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## On the Hurwitz Zeta Function

Lazhar Fekih-Ahmed

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**Abstract** We give new integral and series representations of the Hurwitz zeta function. We also provide a closed-form expression of the coefficients of the Laurent expansion of the Hurwitz-zeta function about any point in the complex plane.

**Keywords** Number Theory · Hurwitz Zeta function

**Mathematics Subject Classification (2000)** 11M06 · 11M35

### 1 Introduction

The Hurwitz zeta function defined by the series

$$\zeta(s, a) = \frac{1}{a^s} + \frac{1}{(1+a)^s} + \cdots + \frac{1}{(k-1+a)^s} + \cdots = \sum_{n=1}^{\infty} (n-1+a)^{-s}, \quad (1.1)$$

where  $0 < a \leq 1$ , is a well-defined series when  $\Re(s) > 1$ , and can be analytically continued to the whole complex plane with one singularity, a simple pole with residue 1 at  $s = 1$ .

The Hurwitz zeta function has also the following integral representation

$$\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(a-1)t}}{e^t - 1} t^{s-1} dt, \quad (1.2)$$

valid for  $\Re(s) > 1$  [6]. Moreover, it has the following analytic continuation represented by the following contour integral

$$\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{e^{-(a-1)t}}{e^t - 1} t^{s-1} dt, \quad (1.3)$$

where  $\mathcal{C}$  is the Hankel contour consisting of the three parts  $C = C_- \cup C_\epsilon \cup C_+$ : a path which extends from  $(-\infty, -\epsilon)$ , around the origin counter clockwise on a circle of

center the origin and of radius  $\epsilon$  and back to  $(-\epsilon, -\infty)$ , where  $\epsilon$  is an arbitrarily small positive number.

The integral (1.3) defines  $\zeta(s, a)$  for all  $s \in \mathbb{C}$  with a single pole at  $s = 1$ .

## 2 New Integral and Series representations of $\zeta(s, a)$

The new representations of  $\zeta(s, a)$  are based on the alternating sums defined by

$$S_n(s, a) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+a)^{-s} \text{ for } n \geq 2, \quad (2.1)$$

and by  $S_1(s, a) = 1$  for  $n = 1$ , and on the real function  $\psi(t)$  defined by

$$\psi(t) = \frac{te^t}{(e^t - 1)^2} - \frac{1}{e^t - 1} - \frac{(a-1)t}{e^t - 1}. \quad (2.2)$$

We prove the following theorem

**Theorem 2.1** *For all  $s$  such that  $\Re(s) > 0$  and all  $0 < a \leq 1$ , we have*

$$\begin{aligned} \text{(A)} \quad & (s-1)\zeta(s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \psi(t) e^{-(a-1)t} t^{s-1} dt. \\ \text{(B)} \quad & (s-1)\zeta(s, a) = \sum_{n=1}^\infty S_n(s, a) \left( \frac{1}{n+1} + \frac{a-1}{n} \right). \end{aligned}$$

*Proof* We can rewrite  $S_n(s, a)$  in (2.1) as

$$\begin{aligned} S_n(s, a) &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-(k+a)t} t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty (1 - e^{-t})^{n-1} e^{-at} t^{s-1} dt, \end{aligned} \quad (2.3)$$

since we know that

$$(n+a-1)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(n+a-1)t} t^{s-1} dt, \quad (2.4)$$

a valid formula for  $\Re(s) > 0$ , and since

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-(k+a)t} = e^{-at} (1 - e^{-t})^{n-1}. \quad (2.5)$$

The proof consists of an evaluation of the sum

$$\sum_{n=1}^\infty \frac{S_n(s, a)}{n+1} + (a-1) \sum_{n=1}^\infty \frac{S_n(s, a)}{n}. \quad (2.6)$$

But before we do so, we need to establish the following two identities valid for  $0 < t < \infty$ :

$$\frac{te^{-t}}{1-e^{-t}} = \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}e^{-t}}{n}. \quad (2.7)$$

$$\frac{te^{-t}}{(1-e^{-t})^2} - \frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}e^{-t}}{n+1}. \quad (2.8)$$

To prove these identities, we start from the series

$$t = -\log(1 - (1 - e^{-t})) = \sum_{n=1}^{\infty} \frac{(1 - e^{-t})^n}{n}. \quad (2.9)$$

This yields

$$\frac{t}{1-e^{-t}} = \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n}, \quad (2.10)$$

and

$$\frac{t}{(1-e^{-t})^2} = \frac{1}{1-e^{-t}} + \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1}, \quad (2.11)$$

from which the two identities follow easily.

Without worrying about interchanging sums and integrals for the moment, we have

$$\sum_{n=1}^{\infty} \frac{S_n(s, a)}{n+1} = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1} e^{-at} t^{s-1} dt \quad (2.12)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1} e^{-at} t^{s-1} dt \quad (2.13)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \left( \frac{te^{-t}}{(1-e^{-t})^2} - \frac{e^{-t}}{1-e^{-t}} \right) e^{-(a-1)t} t^{s-1} dt, \quad (2.14)$$

and similarly

$$\sum_{n=1}^{\infty} \frac{S_n(s, a)}{n} = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(1-e^{-t})^{n-1}}{n} e^{-at} t^{s-1} dt \quad (2.15)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^{n-1}}{n} e^{-at} t^{s-1} dt \quad (2.16)$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{te^{-t}}{1-e^{-t}} e^{-(a-1)t} t^{s-1} dt. \quad (2.17)$$

Therefore,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{S_n(s, a)}{n+1} + (a-1) \sum_{n=1}^{\infty} \frac{S_n(s, a)}{n} = \\ & \frac{1}{\Gamma(s)} \int_0^{\infty} \left( \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1} + \frac{(a-1)t}{e^t-1} \right) e^{-(a-1)t} t^{s-1} dt. \end{aligned} \quad (2.18)$$

This proves that the right hand sides of (A) and (B) in the statement of the theorem are equal. Now, by observing that

$$\frac{d}{dt} \left( \frac{-te^{-(a-1)t}}{e^t-1} \right) = \left( \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1} - \frac{(a-1)t}{e^t-1} \right) e^{-(a-1)t}, \quad (2.19)$$

we can perform an integration by parts in (2.18) when  $\Re(s) > 1$ . The integral in the right hand side of (2.18) becomes

$$\frac{s-1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(a-1)t}}{e^t-1} t^{s-1} dt = (s-1)\zeta(s, a). \quad (2.20)$$

Thus,

$$\sum_{n=1}^{\infty} \frac{S_n(s, a)}{n+1} + (a-1) \sum_{n=1}^{\infty} \frac{S_n(s, a)}{n} = (s-1)\zeta(s, a), \quad (2.21)$$

and this prove the theorem when  $\Re(s) > 1$ . However, formula (2.21) remains valid for  $\Re(s) > 0$  since the integral (2.18) is well-defined for  $\Re(s) > 0$ .

To finish the proof, we now need to justify the interchange of summation and integration in both equation (2.13) and equation (2.16). We show this justification for equation (2.13) only, the other is similar.

The interchange in (2.13) is indeed valid because the series

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1} e^{-at} t^{s-1} dt \quad (2.22)$$

converges absolutely and uniformly for  $0 < t < \infty$ . To prove this, it suffices to show uniform convergence for the dominating series

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{(1-e^{-t})^{n-1}}{n+1} e^{-at} t^{\sigma-1} dt, \quad (2.23)$$

where  $\sigma = \Re(s)$ .

Indeed, let  $K = \max((1-e^{-t})^{n-1} e^{-t/2}), 0 < t < \infty$ . A straightforward calculation of the derivative shows that

$$K = \left(1 - \frac{1}{2n-1}\right)^{n-1} \frac{1}{\sqrt{2n-1}} \quad (2.24)$$

and is attained when  $e^{-t} = \frac{1}{2n-1}$ . Now, for  $n \geq 2$ , we have

$$\begin{aligned} \int_0^\infty (1 - e^{-t})^{n-1} e^{-at} t^{\sigma-1} dt &= \int_0^\infty (1 - e^{-t})^{n-1} e^{-t/2} (e^{-(1/2+a)t} t^{\sigma-1}) dt \\ &\leq K \int_0^\infty e^{-(1/2+a)t} t^{\sigma-1} dt \\ &= \left(1 - \frac{1}{2n-1}\right)^{n-1} \frac{\Gamma(\sigma)}{\sqrt{2n-1}(-1/2+a)^\sigma} \\ &\leq \frac{K'}{\sqrt{2n-1}}. \end{aligned} \quad (2.25)$$

The last inequality implies that each term of the dominating series is bounded by  $K'/(n+1)\sqrt{2n-1}$ . Thus the dominating series converges by the comparison test. This completes the proof of the theorem.  $\square$

With  $\mathcal{C}$  being the Hankel contour defined previously, an immediate corollary of Theorem 2.1 is the following

**Corollary 2.1** *For  $0 < a \leq 1$  and for all  $s \in \mathbb{C}$ , we have*

$$\begin{aligned} \text{(C)} \quad (s-1)\zeta(s, a) &= \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \psi(t) e^{-(a-1)t} t^{s-1} dt. \\ \text{(D)} \quad (s-1)\zeta(s, a) &= \sum_{n=1}^{\infty} S_n(s, a) \left( \frac{1}{n+1} + \frac{a-1}{n} \right). \end{aligned}$$

*Proof* The proof the first statement follows the same steps as in [6] for example. As for the second statement, we can either proceed as in [5] or as in [3]<sup>1</sup>. In [3], the proof was given for  $\zeta(s)$  (i.e.  $a = 1$ ) and uses an estimate of the exact asymptotic order of growth of  $S_n(s, 1)$  when  $n$  is large. By looking at the definition of  $S_n(s, a)$ , we can see that  $S_n(s, 1)$  are the Stirling numbers of the second kind modulo a multiplicative factor when  $s \in \{0, -1, -2, \dots\}$ . Therefore, when  $s = -k$ ,  $k$  a positive integer,  $S_n(-k, 1)$  are eventually zero for  $n$  large enough. Similarly,  $S_n(s, a)$  are the generalized Stirling numbers of the second kind whose generating function is given by [2]:

$$\sum_{k=0}^{\infty} S_n(-k, a) \frac{t^k}{k!} = e^{at} (1 - e^t)^n. \quad (2.26)$$

It follows that

$$S_n(-k, a) = \frac{d^k}{dt^k} \left\{ e^{at} (1 - e^t)^n \right\} \Big|_{t=0}, \quad (2.27)$$

and hence  $S_n(-k, a)$  are eventually zero for  $n$  large enough.

For  $s \notin \{0, -1, -2, \dots\}$ , an asymptotic estimate of  $S_n(s, a)$  can be obtained for  $n$  large. Indeed, we have by definition

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<sup>1</sup> The notation used in [3] may cause some confusion. Contrary to the notation of this paper in which  $\zeta(s, a)$  denotes the Hurwitz zeta function,  $\zeta(s, x)$  in the cited paper is a power series associated with  $\zeta(s)$  and has nothing to do with the Hurwitz zeta function.

$$S_n(s, a) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (k+a)^{-s}. \quad (2.28)$$

By making the change of variable,  $k = m - 1$ , the sum can obviously be put into the form

$$S_n(s, a) = \frac{-1}{n} \sum_{m=1}^n \binom{n}{m} (-1)^m \frac{m}{(m+a-1)^s}. \quad (2.29)$$

Asymptotic expansions of sums of the form (2.29) have been thoroughly studied in [4]. The function  $\frac{z}{(z+a-1)^s}$  has a non-integral algebraic singularity at  $s_0 = 1 - a$  since  $0 < a < 1$ ; thus, when  $s$  is nonintegral,  $S_n(s, a)$  has the following asymptotics when  $n$  is large

$$S_n(s, a) \sim -\frac{\Gamma(1-a)n^{1-a}(\log n)^{s-1}}{n\Gamma(s)} = -\frac{\Gamma(1-a)}{n^a(\log n)^{1-s}\Gamma(s)}, \quad (2.30)$$

and when  $s = k \in \{1, 2, \dots\}$  the following expansion

$$S_n(k, a) \sim -\frac{\Gamma(1-a)(\log n)^{k-1}}{n^a(k-1)!}. \quad (2.31)$$

For the case  $a = 1$  and  $s$  nonintegral, the numerator becomes equal to 1 and we obtain

$$S_n(s) \sim -\frac{1}{n(\log n)^{1-s}\Gamma(s)}. \quad (2.32)$$

The asymptotic estimates (2.30), (2.31) and (2.32) are valid for  $n$  large enough and for all  $s$  such that  $\Re(s) > 0$ . To finish the proof of the corollary, we note that the logarithmic test of series implies that our series is dominated by a uniformly convergent series for all finite  $s$  such that  $\Re(s) > 0$ . Now, by Weierstrass theorem of the uniqueness of analytic continuation, the function  $(s-1)\zeta(s, a)$  can be extended outside of the domain  $\Re(s) > 1$  and that it does not have any singularity when  $\Re(s) > 0$ . Moreover, by repeating the same process for  $\Re(s) > -k$ ,  $k \in \mathbb{N}$ , it is clear that the series defines an analytic continuation of  $\zeta(s)$  valid for all  $s \in \mathbb{C}$ .

□

We can also obtain the analytic continuation of  $\zeta(s, a)$  to the whole complex plane via the following

**Corollary 2.2** *For all  $s$  such that  $\Re(s) > -k$  and all  $0 < a \leq 1$ , we have*

$$(s-1)\zeta(s, a) = \frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty \frac{d^k}{dt^k} (\psi(t)e^{-(a-1)t}) t^{s+k-1} dt. \quad (2.33)$$

*Proof* For all  $k$  and all  $0 < a \leq 1$ , the integrand is bounded and has finite values at the limits of integration. Repeated integration by parts proves the corollary. □

Now if we consider the function  $\zeta(s, a) - a^{-s}$  dealt with in [1], we obtain the following corollary

**Corollary 2.3** Let  $\eta(t) = \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1} - \frac{at}{e^t-1}$ . Then, for all  $s$  such that  $\Re(s) > 0$  and all  $0 \leq a \leq 1$ , we have

$$(E) \quad (s-1)(\zeta(s, a) - a^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \eta(t) e^{-at} t^{s-1} dt.$$

$$(F) \quad (s-1)(\zeta(s, a) - a^{-s}) = \sum_{n=1}^{\infty} S_n(s, a) \left( \frac{1}{n+1} - \frac{a}{n} \right).$$

*Proof* The proof follows the same lines as the proof of Theorem 2.1.  $\square$

We note that when  $a = 0$ ,  $\zeta(s, a) - a^{-s} = \zeta(s)$ . The formulas of Corollary 2.3 reduce to the formulas found in [3].

### 3 A formula for the Laurent coefficients

Berndt [1] (see also the references therein) derived expressions for the coefficients of the Laurent expansion of the Hurwitz zeta-function  $\zeta(s, a)$  about  $s = 1$ . He also provided a method to calculate and estimate these coefficients. In this section, we give exact estimates of the coefficients of  $(s-1)\zeta(s, a)\Gamma(s)$  for any point of the complex plane, and from these coefficients the Laurent coefficients of  $\zeta(s, a)$  about the same point can be easily calculated.

We will only consider expansions around a point  $s_0 = x + iy$  in the right half plane, i.e.  $x > 0$ . For the other points, the reader will realize that the extension can be easily accomplished.

The idea is to simply use the integral formula given in formula (A) of Theorem 2.1. From this formula, the analytic function  $(s-1)\zeta(s)\Gamma(s)$  can be represented by a Taylor series around any point  $s_0$

$$(s-1)\zeta(s, a)\Gamma(s) = \sum_{n=0}^{\infty} a_n (s-s_0)^n \quad (3.1)$$

where the coefficients  $a_0 = (s_0-1)\zeta(s_0, a)\Gamma(s_0)$  and  $a_n$  are given by

$$\begin{aligned} a_n &= \frac{1}{n!} \lim_{s \rightarrow s_0} \frac{d^n}{ds^n} \left\{ \int_0^\infty \psi(t) e^{-(a-1)t} t^{s-1} dt \right\} \\ &= \frac{1}{n!} \int_0^\infty \psi(t) e^{-(a-1)t} (\log t)^n t^{s_0-1} dt, \end{aligned} \quad (3.2)$$

with  $\psi(t)$  being given in equation (2.2).

Thus, following [1]<sup>2</sup>, if we set

$$(s-1)\zeta(s, a) = \sum_{n=0}^{\infty} \gamma_n(a, s_0) (s-s_0)^n, \quad (3.3)$$

and if we expand  $\Gamma(s)$  around the point  $s_0$  into a Taylor series of the form

<sup>2</sup> Actually the coefficients  $\gamma_n(a)$  when  $s_0 = 1$  differ slightly from ours since in [1] they were defined by  $(s-1)\zeta(s, a) = \sum_{n=0}^{\infty} \gamma_n(a) (s-1)^{n+1}$ .



$$\Gamma(s) = \sum_{n=0}^{\infty} c_n (s - s_0)^n, \quad (3.4)$$

where  $c_0 = \Gamma(s_0)$  and

$$\begin{aligned} c_n &= \frac{1}{n!} \lim_{s \rightarrow s_0} \frac{d^n}{ds^n} \left\{ \int_0^{\infty} e^{-t} t^{s-1} dt \right\} \\ &= \frac{1}{n!} \int_0^{\infty} e^{-t} (\log t)^n t^{s_0-1} dt, \end{aligned} \quad (3.5)$$

we easily obtain

$$\gamma_n(a, s_0) = -\frac{1}{c_0} \sum_{k=1}^n \gamma_{n-k}(a, s_0) c_k + a_n. \quad (3.6)$$

To get the last formula, we have merely used the coefficient formula of division of power series. The formula is recursive and can be used to calculate  $\gamma_n(a, s_0)$  for any  $0 < a \leq 1$  and any  $s_0$  in the right half plane. If  $s_0$  is not in the right half plane, the integral formula (2.33) of Corollary 2.2 can be used instead.

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