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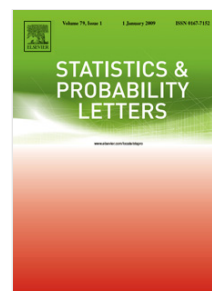
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No-information secretary problems with cardinal payoffs and Poisson arrivals

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Abstract

No-information secretary problem with Poisson stream of applicants is considered. The values of the applicants are random variables drawn from uniform distribution. The goal is to maximize the expectation of the value of the applicant under the condition that the decision maker can only stop on a candidate best so far. We also consider two modifications of this problem.

Key words: Poisson process, Cowan-Zabczyk-Bruss model, relative rank, random horizon, inverse Robbins' problem

1 Introduction

The no-information best choice problem (secretary problem) was considered by many authors (see Ferguson (1989) or Szajowski (2009) for an extensive bibliography). In the paper we consider generalizations of the problem presented in Bearden (2006): There is a single secretarial position available. The applicants are interviewed sequentially at times τ_1, τ_2, \dots of jumps of a Poisson process. The time of observations is limited by T , where T is finite and known. This means that the number of interviewed secretaries is random. Moreover, rejected applicants cannot be recalled. The values X_1, X_2, \dots of subsequent of the applicants are i.i.d. random variables with uniform distribution on $(0, 1)$

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and they are unobservable. The decision about acceptance or rejecting should be made during the interviewing process based only on relative ranks of the applicants interviewed so far. The goal is to maximize the expectation of the reward. In the paper we consider two kind of rewards:

(A) The reward is X_i if the i -th applicant is the best so far and arrived on the interviewing process before time T , and 0 otherwise.

(B) The reward is X_i if the i -th applicant is the best so far and arrived on interviewing process before time T and there were no more then $N - 1$ applicants before her, and 0 otherwise, where N is fixed and known. This means that the decision maker (DM) admitted to the interviewing process only N candidates.

The above models describe the situation when the DM is satisfied only with the best applicant among those interviewed so far or none. Additionally, it is possible that the DM will not have chosen an applicant by time T , because in (A) the number of applicants is Poisson distributed random variable and in (B) it is truncated one.

A generalization of the classical secretary problem to a problem with random number of arrivals was considered in Presman and Sonin (1972). The authors showed that this problem can be reduced to an optimal stopping problem for a certain Markov chain with an infinite number of states and an unbounded number of steps. Then they showed how to find the optimal strategy. Bruss (1984) also considered the problem with random number of candidates and proved the e^{-1} - law of the best choice. Porosiński (1987) showed that if number of candidates has geometrical distribution then an optimal stopping rule is of barrier type and the limiting probability is constant and equal to e^{-1} . An interesting approach to problem with random number of candidates was presented in Samuel-Cahn (1996). She showed that the problem can be translated to a problem with discounting.

The Poisson-arrivals version of secretary problem was introduced by Cowan and Zabczyk (1978) and later generalized to Poisson processes with unknown arrival intensity by Bruss (1987). Stewart (1981) adopted a Bayesian approach to the secretary problem with unknown number of candidates by assuming a prior distribution on N . Secretary problem with costs per observation and Poisson process was considered in Bojdecki (1978). Bruss (2000, 2003) introduced a general class of optimal stopping time problems covering a version of classical secretary problem with Poisson process and found its solution known as the odds-theorem and odds-algorithm.

Elfving (1967) considered the best choice problem with a Poisson process and discounting. He found a differential equation which allows one to compute the optimal mean reward. The Elfving problem was generalized to the problem with random horizon by Ferenstein and Krasnosielska (in press). In the paper authors adopted the theorem concerning stopping problem with random

horizon of Samuel-Cahn by introducing Poisson process. Krasnosielska (2009) modified Elfving problem assuming that the process of decision making starts at random time. Another approach to the best choice problem connected with a Poisson process was presented in Ferenstein and Sierociński (1997), where the authors analyzed the optimal stopping time problem of a risk process.

Bruss and Swan (2009) generalized the other version of the secretary problem known as Robbins' problem introducing Poisson process.

Rasmussen and Pliska (1976) considered no-information secretary problem with a discount penalty function α and known number of observations N . In their model the DM receives the reward α^k if the k -th candidate is chosen and is the best among all candidates, and 0 otherwise. Bearden (2006) analyzed the version of the secretary problem known as inverse Robbins' problem. However, in his problem the reward for the DM from accepting the i -th candidate is X_i if she is the best so far or the last one. The decision is based only on relative ranks of candidates observed so far. Bearden (2006) showed that an optimal strategy is to pass $c - 1$ applicants and to stop at the first $j \in \{c, c + 1, \dots, N - 1\}$ with rank 1, where $c \in \{\lfloor \sqrt{N} \rfloor, \lceil \sqrt{N} \rceil\}$, and if such a j does not exist, then stop at N . Bearden problem was also considered by Samuel-Cahn (2007). She analyzed different kinds of distribution of the random variables X_i and their influence on the optimal strategy and the optimal expected reward. Szajowski (2009) modified the Bearden model introducing discounted cost of stress connected with time.

There also exist experimental studies on the behavior of the DMs in situations similar to those considered in mathematical models related to best choice problems (see Bearden, Rapoport and Murphy (2006), Seale and Rapoport (2000), or Stein, Seale and Rapoport (2003) for an extensive bibliography on the subject). The experiments showed that the DMs have a tendency to shorten the time of observation in comparison with the optimal strategy obtained from a mathematical model.

The above models can be used to solve many problems concerning everyday life like: tourist problem (finding the best hotel room on the highway), asset selling problem, job search problem, marriage problem. In mentioned situations the number of future observations (e.g. the number of apartments which we will meet on our way by the end of the day or the number of future job offers) is unknown and the times when these offers appear are not known either. In some situation the number of observation is random and can not be limited (e.g. the number of future job offers in the Internet). In some other situations we know that there will not be more than N offers, but their exact number is not known (for example the number of hotels on the highway from one city to the other one). Therefore, in this case we should use the model with reward as in problem (B). Hence, the model with a Poisson process or with a Poisson

process and a limited number of applicants better describe the situations of real life and in consequence allow one to find a better strategy of behavior. Our models tell us what to do in order to maximize the expected value of the reward.

The paper is organized as follows: In Section 2, we formulate and solve a model with reward for Problem (A). In Section 3, a model with reward for Problem (B) is considered. Additionally in Appendix, we consider the Bearden problem under the assumption that we are interested only with the best candidate and we compare the result with Bearden problem.

2 Model with Poisson process

In the section we consider two equivalent optimal stopping problems with Poisson stream of offers.

Assumptions. Assume that $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with uniform distribution on $(0, 1)$. The random variable X_i appears at time τ_i of jump of the Poisson process $N(s)$, $s \geq 0$, with intensity 1, $i = 1, 2, \dots$. Moreover, assume that the sequences of X 's and τ 's are independent. Define $Y_n = 1$ if $X_n = \max\{X_1, \dots, X_n\}$, and $Y_n = 0$ otherwise, $n = 1, 2, \dots$

Problem A. Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n, \tau_1, \dots, \tau_n)$. Denote by \mathcal{M} the set of all stopping times t adapted to the sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Let $X_t \mathbb{I}(\tau_t \leq T) \mathbb{I}(Y_t = 1)$ be the reward for the DM making selection at τ_t , where $T > 0$ is finite and known horizon and $\mathbb{I}(A)$ denotes the indicator function of the event A . The aim of the DM is to find the optimal mean reward

$$V = \sup_{t \in \mathcal{M}} E(X_t \mathbb{I}(\tau_t \leq T) \mathbb{I}(Y_t = 1))$$

and an optimal stopping time $\sigma \in \mathcal{M}$, that is $E(X_{\sigma} \mathbb{I}(\tau_{\sigma} \leq T) \mathbb{I}(Y_{\sigma} = 1)) = V$.

Problem A'. Define the sequence of L_n as follows: $L_1 = 1$ and $L_n = \min\{k > L_{n-1} : Y_k = 1\}$ for $n = 2, 3, \dots$. Let $\tilde{\mathcal{F}}_n = \sigma\{L_1, \dots, L_n, \tau_1, \dots, \tau_{L_n}\}$. Denote by $\tilde{\mathcal{M}}$ the set of all stopping times \tilde{t} adapted to the sequence $\{\tilde{\mathcal{F}}_n\}_{n=1}^{\infty}$. Let $X_{L_{\tilde{t}}} \mathbb{I}(\tau_{L_{\tilde{t}}} \leq T)$ be the reward for the DM making selection at $\tau_{L_{\tilde{t}}}$. The aim of the DM is to find the optimal mean reward

$$\tilde{V} = \sup_{\tilde{t} \in \tilde{\mathcal{M}}} E(X_{L_{\tilde{t}}} \mathbb{I}(\tau_{L_{\tilde{t}}} \leq T))$$

and an optimal stopping time $\tilde{\sigma} \in \tilde{\mathcal{M}}$, that is $E(X_{L_{\tilde{\sigma}}} \mathbb{I}(\tau_{L_{\tilde{\sigma}}} \leq T)) = \tilde{V}$.

Note that if $N(T) = 0$, then the payoff of any strategy in the above problems is equal to zero.

Theorem 2.1 *Problems A and A' are equivalent, that is $V = \tilde{V}$.*

Proof. Note that Y_i and Y_j , $i \neq j$, are independent and have the distribution: $P(Y_i = 1) = 1/i$ and $P(Y_i = 0) = (i - 1)/i$ (Rényi (1962)). Moreover, X_i is independent of Y_k for $k = 1, 2, \dots, i - 1$, and the sequences of Y 's and τ 's are also independent. Since the reward $X_i \mathbb{I}(\tau_i \leq T) \mathbb{I}(Y_i = 1)$ is not adapted to the filtration \mathcal{F}_i we can replace it by $\tilde{X}_i = E(X_i \mathbb{I}(\tau_i \leq T) \mathbb{I}(Y_i = 1) | \mathcal{F}_i) = i/(i + 1) \mathbb{I}(\tau_i \leq T) \mathbb{I}(Y_i = 1)$. Since $\tilde{X}_i = 0$ if $Y_i = 0$, and $E(\tilde{X}_{i+1} | \mathcal{F}_i) = 1/(i + 2)(1 - \exp(-(T - \tau_i))) \mathbb{I}(\tau_i \leq T)$, it is never profitable to stop when $Y_i = 0$ and $\tau_i \leq T$. Hence, we infer that $V = \tilde{V}$. ■

Remark 2.1 *Note that the sequences of L 's and τ 's are independent. Moreover, the reward $X_{L_i} \mathbb{I}(\tau_{L_i} \leq T)$ is not adapted to the filtration $\tilde{\mathcal{F}}_i$, so we can replace it by $\tilde{X}_i = E(X_{L_i} \mathbb{I}(\tau_{L_i} \leq T) | \tilde{\mathcal{F}}_i)$. From the definition of the random variable L_n we have*

$$\tilde{X}_i = \frac{L_i}{(L_i + 1)} \mathbb{I}(\tau_{L_i} \leq T).$$

Therefore, the Problem A' is equivalent to finding an optimal stopping time $\tilde{\sigma} \in \tilde{\mathcal{M}}$, that is $E(\tilde{X}_{\tilde{\sigma}}) = \tilde{V}$.

In what follows we will use the below definition and theorem concerning the monotone case. Let

$$A_n = \{\tilde{X}_n \geq E(\tilde{X}_{n+1} | \tilde{\mathcal{F}}_n)\}, \quad n = 1, 2, \dots$$

Definition 2.1 *We say that we are in the monotone case if $A_1 \subset A_2 \subset \dots$ and $P(\bigcup_{n=1}^{\infty} A_n) = 1$.*

Set

$$s = \inf\{n \geq 1 : \tilde{X}_n \geq E(\tilde{X}_{n+1} | \tilde{\mathcal{F}}_n)\}. \quad (1)$$

Theorem 2.2 *In the monotone case, if $s \in \tilde{\mathcal{M}}$ and*

$$\liminf_{n \rightarrow \infty} \int_{\{s > n\}} \tilde{X}_n^+ dP = 0, \quad (2)$$

then $E(\tilde{X}_s) > E(\tilde{X}_r)$ for all $r \in \tilde{\mathcal{M}}$ for which

$$\liminf_{n \rightarrow \infty} \int_{\{r > n\}} \tilde{X}_n^- dP = 0. \quad (3)$$

Proof. See Theorem 3.3 from Chow, Robbins, and Siegmund (1971, p. 55). ■

Theorem 2.3 *The stopping time*

$$\tilde{\sigma} = \inf\{n \geq 1 : \tau_{L_n} \geq T - \ln(2L_n + 1)\},$$

where $\inf \emptyset = +\infty$, is optimal for Problem A'.

Proof. Note that (L_n, τ_n) is a Markov chain and $P(L_1 = 1) = 1$,

$$P(L_2 = k_2) = P(L_2 = k_2 | L_1 = 1) = \frac{1}{k_2(k_2 - 1)}, \quad (4)$$

$$P(L_n = k_n | L_{n-1} = k_{n-1}) = \frac{k_{n-1}}{k_n(k_n - 1)}, \quad (5)$$

for $1 < k_2 < k_3 < \dots$. Hence,

$$E(\tilde{X}_{n+1} | \tilde{\mathcal{F}}_n) = \frac{2L_n + 1}{2(L_n + 1)} (1 - \exp(-(T - \tau_{L_n}))) \mathbb{I}(\tau_{L_n} \leq T).$$

In Problem A', $\{\tilde{X}_n \geq E(\tilde{X}_{n+1} | \tilde{\mathcal{F}}_n)\} = \{T \geq \tau_{L_n} \geq T - \ln(2L_n + 1)\} \cup \{\tau_{L_n} > T\} = \{\tau_{L_n} \geq T - \ln(2L_n + 1)\}$ because $\{\tau_{L_n} > T\} \subset \{\tau_{L_n} \geq T - \ln(2L_n + 1)\}$. To show that $A_n \subset A_{n+1}$ it is enough to note that if $\tau_{L_n} \geq T - \ln(2L_n + 1)$, then $\tau_{L_{n+1}} \geq T - \ln(2L_{n+1} + 1)$. Moreover,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \lim_{n \rightarrow \infty} P(\tau_{L_n} < T - \ln(2L_n + 1)) \geq 1 - \lim_{n \rightarrow \infty} P(\tau_n < T) = 1.$$

Condition (3) is satisfied for all $r \in \tilde{\mathcal{M}}$ because $\tilde{X}_i \geq 0$ for all $i \geq 1$. Condition (2) is satisfied because of the following inequalities:

$$0 \leq \int_{\{s > n\}} \tilde{X}_n^+ dP \leq P(\tau_{L_n} < T - \ln(2L_n + 1)) \leq P(\tau_n < T).$$

Note that $P(\tilde{\sigma} < \infty) = 1$, hence $\tilde{\sigma}$ is optimal stopping time in $\tilde{\mathcal{M}}$. ■

To compute the optimal mean reward we need to find the distribution of the random variable $L_{\tilde{\sigma}}$. Note that $\{L_{\tilde{\sigma}} = k\} = \{L_{\tilde{\sigma}} = k\} \cap \{\tilde{\sigma} \leq k\}$, hence

$$P(L_{\tilde{\sigma}} = k) = P(L_1 = k, \tilde{\sigma} = 1) + \sum_{i=2}^k P(L_i = k, \tilde{\sigma} = i)$$

$$\begin{aligned}
&= \mathbb{I}(k = 1)\mathbb{I}(T - \ln 3 > 0) \exp(-T + \ln 3) \\
&+ \sum_{i=2}^k \left(\frac{1}{k(k-1)} \sum_{m=1}^{k-1} \left(mP(L_{i-1} = m)\mathbb{I}(T - \ln(2m+1) > 0) \right. \right. \\
&\times \left. \left. \left(1 - \exp(-T + \ln(2m+1)) \sum_{j=0}^m \frac{(T - \ln(2m+1))^j}{j!} \right) \right) \right. \\
&\left. - P(L_i = k)(1 - \exp(-T + \ln(2k+1)))\mathbb{I}(T - \ln(2k+1) > 0) \right). \quad (6)
\end{aligned}$$

The probability $P(L_i = k)$ for $i \in \{1, 2\}$ is obtained from (4), for $i \geq 3$ it can be computed by recursion from the following relation:

$$P(L_i = k) = \sum_{n=1}^{k-1} \frac{n}{k(k-1)} P(L_{i-1} = n).$$

Theorem 2.4 *The optimal mean reward*

$$\tilde{V} = \sum_{k=1}^{\infty} \left[\frac{k}{k+1} \left(1 - \exp(-T) \sum_{i=1}^{k-1} \frac{T}{i!} \right) P(L_{\tilde{\sigma}} = k) \right],$$

where $P(L_{\tilde{\sigma}} = k)$ is given by (6).

Proof. Proof follows from equality $\tilde{V} = E(\tilde{X}_{\tilde{\sigma}}) = E\left(\frac{L_{\tilde{\sigma}}}{L_{\tilde{\sigma}}+1} E(\mathbb{I}(\tau_{L_{\tilde{\sigma}}} \leq T) | L_{\tilde{\sigma}})\right)$ and Remark 2.1. ■

Proposition 2.1 *The stopping time $\sigma = L_{\tilde{\sigma}}$ is optimal for Problem A.*

3 Model with Poisson process and limited number of applicants

In the section we consider two equivalent optimal stopping problems with Poisson stream of offers and limited number of observations.

Let the assumptions of Section 2 be satisfied. Let \mathcal{F}_n , $\tilde{\mathcal{F}}_n$, \mathcal{M} , $\tilde{\mathcal{M}}$ and L_n be defined as in Section 2. Moreover, assume that the number N of available applicants is known and fixed. We will assume that $N \geq 2$, because the case $N = 1$ is trivial.

Problem B. Let $X_i \mathbb{I}(\tau_i \leq T) \mathbb{I}(Y_i = 1) \mathbb{I}(i \leq N)$ be the reward for the DM making selection at τ_i . The aim of the DM is to find

$$V^N = \sup_{t \in \mathcal{M}} E(X_t \mathbb{I}(\tau_t \leq T) \mathbb{I}(Y_t = 1) \mathbb{I}(t \leq N))$$

and $\sigma_N \in \mathcal{M}$ such that $E(X_{\sigma_N} \mathbb{I}(\tau_{\sigma_N} \leq T) \mathbb{I}(\sigma_N \leq N) \mathbb{I}(Y_{\sigma_N} = 1)) = V^N$.

Problem B'. Let $\tilde{X}_i^N = L_i/(L_i + 1) \mathbb{I}(\tau_{L_i} \leq T) \mathbb{I}(L_i \leq N)$ be the reward for the DM making selection at τ_i . The aim of the DM is to find

$$\tilde{V}^N = \sup_{\tilde{t} \in \tilde{\mathcal{M}}} E(\tilde{X}_{\tilde{t}}^N),$$

and $\tilde{\sigma}_N \in \tilde{\mathcal{M}}$ such that $E(\tilde{X}_{\tilde{\sigma}_N}^N) = \tilde{V}^N$.

Note that if $N(T) = 0$, then the payoff of any strategy in the above problems is equal to zero.

Theorem 3.1 *Problems B and B' are equivalent, that is $V^N = \tilde{V}^N$.*

Theorem 3.2 *The stopping time*

$$\tilde{\sigma}_N = \inf\{n \geq 1 : L_n \geq N \text{ or } \tau_{L_n} > T \text{ or } g_N(L_n) \geq 1 - \exp(-(T - \tau_{L_n}))\},$$

where $\inf \emptyset = +\infty$, is optimal for Problem B'.

Proof. Note that

$$\begin{aligned} E(\tilde{X}_{n+1}^N | \tilde{\mathcal{F}}_n) &= \frac{L_n}{2} \left(\frac{1}{L_n} + \frac{1}{L_n + 1} - \frac{1}{N} - \frac{1}{N + 1} \right) \\ &\times (1 - \exp(-(T - \tau_{L_n}))) \mathbb{I}(\tau_{L_n} \leq T) \mathbb{I}(L_n \leq N - 1). \end{aligned}$$

Hence,

$$\begin{aligned} &\{\tilde{X}_n^N \geq E(\tilde{X}_{n+1}^N | \tilde{\mathcal{F}}_n)\} \\ &= \{L_n \geq N\} \cup \{\tau_{L_n} > T\} \cup \{g_N(L_n) \geq 1 - \exp(-(T - \tau_{L_n}))\}, \end{aligned}$$

where $g_N(L_n) = 2L_n(2L_n + 1 - L_n(L_n + 1)(1/N + 1/(N + 1)))^{-1}$. Let us note that the random variable $g_N(L_n)$ is increasing with respect to n because it is increasing with respect to L_n and L_n is increasing with respect to n , and $g_N(L_n)$ is greater than $\frac{2}{3}$ for all n such that $L_n < N$, a.e.. Moreover, $1 - \exp(-(T - \tau_{L_n}))$ is decreasing with respect to n , a.e.. Considerations similar to those in Section 2 show that we are in the monotone case. To show it, it is enough to notice that $P(\tilde{X}_n^N \geq E(\tilde{X}_{n+1}^N | \tilde{\mathcal{F}}_n)) \geq P(\tau_{L_n} > T) = 1 - P(\tau_{L_n} \leq T)$ and $\tilde{X}_i^N \leq \tilde{X}_i$. Note that $P(\tilde{\sigma}_N < \infty) = 1$, hence $\tilde{\sigma}_N$ is optimal stopping time in $\tilde{\mathcal{M}}$. ■

To compute the optimal mean reward we need to find the distribution of the random variable $L_{\tilde{\sigma}_N}$. Note that

$$\begin{aligned} P(L_{\tilde{\sigma}_N} = k) &= \mathbb{I}(L_1 = k)P(\{g_N(1) \geq 1 - \exp(-(T - \tau_1))\} \cup \{\tau_1 > T\}) \\ &+ \sum_{i=2}^k P(L_i = k, L_{i-1} < N, \tau_{L_{i-1}} \leq T, g_N(L_{i-1}) < 1 - \exp(-(T - \tau_{L_{i-1}})), \\ &(\{L_i \geq N\} \cup \{\tau_{L_i} > T\} \cup \{g_N(L_i) \geq 1 - \exp(-(T - \tau_{L_i}))\})). \end{aligned}$$

The above probabilities satisfy

$$\begin{aligned} &P(\{g_N(1) \geq 1 - \exp(-(T - \tau_1))\} \cup \{\tau_1 > T\}) \\ &= \mathbb{I}(N \in \{2, 3\}) + \mathbb{I}(N \geq 4) \exp(-((T + \ln(1 - g_N(1))) \vee 0)), \end{aligned}$$

and

$$\begin{aligned} &P(L_i = k, L_{i-1} < N, \tau_{L_{i-1}} \leq T, g_N(L_{i-1}) < 1 - \exp(-(T - \tau_{L_{i-1}})), \\ &(\{L_i \geq N\} \cup \{\tau_{L_i} > T\} \cup \{g_N(L_i) \geq 1 - \exp(-(T - \tau_{L_i}))\})) \\ &= \sum_{m=1}^{N-1} \left(\frac{mP(L_{i-1} = m)}{k(k+1)} \mathbb{I}\left(m(m+1) < \frac{N(N+1)}{2N+1}\right) (1 - \exp(h_N(m, T))) \right. \\ &\times \sum_{n=0}^{m-1} \frac{h_N(m, T)^n}{n!} \left. - \mathbb{I}(k < N) P(L_i = k) \mathbb{I}\left(k(k+1) < \frac{N(N+1)}{2N+1}\right) \right. \\ &\times (1 - \exp(h_N(k, T))) \sum_{n=0}^{k-1} \frac{h_N(k, T)^n}{n!}, \end{aligned} \quad (7)$$

where $h_N(k, T) = (T \wedge (T + \ln(1 - g_N(k)))) \vee 0$.

Theorem 3.3 *The optimal mean reward*

$$\tilde{V}^N = \sum_{k=1}^N \left[\frac{k}{k+1} \left(1 - \exp(-T) \sum_{i=1}^{k-1} \frac{T^i}{i!} \right) P(L_{\tilde{\sigma}_N} = k) \right],$$

where the distribution of $L_{\tilde{\sigma}_N}$ is given by (7).

Proposition 3.1 *The stopping time $\sigma_N = L_{\tilde{\sigma}_N}$ is optimal for Problem B.*

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Appendix

In Appendix we consider two equivalent optimal stopping problems without a Poisson process.

Assumptions. Assume that $\{X_i\}_{i=1}^\infty$ is the sequence of i.i.d. random variables with uniform distribution on $(0, 1)$. Let Y_n and L_n , $n = 1, 2, \dots$ be defined as in Section 2. Moreover, assume that the number N of available applicants is known and fixed and the applicants are interviewed sequentially in a random order and all orders are equally likely.

Problem C. Let $\mathcal{F}_n^* = \sigma(Y_1, \dots, Y_n)$. Denote by \mathcal{M}^* the set of all stopping times t^* adapted to the sequence $\{\mathcal{F}_n^*\}_{n=1}^\infty$. Let $X_i \mathbb{I}(Y_i = 1) \mathbb{I}(i \leq N)$ be the reward for the DM for accepting the i -th applicant. The aim of the DM is to find the optimal mean reward

$$V^* = \sup_{t^* \in \mathcal{M}^*} E(X_{t^*} \mathbb{I}(Y_{t^*} = 1) \mathbb{I}(t^* \leq N))$$

and an optimal stopping time $\sigma^* \in \mathcal{M}^*$, that is $E(X_{\sigma^*} \mathbb{I}(\sigma^* \leq N) \mathbb{I}(Y_{\sigma^*} = 1)) = V^*$.

Problem C'. Let $\tilde{\mathcal{F}}_n^* = \sigma\{L_1, \dots, L_n\}$. Denote by $\tilde{\mathcal{M}}^*$ the set of all stopping times \tilde{t}^* adapted to the sequence $\{\tilde{\mathcal{F}}_n^*\}_{n=1}^\infty$. Let $\tilde{X}_i^* = L_i / (L_i + 1) \mathbb{I}(L_i \leq N)$. The aim is to find the optimal mean reward

$$\tilde{V}^* = \sup_{\tilde{t}^* \in \tilde{\mathcal{M}}^*} E(\tilde{X}_{\tilde{t}^*}^*)$$

and an optimal stopping time $\tilde{\sigma}^* \in \tilde{\mathcal{M}}^*$ such that $E(\tilde{X}_{\tilde{\sigma}^*}^*) = \tilde{V}^*$.

Theorem 3.4 *Problems C and C' are equivalent, that is $V^* = \tilde{V}^*$.*

Let us remind that the difference between this model and Bearden' model is that we are interested in stopping on the best applicant so far while the DM in Bearden' model is interested in stopping on the best applicant so far or on the last one. Hence, we analyze the influence of the decision making in

the last step. The problem can be applied in situation where we know the number of observations and we are interested in choosing the best one from the interviewed to that time or none.

Theorem 3.5 *The stopping time*

$$\tilde{\sigma}^* = \inf \{n \geq 1 : L_n \geq n_0\},$$

where

$$n_0 = \left\lceil \sqrt{\frac{N(N+1)}{(2N+1)} + \frac{1}{4}} - \frac{1}{2} \right\rceil$$

and $\inf \emptyset = +\infty$, is optimal for Problem C'.

Proof. Note that

$$E(\tilde{X}_{n+1}^* | \tilde{\mathcal{F}}_n^*) = \frac{L_n}{2} \left(\frac{1}{L_n} + \frac{1}{L_n+1} - \frac{1}{N} - \frac{1}{N+1} \right) \mathbb{I}(L_n \leq N-1).$$

Hence,

$$\begin{aligned} \{\tilde{X}_n^* \geq E(\tilde{X}_{n+1}^* | \tilde{\mathcal{F}}_n^*)\} &= \left\{ L_n(L_n+1) \geq \frac{N(N+1)}{2N+1} \right\} \cup \{L_n \geq N\} \\ &= \left\{ L_n(L_n+1) \geq \frac{N(N+1)}{2N+1} \right\} = \{L_n \geq n_0\}. \end{aligned}$$

Note that $P(\tilde{\sigma}^* < \infty) = 1$ and all assumptions of Theorem 2.2 are satisfied, hence $\tilde{\sigma}^*$ is optimal stopping time in $\tilde{\mathcal{M}}^*$. ■

Proposition 3.2 *The stopping time $\sigma^* = \inf\{n \geq n_0 : Y_n = 1\}$ is optimal for Problem C.*

Theorem 3.6 *For Problem C,*

$$V^* = \frac{1}{2} + \frac{n_0 - 1}{2} \left(\frac{1}{n_0} - \frac{1}{N} - \frac{1}{N+1} \right)$$

is the optimal mean reward.

Proof. The proof follows from the observation that

$$V^* = \sum_{k=n_0}^N P(Y_k = 1) E(X_k | Y_k = 1) \prod_{i=n_0}^{k-1} P(Y_i = 0) = \sum_{k=n_0}^N \left(\frac{1}{k+1} \right) \left(\frac{n_0 - 1}{k-1} \right). \blacksquare$$

If we are interested in finding the best candidate among those interviewed so far or none at all, we should omit the first n_0 candidates and hire the first who is the best so far. Comparing this problem with the problem considered in Bearden (2006), using our procedure we will reject about $0.7\sqrt{N}$ candidates, while in Bearden (2006) we reject about \sqrt{N} applicants.

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