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# Exploring the polycentric city with multi-worker households: an agent-based microeconomic model

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## Abstract

We propose an agent-based dynamics which leads an urban system to the standard equilibrium of the Alonso, Muth, Mills (AMM) framework. Starting for instance from a random initialization, agents move and bid for land, performing a kind of local search and finally leading the system to equilibrium rent, density and land use. Agreement with continuous analytical results is limited only by the discreteness of simulations. We then study polycentrism in cities with this tool. Two job centers are introduced, and the economic, social and environmental outcomes of various polycentric spatial structures are presented. We also introduce two-worker households whose partners may work in different job centers. When various two-worker households are mixed, polycentrism is desirable, as long as the centers are not too distant from each other. The environmental outcome is also positive, but housing surfaces increase.

*Keywords:* urban economics, location choice, polycentric city, two-worker households, agent-based model

*JEL:* R140, R200, C630

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## 1. Introduction

The Earth's population is now predominantly urban, and urban areas are rapidly expanding (Seto et al. (2011)). In addition to social and economic issues, this process raises environmental concerns regarding biodiversity conservation, loss of carbon sinks and energy use. Empirical evidence shows that various urban development patterns significantly influence carbon dioxide emissions (Glaeser and Kahn (2010)). Low density results in increased vehicle usage, while both low density and increased vehicle usage lead to increased fuel consumption (Brownstone and Golob (2009)). Compact urban development would be the natural answer to these issues, but the debate regarding welfare, distributive and environmental aspects is fierce between the opponents and promoters of compact cities (see e.g. Gordon and Richardson (1997); Ewing (1997)). The issue of spatial and social structure and operation of cities has never been so acute, and there is an obvious need to better understand city spatial development (Anas et al. (1998)).

Here, we present an agent-based simulation model that answers the need to overcome the issue of analytical tractability and to consider spatial dynamics and heterogeneity, while being explicitly based

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on microeconomic behavior of agents (Irwin (2010)). This model is grounded in the classic urban bid-rent framework (hereafter referred to as the AMM model): Alonso (1964)'s monocentric model of land market, Muth (1969)'s introduction of housing industry and Mills (1967, 1972) model. This analytical framework has proved its robustness in describing the higher densities, land and housing rents in city centers (Spivey (2008); Mills (2000)), despite its limitations – among others, the monocentric assumption.

Polycentrism (that is the clustering of economic activities in subcenters along with the main center) is indeed a reality, as shown by empirical evidence (for instance Giuliano and Small (1991)). However, introducing polycentrism in the AMM model proves difficult from the point of view of analytical tractability. Wheaton (2004) challenges monocentrism, based on empirical evidence from US cities, which shows that employment is almost as dispersed as residences. However, in this work, simplifying hypotheses are needed for analytical tractability, such as an exogenous density (consumption of land per worker is fixed and independent of location). In other approaches, centers (and sometimes subcenters) are given exogenously (Hartwick and Hartwick (1974); White (1976, 1988); Sullivan (1986); Wieand (1987); Yinger (1992)). In Fujita and Ogawa (1982), no centers are specified a priori and multiple equilibria are shown (monocentric, multicentric or dispersed patterns). Here again, since the model is not analytically tractable, simplifying hypotheses are required (e.g. lot size is fixed). Lucas and Rossi-Hansberg (2002) go further into endogenous polycentrism. A well-shared conclusion of these papers is that numerical simulations are needed.

Regarding income heterogeneity of residents, Straszheim (1987) points out that with multiple classes of bidders, it is difficult to find realistic specifications of income distribution functions which yield tractable results and, again, this requires numerical solutions. Fujita (1989) describes a principle of numerical resolution when the population is divided into several income groups.

Different agent-based models are used in urban economics to study complex models with heterogeneous agents and space, and sometimes applied to real data. Benenson (1998) introduces an (economic) agent-based model of population dynamics in a city but without bid-rent mechanism. Caruso et al. (2007) integrate urban economics with cellular automata in order to simulate peri-urbanization. Huang et al. (2013) use a similar model, with constant density, no relocation and complete market information, to simulate the effects of agent heterogeneity in interaction with the land market. The way land price formation is modeled is crucial as pointed by Chen et al. (2011). Parker and Filatova (2008) design a bilateral landmarket, where the gains of trade are shared between buyers and sellers, which is implemented by Filatova et al. (2009). However in their model lot size is fixed. Ettema (2011) proposes an endogenous modelling of demand, supply and price setting in housing market, but his model is not yet spatially explicit regarding housing location. Huang et al. (2014) propose a classification of agent-based models in urban economics, according to which our model implements agent heterogeneity, explicit land-market representation with bidding and budget constraint, and socioeconomic outcomes. Moreover, compared to previous agent-based models, our own model includes variable endogenous density, agent relocation and imperfect information.

The main methodological innovation of this work is the approach by which we find the equilibrium of urban economic models. Departing from previous agent-based systems in urban economics, we use a method inspired by local search optimization algorithms in computer science (Lenstra (2003)). Starting from a random configuration, the system is led towards the optimum with local moves. Local search algorithms can usually be defined simply in a few words, but proved very efficient in solving complex optimization problems. They are used in combinatorial optimization, and linked more generally with asynchronous dynamics in game theory or statistical physics, but we adapt the method here to the framework of urban economics. There is already some analytical and simulation work in the literature

on simple urban models, close to game theory, like Schelling's spatial segregation model (see for instance Zhang (2011); Grauwin et al. (2012)) or related urban models with a simple description of price (Lemoy et al. (2011)). However, to our knowledge, this is the first time that such ideas are used with agent-based systems in urban economics. These ideas allow us to develop a robust method for solving urban economic problems.

Two research questions are explored in this work. The first one consists in finding what kind of simple agent-based dynamics can lead to the equilibrium of the standard urban economic model. By adapting local search methods to urban economics, we find such dynamics, and we use it to tackle our second research question: what are the socio-economic and environmental outcomes of the polycentric city?

Regarding the first research question, we use here an agent-based model to find which kind of simple dynamics could underlie the standard equilibrium urban economic model. Indeed, cities are dynamic systems, in constant evolution. Therefore, the urban economic equilibrium can be seen as the result of some dynamics. It is clear that this dynamics corresponds to a bidding process, as land goes to the highest bidder in the AMM model. Another feature is that agents move between locations to increase their utility: at equilibrium, no move can allow an agent to increase his utility. On the basis of these points, we propose a simple asynchronous dynamics inspired by local search methods, that leads the urban system from any configuration to the standard AMM equilibrium. Note that while aiming for parsimony and simplicity, we also try to keep the dynamics as realistic as possible.

To answer the second question, we introduce more complexity by adding various components, in keeping with a parsimony principle. The first component is agent heterogeneity through income groups, in order to test the agreement between agent-based and analytical results. The second component is exogenous multiple centers, illustrated by two job centers at various distances from each other, which interact through competition of agents for housing. These exogenous centers may occur "naturally" when dispersive forces such as congestion or other costs of concentration overcome agglomeration economies or may originate from a government assisting subcenter formation, such as "new towns" (Anas et al. (1998)). The third component is another kind of agent heterogeneity, with two-worker households whose partners may work in different job centers.

The economic outcome of the introduction of two centers is shown to be positive, as agents' utility increases when the distance between the centers increases. However, pollution linked to commuting distances decreases first when centers are taken away from each other but then increases again. Simultaneously, the decreasing competition for land results in increasing housing surfaces and thus city size. Moreover, the existence and uniqueness of equilibria in these polycentric models are discussed and various arguments are elaborated to support these features.

The remainder of the paper is structured as follows. Section 2 describes the agent-based model implementation with the microeconomic behavior of agents. Section 3 compares the simulations results with the analytical ones of the AMM model and illustrates the dynamic feature of the model. Section 4 presents the polycentric urban forms with two-worker households and their economic, social and environmental outcomes. In addition, section 4 of the supplementary material discusses the existence and uniqueness of the equilibrium in these models.

## 2. Description of the framework

### 2.1. Urban economic model

The AMM model was developed to study the location choices of economic agents in an urban space, with agents competing for housing (identified with land in the simplest version of the model). Agents

have a transport cost to commute for work. Their workplace is located in a central business district (CBD), which is represented by a point in the urban space. Agents usually represent single workers, but they can also be used to describe households, which can be made more complex in further versions of the model. Housing is rented by absentee landowners who rent to the highest bidder, which introduces a competition for housing between agents. They also compete with an agricultural use of land, which is represented by an agricultural rent  $R_a$ .

Agents have a utility function expressing their trade-off between housing surface and other goods (Fujita (1989)), taken here as a Cobb-Douglas function

$$U = z^\alpha s^\beta, \quad (1)$$

where  $z$  is a composite good representing all consumer goods except housing and transport (whose price is the same throughout the city),  $s$  is the surface of housing,  $\alpha$  and  $\beta$  are parameters describing agents' preferences for composite good and housing surface, with  $\alpha + \beta = 1$ . Agents also have a budget constraint

$$Y = z + tx + ps, \quad (2)$$

where  $Y$  is their income,  $t$  the transport cost per unit distance,  $x$  the distance from their housing location to the CBD and  $p$  the price of a unit surface of land at location  $x$ . See Fujita (1989) for a detailed presentation of the general model.

The analytical model reproduced in this work with agent-based simulations is a closed city model, where the population size  $N$  is chosen exogenously and remains constant during a simulation<sup>1</sup>. This model can be solved analytically in a two-dimensional space if  $R_a = 0$ . For  $R_a > 0$ , a simple numerical resolution can be performed with one income group (Fujita (1989)). With a population divided into several income groups, a specific algorithm is needed for the resolution of the model.

## 2.2. Agent-based implementation

Our agent-based system uses the framework of asynchronous dynamics and local search optimization, which can be broadly described as follows. An algorithm is designed to find the optimum of a function  $F$  – for instance an energy function. This function  $F$  depends on the state  $S = (s_1, \dots, s_N)$  of  $N$  individual components  $s_1, \dots, s_N$ , which are variables describing particles or agents for instance. A generic algorithm is described here in pseudocode:

- 1:  $S$  = random assignment of values to the variables
- 2: **while**  $S$  is not an optimum, and a maximal search time  $T$  is not reached **do**
- 3:    $s_i$  = a variable  $i \in [1, N]$  selected uniformly at random
- 4:    $\Delta F$  = change in the function  $F$  to optimize if variable  $s_i$  is changed in  $S$
- 5:   With a probability  $p(\Delta F)$  depending on  $\Delta F$ : change variable  $s_i$  in  $S$
- 6: **end while**

The agent-based model presented in this work uses this same framework to reach the equilibrium of the AMM model. This equilibrium corresponds to a maximum homogenous utility for all agents. However in our case, the change of variable  $s_i$  in the fifth step described above is not probabilistic but deterministic. Making it probabilistic with a logit rule, for instance, is a possible perspective of work. The "local" characteristic of local search optimization corresponds to the fact that at each time step, changes in the

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<sup>1</sup>We depart in this from Wu and Plantinga (2003) and Tajibaeva et al. (2008).

system are very localized (i.e. one agent moving or none), so that the states of the system before and after the move are neighbors in the space of all states of the system.

We now describe our agent-based implementation of the standard monocentric AMM model. The simulation space is a two-dimensional grid where each cell can be inhabited by one or several agents, or used for agriculture. These cells correspond to our fixed discretization of the two-dimensional space. The cells have a fixed exogenous land surface  $s^{\text{tot}}$  and the unit of distance in the model is taken as the side length of a cell. The CBD is a point at the center of the grid. Prices and distances to the center are defined only with respect to each cell. However, because several agents may reside in a cell and the housing lot size is endogenous, the density is also endogenous.

At the initialization, a population of  $N$  agents placed at random locations is created. The initial land price in each cell  $p_0$  is equal to the agricultural rent  $R_a$ . At a given location  $x$ , agents occupy an endogenous quantity of land which is the optimal consumption of land conditional on price  $p$  (Fujita (1989)):

$$s = \beta \frac{Y - tx}{p}. \quad (3)$$

This allows us to determine the quantity of composite good they consume, and their utility, using equations (1) and (2).

### 2.2.1. Dynamics of moves and bids for rent

The main feature of the model consists of agent-based dynamics of moves and bids in the urban space. The rules defining agents' moves are inspired by local search methods and asynchronous dynamics described above, but such models do not have price formation mechanisms. Hence, we introduce price formation in the spirit of tâtonnement processes of Walrasian auctions with successive price adjustments: prices are lowered for goods with excess supply, and raised for goods with excess demand. However, there is no auctioneer as the dynamics is defined at the agent level and the information is very limited. By construction, demand and supply of land are always matched, at any given time. The main aim of the dynamics is that it finds the standard urban economic equilibrium (equilibrium rent, density and utility), even though only the utility function and the demand function for land (optimal consumption of land conditional on price and distance to the center) are inputs of the model. To keep the behaviour rules as parsimonious as possible, and in line with local search methods where variables are often updated considering only minimal information, we assume that agents have a very limited knowledge of the market: when deciding to move or not, the only information available to them is their current utility, and the utility they would have in the new location they consider. In section 3, we show that this is enough to reach the standard equilibrium: (spatially) myopic agents moving based on minimal information manage to find the optimum, which can also be linked to self-organization and swarm intelligence ideas (Dorigo et al. (2008)).

Let us describe an iteration  $n$  of the algorithm, changing the variables from their value at step  $n$  to their value at step  $n + 1$ . An agent, which will be candidate to a move, and a cell are chosen randomly. The price per unit of surface in this cell, located at a distance  $x$  of the center, is  $p_n$  at step  $n$  ( $p_0 = R_a$  at initialization). The optimal housing surface that the agent can choose in the candidate cell is given by equation (3)  $s_n = \beta \frac{Y - tx}{p_n}$ , which allows us to compute her composite good consumption and the utility that she would get from the move, using equations (1) and (2). Agents move with no cost, as in the AMM model (see Fujita (1989)).

The bidding process for renting by the agent is chosen as follows. If the candidate agent can have a

utility gain  $\Delta U > 0$  by moving into the candidate cell, then she bids for renting at the price

$$p_{n+1} = p_n \left( 1 + \epsilon \frac{s_n^{\text{occ}}}{s^{\text{tot}}} \frac{\Delta U}{U} \right), \quad (4)$$

where  $\epsilon$  is a parameter that we introduce to control the magnitude of this bid. Since landowners rent to the highest bidder, the price of the candidate cell is raised. The higher parameter  $\epsilon$  is, the faster cell prices evolve.  $s_n^{\text{occ}}$  is the surface of land occupied by other agents in the cell at step  $n$ , and  $s^{\text{tot}}$  the total land surface of the cell. The factor  $s_n^{\text{occ}}/s^{\text{tot}}$ , smaller than 1, is the occupancy ratio of the cell, and makes the bid lower if the cell is less occupied, in other words less attractive. Because of this factor, the first agent to move in an empty cell does not raise the price. This factor can however be removed, at the cost of a slowing down in the convergence to equilibrium.

The price is a price per unit surface, linked to a cell. When an agent bids higher, we assume that the price is immediately changed for all agents in the target cell. Their consumption of land is also changed (decreased) according to equation (3)  $s_{n+1} = \beta \frac{Y - tx}{p_{n+1}}$ , and their utility is computed again. This feature of the model defines a competition for land between agents and a market price, as in the standard analytical model.

If there is enough space for the candidate agent in the target cell (that is, if  $s_{n+1} \leq s^{\text{tot}} - s_{n+1}^{\text{occ}}$ ), then the agent moves in and occupies a share  $s_{n+1}$  of the cell surface. In case the cell is already full ( $s_{n+1} > s^{\text{tot}} - s_{n+1}^{\text{occ}}$ ), the candidate agent does not move in, but the price is still raised, and housing surfaces and utility of agents are updated.

### 2.2.2. Surface constraint, time decrease of price

We described how prices increase in the model for locations with high demand. Conversely, locations with low demand should have a decreasing price. Indeed, because of the stochastic choice of agents and cells, prices can rise above their equilibrium level at some locations, making the corresponding cells unattractive. For instance, the price of a cell where several agents move in successively can increase so much that it provides agents with a low utility. In this case, agents living there will progressively leave the cell for more attractive locations (through the dynamics described in section 2.2.1).

To reflect the behavior of landlords who may have to decrease the price of their property to attract tenants again, we assume that the price of cells where there is free place to accommodate one or more agents decreases exponentially. This is done according to

$$p_{n+1} = \max \left( p_n - \frac{(p_n - R_a \times 0.9)}{T_p} \cdot \frac{s_n^{\text{av}}}{s^{\text{tot}}}, R_a \right), \quad (5)$$

where  $T_p$  is a parameter determining the speed of decrease of prices and  $s_n^{\text{av}} = s^{\text{tot}} - s_n^{\text{occ}}$  is the available (nonoccupied) surface at step  $n$  in the cell. If no agent moves in, the price decreases according to the first argument of the right-hand side of equation (5) until it reaches the agricultural rent, which occurs after a finite time because of the form used, where the asymptotic price is  $R_a \times 0.9$  instead of  $R_a$ . This form for the decrease, slower when the price is closer to the agricultural rent, is chosen empirically to ensure a smooth convergence to the equilibrium of the model. The factor  $s_n^{\text{av}}/s^{\text{tot}}$ , smaller than 1, makes this time decrease slower when the cell is closer to be full, and thus more attractive. This factor can also be removed at the cost of a slower convergence to equilibrium.

The whole agent-based mechanism leading the system to the standard equilibrium is actually contained in equations (4) and (5), which introduce one additional parameter each ( $\epsilon$  and  $T_p$ ) to the standard model.

Parameters	Description	Default value
$\alpha, \beta$	Preferences for composite good and housing	0.75; 0.25
$Y_p, Y_r$	Incomes of poor and rich agents	300, 450
$t$	Transport cost (unit distance)	10
$N$	Total population size	10000
$R_a$	Agricultural rent	10
$s^{\text{tot}}$	Surface of a cell	30
$\epsilon$	Bidding parameter	0.5
$T_p$	Time decrease of the price of non-full cells	30

Table 1: Parameters of the simulations

### 2.2.3. Parameters

The different parameters of the model are listed in Table 1. Most parameters belong to the AMM model itself:  $\alpha$ ,  $\beta$ ,  $Y$ ,  $t$ ,  $N$ ,  $R_a$ ,  $s^{\text{tot}}$ . Their values are chosen arbitrarily, because the model is not calibrated on real data. However, it can be noted that a high population  $N$  could have improved the agreement between the analytical and the agent-based model, but it would have been costly in terms of computation time. Parameters  $\epsilon$  and  $T_p$  are specific to the agent-based model. They determine the scales of price increase and decrease, respectively. They may also be seen as reflecting the relative market power of landlords and tenants. Their values have been chosen such that the competition between agents in the housing market is efficient and the system reaches the equilibrium for the whole city, as this is our primary goal. This will be discussed in more detail in section 3.3.

The present study focuses on the equilibrium of the agent-based model, which is shown to correspond to the equilibrium of the analytical AMM model. Thus, the agent-based dynamics is mainly presented in this work as a resolution method. Comparison with the dynamics of real urban housing markets is a perspective for further work.

### 2.3. Socioeconomic outcomes

To study the urban social structure and the socioeconomic outcomes of the various models developed here, we focus on some variables of the model, which characterize these outcomes. Our benchmark is a reference simulation of a monocentric city.

Three kinds of outcomes are studied. The utility of individuals is associated to their welfare and gives an economic outcome of the models. The cumulated distances of agents' commuting to work  $D_{\text{tot}}$ , associated with cumulated housing surfaces  $S_{\text{tot}}$ , give their environmental outcome, which could be conveyed for instance in terms of greenhouse gases emissions associated to transport, land use and housing (heating and cooling). The evolution of social inequalities can be given by the difference in the utility of individuals belonging to different income groups. The agent-based framework gives an easy access to any other individual or global variable of the model, such as land rents for instance.

## 3. Comparison with the analytical model and temporal evolution

### 3.1. Results with two income groups: model 1

The simulations allow us to reach the equilibrium of the AMM model (or more precisely the Muth-Mills equilibrium, see Brueckner (1987)), as can be seen on figure 1. This equilibrium corresponds to a configuration where no agent can raise her utility by moving. In each income group, individuals have an identical utility across the city. With two income groups, the utility of "rich" agents is still higher than that of "poor" agents, because they do not have the same income. A gap can be observed on the



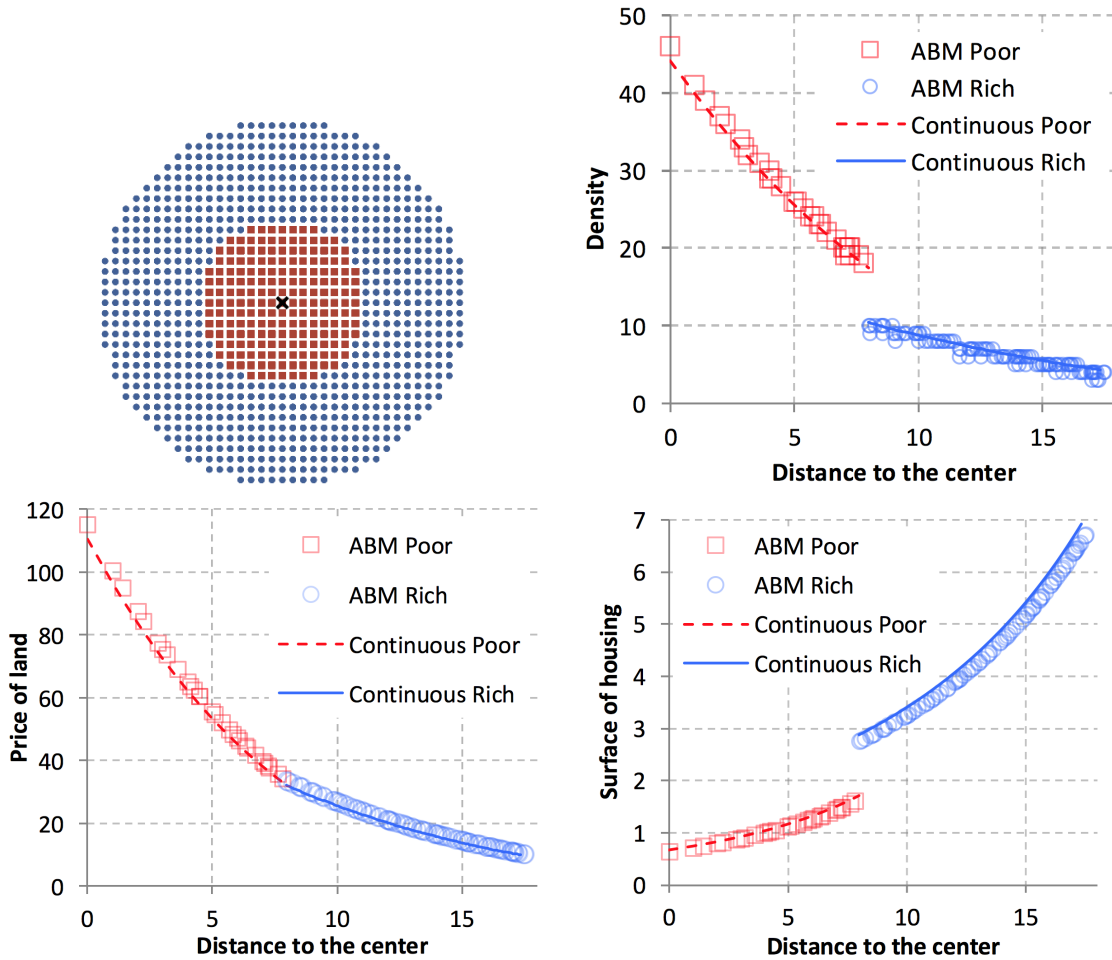


Figure 1: Top left panel, shape of the city. Poor agents are represented by squares, rich agents by circles. Other panels: comparison between the results of the agent-based model (ABM) and the continuous results. Density (number of agents per cell), land rents and housing surfaces as functions of the distance to the center. The lines represent the continuous results, whereas the symbols indicate the results of the agent-based model. Parameter values are given in table 1.

density and housing surface curves, because of this discrete difference in income, and equation (3). As in the analytical equilibrium, rich agents are located at the periphery of the city, where they pay lower land prices and have higher housing surfaces, but also with higher transport costs. This is the standard result of the AMM model and it reproduces the socio-spatial pattern observed in new North-American cities (Glaeser et al. (2008)). The equilibrium of the agent-based model is described in more detail in the following sections. Let us first depict rapidly the temporal evolution of the simulation.

### 3.2. Dynamics

The evolution of the agent-based model shows how a "city" emerges from the interactions between individuals during a simulation. Initially, agents are located at random and all prices are equal to the agricultural rent. Density is quite low as agents are dispersed over the simulation space. Clearly, cells are far from being full, space is not optimally used, and utility is not uniform (see Figure 2 of the supplementary material). The agents then move mainly towards the CBD as shown on Figure 2 and bid higher, so that the rent curve evolves from a flat rent to the equilibrium rent. At the beginning of

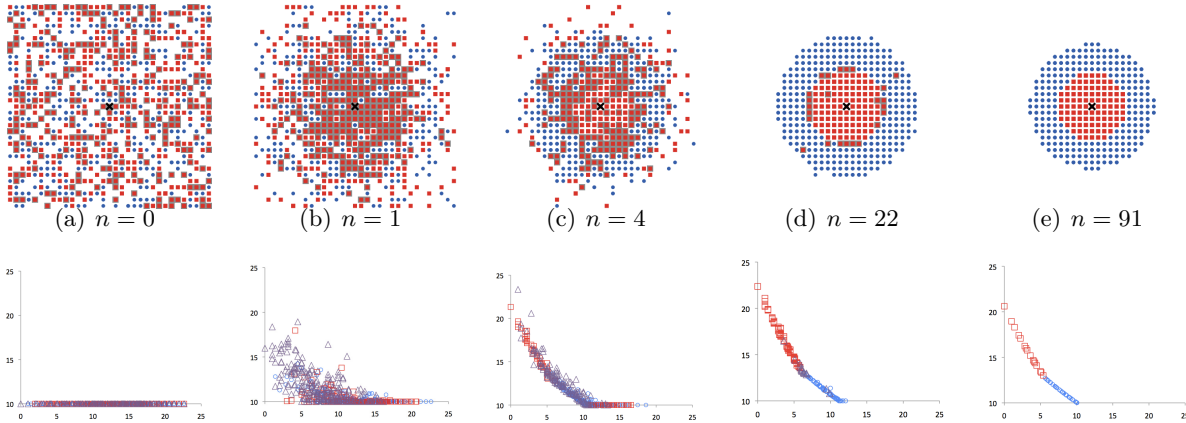


Figure 2: Evolution of the shape of the city (first row) and of the price of land as a function of the distance to the center (second row) during a simulation. Same symbols as Figure 1. On the first row, cells whose background is grey indicate that poor and rich agents live there; these cells are displayed as triangular symbols on the second row. At the equilibrium, the city is completely segregated and there are no more such cells.  $n$  indicates here the mean number of moves per agent since the beginning of the simulation.

the simulation (Figures 2(a) and 2(b)), agents gather at the city center without competing much for land, because many cells close to the center are still not full. However, when all agents are concentrated around the center (from Figure 2(d) on), most bids do not result in an agent moving, because few cells have a sufficient available surface to allow an agent to move in with an interesting utility.

The main variable that indicates the proximity to the equilibrium is the homogeneity of the utility of agents. To describe this homogeneity, we use the relative inhomogeneity of the utility defined as  $\Delta U_{\max} = (U_{\max} - U_{\min})/U_{\max}$ , where  $U_{\max}$  (resp.  $U_{\min}$ ) is the highest (resp. lowest) utility among agents. With two income groups, we compute this variable within each income group and keep the highest of both values. During a simulation,  $\Delta U_{\max}$  has a decreasing value. We choose to stop the simulations when the relative variations of utility within the income groups are smaller than  $10^{-6}$ , which means that  $\Delta U_{\max}$  has decreased by approximately five orders of magnitude, as shown in Figure 1 of the supplementary material.

### 3.3. Analytical and agent-based equilibria

In section 4, we will present our simple polycentric models and their results, and in section 4 of the supplementary material, we argue that the analytical equilibrium of each of these models exists and is unique. Thus, we only need to ensure that the agent-based model can reach a discrete version of this equilibrium.

Figure 1 shows that it is the case for the standard monocentric AMM model with two income groups. The main difference with the analytical AMM model then lies in the discreteness of the simulated model, as opposed to the continuous analytical model. This provides an illustration of a discrete model converging to the continuous AMM model for large population sizes, which can be related to a discussion in the literature (Asami et al. (1991); Berliant and Sabarwal (2008)). Let us now describe more precisely the hypotheses ensuring that the agent-based model reaches an equilibrium that is similar to the analytical one. Section 3.2 defines the equilibrium of the agent-based model as a situation in which utility is homogeneous within each group, ensuring that no agent has an incentive to move. However, this condition alone does not guarantee that the equilibrium is reached, as shown in section 3 of the

supplementary material. Indeed, an additional condition is needed: that every cell is optimally used, either for agriculture or for housing.

From the comparison with the analytical equilibrium, it follows that only two situations should be observed at equilibrium for the cells of the agent-based model. The first is the case of an agricultural cell, whose price should be equal to the agricultural rent, and where no agent should reside. The second case is a residential cell, where no space should be left for another agent to move in. Indeed, if the cell can accommodate (at least) one more agent, it indicates that equilibrium is not reached as the city could be made more compact, providing a higher utility for agents.

To monitor the share of urban land that is not optimally used, we use the ratio of a surface we call "empty"  $S_{\text{empty}}$  to the total housing surface of agents  $S_{\text{tot}}$  (i.e. the total surface of the city). Let us now describe how this "empty" surface is computed. Each cell of the simulation space is visited. If the cell has inhabitants, the smallest housing surface of inhabitants,  $s_{\text{min}}$ , is stored. If the surface still available in the cell  $s_{\text{av}}$  is greater than  $s_{\text{min}}$ , then a part of the cell is considered "empty". To determine the exact amount of available surface, we compute how many agents with housing surface  $s_{\text{min}}$  could still fit in the cell. The corresponding surface is considered "empty". The values of parameters  $\epsilon$  and  $T_p$  are chosen to minimize the quotient  $S_{\text{empty}}/S_{\text{tot}}$  within an acceptable simulation time. This quotient is checked to be smaller than 0.5% at the equilibrium of the simulations presented in this paper. We also check that every cell without inhabitants has a price that is equal to the agricultural rent.

With these conditions, the system converges to a unique equilibrium, as described in section 2 of the supplementary material, whose section 3 presents, as an illustration, a simulation using values of parameters  $\epsilon$  and  $T_p$  that do not allow the system to reach a state where  $S_{\text{empty}}/S_{\text{tot}} < 0.5\%$ .

## 4. Polycentric city and two-worker households: model 2

The agent-based mechanism introduced in this work is robust enough for us to study phenomena which are difficult to treat analytically, like polycentrism. This agent-based dynamics is surely not the only way to simulate the following simple polycentric models. It is robust enough to deal with more complex models, with more heterogeneity and more endogenous mechanisms, like building construction or the location of firms. However, with the goal of a methodical and parsimonious approach to urban modelling, some comparatively simple results about the polycentric city should be established first, to serve as benchmarks to build on for further work. We aim to do this with the simple exogenous polycentric models presented here.

### 4.1. Introducing two job centers

The simplest way to study a polycentric city consists in defining two employment centers, separated by a distance  $d$ . Agents work at the center which is the closest to their housing, and as a consequence, can change jobs as they move. This last feature seems unrealistic but prevents market frictions, to reach the equilibrium more rapidly. The results of this model are presented on the first row of figure 3, keeping only one income group for simplicity. The city with two centers in this model has the shape of two discs, centered on one employment center each, which partly overlap – meaning that both housing markets interact spatially – if the centers are close to one another.

Figure 4 shows the evolution of different variables characterizing the outcomes of this polycentric model, such as agents' utility  $U_{\text{mean}}$ , the total commuting distance of agents  $D_{\text{tot}}$ , the total rent  $R_{\text{tot}}$  paid in the city, the mean price  $P_{\text{mean}}$  and the total surface of the city  $S_{\text{tot}}$ , compared with the reference monocentric configuration given by the case  $d = 0$ . The evolution of the mean density is given by the inverse of the total surface, as the population is fixed.

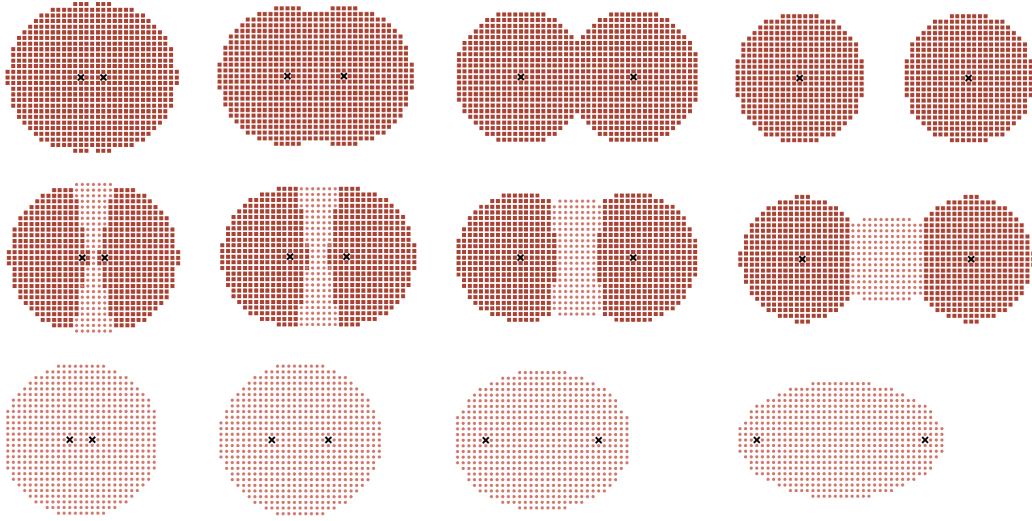


Figure 3: Shape of the city with two-worker households (model 2), with  $m = 0$  (first row – all agents work in the closest center),  $m = 0.2$  (second row) and  $m = 1$  (third row – all households have one worker in each center). Centers are represented by black crosses. Agents of the "common" group are represented by dark red squares, agents of the "split" group by light red circles. The different columns correspond to different values of the distance  $d$  between centers:  $d = 4, 10, 20, 30$  from left to right. Parameters values are given in Table 1.

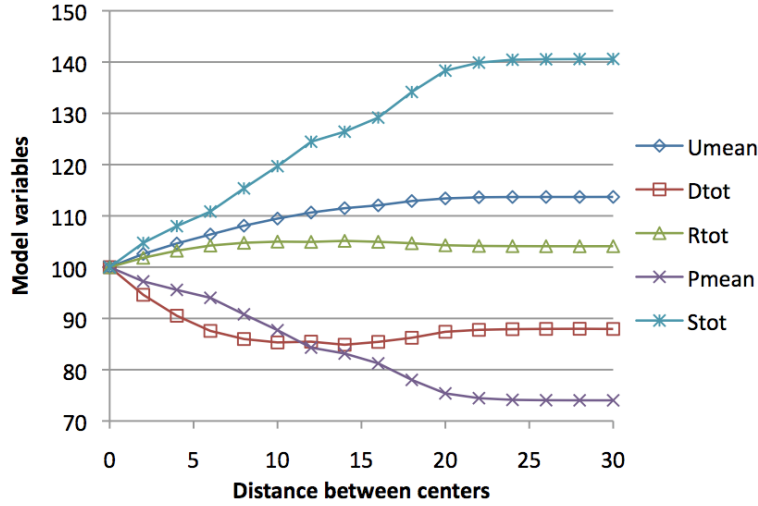


Figure 4: Evolution of the variables characterizing the simplest polycentric model (corresponding to the first row of Figure 3,  $m = 0$ ) as a function of the distance between centers. Variables: agents' utility  $U_{\text{mean}}$ , total commuting distance of agents  $D_{\text{tot}}$ , total rent  $R_{\text{tot}}$ , mean price  $P_{\text{mean}}$  and total surface of the city  $S_{\text{tot}}$ . Parameters values are given in Table 1.

Increasing the number of centers and the distance between them amounts to increasing the surface available at a given commuting distance in the city, thus simultaneously reducing competition for land and transport costs. Agents have greater housing surfaces ( $S_{\text{tot}}$ ), which spreads the city, smaller commuting distances ( $D_{\text{tot}}$ ) and a higher utility ( $U_{\text{mean}}$ ). The total rent  $R_{\text{tot}}$  increases, which seems surprising but is explained by the fact that housing surfaces are greater, which offsets the decrease in prices ( $P_{\text{mean}}$ ).

The economic outcome of the introduction of two centers in this model is positive, because agents' utility increases when the distance between the centers increases. However, the environmental outcome is more difficult to assess. Indeed, as Figure 4 shows, commuting distances  $D_{\text{tot}}$  decrease first when the centers are moved away from each other, which means a reduction of pollution linked to commuting, but then, they increase again. Simultaneously, the decreasing competition for land results in increasing housing surfaces, and thus city size. Bigger housing surfaces result in greater heating (and cooling) needs, which are a major source of energy needs and greenhouse gas emission.

In this simple model, both halves of the city stop interacting if the centers are distant from each other. This can be seen in Figure 4, where all curves are flat for distances greater than 25.

#### 4.2. Introducing two-worker households

The situation where two adjacent cities are not interacting seems unrealistic. One of the reasons why this does not happen in reality is that some households have two workers working in different locations, for instance in two employment centers close to each other. Keeping the same framework, we study this phenomenon and add more realism to the city as a whole by introducing two-worker households. This is also a research question in the literature – see for instance Madden and White (1980); Kohlhase (1986); Hotchkiss and White (1993). Each agent of the previous section (4.1) represents now a household composed of two workers. The transport cost per unit distance for one worker is now  $\tilde{t} = t/2$ . Thus, agents of section 4.1 can be seen as such households with both workers going to the same job center. For simplicity, we consider a city with only two centers. Households are divided in two groups. In the first group, which we denote by "common", both persons in the household work at the same employment center, which corresponds to the first row of Figure 3. In the second group, which we denote by "split", they work in different centers. This is imposed exogenously and does not change during a simulation.

We study the outcome of this model depending on two variables: the distance  $d$  between centers, and the share  $m \in [0, 1]$  of households of the "split" group. Let us label employment centers by "East" and "West", and note  $d_E$  and  $d_W$  the distances between a given household's location and centers East and West, respectively. Then, if both persons in this household work at the same employment center ("common" group), the East center for instance, the transport cost associated with the commuting of the household is  $2 \times \tilde{t} \times d_E = t \times d_E$ . If they work at different centers ("split" group), their transport cost is  $\tilde{t} \times (d_E + d_W) = t \times (d_E + d_W)/2$ .

It is impossible for households to be located simultaneously in both employment centers. One important consequence of the new ingredient added here is then that a minimal commuting distance of  $d$  (or equivalently, a minimal transport cost of  $\tilde{t}d$ ) is imposed for all households of the "split" group. It is their overall commuting distance if they are located on the segment linking both employment centers. Thus, the minimal total commuting distance  $D_{\text{min}}^{\text{tot}}$  of agents in the city is  $D_{\text{min}}^{\text{tot}} = d \times m \times N$ . This minimal distance is exogenously imposed, and is a special feature of model 2. It is a factor of discrepancy between actual commuting patterns and the ones predicted by the standard monocentric model. This discrepancy is better known as "wasteful commuting" as referred to by Hamilton and Röell (1982).

To begin with, let us study what happens in the case  $m = 1$ , where all households belong to the "split" group. To minimize their transport cost, agents choose their location by minimizing  $d_E + d_W$ . Consequently, the shape of the city is elliptic with both employment centers as focal points, as shown in the third row in Figure 3. Indeed, the figure defined by the set of points verifying  $d_E + d_W = k$ , with  $k$  a constant, is an ellipse. The effect of increasing  $d$  on the transport cost of agents can be described as follows. Because of the increasing minimal commuting distance described previously, the transport cost is increased – everywhere, except at both employment centers themselves, where it does not change

when compared with the monocentric ( $d = 0$ ) case. Simultaneously, the center of the city, seen as the place where transport cost is minimal, is spread on a segment linking both employment centers.

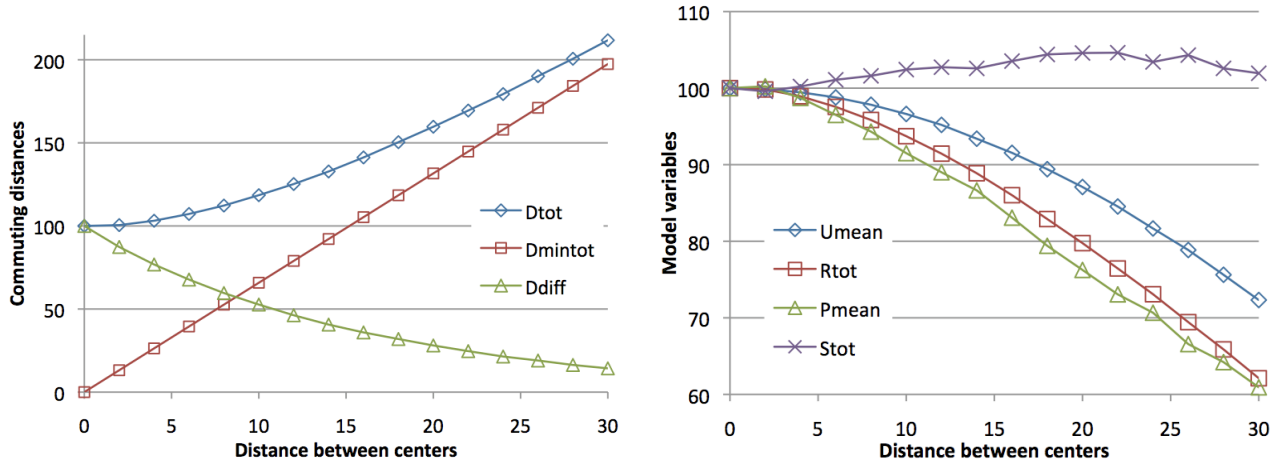


Figure 5: Outcomes of model 2 with  $m = 1$  as functions of the distance  $d$  between centers. Left panel: evolution of the total commuting distance  $D_{tot}$ , the minimal total commuting distance  $D_{min}^{tot}$  and their difference  $D_{diff}$ . Right panel: evolution of agents' utility  $U_{mean}$ , of the total rent  $R_{tot}$ , of the mean price  $P_{mean}$  and of the total surface of the city  $S_{tot}$ . Parameters values are given in Table 1.

Consequently, the total commuting distance  $D_{tot}$  of agents increases when the centers are moved apart, mainly because of the contribution of the minimal commuting distance  $D_{min}^{tot}$ , as shown in the left panel of Figure 5.  $D_{diff} = D_{tot} - D_{min}^{tot}$  is also indicated: its decrease when  $d$  increases shows that agents are gathering around the segment linking both centers. The variables are given on the basis of their value in a reference simulation with  $d = 0$  (corresponding to model 1 with only one income group) for an easy comparison. The utility of agents  $U_{mean}$  decreases when  $d$  increases, very slowly when the centers are close to each other and then more rapidly. The total surface of the city  $S_{tot}$  is always bigger than in the reference (monocentric) simulation, but it decreases when  $d$  is high. The mean price of housing  $P_{mean}$  and the total rent  $R_{tot}$  decrease when  $d$  increases, as the share of income used for transport increases.

In this model with "split" population, polycentrism is undesirable. It has both a negative economic outcome with the decreasing utility of agents and a negative environmental outcome as housing surfaces increase and commuting distances increase. However, it should be noted that commuting distances increase mainly because of the minimal commuting distance shown on the left panel of Figure 5. This effect could be seen as the worst case scenario of the thought experiment in which a monocentric city is transformed into a polycentric one: all households increase their travel distances accordingly. A more realistic scenario is given by the case where only a part of the households increase their travel distances, which we study now.

#### 4.3. Mixing various two-worker households

When  $0 < m < 1$ , simulations show that the utility of agents of the "common" group is always higher than that of agents of the "split" group. This is expected, as households of the "split" group have more constraints: they want to live close to two centers instead of one. The outcomes of this model with  $0 < m < 1$  are intermediate between the two previous cases. The second row of Figure 3 gives the shape of the city with  $m = 0.2$  for different values of  $d$ , and Figure 6 gives the corresponding outcomes

of model 2. These results obtained with  $m = 0.2$  are representative of the general case of  $m$  strictly between 0 and 1. Hence, we present only this value. The city shape is intermediate between  $m = 0$  and  $m = 1$ , in other words between two disks and an ellipse. Agents of the "split" group (in a paler shade) are located between both centers, separating agents of the "common" group into two parts.

As shown in Figure 6, the total commuting distance  $D_{\text{tot}}$  decreases at first when  $d$  increases, and then increases again, mainly because of the contribution of the minimal commuting distance imposed on agents of the "split" group. The utility  $U_0$  of "common" agents increases with  $d$ , as their competition for land with "split" agents decreases. The utility  $U_1$  of "split" agents increases at first when  $d$  increases, and then decreases again, below its value at  $d = 0$ . The total surface of the city  $S_{\text{tot}}$  increases with  $d$ , while the mean price of land  $P_{\text{mean}}$  decreases. The total rent  $R_{\text{tot}}$  increases at first when  $d$  increases, and then decreases below its value at  $d = 0$ .

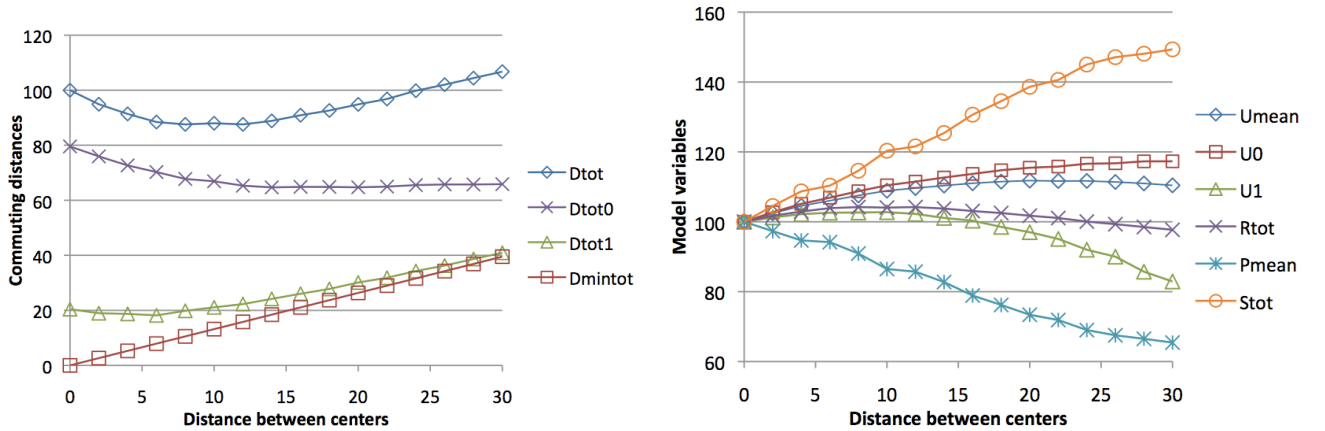


Figure 6: Outcomes of model 2 with  $m = 0.2$  as functions of the distance  $d$  between centers. Same variables as on Figure 5. On the left panel, the total commuting distances  $D_{\text{tot}}^0$  and  $D_{\text{tot}}^1$  of agents of both groups are presented. On the right panel, the mean utility of agents  $U_{\text{mean}}$ , the utility  $U_0$  of "common" agents and  $U_1$  of "split" agents are given. Parameters values are given by Table 1.

In this case, which seems more realistic than the same model with  $m = 0$  or  $m = 1$ , polycentrism is desirable, as long as centers are not moved too distant from each other. Indeed, the utility of agents of both groups ( $U_0$  and  $U_1$ ) increases when  $d$  increases for small values of  $d$ , which gives a positive economic outcome of this model. The environmental outcome is also positive for small values of  $d$ , as the total commuting distance  $D_{\text{tot}}$  decreases when  $d$  increases. However, this positive effect is mitigated by the fact that housing surfaces increase, which tends to increase emissions of greenhouse gases from heating and cooling. Thus, this more realistic model confirms the conclusions reached with only "common" households, as long as the centers are kept not too distant from each other.

## 5. Conclusion and perspectives

Building on the standard urban economic theory, we run simulations of a microeconomic model with agents interacting in an urban area. The dynamics of the model is inspired by local search methods and asynchronous dynamics in game theory and statistical physics. It consist mainly of agents moving and bidding on the urban housing market. This defines price formation in the model, insures matching of supply and demand for land, and pushes the system in the direction of the spatial equilibrium. This equilibrium corresponds to a discrete version of the analytical equilibrium of the AMM model. A

comparison shows the very good agreement of the analytical and the agent-based monocentric models with two income groups.

Next, we study the evolution of this equilibrium when the monocentric hypothesis is abandoned to explore polycentric cities. Our results present economic and environmental outcomes of simple polycentric forms within the agent-based model. Note that this is a partial welfare analysis, because our model ignores economies of agglomeration, especially for firms.

The introduction of several centers, when compared to the monocentric city model, has a positive effect on agents' welfare, as transport expenses and competition on the housing market decrease. Commuting distances are reduced, which gives a positive environmental outcome of the polycentric city in this model. However, the increase in housing surfaces may counterbalance this decrease in emission of greenhouse gases. Although the global effect of a reduction of competition for land between agents is clear, its effect on the different variables of this simple urban model and on different income groups is not obvious, as the results show.

Next, we introduce two-worker households in the polycentric setting. If the whole population is "split" (that is, the two partners work in different job centers) polycentrism is shown to be undesirable. More realistically, when mixing various two-worker households (including households where both partners work in the same job center), polycentrism is again desirable as long as the centers are not moved too distant from each other, rejoining partly the previous conclusions reached with one-worker households.

Three long-term perspectives of work can be considered: first, a research perspective is to study dynamic urban models, which are difficult to treat analytically. For instance, once the models presented here have reached an equilibrium, a parameter value is changed (e.g. an increase in transport cost) and the consecutive dynamic changes on the urban systems can be studied, until another equilibrium is reached. A second research perspective is to introduce along with two-worker households, other factors of heterogeneity such as job specialization or preferences for social neighborhoods – factors which altogether, as suggested by Anas et al. (1998), might explain the discrepancy between actual residential locations and locations predicted by the standard monocentric model. Third, a more applied perspective is to design simulation models that could be used by city planners to help decision-making. Using the robustness of the agent-based dynamics presented here, and applying it to real-world data, for instance various employment centers with transport networks, simulation models could indeed be designed to study economic, environmental and social consequences of different urban planning policies.

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## Appendix A. Evolution of the utility

Figure A.7 shows how the utility becomes homogeneous during a simulation. The relative inhomogeneity of utility  $\Delta U_{\max}$  defined in section 3.2 (main text) decreases with time. The standard deviation

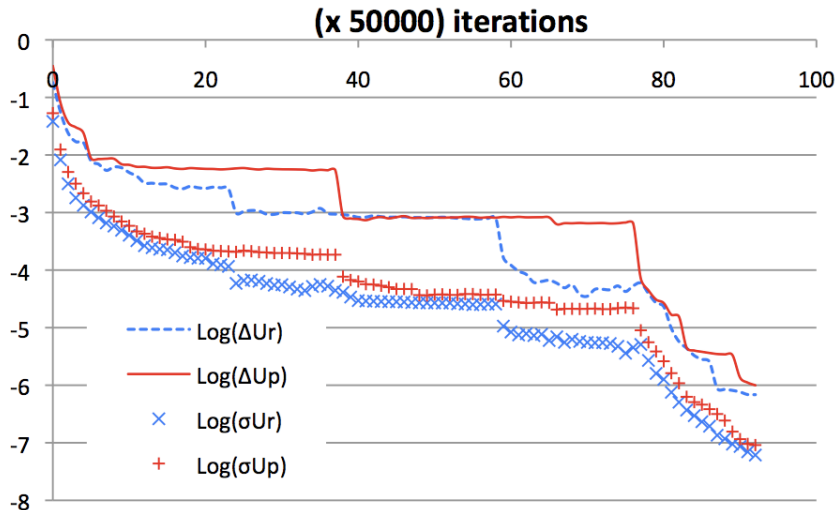


Figure A.7: Evolution of the decimal logarithms of the relative inhomogeneity  $\Delta U_p$  ( $\Delta U_r$ ) and of the relative standard deviation  $\sigma U_p$  ( $\sigma U_r$ ) of poor (rich) agents' utility during a simulation. One unit of the abscissa represents 50000 iterations.

of the utility can also be computed. It gives a more accurate idea of the variations of utility in the model. Figure A.7 shows the evolution of the relative inhomogeneity of utility in each income group, and the corresponding evolution of the standard deviation of utility in each income group (also divided by the maximal utility) during a simulation. The latter is always smaller than the former, as the case should be. This evolution is given as an illustration: because of the stochastic dynamics of the model, it varies across simulations.

## Appendix B. Reproducibility of the results

To confirm that the equilibrium reached by the agent-based model is unique, we perform the same simulation for 15 times. Despite the stochasticity of the dynamics of the model, each run converges to the same equilibrium, in a sense which is defined more precisely here.

The simulations are stopped only when the two conditions ensuring that the equilibrium is reached, described in section 3 (main text), are verified: the relative inhomogeneity of utility  $\Delta U_{\max}$  is smaller than  $10^{-6}$  (section 3.2, main text) and the share of "empty" surface  $S_{\text{empty}}/S_{\text{tot}}$  is smaller than 0.5% (section 3.3, main text).

The results of these simulations are given in Table B.2 for the two models presented in this work: the first part corresponds to model 1, the reference monocentric model with two income groups. The second corresponds to model 2 with  $d = 9$  and  $m = 0.2$ . The equilibrium values of the variables characterizing the models have only very small variations across different simulations. The maximal variation observed, computed for variable  $X$  as  $(X_{\max} - X_{\min})/X_{\min}$ , is of approximately 0.1% under the two previous conditions.

Model 1	$U_r$	$U_p$	$U_r - U_p$	$D_{\text{mean}}^r$	$D_{\text{mean}}^p$	$D_{\text{tot}}$	$R_{\text{tot}}$	$p_{\text{mean}}$	$S_{\text{tot}}$
Variations (in %)	0.009	0.001	0.02	0.09	0.06	0.08	0.02	0.11	0.08
Model 2	$U_p$	$U_0$	$U_1$	$D_{\text{mean}}^0$	$D_{\text{mean}}^1$	$D_{\text{tot}}$	$R_{\text{tot}}$	$p_{\text{mean}}$	$S_{\text{tot}}$
Variations (in %)	0.008	0.01	0.02	0.09	0.05	0.06	0.02	0.09	0.08

Table B.2: Reproducibility of the results: maximal relative variations of the variables characterizing the models across 15 runs of the same simulation.

### Appendix C. Parameters of the agent-based model

In this section, we give an example of stationary configuration<sup>2</sup> of the agent-based model when the parameters specific to the agent-based model,  $\epsilon$  and  $T_p$ , are not chosen to minimize the inhomogeneity of utility  $\Delta U_{\text{max}}$  and the share of "empty" surface  $S_{\text{empty}}/S_{\text{tot}}$ . Consequently, the system does not reach an equilibrium that corresponds to the analytical one.

We retain the values of parameters given in Table 1 (main text), except  $T_p$ , which we take as  $T_p = 3000$ . The results of this simulation are shown in Figure C.8. They can be compared with the results of figure 1 (main text). Because of this much higher value of  $T_p$ , the price of vacant cells decreases

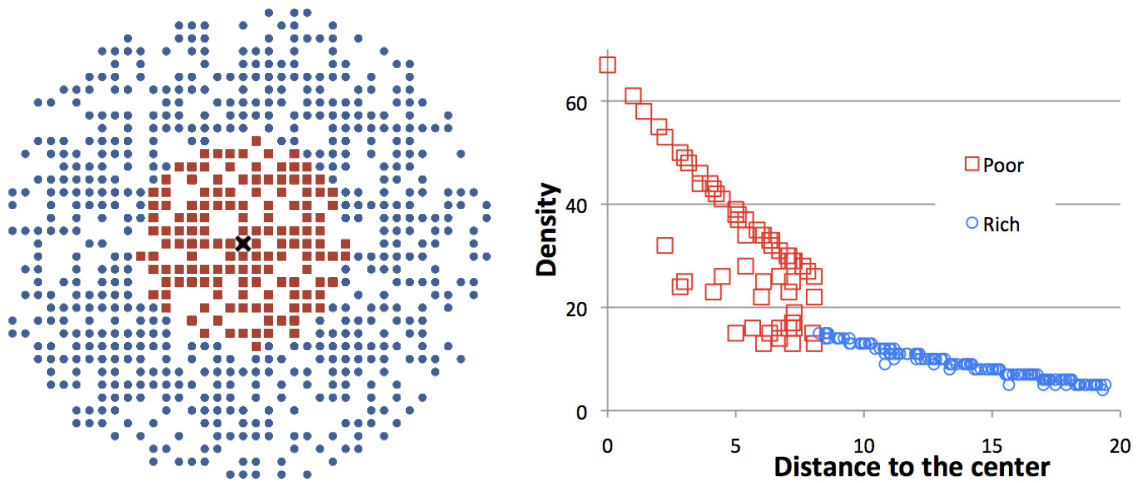


Figure C.8: Monocentric city with two income groups, with  $T_p = 3000$ . Left panel: shape of the city. Right panel: density as a function of the distance to the center. Same symbols as figure 1 (main text).

very slowly. It decreases too slowly to compensate for price increase due to agents' bids, which prevents the system from reaching an equilibrium corresponding to the analytical one. Indeed, as shown in the left panel of Figure C.8, as the price of cells decreases too slowly, some cells, even close to the CBD, are left vacant after their price has increased too much. The bid mechanism still manages to bring the system to a state with homogeneous utility, where  $\Delta U_{\text{max}} < 10^{-6}$ . However, a large part of the urban space is "not optimally used", which is indicated by the value of the share of "empty" surface  $S_{\text{empty}}/S_{\text{tot}} \simeq 130\%$  (in the other simulations presented in this work, this variable is smaller than 0.5% – see section 3.3, main text).

<sup>2</sup>This configuration corresponds actually to a state of the system where the evolution is so slow that the configuration seems stationary. We do not push this study further here and present it as an illustration of a simulation not converging to an equilibrium corresponding to the analytical one.

## Appendix D. Existence and uniqueness of the analytical equilibrium

In this section, we extend the results of Fujita (1985) to some polycentric models, like the ones studied in this paper. Fujita (1985) shows that there exists one unique equilibrium for the standard monocentric AMM model with one or more income groups. The proof of this result is based on boundary rent curves between income groups and between agricultural and residential land uses. Although this result seems difficult to extend to any polycentric city, we show how this result can be extended to some of the models studied in this paper.

The existence of (at least) one equilibrium for the polycentric models studied here is proved in Fujita and Smith (1987) by using fixed-point methods. Hence, it remains to be argued that with the polycentric changes added here to the standard monocentric model, there is no apparition of multiple equilibria, contrary to the results seen with more complicated models, for instance Brueckner et al. (1999) or Fujita and Ogawa (1982). Note that these works add important changes to the standard model by adding variables to the utility function, while our work only considers a more complex transport cost function. Indeed, introducing several employment centers breaks the circular symmetry of the transport cost function compared to the monocentric city model.

Let us first consider the simplest polycentric model we study (model 2 with  $m = 0$ ), a model with two centers separated by distance  $d$ , where agents work at the employment center that is closest to their housing location. When the centers are sufficiently distant, two cities are present and do not interact, with an equal share of the whole population residing in each city. In this case, the result of Fujita (1985) ensuring existence and uniqueness of the equilibrium is clearly still valid. When the centers are brought closer and cities begin to interact, the situation becomes slightly more complicated.

Our approach then consists in mapping this simple polycentric model onto a fictitious monocentric model verifying the assumptions required in Fujita (1985)<sup>3</sup> to ensure existence and uniqueness of the equilibrium. This mapping allows us to prove the existence and uniqueness of the equilibrium for the polycentric model, because of the corresponding result for the fictitious monocentric model. Indeed, in the urban models studied in Fujita (1985), as well as in our model 2 with  $m = 0$ , a given location is completely characterized by the distance of commuting for an agent residing in this place. Equivalently, space is characterized by the amount of land available at each commuting distance  $x$ . Let us note  $L(x)dx$  the amount of land available between commuting distances  $x$  and  $x + dx$ . A monocentric model with a distribution of land equivalent to that of our simple duocentric model would have

$$L(x) = \begin{cases} 4\pi x & \text{for } 0 \leq x \leq d/2 \\ 4x(\pi - \arccos(\frac{d}{2x})) & \text{for } x \geq d/2. \end{cases} \quad (\text{D.1})$$

This fictitious monocentric model verifies all assumptions ensuring that it has one unique equilibrium, and a bijection between this monocentric space and our polycentric one conserving the amount of land  $L(x)dx$  available between  $x$  and  $x + dx$  is easy to define. Consequently, our simple duocentric model has also one unique equilibrium. This result could be extended to models with 3 or more centers, as this would only make the function  $L(x)$  (and the bijection) more complicated. With several income groups, the result still holds true.

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<sup>3</sup>The first assumption is that  $L(x)$  is a piecewise continuous non-negative function on  $\mathbb{R}^+$ , strictly positive on  $[0, x_1]$  with  $x_1$  a positive number, and  $L(x) \leq L_0(x)$  on  $\mathbb{R}^+$ , where  $L_0(x)$  is any continuous function on  $\mathbb{R}^+$ . The second assumption is that the bid rent and housing surface functions are "well behaved", and that  $R_a > 0$ . Finally, the third assumption corresponds for instance to the case of several income groups, whose bid rent functions can be ordered by their steepness.

The case of model 2 with  $m = 1$  is almost similar. In this model, two employment centers (East and West) separated by a distance  $d$  are considered, and each agent represents a two-worker household, with each worker of the household working at a different center from the "mate" (see section 4, main text). Thus, the total commuting distance of the household is the sum  $d_E + d_W$  of distances between the household's housing location and both centers. The function  $L(x)$  of the corresponding monocentric model is now

$$L(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq d \\ 2xE(d/x) & \text{for } x \geq d, \end{cases} \quad (\text{D.2})$$

with  $E(e)$  the complete elliptic integral of the second kind. This last formula corresponds to the circumference of an ellipse of major axis  $x$ , distance between focal points  $d$ , and eccentricity  $e = d/x$ . A similar argument of correspondence proves the uniqueness of the equilibrium in this case, and is still valid with several income groups. Defining such a model with more than two employment centers is not obvious, and outside the scope of the present work.

In the case of model 2 with  $0 < m < 1$ , the previous arguments supporting the uniqueness of the analytical equilibrium seem difficult to reproduce, and we leave this proof to further work. However it remains true that no important change occurs in the utility function when compared with the standard monocentric model. Only the transport cost (seen as a function defined on the two-dimensional space of the model, for each group of agents) is changed. In addition, an important argument in favor of this uniqueness is the fact that for all models presented here, the agent-based model converges to the same equilibrium situation for every run of a simulation, as shown in section Appendix B.