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A NOTE ON H_w^p -BOUNDEDNESS OF RIESZ TRANSFORMS AND θ -CALDERÓN-ZYGMUND OPERATORS THROUGH MOLECULAR CHARACTERIZATION

LUONG DANG KY

ABSTRACT. Let $0 < p \leq 1$ and w in the Muckenhoupt class A_1 . Recently, by using the weighted atomic decomposition and molecular characterization; Lee, Lin and Yang [11] (J. Math. Anal. Appl. 301 (2005), 394–400) established that the Riesz transforms $R_j, j = 1, 2, \dots, n$, are bounded on $H_w^p(\mathbb{R}^n)$. In this note we extend this to the general case of weight w in the Muckenhoupt class A_∞ through molecular characterization. One difficulty, which has not been taken care in [11], consists in passing from atoms to all functions in $H_w^p(\mathbb{R}^n)$. Furthermore, the H_w^p -boundedness of θ -Calderón-Zygmund operators are also given through molecular characterization and atomic decomposition.

1. INTRODUCTION

Calderón-Zygmund operators and their generalizations on Euclidean space \mathbb{R}^n have been extensively studied, see for example [7, 14, 18, 15]. In particular, Yabuta [18] introduced certain θ -Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operator.

Definition 1.1. Let θ be a nonnegative nondecreasing function on $(0, \infty)$ satisfying $\int_0^1 \frac{\theta(t)}{t} dt < \infty$. A continuous function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$ is said to be a θ -Calderón-Zygmund singular integral kernel if there exists a constant $C > 0$ such that

$$|K(x, y)| \leq \frac{C}{|x - y|^n}$$

for all $x \neq y$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{1}{|x - y|^n} \theta\left(\frac{|x - x'|}{|x - y|}\right)$$

for all $2|x - x'| \leq |x - y|$.

A linear operator $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is said to be a θ -Calderón-Zygmund operator if T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$ and there exists

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a θ -Calderon-Zygmund singular integral kernel K such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and all $x \notin \text{supp } f$, we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy.$$

When $K_j(x, y) = \pi^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \frac{x_j - y_j}{|x - y|^{n+1}}$, $j = 1, 2, \dots, n$, then they are the classical Riesz transforms denoted by R_j .

It is well-known that the Riesz transforms R_j , $j = 1, 2, \dots, n$, are bounded on unweighted Hardy spaces $H^p(\mathbb{R}^n)$. There are many different approaches to prove this classical result (see [11, 9]). Recently, by using the weighted molecular theory (see [10]) and combined with García-Cuerva's atomic decomposition [5] for weighted Hardy spaces $H_w^p(\mathbb{R}^n)$, the authors in [11] established that the Riesz transforms R_j , $j = 1, 2, \dots, n$, are bounded on $H_w^p(\mathbb{R}^n)$. More precisely, they proved that $\|R_j f\|_{H_w^p} \leq C$ for every w -($p, \infty, ts - 1$)-atom where $s, t \in \mathbb{N}$ satisfy $n/(n+s) < p \leq n/(n+s-1)$ and $((s-1)r_w + n)/(s(r_w - 1))$ with r_w is the critical index of w for the reverse Hölder condition. Remark that this leaves a gap in the proof. Similar gaps exist in some literatures, for instance in [10, 15] when the authors establish H_w^p -boundedness of Calderón-Zygmund type operators. Indeed, it is now well-known that (see [1]) the argument "the operator T is uniformly bounded in $H_w^p(\mathbb{R}^n)$ on w -(p, ∞, r)-atoms, and hence it extends to a bounded operator on $H_w^p(\mathbb{R}^n)$ " is wrong in general. However, Meda, Sjögren and Vallarino [13] establishes that (in the setting of unweighted Hardy spaces) this is correct if one replaces L^∞ -atoms by L^q -atoms with $1 < q < \infty$. See also [19] for L^2 -atoms with a different method from [13]. Later, the authors in [2] extended these results to the weighted anisotropic Hardy spaces. More precisely, it is claimed in [2] that the operator T can be extended to a bounded operator on $H_w^p(\mathbb{R}^n)$ if it is uniformly bounded on w -(p, q, r)-atoms for $q_w < q < \infty, r \geq [n(q_w/p - 1)]$ where q_w is the critical index of w .

Motivated by [11, 10, 15, 1, 2], in this paper, we extend Theorem 1 in [11] to A_∞ weights (see Theorem 1.1); Theorem 4 in [10] (see Theorem 1.2), Theorem 3 in [15] (see Theorem 3.1) to θ -Calderón-Zygmund operators; and fill the gaps of the proofs by using the atomic decomposition and molecular characterization of $H_w^p(\mathbb{R}^n)$ as in [11].

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In \mathbb{R}^n , we denote by $B = B(x, r)$ an open ball with center x and radius $r > 0$. For any measurable set E , we denote by $|E|$ its Lebesgue measure, and by E^c the set $\mathbb{R}^n \setminus E$.

Let us first recall some notations, definitions and well-known results.

Let $1 \leq p < \infty$. A nonnegative locally integrable function w belongs to the Muckenhoupt class A_p , say $w \in A_p$, if there exists a positive constant C so

that

$$\frac{1}{|B|} \int_B w(x) dx \left(\frac{1}{|B|} \int_B (w(x))^{-1/(p-1)} dx \right)^{p-1} \leq C, \quad \text{if } 1 < p < \infty,$$

and

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess-inf}_{x \in B} w(x), \quad \text{if } p = 1,$$

for all balls B in \mathbb{R}^n . We say that $w \in A_\infty$ if $w \in A_p$ for some $p \in [1, \infty)$.

It is well known that $w \in A_p$, $1 \leq p < \infty$, implies $w \in A_q$ for all $q > p$. Also, if $w \in A_p$, $1 < p < \infty$, then $w \in A_q$ for some $q \in [1, p)$. We thus write $q_w := \inf\{p \geq 1 : w \in A_p\}$ to denote the critical index of w . For a measurable set E , we note $w(E) = \int_E w(x) dx$ its weighted measure.

The following lemma gives a characterization of the class A_p , $1 \leq p < \infty$. It can be found in [6].

Lemma A. The function $w \in A_p$, $1 \leq p < \infty$, if and only if, for all non-negative functions and all balls B ,

$$\left(\frac{1}{|B|} \int_B f(x) dx \right)^p \leq C \frac{1}{w(B)} \int_B f(x)^p w(x) dx.$$

A close relation to A_p is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B w^r(x) dx \right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B w(x) dx \right) \quad \text{for every ball } B \subset \mathbb{R}^n,$$

we say that w satisfies *reverse Hölder condition of order r* and write $w \in RH_r$. It is known that if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w := \sup\{r > 1 : w \in RH_r\}$ to denote the *critical index of w for the reverse Hölder condition*.

The following result provides us the comparison between the Lebesgue measure of a set E and its weighted measure $w(E)$. It also can be found in [6].

Lemma B. Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 \left(\frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left(\frac{|E|}{|B|} \right)^{(r-1)/r},$$

for all cubes B and measurable subsets $E \subset B$.

Given a weight function w on \mathbb{R}^n , as usual we denote by $L_w^q(\mathbb{R}^n)$ the space of all functions f satisfying $\|f\|_{L_w^q} := (\int_{\mathbb{R}^n} |f(x)|^q w(x) dx)^{1/q} < \infty$. When $q = \infty$, $L_w^\infty(\mathbb{R}^n)$ is $L^\infty(\mathbb{R}^n)$ and $\|f\|_{L_w^\infty} = \|f\|_{L^\infty}$. Analogously to the classical Hardy spaces, the *weighted Hardy spaces* $H_w^p(\mathbb{R}^n)$, $p > 0$, can be defined in terms of

maximal functions. Namely, let ϕ be a function in $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing smooth functions, satisfying $\int_{\mathbb{R}^n} \phi(x) dx = 1$. Define

$$\phi_t(x) = t^{-n} \phi(x/t), \quad t > 0, x \in \mathbb{R}^n,$$

and the maximal function f^* by

$$f^*(x) = \sup_{t>0} |f * \phi_t(x)|, \quad x \in \mathbb{R}^n.$$

Then $H_w^p(\mathbb{R}^n)$ consists of those tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $f^* \in L_w^p(\mathbb{R}^n)$ with the (quasi-)norm

$$\|f\|_{H_w^p} = \|f^*\|_{L_w^p}.$$

In order to show the H_w^p -boundedness of Riesz transforms, we characterize weighted Hardy spaces in terms of atoms and molecules in the following way.

Definition of a weighted atom. Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$ such that $w \in A_q$. Let q_w be the critical index of w . Set $[\cdot]$ the integer function. For $s \in \mathbb{N}$ satisfying $s \geq [n(q_w/p - 1)]$, a function $a \in L_w^q(\mathbb{R}^n)$ is called w -(p, q, s)-atom centered at x_0 if

- (i) $\text{supp } a \subset B$ for some ball B centered at x_0 ,
- (ii) $\|a\|_{L_w^q} \leq w(B)^{1/q-1/p}$,
- (iii) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq s$.

Let $H_w^{p,q,s}(\mathbb{R}^n)$ denote the space consisting of tempered distributions admitting a decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where a_j 's are w -(p, q, s)-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. And for every $f \in H_w^{p,q,s}(\mathbb{R}^n)$, we consider the (quasi-)norm

$$\|f\|_{H_w^{p,q,s}} = \inf \left\{ \left(\sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} : f \stackrel{\mathcal{S}'}{=} \sum_{j=1}^\infty \lambda_j a_j, \{a_j\}_{j=1}^\infty \text{ are } w\text{-(}p, q, s\text{)-atoms} \right\}.$$

Denote by $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$ the vector space of all finite linear combinations of w -(p, q, s)-atoms, and the (quasi-)norm of f in $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H_{w,\text{fin}}^{p,q,s}} := \inf \left\{ \left(\sum_{j=1}^k |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^k \lambda_j a_j, k \in \mathbb{N}, \{a_j\}_{j=1}^k \text{ are } w\text{-(}p, q, s\text{)-atoms} \right\}.$$

We have the following atomic decomposition for $H_w^p(\mathbb{R}^n)$. It can be found in [5] (see also [2, 8]).

Theorem A. If the triplet (p, q, s) satisfies the conditions of w -(p, q, s)-atoms, then $H_w^p(\mathbb{R}^n) = H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$ with equivalent norms.

The molecules corresponding to the atoms mentioned above can be defined as follows.

Definition of a weighted molecule. For $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$, let $w \in A_q$ with critical index q_w and critical index r_w for the reverse Hölder

condition. Set $s \geq [n(q_w/p - 1)]$, $\varepsilon > \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, 1/p - 1\}$, $a = 1 - 1/p + \varepsilon$, and $b = 1 - 1/q + \varepsilon$. A w -(p, q, s, ε)-molecule centered at x_0 is a function $M \in L_w^q(\mathbb{R}^n)$ satisfying

- (i) $M.w(B(x_0, \cdot - x_0))^b \in L_w^q(\mathbb{R}^n)$,
- (ii) $\|M\|_{L_w^q}^{a/b} \|M.w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^{1-a/b} \equiv \mathfrak{N}_w(M) < \infty$,
- (iii) $\int_{\mathbb{R}^n} M(x)x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq s$.

The above quantity $\mathfrak{N}_w(M)$ is called the w -molecular norm of M .

In [10], Lee and Lin proved that every weighted molecule belongs to the weighted Hardy space $H_w^p(\mathbb{R}^n)$, and the embedding is continuous.

Theorem B. Let $0 < p \leq 1 \leq q \leq \infty$ and $p \neq q$, $w \in A_q$, and (p, q, s, ε) be the quadruple in the definition of molecule. Then, every w -(p, q, s, ε)-molecule M centered at any point in \mathbb{R}^n is in $H_w^p(\mathbb{R}^n)$, and $\|M\|_{H_w^p} \leq C\mathfrak{N}_w(M)$ where the constant C is independent of the molecule.

Although, in general, one cannot conclude that an operator T is bounded on $H_w^p(\mathbb{R}^n)$ by checking that their norms have uniform bound on all of the corresponding w -(p, ∞, s)-atoms (cf. [1]). However, this is correct when dealing with w -(p, q, s)-atoms with $q_w < q < \infty$. Indeed, we have the following result (see [2, Theorem 7.2]).

Theorem C. Let $0 < p \leq 1$, $w \in A_\infty$, $q \in (q_w, \infty)$ and $s \in \mathbb{Z}$ satisfying $s \geq [n(q_w/p - 1)]$. Suppose that $T : H_{w, \text{fin}}^{p, q, s}(\mathbb{R}^n) \rightarrow H_w^p(\mathbb{R}^n)$ is a linear operator satisfying

$$\sup\{\|Ta\|_{H_w^p} : a \text{ is any } w\text{-(}p, q, s\text{)-atom}\} < \infty.$$

Then T can be extended to a bounded linear operator on $H_w^p(\mathbb{R}^n)$.

Our first main result, which generalizes *Theorem 1* in [11], is as follows:

Theorem 1.1. Let $0 < p \leq 1$ and $w \in A_\infty$. Then, the Riesz transforms are bounded on $H_w^p(\mathbb{R}^n)$.

For the next result, we need the notion $T^*1 = 0$.

Definition 1.2. Let T be a θ -Calderón-Zygmund operator. We say that $T^*1 = 0$ if $\int_{\mathbb{R}^n} Tf(x)dx = 0$ for all $f \in L^q(\mathbb{R}^n)$, $1 < q \leq \infty$, with compact support and $\int_{\mathbb{R}^n} f(x)dx = 0$.

We now can give the H_w^p -boundedness of θ -Calderón-Zygmund type operators, which generalizes *Theorem 4* in [10] by taking $q = 1$ and $\theta(t) = t^\delta$, as follows:

Theorem 1.2. Given $\delta \in (0, 1]$, $n/(n + \delta) < p \leq 1$, and $w \in A_q \cap RH_r$ with $1 \leq q < p(n + \delta)/n$, $(n + \delta)/(n + \delta - nq) < r$. Let θ be a nonnegative nondecreasing function on $(0, \infty)$ with $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$, and T be a θ -Calderón-Zygmund operator satisfying $T^*1 = 0$. Then T is bounded on $H_w^p(\mathbb{R}^n)$.

2. PROOF OF THEOREM 1.1

In order to prove the main theorems, we need the following lemma (see [6, page 412]).

Lemma C. Let $w \in A_r, r > 1$. Then there exists a constant $C > 0$ such that

$$\int_{B^c} \frac{1}{|x - x_0|^{nr}} w(x) dx \leq C \frac{1}{\sigma^{nr}} w(B)$$

for all balls $B = B(x_0, \sigma)$ in \mathbb{R}^n .

Proof of Theorem 1.1. For $q = 2(q_w + 1) \in (q_w, \infty)$, then $s := [n(q/p - 1)] \geq [n(q_w/p - 1)]$. We now choose (and fix) a positive number ε satisfying

$$(2.1) \quad \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < \varepsilon < t(s + 1)(nq)^{-1} + q^{-1} - 1,$$

for some $t \in \mathbb{N}, t \geq 1$ and $\max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < t(s + 1)(nq)^{-1} + q^{-1} - 1$.

Clearly, $\ell := t(s + 1) - 1 \geq s \geq [n(q_w/p - 1)]$. Hence, by Theorem B and Theorem C, it is sufficient to show that for every w -(p, q, ℓ)-atom f centered at x_0 and supported in ball $B = B(x_0, \sigma)$, the Riesz transforms $R_j f = K_j * f$, $j = 1, 2, \dots, n$, are w -(p, q, s, ε)-molecules with the norm $\mathfrak{N}_w(R_j f) \leq C$.

Indeed, as $w \in A_q$ by $q = 2(q_w + 1) \in (q_w, \infty)$. It follows from L_w^q -boundedness of Riesz transforms that

$$(2.2) \quad \|R_j f\|_{L_w^q} \leq \|R_j\|_{L_w^q \rightarrow L_w^q} \|f\|_{L_w^q} \leq C w(B)^{1/q-1/p}.$$

To estimate $\|R_j f \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$ where $b = 1 - 1/q + \varepsilon$, we write

$$\begin{aligned} \|R_j f \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &\quad + \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &= I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sqrt{n}\sigma))^{bq} \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(x) dx \\ &\leq C w(B)^{bq} \|R_j\|_{L_w^q \rightarrow L_w^q}^q \|f\|_{L_w^q}^q \leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II, as f is w -(p, q, ℓ)-atom, by the Taylor's fomular and Lemma A, we get

$$\begin{aligned} |K_j * f(x)| &= \left| \int_{|y-x_0| \leq \sigma} \left(K_j(x-y) - \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} D^\alpha K_j(x-x_0)(x_0-y)^\alpha \right) f(y) dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{\sigma^{\ell+1}}{|x-x_0|^{n+\ell+1}} |f(y)| dy \\ &\leq C \frac{\sigma^{n+\ell+1}}{|x-x_0|^{n+\ell+1}} w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all $x \in (B(x_0, 2\sqrt{n}\sigma))^c$. As $b = 1 - 1/q + \varepsilon$, it follows from (2.1) that $(n + \ell + 1)q - q^2nb > nq$. Therefore, by combining the above inequality, Lemma B and Lemma C, we obtain

$$\begin{aligned} II &= \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q} w(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q}} w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q - q^2nb} w(B)^{(b-1/p)q} \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q - q^2nb}} w(x) dx \\ &\leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

Thus,

$$(2.3) \quad \|R_j f \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q} = (I + II)^{1/q} \leq C w(B)^{b+1/q-1/p}.$$

Remark that $a = 1 - 1/p + \varepsilon$. Combining (2.2) and (2.3), we obtain

$$\mathfrak{N}_w(R_j f) \leq C w(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

The proof will be concluded if we establish the vanishing moment conditions of $R_j f$. One first consider the following lemma.

Lemma. For every classical atom $(p, 2, \ell)$ -atom g centered at x_0 , we have

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0 \quad \text{for } 0 \leq |\alpha| \leq s, 1 \leq j \leq n.$$

Proof of the Lemma. Since $b = 1 - 1/q + \varepsilon < (\ell + 1)(nq)^{-1} < (\ell + 1)n^{-1}$, we obtain $2(n + \ell + 1) - 2nb > n$. It is similar to the previous argument, we also obtain that $R_j g$ and $R_j g \cdot |\cdot - x_0|^{nb}$ belong to $L^2(\mathbb{R}^n)$. Now, we establish that $R_j g \cdot (\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$ for every multi-index α with $|\alpha| \leq s$. Indeed,

since $\varepsilon > q/p - 1$ by (2.1), implies that $2(s - nb) < (s - nb)q' < -n$ by $q = 2(q_w + 1) > 2$, where $1/q + 1/q' = 1$. We use Schwartz inequality to get

$$\begin{aligned} \int_{B(x_0, 1)^c} |R_j g(x)(x - x_0)^\alpha| dx &\leq \int_{B(x_0, 1)^c} |R_j g(x)| |x - x_0|^s dx \\ &\leq \left(\int_{B(x_0, 1)^c} |R_j g(x)|^2 |x - x_0|^{2nb} dx \right)^{1/2} \left(\int_{B(x_0, 1)^c} |x - x_0|^{2(s-nb)} dx \right)^{1/2} \\ &\leq C \|R_j g\| \cdot \|x - x_0\|^{nb} \|x - x_0\|^{2(s-nb)} < \infty, \end{aligned}$$

and

$$\int_{B(x_0, 1)} |R_j g(x)(x - x_0)^\alpha| dx \leq |B(x_0, 1)|^{1/2} \left(\int_{B(x_0, 1)} |R_j g(x)|^2 dx \right)^{1/2} < \infty.$$

Thus, $R_j g(\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$ for any $|\alpha| \leq s$. Deduce that $R_j g(x) x^\alpha \in L^1(\mathbb{R}^n)$ for any $|\alpha| \leq s$. Therefore,

$$(R_j g(x) x^\alpha)^\wedge(\xi) = C_\alpha \cdot \widehat{D^\alpha(R_j g)}(\xi)$$

is continuous, with $|C_\alpha| \leq C_s$ (C_s depends only on s) for any $|\alpha| \leq s$, where \hat{h} is used to denote the fourier transform of h . Consequently,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = C_\alpha \cdot \widehat{D^\alpha(R_j g)}(0) = C_\alpha \cdot D^\alpha(m_j \hat{g})(0),$$

where $m_j(x) = -ix_j/|x|$. Moreover, since g is a classical $(p, 2, \ell)$ -atom, it follows from [17, Lemma 9.1] that \hat{g} is ℓ th order differentiable and $\hat{g}(\xi) = O(|\xi|^{\ell+1})$ as $\xi \rightarrow 0$. We write e_j to be the j th standard basis vector of \mathbb{R}^n , $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index of nonnegative integers α_j , $\Delta_{he_j} \phi(x) = \phi(x) - \phi(x - he_j)$, $\Delta_{he_j}^{\alpha_j} \phi(x) = \Delta_{he_j}^{\alpha_j-1} \phi(x) - \Delta_{he_j}^{\alpha_j-1} \phi(x - he_j)$ for $\alpha_j \geq 2$, $\Delta_{he_j}^0 \phi(x) = \phi(x)$, and $\Delta_h^\alpha = \Delta_{he_1}^{\alpha_1} \dots \Delta_{he_n}^{\alpha_n}$. Then, the boundedness of m_j , and $|C_\alpha| \leq C_s$ for $|\alpha| \leq s$, implies

$$\begin{aligned} \left| \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx \right| &= |C_\alpha| \left| \lim_{h \rightarrow 0} |h|^{-|\alpha|} \Delta_h^\alpha(m_j \hat{g})(0) \right| \\ &\leq C \lim_{h \rightarrow 0} |h|^{\ell+1-|\alpha|} = 0, \end{aligned}$$

for $|\alpha| \leq s$ by $s \leq \ell$. Thus, for any $j = 1, 2, \dots, n$, and $|\alpha| \leq s$,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0.$$

This complete the proof of the lemma.

Let us come back to the proof of Theorem 1.1. As $q/2 = q_w + 1 > q_w$, by Lemma A,

$$\left(\frac{1}{|B|} \int_B |f(x)|^2 dx \right)^{q/2} \leq C \frac{1}{w(B)} \int_B |f(x)|^q w(x) dx.$$

Therefore, $g := C^{-1/q} |B|^{-1/p} w(B)^{1/p} f$ is a classical $(p, 2, \ell)$ -atom since f is w -(p, q, ℓ)-atom associated with ball B . Consequently, by the above lemma,

$$\int_{\mathbb{R}^n} R_j f(x) x^\alpha dx = C^{1/q} |B|^{1/p} w(B)^{-1/p} \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0$$

for all $j = 1, 2, \dots, n$ and $|\alpha| \leq s$. Thus, the theorem is proved. \square

Following a similar but easier argument, we also have the following H_w^p -boundedness of Hilbert transform. We leave details to readers.

Theorem 2.1. *Let $0 < p \leq 1$ and $w \in A_\infty$. Then, the Hilbert transform is bounded on $H_w^p(\mathbb{R})$.*

3. PROOF OF THEOREM 1.2

We first consider the following lemma

Lemma 3.1. *Let $p \in (0, 1]$, $w \in A_q$, $1 < q < \infty$, and T be a θ -Calderón-Zygmund operator satisfying $T^*1 = 0$. Then, $\int_{\mathbb{R}^n} T f(x) dx = 0$ for all w -($p, q, 0$)-atoms f .*

Proof of Lemma 3.1. Let f be an arbitrary w -($p, q, 0$)-atom associated with ball B . It is well-known that there exists $1 < r < q$ such that $w \in A_r$. Therefore, it follows from Lemma A that

$$\int_B |f(x)|^{q/r} dx \leq C |B| w(B)^{1/r} \|f\|_{L_w^q}^{q/r} < \infty.$$

We deduce that f is a multiple of classical $(p, q/r, 0)$ -atom, and thus the condition $T^*1 = 0$ implies $\int_{\mathbb{R}^n} T f(x) dx = 0$. \square

Proof of Theorem 1.2. Because of the hypothesis, without loss of generality we can assume $q > 1$. Furthermore, it is clear that $[n(q_w/p - 1)] = 0$, and there exists a positive constant ε such that

$$(3.1) \quad \max \left\{ \frac{1}{r_w - 1}, \frac{1}{p} - 1 \right\} < \varepsilon < \frac{n + \delta}{nq} - 1.$$

Similarly to the arguments in Theorem 1.1, it is sufficient to show that, for every w -($p, q, 0$)-atom f centered at x_0 and supported in ball $B = B(x_0, \sigma)$,

Tf is a w -($p, q, 0, \varepsilon$)-molecule with the norm $\mathfrak{N}_w(Tf) \leq C$. One first observe that $\int_{\mathbb{R}^n} Tf(x)dx = 0$ by Lemma 3.1, and

$$\sum_{k=0}^{\infty} \theta(2^{-k})2^{knbq} < \infty,$$

where $b = 1 - 1/q + \varepsilon$, by $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$ and (3.1). We deduce that

$$(3.2) \quad \sum_{k=0}^{\infty} \left(\theta(2^{-k})2^{knbq} \right)^q < \infty.$$

As $w \in A_q$, $1 < q < \infty$, it follows from [18, Theorem 2.4] that

$$(3.3) \quad \|Tf\|_{L_w^q} \leq C\|f\|_{L_w^q} \leq Cw(B)^{1/q-1/p}.$$

To estimate $\|Tf.w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$, we write

$$\begin{aligned} \|Tf.w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &+ \int_{|x-x_0| > 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx = I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sigma))^{bq} \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(x) dx \\ &\leq Cw(B)^{bq} \|f\|_{L_w^q}^q \leq Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II , since f is of mean zero, by Lemma A, we have

$$\begin{aligned} |Tf(x)| &= \left| \int_{|y-x_0| \leq \sigma} (K(x, y) - K(x, x_0))f(y)dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{1}{|x - x_0|^n} \theta\left(\frac{|y - x_0|}{|x - x_0|}\right) |f(y)| dy \\ &\leq C \frac{\sigma^n}{|x - x_0|^n} \theta\left(\frac{\sigma}{|x - x_0|}\right) w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all $x \in (B(x_0, 2\sigma))^c$. Therefore, by combining the above inequality, Lemma B and (3.2), we obtain

$$\begin{aligned}
II &= \int_{|x-x_0|>2\sigma} |Tf(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
&\leq Cw(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0|>2\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left(\theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
&\leq Cw(B)^{-q/p} \sum_{k=1}^{\infty} \int_{2^k\sigma < |x-x_0| \leq 2^{k+1}\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left(\theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
&\leq Cw(B)^{(b+1/q-1/p)q} \sum_{k=0}^{\infty} \left(\theta(2^{-k}) 2^{knbq} \right)^q \leq Cw(B)^{(b+1/q-1/p)q}.
\end{aligned}$$

Thus,

$$(3.4) \quad \|Tf \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q} = (I + II)^{1/q} \leq Cw(B)^{b+1/q-1/p}.$$

Remark that $a = 1 - 1/p + \varepsilon$. Combining (3.3) and (3.4), we obtain

$$\mathfrak{N}_w(Tf) \leq Cw(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

This finishes the proof. \square

It is well-known that the molecular theory of (unweighted) Hardy spaces of Taibleson and Weiss [17] is one of useful tools to establish boundedness of operators in Hardy spaces (cf. [17, 12]). In the setting of Muckenhoupt weight, this theory has been considered by the authors in [10], since then, they have been well used to establish boundedness of operators in weighted Hardy spaces (cf. [10, 11, 3]). However in some cases, the *weighted molecular characterization*, which obtained in [10], does not give the best possible results. For Calderón-Zygmund type operators in Theorem 1.2, for instance, it involves assumption on the *critical index of w for the reverse Hölder condition* as the following theorem does not.

Theorem 3.1. *Given $\delta \in (0, 1]$, $n/(n + \delta) < p \leq 1$, and $w \in A_q$ with $1 \leq q < p(n + \delta)/n$. Let θ be a nonnegative nondecreasing function on $(0, \infty)$ with $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$, and T be a θ -Calderón-Zygmund operator satisfying $T^*1 = 0$. Then T is bounded on $H_w^p(\mathbb{R}^n)$.*

The following corollary give the boundedness of the classical Calderón-Zygmund type operators on weighted Hardy spaces (see [15, Theorem 3]).

Corollary 3.1. *Let $0 < \delta \leq 1$ and T be the classical δ -Calderón-Zygmund operator, i.e. $\theta(t) = t^\delta$, satisfying $T^*1 = 0$. If $n/(n + \delta) < p \leq 1$ and $w \in A_q$ with $1 \leq q < p(n + \delta)/n$, then T is bounded on $H_w^p(\mathbb{R}^n)$.*

Proof of Corollary 3.1. By taking $\delta' \in (0, \delta)$ which is close enough δ . Then, we apply Theorem 3.1 with δ' instead of δ . \square

Proof of Theorem 3.1. Without loss of generality we can assume $1 < q < p(n + \delta)/n$. Fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$. By Theorem C, it is sufficient to show that for every w -($p, q, 0$)-atom f centered at x_0 and supported in ball $B = B(x_0, \sigma)$, $\|(Tf)^*\|_{L_w^p} \leq C$. In order to do this, one write

$$\begin{aligned} \|(Tf)^*\|_{L_w^p}^p &= \int_{|x-x_0| \leq 4\sigma} \left((Tf)^*(x)\right)^p w(x) dx + \int_{|x-x_0| > 4\sigma} \left((Tf)^*(x)\right)^p w(x) dx \\ &= L_1 + L_2. \end{aligned}$$

By Hölder inequality, L_w^q -boundedness of the maximal function and Lemma B, we get

$$\begin{aligned} L_1 &\leq \left(\int_{|x-x_0| \leq 4\sigma} \left((Tf)^*(x)\right)^q w(x) dx \right)^{p/q} \left(\int_{|x-x_0| \leq 4\sigma} w(x) dx \right)^{1-p/q} \\ &\leq C \|f\|_{L_w^q}^p w(B(x_0, 4\sigma))^{1-p/q} \leq C. \end{aligned}$$

To estimate L_2 , we first estimate $(Tf)^*(x)$ for $|x - x_0| > 4\sigma$. For any $t > 0$, since $\int_{\mathbb{R}^n} Tf(x) dx = 0$ by Lemma 3.1, we get

$$\begin{aligned} |Tf * \phi_t(x)| &= \left| \int_{\mathbb{R}^n} Tf(y) \frac{1}{t^n} \left(\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right) dy \right| \\ &\leq \frac{1}{t^n} \int_{|y-x_0| < 2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\ &\quad + \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \dots + \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \dots \\ &= E_1(t) + E_2(t) + E_3(t). \end{aligned}$$

As $|x - x_0| > 4\sigma$, by the mean value theorem, Lemma A and Lemma B, we get

$$\begin{aligned}
E_1(t) &= \frac{1}{t^n} \int_{|y-x_0| < 2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\
&\leq \frac{1}{t^n} \int_{|y-x_0| < 2\sigma} |Tf(y)| \frac{|y-x_0|}{t} \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0 + \lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} \int_{|y-x_0| < 2\sigma} |Tf(y)| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} |B(x_0, 2\sigma)| w(B(x_0, 2\sigma))^{-1/q} \|Tf\|_{L_w^q} \\
&\leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/q} \|f\|_{L_w^q} \leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/p}.
\end{aligned}$$

Similarly, we also get

$$\begin{aligned}
E_2(t) &\leq \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) \left(K(y, z) - K(y, x_0) \right) dz \right| \frac{|y-x_0|}{t} \\
&\quad \times \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0 + \lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{1}{|x-x_0|^{n+1}} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} |y-x_0| \int_{|z-x_0| < \sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^{n+1} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^2} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^{n+1} \left(\frac{|x-x_0|}{2\sigma} \right)^{1-\delta} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^{1+\delta}} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Next, let us look at L_3 . Similarly, we also have

$$\begin{aligned}
E_3(t) &\leq \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) \left(K(y, z) - K(y, x_0) \right) dz \right| \left(\left| \phi\left(\frac{y-x_0}{t}\right) \right| + 2 \left| \phi\left(\frac{x-x_0}{t}\right) \right| \right) dy \\
&\leq C \frac{1}{|x-x_0|^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \int_{|z-x_0| < \sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t^{1+\delta}} dt \left(\frac{2\sigma}{|x-x_0|} \right)^\delta w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|} \right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Therefore, for all $|x-x_0| > 4\sigma$,

$$(Tf)^*(x) = \sup_{t>0} (E_1(t) + E_2(t) + E_3(t)) \leq C \left(\frac{\sigma}{|x-x_0|} \right)^{n+\delta} w(B)^{-1/p}.$$

Combining this, Lemma C and Lemma B, we obtain that

$$\begin{aligned}
L_2 = \int_{|x-x_0| > 4\sigma} \left((Tf)^*(x) \right)^p w(x) dx &\leq C \int_{|x-x_0| > 4\sigma} \frac{\sigma^{(n+\delta)p}}{|x-x_0|^{(n+\delta)p}} w(B)^{-1} w(x) dx \\
&\leq C w(B)^{-1} w(B(x_0, 4\sigma)) \leq C,
\end{aligned}$$

since $(n+\delta)p > nq$. This finishes the proof. \square

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