy \\ &\ll_\varepsilon \frac{1}{H} \int_x^{x+H} \{|D_k(x) - D_k(y)| + O(x^\varepsilon)\} \, dy \\ &\ll_\varepsilon H x^\varepsilon. \end{aligned}$$

This gives, by the Cauchy-Schwarz inequality for integrals and (1.3),

$$\begin{aligned} \Delta_k(x) &\ll_\varepsilon \frac{1}{H} \int_x^{x+H} |\Delta_k(y)| \, dy + H x^\varepsilon \\ &\ll_\varepsilon x^{1+\beta_k+\varepsilon} H^{-1} + H x^\varepsilon \\ &\ll_\varepsilon x^{(1+\beta_k)/2+\varepsilon} \end{aligned}$$

with  $H = x^{(1+\beta_k)/2}$ . Hence

$$\Delta_k(x) \ll_\varepsilon x^{(1+\beta_k)/2+\varepsilon}$$

and (3.13) follows. Now in (3.12) we have  $HN^{\alpha_k} \leq H^2$  for  $H \geq N^{\alpha_k}$ . If  $H \leq N^{\alpha_k}$ , then  $HN^{\alpha_k} \leq N^{2\alpha_k} \leq N^{1+\beta_k}$  by (3.13). Thus the term  $HN^{\alpha_k}$  in (3.12) can be discarded, and (2.6) follows. This completes the proof of the Theorem.

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