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# A NOTE ON $H_w^p$ -BOUNDEDNESS OF RIESZ TRANSFORMS AND $\theta$ -CALDERÓN-ZYGMUND OPERATORS THROUGH MOLECULAR CHARACTERIZATION

LUONG DANG KY

ABSTRACT. Let  $0 < p \leq 1$  and  $w$  in the Muckenhoupt class  $A_1$ . Recently, by using the weighted atomic decomposition and molecular characterization; Lee, Lin and Yang [11] (J. Math. Anal. Appl. 301 (2005), 394–400) established that the Riesz transforms  $R_j, j = 1, 2, \dots, n$ , are bounded on  $H_w^p(\mathbb{R}^n)$ . In this note we extend this to the general case of weight  $w$  in the Muckenhoupt class  $A_\infty$  through molecular characterization. One difficulty, which has not been taken care in [11], consists in passing from atoms to all functions in  $H_w^p(\mathbb{R}^n)$ . Furthermore, the  $H_w^p$ -boundedness of  $\theta$ -Calderón-Zygmund operators are also given through molecular characterization and atomic decomposition.

## 1. INTRODUCTION

Calderón-Zygmund operators and their generalizations on Euclidean space  $\mathbb{R}^n$  have been extensively studied, see for example [7, 14, 18, 15]. In particular, Yabuta [18] introduced certain  $\theta$ -Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operator.

**Definition 1.1.** Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  satisfying  $\int_0^1 \frac{\theta(t)}{t} dt < \infty$ . A continuous function  $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$  is said to be a  $\theta$ -Calderón-Zygmund singular integral kernel if there exists a constant  $C > 0$  such that

$$|K(x, y)| \leq \frac{C}{|x - y|^n}$$

for all  $x \neq y$ ,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{1}{|x - y|^n} \theta\left(\frac{|x - x'|}{|x - y|}\right)$$

for all  $2|x - x'| \leq |x - y|$ .

A linear operator  $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$  is said to be a  $\theta$ -Calderón-Zygmund operator if  $T$  can be extended to a bounded operator on  $L^2(\mathbb{R}^n)$  and there exists

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a  $\theta$ -Calderon-Zygmund singular integral kernel  $K$  such that for all  $f \in C_c^\infty(\mathbb{R}^n)$  and all  $x \notin \text{supp } f$ , we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy.$$

When  $K_j(x, y) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)\frac{x_j-y_j}{|x-y|^{n+1}}$ ,  $j = 1, 2, \dots, n$ , then they are the classical Riesz transforms denoted by  $R_j$ .

It is well-known that the Riesz transforms  $R_j$ ,  $j = 1, 2, \dots, n$ , are bounded on unweighted Hardy spaces  $H^p(\mathbb{R}^n)$ . There are many different approaches to prove this classical result (see [11, 9]). Recently, by using the weighted molecular theory (see [10]) and combined with García-Cuerva's atomic decomposition [5] for weighted Hardy spaces  $H_w^p(\mathbb{R}^n)$ , the authors in [11] established that the Riesz transforms  $R_j$ ,  $j = 1, 2, \dots, n$ , are bounded on  $H_w^p(\mathbb{R}^n)$ . More precisely, they proved that  $\|R_j f\|_{H_w^p} \leq C$  for every  $w$ - $(p, \infty, ts-1)$ -atom where  $s, t \in \mathbb{N}$  satisfy  $n/(n+s) < p \leq n/(n+s-1)$  and  $((s-1)r_w+n)/(s(r_w-1))$  with  $r_w$  is the *critical index of  $w$  for the reverse Hölder condition*. Remark that this leaves a gap in the proof. Similar gaps exist in some literatures, for instance in [10, 15] when the authors establish  $H_w^p$ -boundedness of Calderón-Zygmund type operators. Indeed, it is now well-known that (see [1]) the argument "*the operator  $T$  is uniformly bounded in  $H_w^p(\mathbb{R}^n)$  on  $w$ - $(p, \infty, r)$ -atoms, and hence it extends to a bounded operator on  $H_w^p(\mathbb{R}^n)$* " is wrong in general. However, Meda, Sjögren and Vallarino [13] establishes that (in the setting of unweighted Hardy spaces) this is correct if one replaces  $L^\infty$ -atoms by  $L^q$ -atoms with  $1 < q < \infty$ . See also [19] for  $L^2$ -atoms with a different method from [13]. Later, the authors in [2] extended these results to the weighted anisotropic Hardy spaces. More precisely, it is claimed in [2] that the operator  $T$  can be extended to a bounded operator on  $H_w^p(\mathbb{R}^n)$  if it is uniformly bounded on  $w$ - $(p, q, r)$ -atoms for  $q_w < q < \infty, r \geq [n(q_w/p - 1)]$  where  $q_w$  is the *critical index of  $w$* .

Motivated by [11, 10, 15, 1, 2], in this paper, we extend *Theorem 1* in [11] to  $A_\infty$  weights (see *Theorem 1.1*); *Theorem 4* in [10] (see *Theorem 1.2*), *Theorem 3* in [15] (see *Theorem 3.1*) to  $\theta$ -Calderón-Zygmund operators; and fill the gaps of the proofs by using the atomic decomposition and molecular characterization of  $H_w^p(\mathbb{R}^n)$  as in [11].

Throughout the whole paper,  $C$  denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In  $\mathbb{R}^n$ , we denote by  $B = B(x, r)$  an open ball with center  $x$  and radius  $r > 0$ . For any measurable set  $E$ , we denote by  $|E|$  its Lebesgue measure, and by  $E^c$  the set  $\mathbb{R}^n \setminus E$ .

Let us first recall some notations, definitions and well-known results.

Let  $1 \leq p < \infty$ . A nonnegative locally integrable function  $w$  belongs to the *Muckenhoupt class  $A_p$* , say  $w \in A_p$ , if there exists a positive constant  $C$  so

that

$$\frac{1}{|B|} \int_B w(x) dx \left( \frac{1}{|B|} \int_B (w(x))^{-1/(p-1)} dx \right)^{p-1} \leq C, \quad \text{if } 1 < p < \infty,$$

and

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess-inf}_{x \in B} w(x), \quad \text{if } p = 1,$$

for all balls  $B$  in  $\mathbb{R}^n$ . We say that  $w \in A_\infty$  if  $w \in A_p$  for some  $p \in [1, \infty)$ .

It is well known that  $w \in A_p$ ,  $1 \leq p < \infty$ , implies  $w \in A_q$  for all  $q > p$ . Also, if  $w \in A_p$ ,  $1 < p < \infty$ , then  $w \in A_q$  for some  $q \in [1, p)$ . We thus write  $q_w := \inf\{p \geq 1 : w \in A_p\}$  to denote the critical index of  $w$ . For a measurable set  $E$ , we note  $w(E) = \int_E w(x) dx$  its weighted measure.

The following lemma gives a characterization of the class  $A_p$ ,  $1 \leq p < \infty$ . It can be found in [6].

**Lemma A.** The function  $w \in A_p$ ,  $1 \leq p < \infty$ , if and only if, for all non-negative functions and all balls  $B$ ,

$$\left( \frac{1}{|B|} \int_B f(x) dx \right)^p \leq C \frac{1}{w(B)} \int_B f(x)^p w(x) dx.$$

A close relation to  $A_p$  is the reverse Hölder condition. If there exist  $r > 1$  and a fixed constant  $C > 0$  such that

$$\left( \frac{1}{|B|} \int_B w^r(x) dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) dx \right) \quad \text{for every ball } B \subset \mathbb{R}^n,$$

we say that  $w$  satisfies *reverse Hölder condition of order  $r$*  and write  $w \in RH_r$ .

It is known that if  $w \in RH_r$ ,  $r > 1$ , then  $w \in RH_{r+\varepsilon}$  for some  $\varepsilon > 0$ . We thus write  $r_w := \sup\{r > 1 : w \in RH_r\}$  to denote the *critical index of  $w$  for the reverse Hölder condition*.

The following result provides us the comparison between the Lebesgue measure of a set  $E$  and its weighted measure  $w(E)$ . It also can be found in [6].

**Lemma B.** Let  $w \in A_p \cap RH_r$ ,  $p \geq 1$  and  $r > 1$ . Then there exist constants  $C_1, C_2 > 0$  such that

$$C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r},$$

for all cubes  $B$  and measurable subsets  $E \subset B$ .

Given a weight function  $w$  on  $\mathbb{R}^n$ , as usual we denote by  $L_w^q(\mathbb{R}^n)$  the space of all functions  $f$  satisfying  $\|f\|_{L_w^q} := \left( \int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right)^{1/q} < \infty$ . When  $q = \infty$ ,  $L_w^\infty(\mathbb{R}^n)$  is  $L^\infty(\mathbb{R}^n)$  and  $\|f\|_{L_w^\infty} = \|f\|_{L^\infty}$ . Analogously to the classical Hardy spaces, the *weighted Hardy spaces*  $H_w^p(\mathbb{R}^n)$ ,  $p > 0$ , can be defined in terms of

maximal functions. Namely, let  $\phi$  be a function in  $\mathcal{S}(\mathbb{R}^n)$ , the Schwartz space of rapidly decreasing smooth functions, satisfying  $\int_{\mathbb{R}^n} \phi(x) dx = 1$ . Define

$$\phi_t(x) = t^{-n} \phi(x/t), \quad t > 0, x \in \mathbb{R}^n,$$

and the maximal function  $f^*$  by

$$f^*(x) = \sup_{t>0} |f * \phi_t(x)|, \quad x \in \mathbb{R}^n.$$

Then  $H_w^p(\mathbb{R}^n)$  consists of those tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^n)$  for which  $f^* \in L_w^p(\mathbb{R}^n)$  with the (quasi-)norm

$$\|f\|_{H_w^p} = \|f^*\|_{L_w^p}.$$

In order to show the  $H_w^p$ -boundedness of Riesz transforms, we characterize weighted Hardy spaces in terms of atoms and molecules in the following way.

**Definition of a weighted atom.** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$  such that  $w \in A_q$ . Let  $q_w$  be the critical index of  $w$ . Set  $[\cdot]$  the integer function. For  $s \in \mathbb{N}$  satisfying  $s \geq [n(q_w/p - 1)]$ , a function  $a \in L_w^q(\mathbb{R}^n)$  is called  $w$ - $(p, q, s)$ -atom centered at  $x_0$  if

- (i)  $\text{supp } a \subset B$  for some ball  $B$  centered at  $x_0$ ,
- (ii)  $\|a\|_{L_w^q} \leq w(B)^{1/q-1/p}$ ,
- (iii)  $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ .

Let  $H_w^{p,q,s}(\mathbb{R}^n)$  denote the space consisting of tempered distributions admitting a decomposition  $f = \sum_{j=1}^{\infty} \lambda_j a_j$  in  $\mathcal{S}'(\mathbb{R}^n)$ , where  $a_j$ 's are  $w$ - $(p, q, s)$ -atoms and  $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ . And for every  $f \in H_w^{p,q,s}(\mathbb{R}^n)$ , we consider the (quasi-)norm

$$\|f\|_{H_w^{p,q,s}} = \inf \left\{ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j, \{a_j\}_{j=1}^{\infty} \text{ are } w\text{-}(p, q, s)\text{-atoms} \right\}.$$

Denote by  $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$  the vector space of all finite linear combinations of  $w$ - $(p, q, s)$ -atoms, and the (quasi-)norm of  $f$  in  $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$  is defined by

$$\|f\|_{H_{w,\text{fin}}^{p,q,s}} := \inf \left\{ \left( \sum_{j=1}^k |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^k \lambda_j a_j, k \in \mathbb{N}, \{a_j\}_{j=1}^k \text{ are } w\text{-}(p, q, s)\text{-atoms} \right\}.$$

We have the following atomic decomposition for  $H_w^p(\mathbb{R}^n)$ . It can be found in [5] (see also [2, 8]).

**Theorem A.** If the triplet  $(p, q, s)$  satisfies the conditions of  $w$ - $(p, q, s)$ -atoms, then  $H_w^p(\mathbb{R}^n) = H_w^{p,q,s}(\mathbb{R}^n)$  with equivalent norms.

The molecules corresponding to the atoms mentioned above can be defined as follows.

**Definition of a weighted molecule.** For  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$ , let  $w \in A_q$  with critical index  $q_w$  and critical index  $r_w$  for the reverse Hölder

condition. Set  $s \geq [n(q_w/p - 1)]$ ,  $\varepsilon > \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, 1/p - 1\}$ ,  $a = 1 - 1/p + \varepsilon$ , and  $b = 1 - 1/q + \varepsilon$ . A  $w$ - $(p, q, s, \varepsilon)$ -molecule centered at  $x_0$  is a function  $M \in L_w^q(\mathbb{R}^n)$  satisfying

- (i)  $M.w(B(x_0, \cdot - x_0))^b \in L_w^q(\mathbb{R}^n)$ ,
- (ii)  $\|M\|_{L_w^q}^{a/b} \|M.w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^{1-a/b} \equiv \mathfrak{N}_w(M) < \infty$ ,
- (iii)  $\int_{\mathbb{R}^n} M(x)x^\alpha dx = 0$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ .

The above quantity  $\mathfrak{N}_w(M)$  is called the  $w$ -molecular norm of  $M$ .

In [10], Lee and Lin proved that every weighted molecule belongs to the weighted Hardy space  $H_w^p(\mathbb{R}^n)$ , and the embedding is continuous.

**Theorem B.** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$ ,  $w \in A_q$ , and  $(p, q, s, \varepsilon)$  be the quadruple in the definition of molecule. Then, every  $w$ - $(p, q, s, \varepsilon)$ -molecule  $M$  centered at any point in  $\mathbb{R}^n$  is in  $H_w^p(\mathbb{R}^n)$ , and  $\|M\|_{H_w^p} \leq C\mathfrak{N}_w(M)$  where the constant  $C$  is independent of the molecule.

Although, in general, one cannot conclude that an operator  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$  by checking that their norms have uniform bound on all of the corresponding  $w$ - $(p, \infty, s)$ -atoms (cf. [1]). However, this is correct when dealing with  $w$ - $(p, q, s)$ -atoms with  $q_w < q < \infty$ . Indeed, we have the following result (see [2, Theorem 7.2]).

**Theorem C.** Let  $0 < p \leq 1$ ,  $w \in A_\infty$ ,  $q \in (q_w, \infty)$  and  $s \in \mathbb{Z}$  satisfying  $s \geq [n(q_w/p - 1)]$ . Suppose that  $T : H_{w, \text{fin}}^{p, q, s}(\mathbb{R}^n) \rightarrow H_w^p(\mathbb{R}^n)$  is a linear operator satisfying

$$\sup\{\|Ta\|_{H_w^p} : a \text{ is any } w\text{-}(p, q, s)\text{-atom}\} < \infty.$$

Then  $T$  can be extended to a bounded linear operator on  $H_w^p(\mathbb{R}^n)$ .

Our first main result, which generalizes *Theorem 1* in [11], is as follows:

**Theorem 1.1.** Let  $0 < p \leq 1$  and  $w \in A_\infty$ . Then, the Riesz transforms are bounded on  $H_w^p(\mathbb{R}^n)$ .

For the next result, we need the notion  $T^*1 = 0$ .

**Definition 1.2.** Let  $T$  be a  $\theta$ -Calderón-Zygmund operator. We say that  $T^*1 = 0$  if  $\int_{\mathbb{R}^n} Tf(x)dx = 0$  for all  $f \in L^q(\mathbb{R}^n)$ ,  $1 < q \leq \infty$ , with compact support and  $\int_{\mathbb{R}^n} f(x)dx = 0$ .

We now can give the  $H_w^p$ -boundedness of  $\theta$ -Calderón-Zygmund type operators, which generalizes *Theorem 4* in [10] by taking  $q = 1$  and  $\theta(t) = t^\delta$ , as follows:

**Theorem 1.2.** Given  $\delta \in (0, 1]$ ,  $n/(n + \delta) < p \leq 1$ , and  $w \in A_q \cap RH_r$  with  $1 \leq q < p(n + \delta)/n$ ,  $(n + \delta)/(n + \delta - nq) < r$ . Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  with  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .

## 2. PROOF OF THEOREM 1.1

In order to prove the main theorems, we need the following lemma (see [6, page 412]).

**Lemma C.** Let  $w \in A_r, r > 1$ . Then there exists a constant  $C > 0$  such that

$$\int_{B^c} \frac{1}{|x - x_0|^{nr}} w(x) dx \leq C \frac{1}{\sigma^{nr}} w(B)$$

for all balls  $B = B(x_0, \sigma)$  in  $\mathbb{R}^n$ .

*Proof of Theorem 1.1.* For  $q = 2(q_w + 1) \in (q_w, \infty)$ , then  $s := [n(q/p - 1)] \geq [n(q_w/p - 1)]$ . We now choose (and fix) a positive number  $\varepsilon$  satisfying

$$(2.1) \quad \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < \varepsilon < t(s + 1)(nq)^{-1} + q^{-1} - 1,$$

for some  $t \in \mathbb{N}, t \geq 1$  and  $\max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < t(s + 1)(nq)^{-1} + q^{-1} - 1$ .

Clearly,  $\ell := t(s + 1) - 1 \geq s \geq [n(q_w/p - 1)]$ . Hence, by Theorem B and Theorem C, it is sufficient to show that for every  $w$ - $(p, q, \ell)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ , the Riesz transforms  $R_j f = K_j * f$ ,  $j = 1, 2, \dots, n$ , are  $w$ - $(p, q, s, \varepsilon)$ -molecules with the norm  $\mathfrak{N}_w(R_j f) \leq C$ .

Indeed, as  $w \in A_q$  by  $q = 2(q_w + 1) \in (q_w, \infty)$ . It follows from  $L_w^q$ -boundedness of Riesz transforms that

$$(2.2) \quad \|R_j f\|_{L_w^q} \leq \|R_j\|_{L_w^q \rightarrow L_w^q} \|f\|_{L_w^q} \leq C w(B)^{1/q-1/p}.$$

To estimate  $\|R_j f \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$  where  $b = 1 - 1/q + \varepsilon$ , we write

$$\begin{aligned} \|R_j f \cdot w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &\quad + \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &= I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sqrt{n}\sigma))^{bq} \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(x) dx \\ &\leq C w(B)^{bq} \|R_j\|_{L_w^q \rightarrow L_w^q}^q \|f\|_{L_w^q}^q \leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II, as  $f$  is  $w$ - $(p, q, \ell)$ -atom, by the Taylor's fomular and Lemma A, we get

$$\begin{aligned} |K_j * f(x)| &= \left| \int_{|y-x_0| \leq \sigma} \left( K_j(x-y) - \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} D^\alpha K_j(x-x_0)(x_0-y)^\alpha \right) f(y) dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{\sigma^{\ell+1}}{|x-x_0|^{n+\ell+1}} |f(y)| dy \\ &\leq C \frac{\sigma^{n+\ell+1}}{|x-x_0|^{n+\ell+1}} w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all  $x \in (B(x_0, 2\sqrt{n}\sigma))^c$ . As  $b = 1 - 1/q + \varepsilon$ , it follows from (2.1) that  $(n + \ell + 1)q - q^2nb > nq$ . Therefore, by combining the above inequality, Lemma B and Lemma C, we obtain

$$\begin{aligned} II &= \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q} w(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q}} w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q - q^2nb} w(B)^{(b-1/p)q} \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q - q^2nb}} w(x) dx \\ &\leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

Thus,

$$(2.3) \quad \|R_j f \cdot w(B(x_0, |\cdot - x_0|))\|_{L_w^q}^b = (I + II)^{1/q} \leq C w(B)^{b+1/q-1/p}.$$

Remark that  $a = 1 - 1/p + \varepsilon$ . Combining (2.2) and (2.3), we obtain

$$\mathfrak{N}_w(R_j f) \leq C w(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

The proof will be concluded if we establish the vanishing moment conditions of  $R_j f$ . One first consider the following lemma.

**Lemma.** For every classical atom  $(p, 2, \ell)$ -atom  $g$  centered at  $x_0$ , we have

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0 \quad \text{for } 0 \leq |\alpha| \leq s, 1 \leq j \leq n.$$

*Proof of the Lemma.* Since  $b = 1 - 1/q + \varepsilon < (\ell + 1)(nq)^{-1} < (\ell + 1)n^{-1}$ , we obtain  $2(n + \ell + 1) - 2nb > n$ . It is similar to the previous argument, we also obtain that  $R_j g$  and  $R_j g \cdot |\cdot - x_0|^{nb}$  belong to  $L^2(\mathbb{R}^n)$ . Now, we establish that  $R_j g \cdot (\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ . Indeed,



since  $\varepsilon > q/p - 1$  by (2.1), implies that  $2(s - nb) < (s - nb)q' < -n$  by  $q = 2(q_w + 1) > 2$ , where  $1/q + 1/q' = 1$ . We use Schwartz inequality to get

$$\begin{aligned} \int_{B(x_0,1)^c} |R_j g(x)(x - x_0)^\alpha| dx &\leq \int_{B(x_0,1)^c} |R_j g(x)| |x - x_0|^s dx \\ &\leq \left( \int_{B(x_0,1)^c} |R_j g(x)|^2 |x - x_0|^{2nb} dx \right)^{1/2} \left( \int_{B(x_0,1)^c} |x - x_0|^{2(s-nb)} dx \right)^{1/2} \\ &\leq C \|R_j g \cdot |\cdot - x_0|^{nb}\|_{L^2} < \infty, \end{aligned}$$

and

$$\int_{B(x_0,1)} |R_j g(x)(x - x_0)^\alpha| dx \leq |B(x_0,1)|^{1/2} \left( \int_{B(x_0,1)} |R_j g(x)|^2 dx \right)^{1/2} < \infty.$$

Thus,  $R_j g(\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$  for any  $|\alpha| \leq s$ . Deduce that  $R_j g(x) x^\alpha \in L^1(\mathbb{R}^n)$  for any  $|\alpha| \leq s$ . Therefore,

$$(R_j g(x) x^\alpha)^\wedge(\xi) = C_\alpha \cdot D^\alpha(\widehat{R_j g})(\xi)$$

is continuous, with  $|C_\alpha| \leq C_s$  ( $C_s$  depends only on  $s$ ) for any  $|\alpha| \leq s$ , where  $\hat{h}$  is used to denote the fourier transform of  $h$ . Consequently,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = C_\alpha \cdot D^\alpha(\widehat{R_j g})(0) = C_\alpha \cdot D^\alpha(m_j \hat{g})(0),$$

where  $m_j(x) = -ix_j/|x|$ . Moreover, since  $g$  is a classical  $(p, 2, \ell)$ -atom, it follows from [17, Lemma 9.1] that  $\hat{g}$  is  $\ell$ th order differentiable and  $\hat{g}(\xi) = O(|\xi|^{\ell+1})$  as  $\xi \rightarrow 0$ . We write  $e_j$  to be the  $j$ th standard basis vector of  $\mathbb{R}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  a multi-index of nonnegative integers  $\alpha_j$ ,  $\Delta_{he_j} \phi(x) = \phi(x) - \phi(x - he_j)$ ,  $\Delta_{he_j}^{\alpha_j} \phi(x) = \Delta_{he_j}^{\alpha_j-1} \phi(x) - \Delta_{he_j}^{\alpha_j-1} \phi(x - he_j)$  for  $\alpha_j \geq 2$ ,  $\Delta_{he_j}^0 \phi(x) = \phi(x)$ , and  $\Delta_h^\alpha = \Delta_{he_1}^{\alpha_1} \dots \Delta_{he_n}^{\alpha_n}$ . Then, the boundedness of  $m_j$ , and  $|C_\alpha| \leq C_s$  for  $|\alpha| \leq s$ , implies

$$\begin{aligned} \left| \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx \right| &= |C_\alpha| \left| \lim_{h \rightarrow 0} |h|^{-|\alpha|} \Delta_h^\alpha(m_j \hat{g})(0) \right| \\ &\leq C \lim_{h \rightarrow 0} |h|^{\ell+1-|\alpha|} = 0, \end{aligned}$$

for  $|\alpha| \leq s$  by  $s \leq \ell$ . Thus, for any  $j = 1, 2, \dots, n$ , and  $|\alpha| \leq s$ ,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0.$$

This complete the proof of the lemma.

Let us come back to the proof of Theorem 1.1. As  $q/2 = q_w + 1 > q_w$ , by Lemma A,

$$\left( \frac{1}{|B|} \int_B |f(x)|^2 dx \right)^{q/2} \leq C \frac{1}{w(B)} \int_B |f(x)|^q w(x) dx.$$

Therefore,  $g := C^{-1/q} |B|^{-1/p} w(B)^{1/p} f$  is a classical  $(p, 2, \ell)$ -atom since  $f$  is  $w$ - $(p, q, \ell)$ -atom associated with ball  $B$ . Consequently, by the above lemma,

$$\int_{\mathbb{R}^n} R_j f(x) x^\alpha dx = C^{1/q} |B|^{1/p} w(B)^{-1/p} \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0$$

for all  $j = 1, 2, \dots, n$  and  $|\alpha| \leq s$ . Thus, the theorem is proved.  $\square$

Following a similar but easier argument, we also have the following  $H_w^p$ -boundedness of Hilbert transform. We leave details to readers.

**Theorem 2.1.** *Let  $0 < p \leq 1$  and  $w \in A_\infty$ . Then, the Hilbert transform is bounded on  $H_w^p(\mathbb{R})$ .*

### 3. PROOF OF THEOREM 1.2

We first consider the following lemma

**Lemma 3.1.** *Let  $p \in (0, 1]$ ,  $w \in A_q$ ,  $1 < q < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then,  $\int_{\mathbb{R}^n} T f(x) dx = 0$  for all  $w$ - $(p, q, 0)$ -atoms  $f$ .*

*Proof of Lemma 3.1.* Let  $f$  be an arbitrary  $w$ - $(p, q, 0)$ -atom associated with ball  $B$ . It is well-known that there exists  $1 < r < q$  such that  $w \in A_r$ . Therefore, it follows from Lemma A that

$$\int_B |f(x)|^{q/r} dx \leq C |B| w(B)^{1/r} \|f\|_{L_w^q}^{q/r} < \infty.$$

We deduce that  $f$  is a multiple of classical  $(p, q/r, 0)$ -atom, and thus the condition  $T^*1 = 0$  implies  $\int_{\mathbb{R}^n} T f(x) dx = 0$ .  $\square$

*Proof of Theorem 1.2.* Because of the hypothesis, without loss of generality we can assume  $q > 1$ . Furthermore, it is clear that  $[n(q_w/p - 1)] = 0$ , and there exists a positive constant  $\varepsilon$  such that

$$(3.1) \quad \max \left\{ \frac{1}{r_w - 1}, \frac{1}{p} - 1 \right\} < \varepsilon < \frac{n + \delta}{nq} - 1.$$

Similarly to the arguments in Theorem 1.1, it is sufficient to show that, for every  $w$ - $(p, q, 0)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ ,

$Tf$  is a  $w$ -( $p, q, 0, \varepsilon$ )-molecule with the norm  $\mathfrak{N}_w(Tf) \leq C$ . One first observe that  $\int_{\mathbb{R}^n} Tf(x)dx = 0$  by Lemma 3.1, and

$$\sum_{k=0}^{\infty} \theta(2^{-k})2^{knbq} < \infty,$$

where  $b = 1 - 1/q + \varepsilon$ , by  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$  and (3.1). We deduce that

$$(3.2) \quad \sum_{k=0}^{\infty} \left( \theta(2^{-k})2^{knbq} \right)^q < \infty.$$

As  $w \in A_q$ ,  $1 < q < \infty$ , it follows from [18, Theorem 2.4] that

$$(3.3) \quad \|Tf\|_{L_w^q} \leq C\|f\|_{L_w^q} \leq Cw(B)^{1/q-1/p}.$$

To estimate  $\|Tf \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$ , we write

$$\begin{aligned} \|Tf \cdot w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &+ \int_{|x-x_0| > 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx = I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sigma))^{bq} \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(x) dx \\ &\leq Cw(B)^{bq} \|f\|_{L_w^q}^q \leq Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate  $II$ , since  $f$  is of mean zero, by Lemma A, we have

$$\begin{aligned} |Tf(x)| &= \left| \int_{|y-x_0| \leq \sigma} (K(x, y) - K(x, x_0))f(y)dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{1}{|x - x_0|^n} \theta\left(\frac{|y - x_0|}{|x - x_0|}\right) |f(y)| dy \\ &\leq C \frac{\sigma^n}{|x - x_0|^n} \theta\left(\frac{\sigma}{|x - x_0|}\right) w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all  $x \in (B(x_0, 2\sigma))^c$ . Therefore, by combining the above inequality, Lemma B and (3.2), we obtain

$$\begin{aligned}
 II &= \int_{|x-x_0|>2\sigma} |Tf(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0|>2\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left( \theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{-q/p} \sum_{k=1}^{\infty} \int_{2^k\sigma < |x-x_0| \leq 2^{k+1}\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left( \theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{(b+1/q-1/p)q} \sum_{k=0}^{\infty} \left( \theta(2^{-k}) 2^{knbq} \right)^q \leq Cw(B)^{(b+1/q-1/p)q}.
 \end{aligned}$$

Thus,

$$(3.4) \quad \|Tf \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q} = (I + II)^{1/q} \leq Cw(B)^{b+1/q-1/p}.$$

Remark that  $a = 1 - 1/p + \varepsilon$ . Combining (3.3) and (3.4), we obtain

$$\mathfrak{N}_w(Tf) \leq Cw(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

This finishes the proof.  $\square$

It is well-known that the molecular theory of (unweighted) Hardy spaces of Taibleson and Weiss [17] is one of useful tools to establish boundedness of operators in Hardy spaces (cf. [17, 12]). In the setting of Muckenhoupt weight, this theory has been considered by the authors in [10], since then, they have been well used to establish boundedness of operators in weighted Hardy spaces (cf. [10, 11, 3]). However in some cases, the *weighted molecular characterization*, which obtained in [10], does not give the best possible results. For Calderón-Zygmund type operators in Theorem 1.2, for instance, it involves assumption on the *critical index of  $w$  for the reverse Hölder condition* as the following theorem does not.

**Theorem 3.1.** *Given  $\delta \in (0, 1]$ ,  $n/(n + \delta) < p \leq 1$ , and  $w \in A_q$  with  $1 \leq q < p(n + \delta)/n$ . Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  with  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .*

The following corollary give the boundedness of the classical Calderón-Zygmund type operators on weighted Hardy spaces (see [15, Theorem 3]).

**Corollary 3.1.** *Let  $0 < \delta \leq 1$  and  $T$  be the classical  $\delta$ -Calderón-Zygmund operator, i.e.  $\theta(t) = t^\delta$ , satisfying  $T^*1 = 0$ . If  $n/(n + \delta) < p \leq 1$  and  $w \in A_q$  with  $1 \leq q < p(n + \delta)/n$ , then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .*

*Proof of Corollary 3.1.* By taking  $\delta' \in (0, \delta)$  which is close enough  $\delta$ . Then, we apply Theorem 3.1 with  $\delta'$  instead of  $\delta$ .  $\square$

*Proof of Theorem 3.1.* Without loss of generality we can assume  $1 < q < p(n + \delta)/n$ . Fix  $\phi \in \mathcal{S}(\mathbb{R}^n)$  with  $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$ . By Theorem C, it is sufficient to show that for every  $w$ - $(p, q, 0)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ ,  $\|(Tf)^*\|_{L_w^p} \leq C$ . In order to do this, one write

$$\begin{aligned} \|(Tf)^*\|_{L_w^p}^p &= \int_{|x-x_0| \leq 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx + \int_{|x-x_0| > 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx \\ &= L_1 + L_2. \end{aligned}$$

By Hölder inequality,  $L_w^q$ -boundedness of the maximal function and Lemma B, we get

$$\begin{aligned} L_1 &\leq \left( \int_{|x-x_0| \leq 4\sigma} \left( (Tf)^*(x) \right)^q w(x) dx \right)^{p/q} \left( \int_{|x-x_0| \leq 4\sigma} w(x) dx \right)^{1-p/q} \\ &\leq C \|f\|_{L_w^q}^p w(B(x_0, 4\sigma))^{1-p/q} \leq C. \end{aligned}$$

To estimate  $L_2$ , we first estimate  $(Tf)^*(x)$  for  $|x - x_0| > 4\sigma$ . For any  $t > 0$ , since  $\int_{\mathbb{R}^n} Tf(x) dx = 0$  by Lemma 3.1, we get

$$\begin{aligned} |Tf * \phi_t(x)| &= \left| \int_{\mathbb{R}^n} Tf(y) \frac{1}{t^n} \left( \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right) dy \right| \\ &\leq \frac{1}{t^n} \int_{|y-x_0| < 2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\ &\quad + \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \dots + \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \dots \\ &= E_1(t) + E_2(t) + E_3(t). \end{aligned}$$

As  $|x - x_0| > 4\sigma$ , by the mean value theorem, Lemma A and Lemma B, we get

$$\begin{aligned}
E_1(t) &= \frac{1}{t^n} \int_{|y-x_0|<2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\
&\leq \frac{1}{t^n} \int_{|y-x_0|<2\sigma} |Tf(y)| \frac{|y-x_0|}{t} \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0+\lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} \int_{|y-x_0|<2\sigma} |Tf(y)| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} |B(x_0, 2\sigma)| w(B(x_0, 2\sigma))^{-1/q} \|Tf\|_{L_w^q} \\
&\leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/q} \|f\|_{L_w^q} \leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/p}.
\end{aligned}$$

Similarly, we also get

$$\begin{aligned}
E_2(t) &\leq \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) (K(y, z) - K(y, x_0)) dz \right| \frac{|y-x_0|}{t} \\
&\quad \times \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0+\lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{1}{|x-x_0|^{n+1}} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} |y-x_0| \int_{|z-x_0|<\sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+1} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^2} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+1} \left(\frac{|x-x_0|}{2\sigma}\right)^{1-\delta} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^{1+\delta}} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Next, let us look at  $L_3$ . Similarly, we also have

$$\begin{aligned}
E_3(t) &\leq \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) \left( K(y, z) - K(y, x_0) \right) dz \right| \left( \left| \phi\left(\frac{y-x_0}{t}\right) \right| + 2 \left| \phi\left(\frac{x-x_0}{t}\right) \right| \right) dy \\
&\leq C \frac{1}{|x-x_0|^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \int_{|z-x_0| < \sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t^{1+\delta}} dt \left(\frac{2\sigma}{|x-x_0|}\right)^\delta w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Therefore, for all  $|x-x_0| > 4\sigma$ ,

$$(Tf)^*(x) = \sup_{t>0} (E_1(t) + E_2(t) + E_3(t)) \leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.$$

Combining this, Lemma C and Lemma B, we obtain that

$$\begin{aligned}
L_2 &= \int_{|x-x_0| > 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx \leq C \int_{|x-x_0| > 4\sigma} \frac{\sigma^{(n+\delta)p}}{|x-x_0|^{(n+\delta)p}} w(B)^{-1} w(x) dx \\
&\leq C w(B)^{-1} w(B(x_0, 4\sigma)) \leq C,
\end{aligned}$$

since  $(n+\delta)p > nq$ . This finishes the proof.  $\square$

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