



**HAL**  
open science

# A note on $H^p_w$ -boundedness of Riesz transforms and $\theta$ -Calderón-Zygmund operators through molecular characterization

Luong Dang Ky

► **To cite this version:**

Luong Dang Ky. A note on  $H^p_w$ -boundedness of Riesz transforms and  $\theta$ -Calderón-Zygmund operators through molecular characterization. 2011. hal-00602760v2

**HAL Id: hal-00602760**

**<https://hal.science/hal-00602760v2>**

Preprint submitted on 21 Oct 2011 (v2), last revised 14 Jan 2012 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A NOTE ON $H_w^p$ -BOUNDEDNESS OF RIESZ TRANSFORMS AND $\theta$ -CALDERÓN-ZYGMUND OPERATORS THROUGH MOLECULAR CHARACTERIZATION

LUONG DANG KY

ABSTRACT. Let  $0 < p \leq 1$  and  $w$  in the Muckenhoupt class  $A_1$ . Recently, by using the weighted atomic decomposition and molecular characterization; Lee, Lin and Yang [11] (J. Math. Anal. Appl. 301 (2005), 394–400) established that the Riesz transforms  $R_j, j = 1, 2, \dots, n$ , are bounded on  $H_w^p(\mathbb{R}^n)$ . In this note we extend this to the general case of weight  $w$  in the Muckenhoupt class  $A_\infty$  through molecular characterization. One difficulty, which has not been taken care in [11], consists in passing from atoms to all functions in  $H_w^p(\mathbb{R}^n)$ . Furthermore, the  $H_w^p$ -boundedness of  $\theta$ -Calderón-Zygmund operators are also given through molecular characterization and atomic decomposition.

## 1. INTRODUCTION

Calderón-Zygmund operators and their generalizations on Euclidean space  $\mathbb{R}^n$  have been extensively studied, see for example [7, 14, 18, 15]. In particular, Yabuta [18] introduced certain  $\theta$ -Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operator.

**Definition 1.1.** Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  satisfying  $\int_0^1 \frac{\theta(t)}{t} dt < \infty$ . A continuous function  $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\} \rightarrow \mathbb{C}$  is said to be a  $\theta$ -Calderón-Zygmund singular integral kernel if there exists a constant  $C > 0$  such that

$$|K(x, y)| \leq \frac{C}{|x - y|^n}$$

for all  $x \neq y$ ,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{1}{|x - y|^n} \theta\left(\frac{|x - x'|}{|x - y|}\right)$$

for all  $2|x - x'| \leq |x - y|$ .

A linear operator  $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$  is said to be a  $\theta$ -Calderón-Zygmund operator if  $T$  can be extended to a bounded operator on  $L^2(\mathbb{R}^n)$  and there exists

---

2010 *Mathematics Subject Classification.* 42B20, 42B25, 42B30.

*Key words and phrases.* Muckenhoupt weights, weighted Hardy spaces, atomic decomposition, molecular characterization, Riesz transforms, Calderón-Zygmund operators.

a  $\theta$ -Calderon-Zygmund singular integral kernel  $K$  such that for all  $f \in C_c^\infty(\mathbb{R}^n)$  and all  $x \notin \text{supp } f$ , we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy.$$

When  $K_j(x, y) = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)\frac{x_j-y_j}{|x-y|^{n+1}}$ ,  $j = 1, 2, \dots, n$ , then they are the classical Riesz transforms denoted by  $R_j$ .

It is well-known that the Riesz transforms  $R_j, j = 1, 2, \dots, n$ , are bounded on unweighted Hardy spaces  $H^p(\mathbb{R}^n)$ . There are many different approaches to prove this classical result (see [11, 9]). Recently, by using the weighted molecular theory (see [10]) and combined with García-Cuerva's atomic decomposition [5] for weighted Hardy spaces  $H_w^p(\mathbb{R}^n)$ , the authors in [11] established that the Riesz transforms  $R_j, j = 1, 2, \dots, n$ , are bounded on  $H_w^p(\mathbb{R}^n)$ . More precisely, they proved that  $\|R_j f\|_{H_w^p} \leq C$  for every  $w$ - $(p, \infty, ts-1)$ -atom where  $s, t \in \mathbb{N}$  satisfy  $n/(n+s) < p \leq n/(n+s-1)$  and  $((s-1)r_w+n)/(s(r_w-1))$  with  $r_w$  is the critical index of  $w$  for the reverse Hölder condition. Remark that this leaves a gap in the proof. Similar gaps exist in some literatures, for instance in [10, 15] when the authors establish  $H_w^p$ -boundedness of Calderón-Zygmund type operators. Indeed, it is now well-known that (see [1]) the argument "the operator  $T$  is uniformly bounded in  $H_w^p(\mathbb{R}^n)$  on  $w$ - $(p, \infty, r)$ -atoms, and hence it extends to a bounded operator on  $H_w^p(\mathbb{R}^n)$ " is wrong in general. However, Meda, Sjögren and Vallarino [13] establishes that (in the setting of unweighted Hardy spaces) this is correct if one replaces  $L^\infty$ -atoms by  $L^q$ -atoms with  $1 < q < \infty$ . See also [19] for  $L^2$ -atoms with a different method from [13]. Later, the authors in [2] extended these results to the weighted anisotropic Hardy spaces. More precisely, it is claimed in [2] that the operator  $T$  can be extended to a bounded operator on  $H_w^p(\mathbb{R}^n)$  if it is uniformly bounded on  $w$ - $(p, q, r)$ -atoms for  $q_w < q < \infty, r \geq [n(q_w/p - 1)]$  where  $q_w$  is the critical index of  $w$ .

Motivated by [11, 10, 15, 1, 2], in this paper, we extend Theorem 1 in [11] to  $A_\infty$  weights (see Theorem 1.1); Theorem 4 in [10] (see Theorem 1.2), Theorem 3 in [15] (see Theorem 3.1) to  $\theta$ -Calderón-Zygmund operators; and fill the gaps of the proofs by using the atomic decomposition and molecular characterization of  $H_w^p(\mathbb{R}^n)$  as in [11].

Throughout the whole paper,  $C$  denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. In  $\mathbb{R}^n$ , we denote by  $B = B(x, r)$  an open ball with center  $x$  and radius  $r > 0$ . For any measurable set  $E$ , we denote by  $|E|$  its Lebesgue measure, and by  $E^c$  the set  $\mathbb{R}^n \setminus E$ .

Let us first recall some notations, definitions and well-known results.

Let  $1 \leq p < \infty$ . A nonnegative locally integrable function  $w$  belongs to the Muckenhoupt class  $A_p$ , say  $w \in A_p$ , if there exists a positive constant  $C$  so

that

$$\frac{1}{|B|} \int_B w(x) dx \left( \frac{1}{|B|} \int_B (w(x))^{-1/(p-1)} dx \right)^{p-1} \leq C, \quad \text{if } 1 < p < \infty,$$

and

$$\frac{1}{|B|} \int_B w(x) dx \leq C \operatorname{ess-inf}_{x \in B} w(x), \quad \text{if } p = 1,$$

for all balls  $B$  in  $\mathbb{R}^n$ . We say that  $w \in A_\infty$  if  $w \in A_p$  for some  $p \in [1, \infty)$ .

It is well known that  $w \in A_p$ ,  $1 \leq p < \infty$ , implies  $w \in A_q$  for all  $q > p$ . Also, if  $w \in A_p$ ,  $1 < p < \infty$ , then  $w \in A_q$  for some  $q \in [1, p)$ . We thus write  $q_w := \inf\{p \geq 1 : w \in A_p\}$  to denote the critical index of  $w$ . For a measurable set  $E$ , we note  $w(E) = \int_E w(x) dx$  its weighted measure.

The following lemma gives a characterization of the class  $A_p$ ,  $1 \leq p < \infty$ . It can be found in [6].

**Lemma A.** The function  $w \in A_p$ ,  $1 \leq p < \infty$ , if and only if, for all non-negative functions and all balls  $B$ ,

$$\left( \frac{1}{|B|} \int_B f(x) dx \right)^p \leq C \frac{1}{w(B)} \int_B f(x)^p w(x) dx.$$

A close relation to  $A_p$  is the reverse Hölder condition. If there exist  $r > 1$  and a fixed constant  $C > 0$  such that

$$\left( \frac{1}{|B|} \int_B w^r(x) dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) dx \right) \quad \text{for every ball } B \subset \mathbb{R}^n,$$

we say that  $w$  satisfies *reverse Hölder condition of order  $r$*  and write  $w \in RH_r$ . It is known that if  $w \in RH_r$ ,  $r > 1$ , then  $w \in RH_{r+\varepsilon}$  for some  $\varepsilon > 0$ . We thus write  $r_w := \sup\{r > 1 : w \in RH_r\}$  to denote the *critical index of  $w$  for the reverse Hölder condition*.

The following result provides us the comparison between the Lebesgue measure of a set  $E$  and its weighted measure  $w(E)$ . It also can be found in [6].

**Lemma B.** Let  $w \in A_p \cap RH_r$ ,  $p \geq 1$  and  $r > 1$ . Then there exist constants  $C_1, C_2 > 0$  such that

$$C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r},$$

for all cubes  $B$  and measurable subsets  $E \subset B$ .

Given a weight function  $w$  on  $\mathbb{R}^n$ , as usual we denote by  $L_w^q(\mathbb{R}^n)$  the space of all functions  $f$  satisfying  $\|f\|_{L_w^q} := \left( \int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right)^{1/q} < \infty$ . When  $q = \infty$ ,  $L_w^\infty(\mathbb{R}^n)$  is  $L^\infty(\mathbb{R}^n)$  and  $\|f\|_{L_w^\infty} = \|f\|_{L^\infty}$ . Analogously to the classical Hardy spaces, the *weighted Hardy spaces*  $H_w^p(\mathbb{R}^n)$ ,  $p > 0$ , can be defined in terms of

maximal functions. Namely, let  $\phi$  be a function in  $\mathcal{S}(\mathbb{R}^n)$ , the Schwartz space of rapidly decreasing smooth functions, satisfying  $\int_{\mathbb{R}^n} \phi(x) dx = 1$ . Define

$$\phi_t(x) = t^{-n} \phi(x/t), \quad t > 0, x \in \mathbb{R}^n,$$

and the maximal function  $f^*$  by

$$f^*(x) = \sup_{t>0} |f * \phi_t(x)|, \quad x \in \mathbb{R}^n.$$

Then  $H_w^p(\mathbb{R}^n)$  consists of those tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^n)$  for which  $f^* \in L_w^p(\mathbb{R}^n)$  with the (quasi-)norm

$$\|f\|_{H_w^p} = \|f^*\|_{L_w^p}.$$

In order to show the  $H_w^p$ -boundedness of Riesz transforms, we characterize weighted Hardy spaces in terms of atoms and molecules in the following way.

**Definition of a weighted atom.** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$  such that  $w \in A_q$ . Let  $q_w$  be the critical index of  $w$ . Set  $[\cdot]$  the integer function. For  $s \in \mathbb{N}$  satisfying  $s \geq [n(q_w/p - 1)]$ , a function  $a \in L_w^q(\mathbb{R}^n)$  is called  $w$ - $(p, q, s)$ -atom centered at  $x_0$  if

- (i)  $\text{supp } a \subset B$  for some ball  $B$  centered at  $x_0$ ,
- (ii)  $\|a\|_{L_w^q} \leq w(B)^{1/q-1/p}$ ,
- (iii)  $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ .

Let  $H_w^{p,q,s}(\mathbb{R}^n)$  denote the space consisting of tempered distributions admitting a decomposition  $f = \sum_{j=1}^{\infty} \lambda_j a_j$  in  $\mathcal{S}'(\mathbb{R}^n)$ , where  $a_j$ 's are  $w$ - $(p, q, s)$ -atoms and  $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ . And for every  $f \in H_w^{p,q,s}(\mathbb{R}^n)$ , we consider the (quasi-)norm

$$\|f\|_{H_w^{p,q,s}} = \inf \left\{ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} : f \stackrel{\mathcal{S}'}{=} \sum_{j=1}^{\infty} \lambda_j a_j, \{a_j\}_{j=1}^{\infty} \text{ are } w\text{-}(p, q, s)\text{-atoms} \right\}.$$

Denote by  $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$  the vector space of all finite linear combinations of  $w$ - $(p, q, s)$ -atoms, and the (quasi-)norm of  $f$  in  $H_{w,\text{fin}}^{p,q,s}(\mathbb{R}^n)$  is defined by

$$\|f\|_{H_{w,\text{fin}}^{p,q,s}} := \inf \left\{ \left( \sum_{j=1}^k |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^k \lambda_j a_j, k \in \mathbb{N}, \{a_j\}_{j=1}^k \text{ are } w\text{-}(p, q, s)\text{-atoms} \right\}.$$

We have the following atomic decomposition for  $H_w^p(\mathbb{R}^n)$ . It can be found in [5] (see also [2, 8]).

**Theorem A.** If the triplet  $(p, q, s)$  satisfies the conditions of  $w$ - $(p, q, s)$ -atoms, then  $H_w^p(\mathbb{R}^n) = H_w^{p,q,s}(\mathbb{R}^n)$  with equivalent norms.

The molecules corresponding to the atoms mentioned above can be defined as follows.

**Definition of a weighted molecule.** For  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$ , let  $w \in A_q$  with critical index  $q_w$  and critical index  $r_w$  for the reverse Hölder

condition. Set  $s \geq [n(q_w/p - 1)]$ ,  $\varepsilon > \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, 1/p - 1\}$ ,  $a = 1 - 1/p + \varepsilon$ , and  $b = 1 - 1/q + \varepsilon$ . A  $w$ - $(p, q, s, \varepsilon)$ -molecule centered at  $x_0$  is a function  $M \in L_w^q(\mathbb{R}^n)$  satisfying

- (i)  $M.w(B(x_0, \cdot - x_0))^b \in L_w^q(\mathbb{R}^n)$ ,
- (ii)  $\|M\|_{L_w^q}^{a/b} \|M.w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^{1-a/b} \equiv \mathfrak{N}_w(M) < \infty$ ,
- (iii)  $\int_{\mathbb{R}^n} M(x)x^\alpha dx = 0$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ .

The above quantity  $\mathfrak{N}_w(M)$  is called the  $w$ -molecular norm of  $M$ .

In [10], Lee and Lin proved that every weighted molecule belongs to the weighted Hardy space  $H_w^p(\mathbb{R}^n)$ , and the embedding is continuous.

**Theorem B.** Let  $0 < p \leq 1 \leq q \leq \infty$  and  $p \neq q$ ,  $w \in A_q$ , and  $(p, q, s, \varepsilon)$  be the quadruple in the definition of molecule. Then, every  $w$ - $(p, q, s, \varepsilon)$ -molecule  $M$  centered at any point in  $\mathbb{R}^n$  is in  $H_w^p(\mathbb{R}^n)$ , and  $\|M\|_{H_w^p} \leq C\mathfrak{N}_w(M)$  where the constant  $C$  is independent of the molecule.

Although, in general, one cannot conclude that an operator  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$  by checking that their norms have uniform bound on all of the corresponding  $w$ - $(p, \infty, s)$ -atoms (cf. [1]). However, this is correct when dealing with  $w$ - $(p, q, s)$ -atoms with  $q_w < q < \infty$ . Indeed, we have the following result (see [2, Theorem 7.2]).

**Theorem C.** Let  $0 < p \leq 1$ ,  $w \in A_\infty$ ,  $q \in (q_w, \infty)$  and  $s \in \mathbb{Z}$  satisfying  $s \geq [n(q_w/p - 1)]$ . Suppose that  $T : H_{w, \text{fin}}^{p, q, s}(\mathbb{R}^n) \rightarrow H_w^p(\mathbb{R}^n)$  is a linear operator satisfying

$$\sup\{\|Ta\|_{H_w^p} : a \text{ is any } w\text{-}(p, q, s)\text{-atom}\} < \infty.$$

Then  $T$  can be extended to a bounded linear operator on  $H_w^p(\mathbb{R}^n)$ .

Our first main result, which generalizes *Theorem 1* in [11], is as follows:

**Theorem 1.1.** Let  $0 < p \leq 1$  and  $w \in A_\infty$ . Then, the Riesz transforms are bounded on  $H_w^p(\mathbb{R}^n)$ .

For the next result, we need the notion  $T^*1 = 0$ .

**Definition 1.2.** Let  $T$  be a  $\theta$ -Calderón-Zygmund operator. We say that  $T^*1 = 0$  if  $\int_{\mathbb{R}^n} Tf(x)dx = 0$  for all  $f \in L^q(\mathbb{R}^n)$ ,  $1 < q \leq \infty$ , with compact support and  $\int_{\mathbb{R}^n} f(x)dx = 0$ .

We now can give the  $H_w^p$ -boundedness of  $\theta$ -Calderón-Zygmund type operators, which generalizes *Theorem 4* in [10] by taking  $q = 1$  and  $\theta(t) = t^\delta$ , as follows:

**Theorem 1.2.** Given  $\delta \in (0, 1]$ ,  $n/(n + \delta) < p \leq 1$ , and  $w \in A_q \cap RH_r$  with  $1 \leq q < p(n + \delta)/n$ ,  $(n + \delta)/(n + \delta - nq) < r$ . Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  with  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .

## 2. PROOF OF THEOREM 1.1

In order to prove the main theorems, we need the following lemma (see [6, page 412]).

**Lemma C.** Let  $w \in A_r, r > 1$ . Then there exists a constant  $C > 0$  such that

$$\int_{B^c} \frac{1}{|x - x_0|^{nr}} w(x) dx \leq C \frac{1}{\sigma^{nr}} w(B)$$

for all balls  $B = B(x_0, \sigma)$  in  $\mathbb{R}^n$ .

*Proof of Theorem 1.1.* For  $q = 2(q_w + 1) \in (q_w, \infty)$ , then  $s := [n(q/p - 1)] \geq [n(q_w/p - 1)]$ . We now choose (and fix) a positive number  $\varepsilon$  satisfying

$$(2.1) \quad \max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < \varepsilon < t(s + 1)(nq)^{-1} + q^{-1} - 1,$$

for some  $t \in \mathbb{N}, t \geq 1$  and  $\max\{sr_w(r_w - 1)^{-1}n^{-1} + (r_w - 1)^{-1}, q/p - 1\} < t(s + 1)(nq)^{-1} + q^{-1} - 1$ .

Clearly,  $\ell := t(s + 1) - 1 \geq s \geq [n(q_w/p - 1)]$ . Hence, by Theorem B and Theorem C, it is sufficient to show that for every  $w$ - $(p, q, \ell)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ , the Riesz transforms  $R_j f = K_j * f$ ,  $j = 1, 2, \dots, n$ , are  $w$ - $(p, q, s, \varepsilon)$ -molecules with the norm  $\mathfrak{R}_w(R_j f) \leq C$ .

Indeed, as  $w \in A_q$  by  $q = 2(q_w + 1) \in (q_w, \infty)$ . It follows from  $L_w^q$ -boundedness of Riesz transforms that

$$(2.2) \quad \|R_j f\|_{L_w^q} \leq \|R_j\|_{L_w^q \rightarrow L_w^q} \|f\|_{L_w^q} \leq C w(B)^{1/q-1/p}.$$

To estimate  $\|R_j f \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$  where  $b = 1 - 1/q + \varepsilon$ , we write

$$\begin{aligned} \|R_j f \cdot w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &\quad + \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &= I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sqrt{n}\sigma))^{bq} \int_{|x-x_0| \leq 2\sqrt{n}\sigma} |R_j f(x)|^q w(x) dx \\ &\leq C w(B)^{bq} \|R_j\|_{L_w^q \rightarrow L_w^q}^q \|f\|_{L_w^q}^q \leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate II, as  $f$  is  $w$ -( $p, q, \ell$ )-atom, by the Taylor's fomular and Lemma A, we get

$$\begin{aligned} |K_j * f(x)| &= \left| \int_{|y-x_0| \leq \sigma} \left( K_j(x-y) - \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} D^\alpha K_j(x-x_0)(x_0-y)^\alpha \right) f(y) dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{\sigma^{\ell+1}}{|x-x_0|^{n+\ell+1}} |f(y)| dy \\ &\leq C \frac{\sigma^{n+\ell+1}}{|x-x_0|^{n+\ell+1}} w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all  $x \in (B(x_0, 2\sqrt{n}\sigma))^c$ . As  $b = 1 - 1/q + \varepsilon$ , it follows from (2.1) that  $(n + \ell + 1)q - q^2nb > nq$ . Therefore, by combining the above inequality, Lemma B and Lemma C, we obtain

$$\begin{aligned} II &= \int_{|x-x_0| > 2\sqrt{n}\sigma} |R_j f(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q} w(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q}} w(B(x_0, |x-x_0|))^{bq} w(x) dx \\ &\leq C \sigma^{(n+\ell+1)q - q^2nb} w(B)^{(b-1/p)q} \int_{|x-x_0| > 2\sqrt{n}\sigma} \frac{1}{|x-x_0|^{(n+\ell+1)q - q^2nb}} w(x) dx \\ &\leq C w(B)^{(b+1/q-1/p)q}. \end{aligned}$$

Thus,

$$(2.3) \quad \|R_j f \cdot w(B(x_0, |\cdot - x_0|))\|_{L_w^q}^b = (I + II)^{1/q} \leq C w(B)^{b+1/q-1/p}.$$

Remark that  $a = 1 - 1/p + \varepsilon$ . Combining (2.2) and (2.3), we obtain

$$\mathfrak{N}_w(R_j f) \leq C w(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

The proof will be concluded if we establish the vanishing moment conditions of  $R_j f$ . One first consider the following lemma.

**Lemma.** For every classical atom ( $p, 2, \ell$ )-atom  $g$  centered at  $x_0$ , we have

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0 \quad \text{for } 0 \leq |\alpha| \leq s, 1 \leq j \leq n.$$

*Proof of the Lemma.* Since  $b = 1 - 1/q + \varepsilon < (\ell + 1)(nq)^{-1} < (\ell + 1)n^{-1}$ , we obtain  $2(n + \ell + 1) - 2nb > n$ . It is similar to the previous argument, we also obtain that  $R_j g$  and  $R_j g \cdot |\cdot - x_0|^{nb}$  belong to  $L^2(\mathbb{R}^n)$ . Now, we establish that  $R_j g \cdot (\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$  for every multi-index  $\alpha$  with  $|\alpha| \leq s$ . Indeed,



since  $\varepsilon > q/p - 1$  by (2.1), implies that  $2(s - nb) < (s - nb)q' < -n$  by  $q = 2(q_w + 1) > 2$ , where  $1/q + 1/q' = 1$ . We use Schwartz inequality to get

$$\begin{aligned} \int_{B(x_0,1)^c} |R_j g(x)(x - x_0)^\alpha| dx &\leq \int_{B(x_0,1)^c} |R_j g(x)| |x - x_0|^s dx \\ &\leq \left( \int_{B(x_0,1)^c} |R_j g(x)|^2 |x - x_0|^{2nb} dx \right)^{1/2} \left( \int_{B(x_0,1)^c} |x - x_0|^{2(s-nb)} dx \right)^{1/2} \\ &\leq C \|R_j g \cdot |\cdot - x_0|^{nb}\|_{L^2} < \infty, \end{aligned}$$

and

$$\int_{B(x_0,1)} |R_j g(x)(x - x_0)^\alpha| dx \leq |B(x_0,1)|^{1/2} \left( \int_{B(x_0,1)} |R_j g(x)|^2 dx \right)^{1/2} < \infty.$$

Thus,  $R_j g(\cdot - x_0)^\alpha \in L^1(\mathbb{R}^n)$  for any  $|\alpha| \leq s$ . Deduce that  $R_j g(x) x^\alpha \in L^1(\mathbb{R}^n)$  for any  $|\alpha| \leq s$ . Therefore,

$$(R_j g(x) x^\alpha)^\wedge(\xi) = C_\alpha \cdot D^\alpha(\widehat{R_j g})(\xi)$$

is continuous, with  $|C_\alpha| \leq C_s$  ( $C_s$  depends only on  $s$ ) for any  $|\alpha| \leq s$ , where  $\hat{h}$  is used to denote the fourier transform of  $h$ . Consequently,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = C_\alpha \cdot D^\alpha(\widehat{R_j g})(0) = C_\alpha \cdot D^\alpha(m_j \hat{g})(0),$$

where  $m_j(x) = -ix_j/|x|$ . Moreover, since  $g$  is a classical  $(p, 2, \ell)$ -atom, it follows from [17, Lemma 9.1] that  $\hat{g}$  is  $\ell$ th order differentiable and  $\hat{g}(\xi) = O(|\xi|^{\ell+1})$  as  $\xi \rightarrow 0$ . We write  $e_j$  to be the  $j$ th standard basis vector of  $\mathbb{R}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$  a multi-index of nonnegative integers  $\alpha_j$ ,  $\Delta_{he_j} \phi(x) = \phi(x) - \phi(x - he_j)$ ,  $\Delta_{he_j}^{\alpha_j} \phi(x) = \Delta_{he_j}^{\alpha_j-1} \phi(x) - \Delta_{he_j}^{\alpha_j-1} \phi(x - he_j)$  for  $\alpha_j \geq 2$ ,  $\Delta_{he_j}^0 \phi(x) = \phi(x)$ , and  $\Delta_h^\alpha = \Delta_{he_1}^{\alpha_1} \dots \Delta_{he_n}^{\alpha_n}$ . Then, the boundedness of  $m_j$ , and  $|C_\alpha| \leq C_s$  for  $|\alpha| \leq s$ , implies

$$\begin{aligned} \left| \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx \right| &= |C_\alpha| \left| \lim_{h \rightarrow 0} |h|^{-|\alpha|} \Delta_h^\alpha(m_j \hat{g})(0) \right| \\ &\leq C \lim_{h \rightarrow 0} |h|^{\ell+1-|\alpha|} = 0, \end{aligned}$$

for  $|\alpha| \leq s$  by  $s \leq \ell$ . Thus, for any  $j = 1, 2, \dots, n$ , and  $|\alpha| \leq s$ ,

$$\int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0.$$

This complete the proof of the lemma.

Let us come back to the proof of Theorem 1.1. As  $q/2 = q_w + 1 > q_w$ , by Lemma A,

$$\left( \frac{1}{|B|} \int_B |f(x)|^2 dx \right)^{q/2} \leq C \frac{1}{w(B)} \int_B |f(x)|^q w(x) dx.$$

Therefore,  $g := C^{-1/q} |B|^{-1/p} w(B)^{1/p} f$  is a classical  $(p, 2, \ell)$ -atom since  $f$  is  $w$ - $(p, q, \ell)$ -atom associated with ball  $B$ . Consequently, by the above lemma,

$$\int_{\mathbb{R}^n} R_j f(x) x^\alpha dx = C^{1/q} |B|^{1/p} w(B)^{-1/p} \int_{\mathbb{R}^n} R_j g(x) x^\alpha dx = 0$$

for all  $j = 1, 2, \dots, n$  and  $|\alpha| \leq s$ . Thus, the theorem is proved.  $\square$

Following a similar but easier argument, we also have the following  $H_w^p$ -boundedness of Hilbert transform. We leave details to readers.

**Theorem 2.1.** *Let  $0 < p \leq 1$  and  $w \in A_\infty$ . Then, the Hilbert transform is bounded on  $H_w^p(\mathbb{R})$ .*

### 3. PROOF OF THEOREM 1.2

We first consider the following lemma

**Lemma 3.1.** *Let  $p \in (0, 1]$ ,  $w \in A_q$ ,  $1 < q < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then,  $\int_{\mathbb{R}^n} T f(x) dx = 0$  for all  $w$ - $(p, q, 0)$ -atoms  $f$ .*

*Proof of Lemma 3.1.* Let  $f$  be an arbitrary  $w$ - $(p, q, 0)$ -atom associated with ball  $B$ . It is well-known that there exists  $1 < r < q$  such that  $w \in A_r$ . Therefore, it follows from Lemma A that

$$\int_B |f(x)|^{q/r} dx \leq C |B| w(B)^{1/r} \|f\|_{L_w^q}^{q/r} < \infty.$$

We deduce that  $f$  is a multiple of classical  $(p, q/r, 0)$ -atom, and thus the condition  $T^*1 = 0$  implies  $\int_{\mathbb{R}^n} T f(x) dx = 0$ .  $\square$

*Proof of Theorem 1.2.* Because of the hypothesis, without loss of generality we can assume  $q > 1$ . Furthermore, it is clear that  $[n(q_w/p - 1)] = 0$ , and there exists a positive constant  $\varepsilon$  such that

$$(3.1) \quad \max \left\{ \frac{1}{r_w - 1}, \frac{1}{p} - 1 \right\} < \varepsilon < \frac{n + \delta}{nq} - 1.$$

Similarly to the arguments in Theorem 1.1, it is sufficient to show that, for every  $w$ - $(p, q, 0)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ ,

$Tf$  is a  $w$ -( $p, q, 0, \varepsilon$ )-molecule with the norm  $\mathfrak{N}_w(Tf) \leq C$ . One first observe that  $\int_{\mathbb{R}^n} Tf(x)dx = 0$  by Lemma 3.1, and

$$\sum_{k=0}^{\infty} \theta(2^{-k})2^{knbq} < \infty,$$

where  $b = 1 - 1/q + \varepsilon$ , by  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$  and (3.1). We deduce that

$$(3.2) \quad \sum_{k=0}^{\infty} \left( \theta(2^{-k})2^{knbq} \right)^q < \infty.$$

As  $w \in A_q$ ,  $1 < q < \infty$ , it follows from [18, Theorem 2.4] that

$$(3.3) \quad \|Tf\|_{L_w^q} \leq C\|f\|_{L_w^q} \leq Cw(B)^{1/q-1/p}.$$

To estimate  $\|Tf \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q}$ , we write

$$\begin{aligned} \|Tf \cdot w(B(x_0, \cdot - x_0))^b\|_{L_w^q}^q &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx + \\ &+ \int_{|x-x_0| > 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx = I + II. \end{aligned}$$

By Lemma B, we have the following estimate,

$$\begin{aligned} I &= \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(B(x_0, |x - x_0|))^{bq} w(x) dx \\ &\leq w(B(x_0, 2\sigma))^{bq} \int_{|x-x_0| \leq 2\sigma} |Tf(x)|^q w(x) dx \\ &\leq Cw(B)^{bq} \|f\|_{L_w^q}^q \leq Cw(B)^{(b+1/q-1/p)q}. \end{aligned}$$

To estimate  $II$ , since  $f$  is of mean zero, by Lemma A, we have

$$\begin{aligned} |Tf(x)| &= \left| \int_{|y-x_0| \leq \sigma} (K(x, y) - K(x, x_0))f(y)dy \right| \\ &\leq C \int_{|y-x_0| \leq \sigma} \frac{1}{|x-x_0|^n} \theta\left(\frac{|y-x_0|}{|x-x_0|}\right) |f(y)| dy \\ &\leq C \frac{\sigma^n}{|x-x_0|^n} \theta\left(\frac{\sigma}{|x-x_0|}\right) w(B)^{-1/q} \|f\|_{L_w^q}, \end{aligned}$$

for all  $x \in (B(x_0, 2\sigma))^c$ . Therefore, by combining the above inequality, Lemma B and (3.2), we obtain

$$\begin{aligned}
 II &= \int_{|x-x_0|>2\sigma} |Tf(x)|^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{-1} \|f\|_{L_w^q}^q \int_{|x-x_0|>2\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left( \theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{-q/p} \sum_{k=1}^{\infty} \int_{2^k\sigma < |x-x_0| \leq 2^{k+1}\sigma} \frac{\sigma^{nq}}{|x-x_0|^{nq}} \left( \theta\left(\frac{\sigma}{|x-x_0|}\right) \right)^q w(B(x_0, |x-x_0|))^{bq} w(x) dx \\
 &\leq Cw(B)^{(b+1/q-1/p)q} \sum_{k=0}^{\infty} \left( \theta(2^{-k}) 2^{knbq} \right)^q \leq Cw(B)^{(b+1/q-1/p)q}.
 \end{aligned}$$

Thus,

$$(3.4) \quad \|Tf \cdot w(B(x_0, |\cdot - x_0|))^b\|_{L_w^q} = (I + II)^{1/q} \leq Cw(B)^{b+1/q-1/p}.$$

Remark that  $a = 1 - 1/p + \varepsilon$ . Combining (3.3) and (3.4), we obtain

$$\mathfrak{N}_w(Tf) \leq Cw(B)^{(1/q-1/p)a/b} w(B)^{(b+1/q-1/p)(1-a/b)} \leq C.$$

This finishes the proof.  $\square$

It is well-known that the molecular theory of (unweighted) Hardy spaces of Taibleson and Weiss [17] is one of useful tools to establish boundedness of operators in Hardy spaces (cf. [17, 12]). In the setting of Muckenhoupt weight, this theory has been considered by the authors in [10], since then, they have been well used to establish boundedness of operators in weighted Hardy spaces (cf. [10, 11, 3]). However in some cases, the *weighted molecular characterization*, which obtained in [10], does not give the best possible results. For Calderón-Zygmund type operators in Theorem 1.2, for instance, it involves assumption on the *critical index of  $w$  for the reverse Hölder condition* as the following theorem does not.

**Theorem 3.1.** *Given  $\delta \in (0, 1]$ ,  $n/(n + \delta) < p \leq 1$ , and  $w \in A_q$  with  $1 \leq q < p(n + \delta)/n$ . Let  $\theta$  be a nonnegative nondecreasing function on  $(0, \infty)$  with  $\int_0^1 \frac{\theta(t)}{t^{1+\delta}} dt < \infty$ , and  $T$  be a  $\theta$ -Calderón-Zygmund operator satisfying  $T^*1 = 0$ . Then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .*

The following corollary give the boundedness of the classical Calderón-Zygmund type operators on weighted Hardy spaces (see [15, Theorem 3]).

**Corollary 3.1.** *Let  $0 < \delta \leq 1$  and  $T$  be the classical  $\delta$ -Calderón-Zygmund operator, i.e.  $\theta(t) = t^\delta$ , satisfying  $T^*1 = 0$ . If  $n/(n + \delta) < p \leq 1$  and  $w \in A_q$  with  $1 \leq q < p(n + \delta)/n$ , then  $T$  is bounded on  $H_w^p(\mathbb{R}^n)$ .*

*Proof of Corollary 3.1.* By taking  $\delta' \in (0, \delta)$  which is close enough  $\delta$ . Then, we apply Theorem 3.1 with  $\delta'$  instead of  $\delta$ .  $\square$

*Proof of Theorem 3.1.* Without loss of generality we can assume  $1 < q < p(n + \delta)/n$ . Fix  $\phi \in \mathcal{S}(\mathbb{R}^n)$  with  $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$ . By Theorem C, it is sufficient to show that for every  $w$ - $(p, q, 0)$ -atom  $f$  centered at  $x_0$  and supported in ball  $B = B(x_0, \sigma)$ ,  $\|(Tf)^*\|_{L_w^p} \leq C$ . In order to do this, one write

$$\begin{aligned} \|(Tf)^*\|_{L_w^p}^p &= \int_{|x-x_0| \leq 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx + \int_{|x-x_0| > 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx \\ &= L_1 + L_2. \end{aligned}$$

By Hölder inequality,  $L_w^q$ -boundedness of the maximal function and Lemma B, we get

$$\begin{aligned} L_1 &\leq \left( \int_{|x-x_0| \leq 4\sigma} \left( (Tf)^*(x) \right)^q w(x) dx \right)^{p/q} \left( \int_{|x-x_0| \leq 4\sigma} w(x) dx \right)^{1-p/q} \\ &\leq C \|f\|_{L_w^q}^p w(B(x_0, 4\sigma))^{1-p/q} \leq C. \end{aligned}$$

To estimate  $L_2$ , we first estimate  $(Tf)^*(x)$  for  $|x - x_0| > 4\sigma$ . For any  $t > 0$ , since  $\int_{\mathbb{R}^n} Tf(x) dx = 0$  by Lemma 3.1, we get

$$\begin{aligned} |Tf * \phi_t(x)| &= \left| \int_{\mathbb{R}^n} Tf(y) \frac{1}{t^n} \left( \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right) dy \right| \\ &\leq \frac{1}{t^n} \int_{|y-x_0| < 2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\ &\quad + \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \dots + \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \dots \\ &= E_1(t) + E_2(t) + E_3(t). \end{aligned}$$

As  $|x - x_0| > 4\sigma$ , by the mean value theorem, Lemma A and Lemma B, we get

$$\begin{aligned}
E_1(t) &= \frac{1}{t^n} \int_{|y-x_0|<2\sigma} |Tf(y)| \left| \phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x-x_0}{t}\right) \right| dy \\
&\leq \frac{1}{t^n} \int_{|y-x_0|<2\sigma} |Tf(y)| \frac{|y-x_0|}{t} \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0+\lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} \int_{|y-x_0|<2\sigma} |Tf(y)| dy \\
&\leq C \frac{\sigma}{|x-x_0|^{n+1}} |B(x_0, 2\sigma)| w(B(x_0, 2\sigma))^{-1/q} \|Tf\|_{L_w^q} \\
&\leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/q} \|f\|_{L_w^q} \leq C \frac{\sigma^{n+1}}{|x-x_0|^{n+1}} w(B)^{-1/p}.
\end{aligned}$$

Similarly, we also get

$$\begin{aligned}
E_2(t) &\leq \frac{1}{t^n} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) (K(y, z) - K(y, x_0)) dz \right| \frac{|y-x_0|}{t} \\
&\quad \times \sup_{\lambda \in (0,1)} \left| \nabla \phi\left(\frac{x-x_0+\lambda(y-x_0)}{t}\right) \right| dy \\
&\leq C \frac{1}{|x-x_0|^{n+1}} \int_{2\sigma \leq |y-x_0| < \frac{|x-x_0|}{2}} |y-x_0| \int_{|z-x_0|<\sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+1} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^2} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+1} \left(\frac{|x-x_0|}{2\sigma}\right)^{1-\delta} \int_{2\sigma/|x-x_0|}^{1/2} \frac{\theta(t)}{t^{1+\delta}} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Next, let us look at  $L_3$ . Similarly, we also have

$$\begin{aligned}
E_3(t) &\leq \frac{1}{t^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \left| \int_{\mathbb{R}^n} f(z) \left( K(y, z) - K(y, x_0) \right) dz \right| \left( \left| \phi\left(\frac{y-x_0}{t}\right) \right| + 2 \left| \phi\left(\frac{x-x_0}{t}\right) \right| \right) dy \\
&\leq C \frac{1}{|x-x_0|^n} \int_{|y-x_0| \geq \frac{|x-x_0|}{2}} \int_{|z-x_0| < \sigma} |f(z)| \frac{1}{|y-x_0|^n} \theta\left(\frac{|z-x_0|}{|y-x_0|}\right) dz dy \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t} dt w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^n \int_0^{2\sigma/|x-x_0|} \frac{\theta(t)}{t^{1+\delta}} dt \left(\frac{2\sigma}{|x-x_0|}\right)^\delta w(B)^{-1/p} \\
&\leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.
\end{aligned}$$

Therefore, for all  $|x-x_0| > 4\sigma$ ,

$$(Tf)^*(x) = \sup_{t>0} (E_1(t) + E_2(t) + E_3(t)) \leq C \left(\frac{\sigma}{|x-x_0|}\right)^{n+\delta} w(B)^{-1/p}.$$

Combining this, Lemma C and Lemma B, we obtain that

$$\begin{aligned}
L_2 &= \int_{|x-x_0| > 4\sigma} \left( (Tf)^*(x) \right)^p w(x) dx \leq C \int_{|x-x_0| > 4\sigma} \frac{\sigma^{(n+\delta)p}}{|x-x_0|^{(n+\delta)p}} w(B)^{-1} w(x) dx \\
&\leq C w(B)^{-1} w(B(x_0, 4\sigma)) \leq C,
\end{aligned}$$

since  $(n+\delta)p > nq$ . This finishes the proof.  $\square$

## REFERENCES

- [1] M. Bownik, Boundedness of operators on Hardy spaces via atomic decompositions. Proc. Amer. Math. Soc. 133 (2005), 3535–3542.
- [2] M. Bownik, B. Li, D. Yang, and Y. Zhou, Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators. Indiana Univ. Math. J. 57 (2008), no. 7, 3065–3100.
- [3] Y. Ding, M.-Y. Lee and C.-C. Lin, Fractional integrals on weighted Hardy spaces. J. Math. Anal. Appl. 282 (2003), 356–368.
- [4] C. Fefferman, E.M. Stein,  $H^p$  spaces of several variables. Acta Math. 129 (1972), 137–193.
- [5] J. García-Cuerva, Weighted  $H^p$  spaces. Dissertations Math. 162 (1979), 1–63.
- [6] J. García-Cuerva and J.L. Rubio de Francia, Weighted norm inequalities and related topics. NorthHolland Math. Stud. 116, 1985.

- [7] J.-L. Journé, Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón. Lecture notes in Math. 994, Springer-Verlag, Berlin, 1983.
- [8] L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators, submitted, arXiv:1103.3757.
- [9] M.-Y. Lee, Different approaches to the  $H^p$  boundedness of Riesz transforms. Commun. Math. Anal. 1 (2006), no. 2, 101–108.
- [10] M.-Y. Lee and C.-C. Lin, The molecular characterization of weighted Hardy spaces. J. Funct. Anal. 188 (2002), 442–460.
- [11] M.-Y. Lee, C.-C. Lin and W.-C. Yang,  $H_w^p$ -boundedness of Riesz transforms. J. Math. Anal. Appl. 301 (2005), 394–400.
- [12] S.Z. Lu, Four lectures on real  $H^p$  spaces. World Scientific Publishing Co. Pte. Ltd, 1995.
- [13] S. Meda, P. Sjögren and M. Vallarino, On the  $H^1-L^1$  boundedness of operators. Proc. Amer. Math. Soc. 136 (2008), 2921–2931.
- [14] Y. Meyer and R. Coifman, Wavelets, Calderón-Zygmund and multilinear operators. Advanced mathematics. Cambridge University Press, 1997.
- [15] T. Quek and D. Yang, Calderón-Zygmund-type operators on weighted weak Hardy spaces over  $\mathbb{R}^n$ , Acta Math. Sin. (Engl. Ser.), 16 (2000), 141–160.
- [16] J.-O. Strömberg and A. Torchinsky, Weighted Hardy spaces. Lecture Notes in Mathematics, Vol. 1381, Springer-Verlag, Berlin/New York, 1989.
- [17] M.H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces; Astérisque. 77 (1980), 67–149.
- [18] K. Yabuta, Generalizations of Calderón-Zygmund operators, Studia Math. 82 (1985) 17–31.
- [19] D. Yang and Y. Zhou, A boundedness criterion via atoms for linear operators in Hardy spaces. Constr. Approx. 29, 207–218 (2009).

MAPMO-UMR 6628, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'ORLÉANS,  
45067 ORLÉANS CEDEX 2, FRANCE

*E-mail address:* dangky@math.cnrs.fr