



HAL
open science

Exploring the polycentric city with an agent-based model

Rémi Lemoy, Charles Raux, Pablo Jensen

► **To cite this version:**

Rémi Lemoy, Charles Raux, Pablo Jensen. Exploring the polycentric city with an agent-based model. 2012. hal-00602087v2

HAL Id: hal-00602087

<https://hal.science/hal-00602087v2>

Preprint submitted on 26 Mar 2012 (v2), last revised 14 Oct 2016 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Exploring the polycentric city with an agent-based model

Rémi Lemoy^{a,b,c}, Charles Raux^{a,*}, Pablo Jensen^{b,c}

^a*LET (Transport Economics Laboratory), CNRS and University of Lyon, 14 avenue Berthelot F-69363 Lyon Cedex 07, France*

^b*Laboratoire de Physique, Ecole Normale Supérieure de Lyon, CNRS and University of Lyon, 46 Allée d'Italie, F-69007 Lyon, France*

^c*IXXI (Institut Rhônalpin des Systèmes Complexes), 5 Rue du Vercors, F-69007 Lyon, France*

Abstract

We present an agent-based model grounded in the Alonso, Muth, Mills (AMM) framework, using microeconomic interactions between heterogeneous agents and able to reach the equilibrium in a dynamic way. This model is shown to reproduce the equilibrium of the AMM model. An illustration is given with two income groups. Two job centers at various distances from each other are introduced. Economic, social and environmental outcomes of these various polycentric spatial structures are studied. Then two-worker households whose partners may work in different job centers are introduced. Regarding welfare we find that polycentrism is desirable, as long as centers are not moved too far apart from each other. The environmental outcome is also positive for small values of this distance but this positive effect is mitigated by the fact that housing surfaces increase.

Keywords: agent-based model, urban economics, location choice, polycentric city, two-worker households

JEL: C63, D01, R11, R14

Introduction

The earth population is now predominantly urban, and urban areas are expanding fast (Seto et al. (2011)). Beside social and economic issues, this process raises environmental concerns regarding biodiversity conservation, loss of carbon sinks and energy use. Empirical evidence shows that various urban development patterns significantly influence carbon dioxide emissions (Glaeser and Kahn (2010)). Low density brings about increasing vehicle usage while both low density and increased vehicle usage bring about increasing fuel consumption (Brownstone and Golob (2009)). Compact urban development would be the natural answer to these issues but the debate regarding welfare, distributive and environmental aspects is fierce between opponents and promoters of compact cities (see e.g. Gordon and Richardson (1997); Ewing (1997)). The issue of spatial and social structure and operation of cities has never been so acute, and there is an obvious need to better understand city spatial development (Anas et al. (1998)).

A good starting point is the well-known urban economics framework: the Alonso-Muth-Mills (AMM) model integrates Alonso (1964) monocentric model of land market followed by Muth (1969) and Mills (1967, 1972) models, which include housing in the residential location model. This analytical framework

*Corresponding author.

Email addresses: remilemoy@gmail.com (Rémi Lemoy), charles.raux@let.ish-lyon.cnrs.fr (Charles Raux), pablo.jensen@ens-lyon.fr (Pablo Jensen)

has proved its robustness in describing the higher densities, land and housing rents in cities centers (Spivey (2008); Mills (2000)), despite the limitations of the model as shown in the summary by Brueckner (1987). These limitations are, among others, the city is mono-centric, urban residents all earn the same income and the commuting cost function is exogenous.

Polycentrism (or multiple centers) is a reality, as shown by empirical evidence (Giuliano and Small (1991)). However, introducing polycentrism in the AMM model proves difficult from the point of view of analytical tractability. Wheaton (2004) also challenges the monocentrism, based on empirical evidence on US cities which shows that employment is almost as dispersed as residences. However, in this work, simplifying hypotheses are needed for analytical tractability, such as an exogenous density (consumption of land per worker is fixed and independent of location). In other approaches, centers (and sometimes subcenters) are given exogenously (White (1976, 1988); Sullivan (1986); Wieand (1987)). In Fujita and Ogawa (1982), no centers are specified a priori and multiple equilibria are shown (monocentric, multicentric or dispersed patterns). Here again, since the model is not analytically tractable, simplifying hypotheses are required (e.g. lot size is fixed). A well shared conclusion of these papers is that numerical simulations are needed.

Regarding income heterogeneity of residents, its introduction in the AMM model seems in theory possible: this heterogeneity would translate into multiple classes of bidders and even a continuous variation in bid-rent gradients among residents. However, as Straszheim (1987) points out, when there are multiple classes of bidders it is difficult to find realistic specifications of distribution functions (of income) which yield tractable results and, again, this requires numerical solutions.

These difficulties in extending the AMM model while retaining analytical solutions to equilibrium, have brought us around another modeling tool i.e. agent-based models. Agent-based models in urban economics are still in the infancy (putting aside the extensive literature on the Schelling model). To our knowledge the first application of agent-based models in urban economics is a paper by Caruso et al. (2007) which integrates urban economics with cellular automata in order to simulate peri-urbanization.

Basically, agent-based models include three main components, i.e. agents, an environment and rules of behavior. The agents have internal states, some fixed and others that can change, like their preferences, and follow rules of behavior. The environment is defined as a two-dimensional space supporting resources and can include a communication network. Rules of behavior determine the interactions between agents, between agents and the environment and within the environment. In our model these rules are grounded in the urban microeconomic framework.

The first advantage of this tool is its flexibility. In agent-based models, agents' states (e.g. heterogeneity in income), microeconomic rules of behavior (e.g. bid-rent function) and the environment (e.g. multiple employment centers or amenities) can be easily handled. Individual and collective behaviors can be monitored in a simple way. In addition the model is dynamic, which is not the case for most analytical (equilibrium) economic models.

However, before exploiting the full range of advantages of agent-based modeling, we need to prove its relevance to model urban equilibria within the AMM framework, and this is the first aim of this paper. The agent-based simulation framework is used to define microeconomic interactions between agents (households) and shown to reproduce the results of the AMM model. The main difference lies in the discreteness of the simulated model, whereas the analytical model is continuous. This provides an illustration of a discrete model converging to the continuous AMM model for large population sizes, which can be related to a discussion in the literature (Asami et al. (1991); Berliant and Sabarwal (2008)).

The simulated model is dynamic: starting from a random initial state, interactions between agents on an urban housing market (land and housing coincide in this model) lead progressively the whole system to an equilibrium state. Illustrations of the time evolution of the urban system are presented.

As the results of the agent-based model are validated by comparison with the analytical model, the model can be made more complex by adding various ingredients. This is done here in keeping with a parsimony principle. The first ingredient is heterogeneity through income groups: an illustration is given with two income groups.

The second ingredient is multiple centers, shown here with two job centers at various distances from each other. Economic, social and environmental outcomes of these various polycentric spatial structures are studied, and this is the second aim of this work. The economic outcome of the introduction of two centers is shown to be positive, as agents' utility increases when the distance between centers increases. However, pollution linked to commuting distances decreases first when centers are taken away from each other but then increases again. At the same time, the decreasing competition for land results in increasing housing surfaces and thus city size.

Then a more realistic ingredient is introduced, that is two-worker households whose partners may work in different job centers. Various distances between job centers and various mix in two-worker households are simulated. Regarding welfare it is shown that polycentrism is desirable, as long as centers are not moved too far apart from each other. The environmental outcome is also positive for small values of this distance but this positive effect is mitigated by the fact that housing surfaces increase, which may increase emissions of greenhouse gases.

Moreover, the existence and uniqueness of equilibria in these polycentric models are discussed and various arguments are elaborated upon to support these features.

The remainder of the paper is structured as follows. Section 1 describes the agent-based implementation with the microeconomic behavior of agents. Section 2 compares the simulations results with the analytical ones of the AMM model and illustrates the dynamic feature of the model. Section 3 presents the polycentric urban forms and their economic, social and environmental outcomes. Section 4 is about the existence and uniqueness of equilibria in these models, while research perspectives of this work are discussed in section 5 before concluding.

1. Description of the model

1.1. *Urban economics model*

The AMM model was developed to study the location choices of economic agents in an urban space, with agents competing for housing (which is identified with land in the simplest version of the model). Agents have a transport cost to commute for work. Their workplace is located in a central business district (CBD), which is represented by a point in the urban space. Agents usually represent single workers, but they can also be used to describe households, which can be made more complex in further versions of the model. Housing is rented by absentee landowners who rent to the highest bidder, which introduces a competition for housing between agents. They also compete with an agricultural use of land, which is represented by an agricultural rent R_a .

Agents have a utility function describing their welfare, which is here a Cobb-Douglas function $U = z^\alpha s^\beta$, where z is a composite good representing all consumer goods except housing and transport (whose price is the same everywhere in the city), s is the surface of housing, α and β are parameters describing agents' preferences for composite good and housing surface, with $\alpha + \beta = 1$. Agents also have a budget constraint $Y = z + tx + ps$, where Y is their income, t the transport cost per unit distance, x the distance of their housing location to the CBD and p the price of a unit surface of land at location x . See Fujita (1989) for a more detailed description of this model.

The analytical model reproduced in this work with agent-based simulations is a closed city model, where the population size N is chosen exogenously and remains constant during a simulation. This

model can be solved analytically in a two-dimensional space if $R_a = 0$. For $R_a > 0$, a simple numerical resolution can be performed with one income group (Fujita (1989)). With a population divided into several income groups, a specific algorithm is needed for the resolution of the model. The principle of this resolution is described in Fujita (1989).

1.2. Agent-based implementation

Let us describe in this section the agent-based implementation of the standard monocentric AMM model. In the agent-based system, the simulation space is a two-dimensional grid where each cell can be inhabited by one or several agents, or used for agriculture. These cells have a fixed land surface s_{tot} . The unit of distance is taken as the side length of a cell. The CBD is a point at the center of the space.

At the initialization, a population of N agents is created. These agents are placed at random locations. The initial land price in each cell p_0 is equal to the agricultural rent R_a . At a given location x , agents occupy a quantity of land which is the optimal consumption of land conditional on price p : $s = \beta \frac{Y-tx}{p}$. This determines the quantity of composite good they consume, and their utility.

1.2.1. Dynamics of moves

The main feature of the model consists in agent-based dynamics of moves and bids in the urban space. The rules defining agents' moves are suggested by the competition for land in the analytical model.

Agents move with no cost, as in the AMM model. Let us describe an iteration n of the simulation, changing the variables from their value at step n to their value at step $n + 1$. An agent which will be candidate to a move and a cell are chosen randomly. The price of this cell, located at a distance x of the center, is p_n at step n . The optimal housing surface that the agent can choose in the candidate cell is $s_n = \beta \frac{Y-tx}{p_n}$, which allows us to compute her composite good consumption and the utility that she would get thanks to the move.

If the agent candidate can have a utility gain $\Delta U > 0$ by moving into the candidate cell, then she bids for renting at the price $p_{n+1} = p_n(1 + \epsilon \frac{s_{\text{occ}}}{s_{\text{tot}}} \frac{\Delta U}{U})$, where ϵ is a parameter that we introduce to control the magnitude of this bid. Since landowners rent to the highest bidder the price of the candidate cell is raised. The higher parameter ϵ is, the faster cell prices evolve. s_{occ} is the surface of land occupied by other agents in the cell and s_{tot} the total land surface of the cell. The factor $\frac{s_{\text{occ}}}{s_{\text{tot}}}$ makes the bid higher if the cell is more occupied, that is to say, more attractive. Because of this factor, the first agent to move in an empty cell does not raise the price.

The price is a price per unit surface, linked to a cell. When an agent bids higher, the price is changed for all agents in the target cell. Their consumption of land is also changed according to $s = \beta \frac{Y-tx}{p_{n+1}}$ and their utility is computed again. This feature of the model defines a competition for land between agents and a market price, as in the standard analytical model¹.

1.2.2. Surface constraint, time decrease of price

We described how prices increase in the model. Due to the stochastic choice of agents and cells, prices can rise above their equilibrium level at some locations, making some cells unattractive. Indeed,

¹Specific situations arise which do not appear in an equilibrium (static) model. For instance, an agent may want to move into a candidate cell that is already full, proposing a higher bid on the price of housing there. Then we make the following choice: the price of housing is raised for all agents living in the cell to the level of this new bid, but the agent candidate does not move in. Then agents' surfaces of housing and utilities are computed again. As the price is raised, housing surfaces are decreased and there is a chance that enough space is freed for the candidate agent to move in, in which case she does. Else, she has to wait until she is proposed another move.

Parameters	Description	Default value
α, β	Preferences for composite good and housing	0.75; 0.25
Y_p, Y_r	Incomes of poor and rich agents	300, 450
t	Transport cost (unit distance)	10
N	Total population size	10000
R_a	Agricultural rent	10
s_{tot}	Surface of a cell	30
ϵ	Bidding parameter	0.5
T_p	Time decrease of the price of non-full cells	30

Table 1: Parameters of the simulations

the price of a cell where several agents move in successively can increase so much that it reaches a value which makes the cell unattractive. In this case, agents living there will progressively leave the cell for more attractive locations.

Therefore we choose to decrease exponentially the price of cells where there is free place to accommodate one or more agents. This is done according to $p_{n+1} = p_n - (p_n - R_a \times 0.9) / T_p \cdot s_{av} / s_{tot}$, where T_p is a parameter determining the speed of decrease of prices, $s_{av} = s_{tot} - s_{occ}$ and s_{tot} are the available (non occupied) surface and the total surface of the cell respectively². If no agent moves in, the price decreases according to this formula until it reaches the agricultural rent, which occurs after a finite time because of the form used. Then the decrease stops, and the cell is used for agriculture. The factor $\frac{s_{av}}{s_{tot}}$ makes this time decrease quicker as the cell is emptier and thus less attractive.

1.2.3. Parameters

The different parameters of the model are listed in table 1. Most parameters belong to the AMM model itself: $\alpha, \beta, Y, t, N, R_a, s_{tot}$. Their values are chosen arbitrarily, as the model is not calibrated on real data for the moment. However, it can be noted that a higher population N could have improved the agreement between the analytical and the agent-based model, but it would have slowed down the simulations. Parameters ϵ and T_p are specific to the agent-based model. Their values have been chosen such that the competition between agents on the housing market is efficient and the system reaches the equilibrium in the whole city. This will be discussed in section 2.3 and Appendix B.

The present study focuses on the equilibrium of the agent-based model, which is shown to correspond to the equilibrium of the analytical AMM model, so that the agent-based dynamics is only presented in this work as a resolution method. Comparison with the dynamics of real urban housing markets is beyond the scope of this paper. At first sight, these parameters ϵ and T_p do not seem to have an immediate correspondence with relevant or measurable variables explaining the dynamics of urban housing markets.

1.3. Socioeconomic outcomes

To study the urban social structure and the socioeconomic outcomes of the various models developed here, we focus on some variables of the model, which characterize these outcomes. Our benchmark is a reference simulation of a monocentric city with two income groups.

Three kinds of outcomes are studied. The utility of individuals is associated to their welfare and gives an economic outcome of the models. The cumulated distances of agents' commuting to work, associated

²With two income groups, it is difficult to determine whether a cell is full or not: we choose to let the price decrease if the smallest optimal surface of housing s_{min} of agents in this cell is smaller than the available surface of the cell s_{av} .

with housing surfaces, give their environmental outcome, which could be conveyed for instance in terms of greenhouse gases emissions associated to transport, land use and housing (heating and cooling). The evolution of social inequalities are given by the difference in the utility of individuals belonging to different income groups.

The use of agent-based systems gives an easy access to any individual or global variable of the model, such as land rents for instance.

2. Comparison with the analytical model and temporal evolution

2.1. Results with two income groups: model 1

The simulations allow us to reach the equilibrium of the AMM model, as can be seen on figure 1. This equilibrium corresponds to a configuration where no agent can raise her utility by moving,

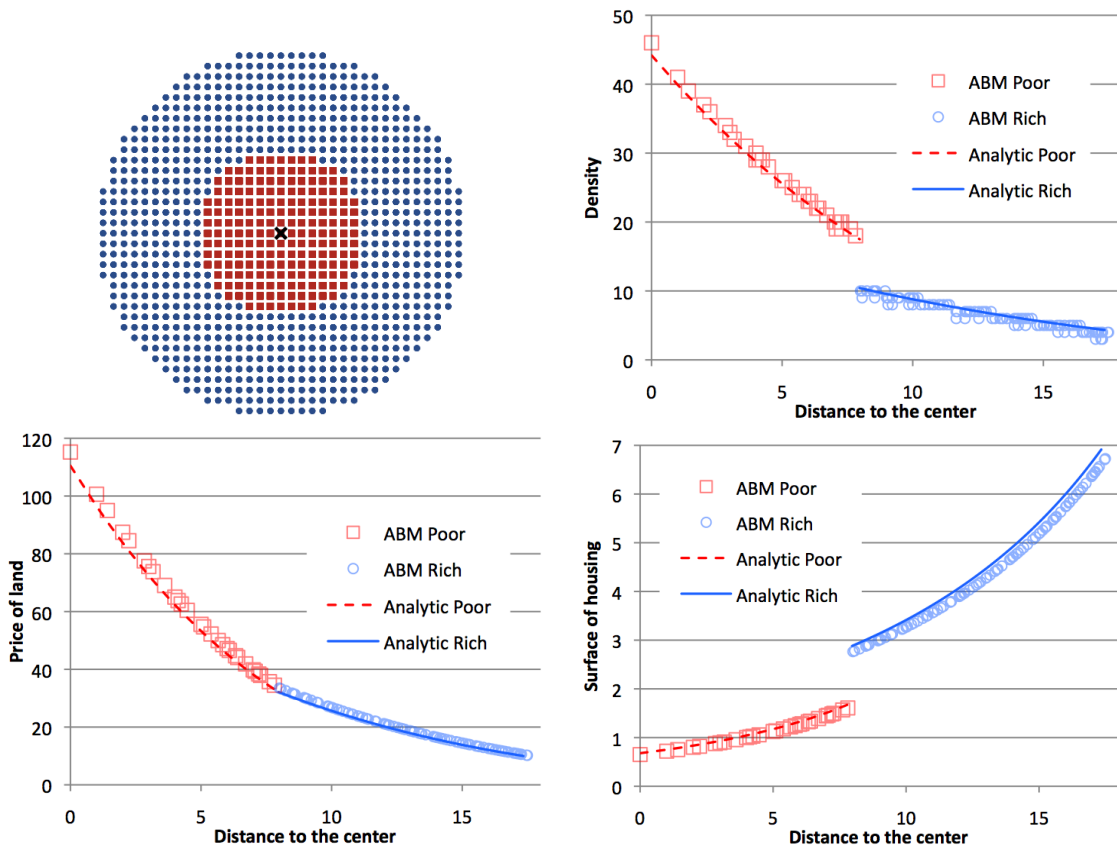


Figure 1: Top left panel, shape of the city. Poor agents are represented by squares, rich agents by circles. Other panels: comparison between the results of the agent-based model (ABM) and the analytical results. Density, land rents and housing surfaces as functions of the distance to the center. The lines represent the analytical results, whereas the symbols indicate the results of the agent-based model. Parameters values are given in table 1.

and therefore no agent has an incentive to do so. In each income group, individuals have an identical utility across the city. With two income groups, the utility of "rich" agents is still higher than that of "poor" agents, because they do not have the same exogenous parameters (they have different incomes, see table 1). A gap can be observed on the density and housing surface curves, because of this discrete difference in income. As in the analytical equilibrium, rich agents are located at the periphery of the

city, where they pay lower land prices and have higher housing surfaces, but also with higher transport costs. This is the standard result of the AMM model and it reproduces the pattern observed in most North-American cities (Fujita (1989)). However, thanks to empirical data Wheaton (1977) argues that the AMM model contributes little to the explanation of American location-income patterns. This issue is no more developed in this paper although the agent model described here can be used to explore it (Lemoy et al. (2010)).

The equilibrium of the agent-based model is described in more detail in the following sections. Let us first depict rapidly the temporal evolution of the simulation.

2.2. Emergence of a city

The evolution of the agent-based model shows how a city emerges from the interactions between individuals during a simulation. Initially agents are located at a random and all prices are equal to the agricultural rent. Density is quite low as agents are dispersed over the simulation space. Then agents move mainly towards the CBD as shown on figure 2 and bid higher, so that the rent curve evolves from a flat rent to the equilibrium rent. At the beginning of the simulation (figures 2(a) and 2(b)), agents

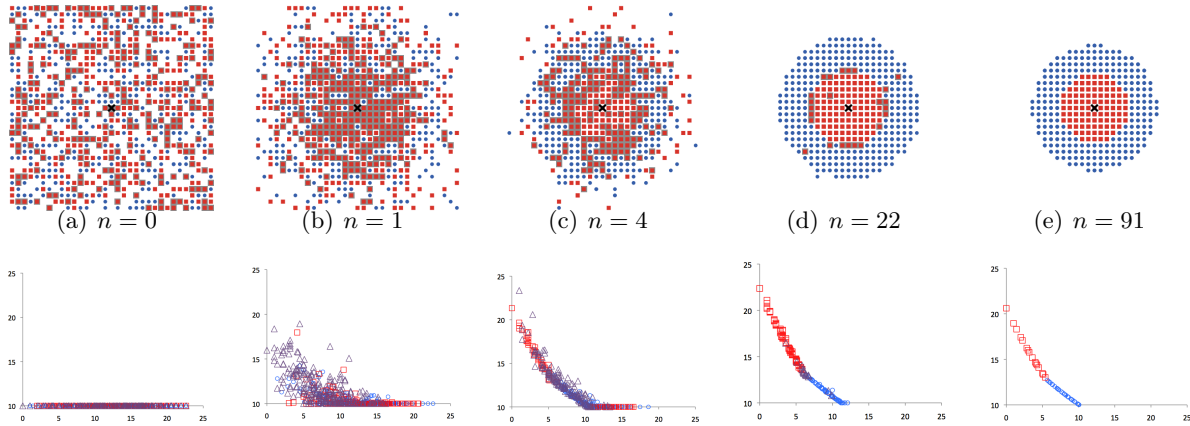


Figure 2: Evolution of the shape of the city (first line) and of the price of land as a function of the distance to the center (second line) during a simulation. Same symbols as figure 1. On the first line, cells whose background is grey indicate that poor and rich agents live there; these cells are displayed as triangular symbols on the second line. At the equilibrium, the city is completely segregated and there are no more such cells. n indicates the mean number of moves per agent since the beginning of the simulation.

gather at the city center without competing much for land, because many cells close to the center are still not full. But when all agents are concentrated around the center (from figure 2(d) on), most bids do not result in an agent moving, for few cells have a sufficient available surface to allow an agent to move in with an interesting utility.

The main variable which indicates the proximity to the equilibrium is the homogeneity of the utility of agents. To describe this homogeneity, we use the relative inhomogeneity of the utility defined as $\Delta U_{\max} = (U_{\max} - U_{\min})/U_{\max}$. With two income groups, we compute this variable within each income group and keep the highest of both values. During a simulation, ΔU_{\max} has a decreasing value. We choose to stop the simulations when the relative variations of utility within income groups are smaller than 10^{-6} , which means that ΔU_{\max} has decreased by approximately five orders of magnitude, as shown on figure 3.

We can test explicitly if no agent has an incentive to move: when $\Delta U_{\max} < 10^{-6}$, each agent tests if she can raise her utility by moving into any other cell, regardless of a sufficient or not sufficient available surface in the cell. The relative possible variations of utility are found to be of the same order of magnitude as ΔU_{\max} .

The standard deviation of the utility can also be computed. It gives a more accurate idea of the variations of utility in the model. On figure 3 are displayed the evolution of the relative inhomogeneity of utility in each income group, and the corresponding evolution of the standard deviation of utility in each income group (also divided by the maximal utility) during a simulation. The latter is always smaller than the former, as should be. This evolution is given as an illustration: because of the stochastic dynamics of the model, it varies across simulations. The equilibrium of the agent-based model is described in

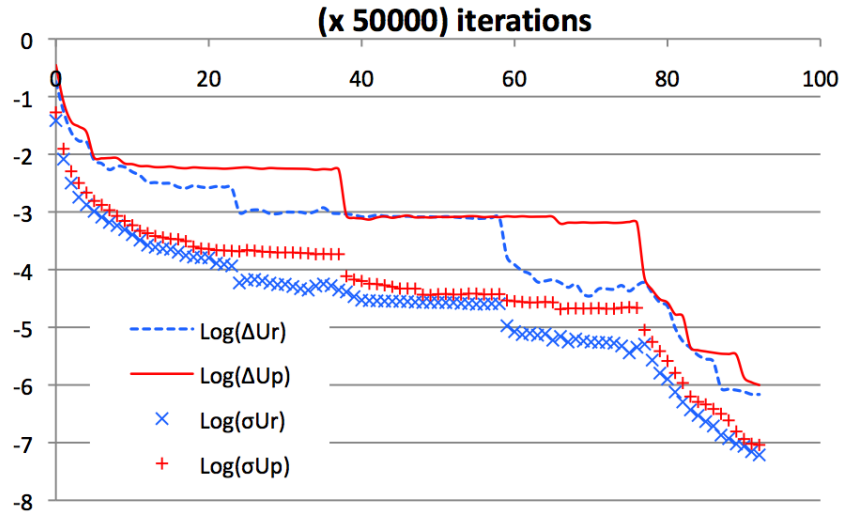


Figure 3: Evolution of the decimal logarithms of the relative inhomogeneity ΔUp (respectively ΔUr) and of the relative standard deviation σUp (resp. σUr) of poor (resp. rich) agents' utility during a simulation. One unit of the abscissa represents 50000 iterations.

more detail in the following section.

2.3. Analytical and agent-based equilibria

In section 3 we will present our simple polycentric models and their results, and in section 4 we argue that the analytical equilibrium of each of these models exists and is unique. Assuming for the moment this existence and uniqueness, it should still be argued that the agent-based model is able to reach a discrete version of this equilibrium.

Figure 1 shows that it is so for the standard monocentric AMM model with two income groups. Let us describe more precisely the hypotheses ensuring that the agent-based model reaches an equilibrium which is similar to the analytical one. Section 2.2 defines the equilibrium of the agent-based model as a situation in which utility is homogeneous within each group, ensuring that no agent has an incentive to move. But this condition alone does not guarantee that the equilibrium is reached, as shown in Appendix B. Indeed, a supplementary condition is needed: that every cell is optimally used, either for agriculture or for housing.

From the comparison with the analytical equilibrium, it follows that only two situations should be observed at equilibrium for the cells of the agent-based model. The first is the case of an agricultural cell, whose price should be equal to the agricultural rent, and where no agent should reside. The second

case is a residential cell, where no space should be left for another agent to move in. Indeed, if the cell can accommodate (at least) one more agent, it indicates that equilibrium is not reached as the city could be made more compact, providing a higher utility for agents.

In order to monitor the number of cells which are not optimally used, we use is the ratio of a surface we call "empty" S_{empty} to the total housing surface of agents S_{tot} . Let us now describe how this "empty" surface is computed. Each cell of the simulation space is visited. If the cell has inhabitants, the smallest housing surface of inhabitants, s_{min} , is stored. If the surface still available in the cell s_{av} is greater than s_{min} , then a part of the cell is considered "empty". To determine how much exactly, it is computed how many agents with housing surface s_{min} could still fit in the cell. The corresponding surface is considered "empty". The values of parameters ϵ and T_p are chosen so as to minimize the quotient $S_{\text{empty}}/S_{\text{tot}}$ within an acceptable simulation time. This quotient is checked to be smaller than 0.5% at the equilibrium of the simulations presented in this paper. We also check that every cell without inhabitants has a price which is equal to the agricultural rent.

With these conditions, the system converges to a unique equilibrium, as described in Appendix A. Appendix B presents as an illustration a simulation using values of parameters ϵ and T_p which do not allow the system to reach a state where $S_{\text{empty}}/S_{\text{tot}} < 0.5\%$.

3. Polycentric city and two-worker households: model 2

The agent-based mechanism introduced in this work to reproduce the results of the AMM model is robust enough for us to introduce effects which are difficult to treat analytically. For instance, several employment centers can be introduced to deal with polycentric cities. This is an important domain of research in the literature (Hartwick and Hartwick (1974); White (1976, 1988); Wieand (1987); Yinger (1992)).

3.1. Introducing two job centers

The simplest way to study a polycentric city consists in defining two employment centers, separated by a distance d . Agents work at the center which is the closest to their housing, and as a consequence, can change jobs as they move. This last feature seems unrealistic but prevents market frictions, in order to reach more rapidly the equilibrium. The results of a such model are presented on the first line of figure 4, keeping only one income group for simplicity.

Figure 5 shows the evolution of different variables characterizing the outcomes of this polycentric model, such as agents' utility U_{mean} , the total commuting distance of agents D_{tot} , the total rent R_{tot} , the mean price P_{mean} and the total surface of the city S_{tot} , compared with the reference monocentric configuration given by the case $d = 0$. The mean density is given by the inverse of the total surface, as the population is fixed.

Raising the number of centers and the distance between them amounts to raising the surface available at a given commuting distance in the city, thus simultaneously reducing competition for land and transport costs. Agents have greater housing surfaces (S_{tot}), which spreads the city, smaller commuting distances (D_{tot}) and a higher utility (U_{mean}). The total rent R_{tot} increases, which seems surprising but is explained by the fact that housing surfaces are greater, which offsets the decrease in prices (P_{mean}).

The economic outcome of the introduction of two centers in this model is positive, as agents' utility increases when the distance between centers increases. However, the environmental outcome is more difficult to assess. Indeed, as figure 5 shows, commuting distances D_{tot} decrease first when centers are taken away from each other, which means a reduction of pollution linked to commuting, but then they increase again. At the same time, the decreasing competition for land results in increasing housing

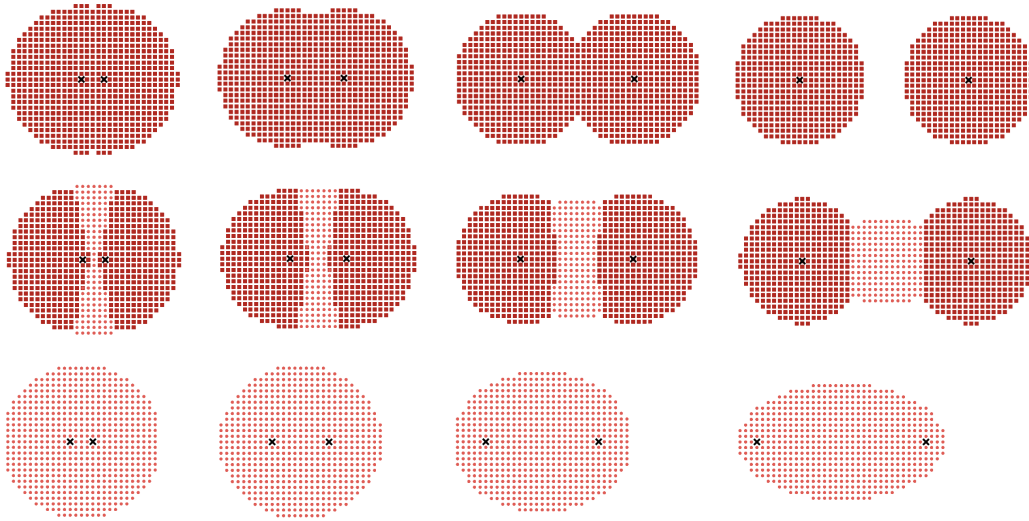


Figure 4: Shape of the city with households (model 2), with $m = 0$ (first line), $m = 0.2$ (second line) and $m = 1$ (third line). The different columns correspond to different values of the distance d between centers: $d = 4, 10, 20, 30$ from left to right. Agents of the "common" group have a darker color than agents of the "split" group. Parameters values are given by table 1.

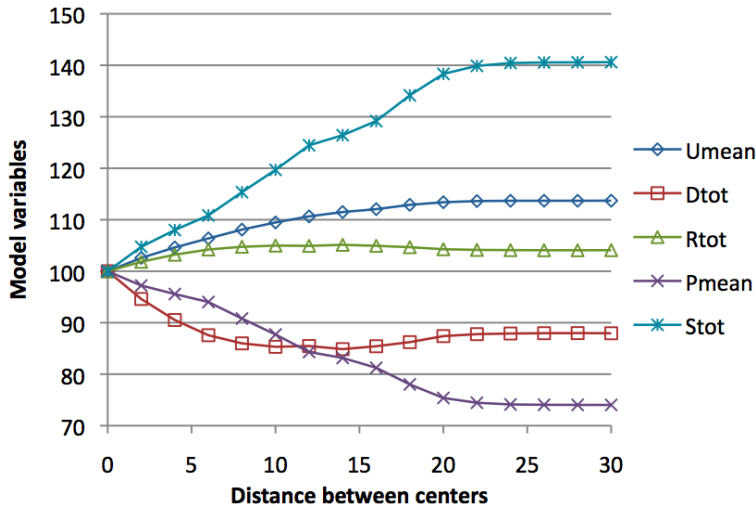


Figure 5: Evolution of the variables characterizing the simplest polycentric model (corresponding to the first line of figure 4) as a function of the distance between centers. Parameters values are given by table 1.

surfaces, and thus city size. Bigger housing surfaces result in greater heating (and cooling) needs, which are a major source of energy needs and greenhouse gases emissions.

In this simple model, both halves of the city stop interacting if the centers are far away from each other. This can be seen on figure 5, where all curves are flat for distances greater than approximately 25 cells.

3.2. Introducing two-worker households

Keeping the same framework, we add more coherence to the city as a whole by studying the behavior of two-worker households. Indeed, this is also a research question in the literature (see Madden and White (1980); Kohlhase (1986); Hotchkiss and White (1993)). Each agent of the previous section (3.1) represents now a household composed of two workers. The transport cost per unit distance for one worker is now $\tilde{t} = t/2$, so that agents of section 3.1 can be seen as such households with both workers going to the same job center. For simplicity, we consider only a city with two centers. Households are divided in two groups. In the first group, which we denote by "common", both persons in the household work at the same employment center, which corresponds to the first line of figure 4. In the second group, which we denote by "split", they work in different centers. This is imposed exogenously and does not change during a simulation.

We study the outcome of this model depending on two variables: the distance d between centers, and the share $m \in [0, 1]$ of households of the "split" group. Let us label employment centers by "East" and "West", and note d_E and d_W the distances between a given household's location and centers East and West. Then if both persons in this household work at the same employment center ("common" group), the East center for instance, the transport cost associated with the commuting of the household is $2 \times \tilde{t} \times d_E = t \times d_E$. If they work at different centers ("split" group), their transport cost is $\tilde{t} \times (d_E + d_W) = t \times (d_E + d_W)/2$.

One important consequence of the new ingredient added here is that a minimal commuting distance of d (or equivalently, a minimal transport cost of $\tilde{t}d$) is imposed for all households of the "split" group. It is their overall commuting distance if they are located on the segment linking both employment centers. So that the minimal total commuting distance D_{\min}^{tot} of agents in the city is $D_{\min}^{\text{tot}} = d \times m \times N$. This minimal distance is exogenously imposed, and is a special feature of this model 2.

To begin with, let us study what happens in the case $m = 1$, where all households belong to the "split" group. To minimize their transport cost, agents choose their location by minimizing $d_E + d_W$. As a result, the shape of the city is elliptic with both employment centers as focal points, as can be seen on the third line of figure 4. Indeed, the figure defined by the set of points verifying $d_E + d_W = k$, with k a given constant, is an ellipse. The effect of increasing d on the transport cost of agents can be described as follows. Because of the increasing minimal commuting distance described previously, the transport cost is increased – everywhere, except at both employment centers themselves, where it does not change when compared with the monocentric ($d = 0$) case. At the same time, the center of the city, seen as the place where transport cost is minimal, is spread on a segment linking both employment centers.

As a consequence, the total commuting distance D_{tot} of agents increases when centers are moved apart, mainly because of the contribution of the minimal commuting distance D_{\min}^{tot} , as can be seen on the left panel of figure 6. $D_{\text{diff}} = D_{\text{tot}} - D_{\min}^{\text{tot}}$ is also indicated: its decrease when d increases shows that agents are gathering around the segment linking both centers. The variables are given on the basis of their value in a reference simulation with $d = 0$ (corresponding to model 1 with only one income group), to have an easy comparison. The utility of agents U_{mean} decreases when d increases, very slowly when centers are close to each other and then more rapidly. The total surface of the city S_{tot} is always bigger than in the reference (monocentric) simulation, but it decreases when d is high. The mean price of housing P_{mean} and the total rent R_{tot} decrease when d increases, as the share of income used for transport increases.

In this model with "split" population, polycentrism is undesirable. It has both a negative economic outcome with the decreasing utility of agents, and a negative environmental outcome, as housing surfaces increase and commuting distances increase. However, it has to be remembered that commuting distances

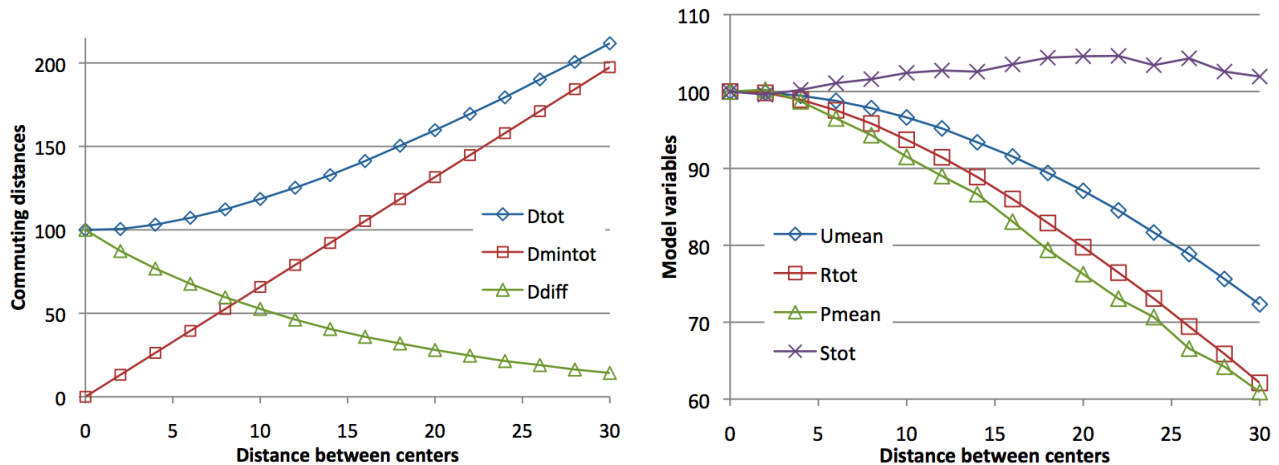


Figure 6: Outcomes of model 2 with $m = 1$ as functions of the distance d between centers. Left panel: evolution of the total commuting distance, the minimal distance and their difference. Right panel: evolution of agents' utility, of the total rent, of the mean price and of the total surface of the city. Parameters values are given by table 1.

increase mainly because of the minimal commuting distance shown on the left panel of figure 6. This effect could be seen as the worst case scenario of a monocentric city evolving towards a polycentric shape: all households increase their travel distances accordingly. A more realistic scenario is given by the case where only a part of the households increase their travel distances, which we study now.

3.3. Mixing various two-worker households

When $0 < m < 1$, simulations show that the utility of agents of the "common" group is always higher than that of agents of the "split" group. This is logical, as agents of the "split" group have more constraints, as they want to stay close to two places. The outcomes of this model with $0 < m < 1$ are intermediate between the two previous cases. The second line of figure 4 gives the shape of the city with $m = 0.2$ for different values of d , and figure 7 gives the corresponding outcomes of model 2. The city shape is intermediate between $m = 0$ and $m = 1$, that is to say, between two disks and an ellipse. Agents of the "split" group (in a paler shade) are located between both centers, separating agents of the "common" group into two parts.

As shown on figure 7, the total commuting distance D_{tot} decreases at first when d increases, and then increases again, mainly because of the contribution of the minimal commuting distance imposed on agents of the "split" group. The utility U_0 of "common" agents increases with d , as their competition for land with "split" agents decreases. The utility U_1 of "split" agents increases at first when d increases, and then decreases again, below its value at $d = 0$. The total surface of the city S_{tot} increases with d , while the mean price of land P_{mean} decreases. The total rent R_{tot} increases at first when d increases, and then decreases below its value at $d = 0$.

In this case, which seems more realistic than the same model with $m = 0$ or $m = 1$, polycentrism is desirable, as long as centers are not moved too far apart from each other. Indeed, the utility of agents of both groups (U_0 and U_1) increases when d increases for small values of d , which gives a positive economic outcome of this model. The environmental outcome is also positive for small values of d , as the total commuting distance D_{tot} decreases when d increases. But this positive effect is mitigated by the fact that housing surfaces increase, which tends to increase emissions of greenhouse gases. Thus this more realistic model tends to confirm the conclusions reached with only "common" households, as

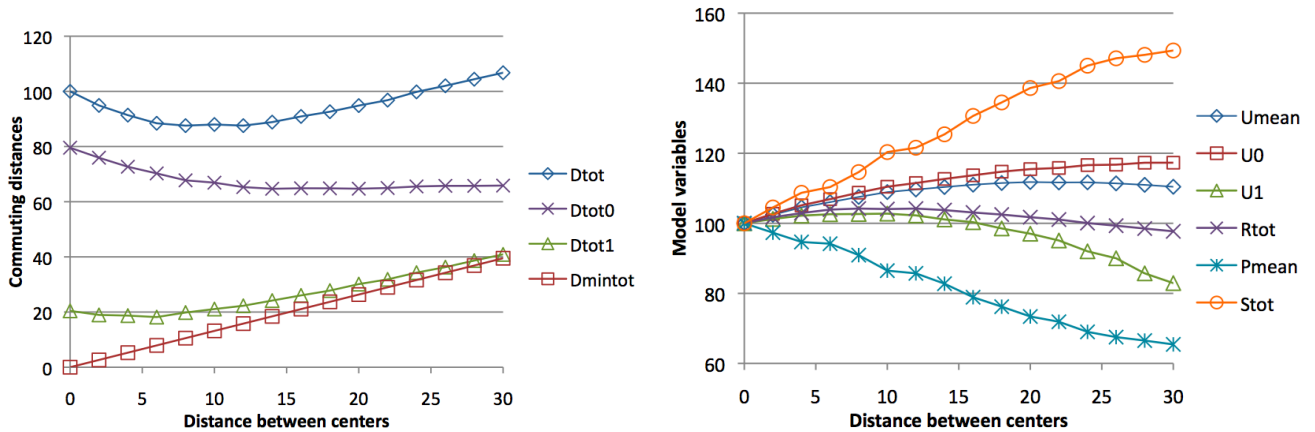


Figure 7: Outcomes of model 2 with $m = 0.2$ as functions of the distance d between centers. Same variables as on figure 6. On the left panel, the total commuting distances D_{tot}^0 and D_{tot}^1 of agents of both groups are presented. On the right panel, the mean utility of agents U_{mean} , the utility U_0 of "common" agents and U_1 of "split" agents are given. Parameters values are given by table 1.

long as centers are kept not too far away from each other.

4. Existence and uniqueness of the analytical equilibrium

It is shown in Fujita (1985) that there exists one unique equilibrium for the standard monocentric AMM model with one or more income groups. The proof of this result is based on boundary rent curves between income groups and between agricultural and residential land uses. Although this result seems difficult to extend to any polycentric city, we give arguments to support the fact that there is also one unique equilibrium in the models which are studied in this paper.

The existence of (at least) one equilibrium for the polycentric models studied here is proved in Fujita and Smith (1987) using fixed-point methods. Hence, it remains to be argued that with the polycentric changes added here to the standard monocentric model, there is no apparition of multiple equilibria, contrary to what can happen with more complicated models, for instance Brueckner et al. (1999) or Fujita and Ogawa (1982). It can be observed that these works add important changes to the standard model by adding variables to the utility function, while our work only considers a more complex transport cost function. Indeed, introducing several employment centers breaks the circular symmetry of the transport cost function compared to the monocentric city model.

Let us first consider the simplest polycentric model we study (model 2 with $m = 0$), a model with two centers separated by a distance d , where agents work at the employment center which is closest to their housing location. When centers are sufficiently far apart, two cities are present and do not interact, with an equal share of the whole population residing in each city. In this case, the result of Fujita (1985) ensuring existence and uniqueness of the equilibrium is clearly still valid. When centers are brought closer and cities begin to interact, the situation is a bit more complicated.

Our approach then consists in mapping this simple polycentric model onto a fictitious monocentric model verifying the assumptions required in Fujita (1985)³ to ensure existence and uniqueness of the

³The first assumption is that $L(x)$ is a piecewise continuous non-negative function on \mathcal{R}^+ , strictly positive on $[0, x_1]$ with x_1 a positive number, and $L(x) \leq L_0(x)$ on \mathcal{R}^+ , where $L_0(x)$ is any continuous function on \mathcal{R}^+ . The second assumption

equilibrium. This mapping allows us to prove the existence and uniqueness of the equilibrium for the polycentric model. Indeed, in the urban models studied in Fujita (1985), as well as in our model 2 with $m = 0$, a given location is completely characterized by the distance of commuting for an agent residing in this place. Equivalently, space is characterized by the amount of land available at each commuting distance x . Let us note $L(x)dx$ the amount of land available between commuting distances x and $x + dx$. A monocentric model with a distribution of land equivalent to that of our simple duocentric model would have $L(x) = 4\pi x$ for $0 \leq x \leq d/2$ and $L(x) = 4x(\pi - \arccos(d/x))$ for $x \geq d/2$. This fictitious monocentric model verifies all assumptions ensuring that it has one unique equilibrium, which is also true for our duocentric model as a consequence. This result could be extended to models with 3 or more centers, as this would only make the function $L(x)$ more complicated. With several income groups, the result still holds.

The case of model 2 with $m = 1$ is almost similar. In this model two employment centers (East and West) separated by a distance d are considered, and each agent represents a two-worker household, with each worker of the household working at a different center from the "mate" (see section 3). Thus the total commuting distance of the household is the sum $d_E + d_W$ of distances between the household's housing location and both centers. The function $L(x)$ of the corresponding monocentric model is now $L(x) = 0$ for $0 \leq x \leq d$ and $L(x) = 2xE(d/x)$ for $x \geq d$, with $E(e)$ the complete elliptic integral of the second kind. This last formula corresponds to the circumference of an ellipse of major axis x , distance between focal points d , and eccentricity d/x . A similar argument of correspondence proves the uniqueness of the equilibrium in this case⁴, and is still valid with several income groups. However, it seems difficult to extend this result to more than two employment centers in this case.

In the cases of model 2 with $0 < m < 1$, the previous arguments supporting the uniqueness of the analytical equilibrium seem difficult to reproduce. But it remains true that no important change is brought to the utility function when compared with the standard monocentric model. Only the transport cost (seen as a function defined on the two-dimensional space of the model, for each group of agents) is changed.

In addition, an important argument in favor of this uniqueness is the fact that for all models presented here, the agent-based model converges to the same equilibrium situation for every run of a simulation, as shown in Appendix A.

5. Perspectives and Discussion

Calibration

In order to get more definite conclusions on the outcomes of the models we study, an important step is to calibrate our models on actual data. This means that the parameters of the agent-based model such as population size, income or transport cost, should be consistent, at least in a rough way, with actual values of a given city (or of a generic city which could be taken as representative of cities of a given size). The results of the simulated city, like density, housing surface, prices, as functions of the distance to the center, could be compared to empirical data.

is that the bid rent and housing surface functions are "well behaved", and that $R_a > 0$. And the third assumption corresponds for instance to the case of several income groups, whose bid rent functions can be ordered by their steepness.

⁴In Fujita (1985), it is assumed that $L(x) > 0$ on $[0, x_1]$ with x_1 a positive number. We assume that the result of existence and uniqueness of the equilibrium is still valid under the condition $L(x) > 0$ on $[x_0, x_1]$ with $0 < x_0 < x_1$, though we do not provide a proof supporting this assertion.

However, such a work seems impossible using Alonso's model, because this model does not take into account vertical housing: land and housing surface are not distinguished, as all agents live on the ground. A simple solution would be to use an exogenous function of available housing surface depending on the distance to the center, which could be inspired from real-world data. But this solution seems unsatisfactory for the modeler.

Building construction can be introduced in the agent-based model, as it is introduced in Muth's model (see Fujita (1989)): building choices of housing industry are modeled in a simple way to determine the housing surface which is built at a given location.

Polycentric city with endogenous centers

This paper presents a study of polycentric models, to explore the outcomes of the AMM model beyond the monocentric framework. But employment centers are still given exogenously, so that the location of jobs can not be studied within this model. It is an interesting perspective of this work to study models with endogenous location of employment centers. The present study can be seen as a first step in this direction. It is indeed important to know what happens with given employment centers, before studying models where these location mechanisms are endogenous.

A model with endogenous location of employment should have a new type of agent which would represent firms, as introduced for instance in Fujita and Ogawa (1982). In such models, firms compete with a residential use of land and try to maximize their profit. Fujita and Thisse (2003) present such analytical models of one-dimensional cities.

Open and closed city models

As stated in section 1, an important choice in this work is the closed city framework, where population size N is fixed. But of course, when looking away from the computer screen to observe economic reality, it is quite obvious that the comparisons made here (corresponding to changes in the models) between mono- and polycentric cities would be in the real world accompanied by changes in population size. This is what is observed in open city models (see Fujita (1989)), where a decrease in transport cost for instance induces an increase in the population (and city) size, but no increase in agents' welfare.

Indeed, in open city models, the utility of agents at equilibrium is completely determined by u_w , the utility "of the rest of the world"⁵, which acts as a chemical potential in statistical physics (Lemoy et al. (2011)).

From a point of view linked to dynamics, such as that conveyed by the agent-based model, agents arrive into the city as long as their utility outside (in the rest of the world) is lower than the utility of agents in the city. The increase in population size increases the competition on the housing market and decreases the utility of agents. This evolution stops once the utility in the city is equal to u_w .

A more realistic description of urban systems lies certainly in between open and closed city frameworks: suppose that a decrease in transport cost such as that provided by the introduction of polycentrism in our models results indeed in an increasing welfare of inhabitants. Then it is likely that some agents will migrate into the city, as predicted by open city models, and that the economic welfare of inhabitants will be globally raised, as predicted by closed city models. A solution to account for both phenomena would be to introduce market frictions in an open city model (for example a cost of moves

⁵It is a drawback of open city models that the parameter determining the population (and city) size is the equilibrium utility, which is difficult to measure to say the least, and is not as intuitive as population size (which is fixed in closed city models). With several income groups, the equilibrium utility levels of the different income groups must be defined.

into the city, preventing some people outside to arrive). Thus, the equilibrium utility of agents in the city would be intermediate between the predictions of open and closed city models.

Broader perspectives

This work is interesting as a complement of analytical works when analytical results are difficult to obtain. It is not meant to come in competition with a mathematical treatment of urban models. A complete analytical study of the different models presented in this work would surely bring other insights on these simple polycentric models.

Hence, two important perspectives can be considered: first, a research perspective is to study dynamic urban models, which are difficult to treat analytically. For instance, once the models presented here have reached an equilibrium, a parameter value is changed (e.g. a raise in transport cost) and the consecutive dynamic changes on the urban systems can be studied, until another equilibrium is reached. Second, a more applied perspective is to design simulation models which could be of an easy use for city planners to help decision-making. Using the robustness of the agent-based dynamics presented here, and applying it to real-world data, for instance various employment centers with transport networks, simulation models could indeed be designed to study economic, environmental and social consequences of different urban planning policies, within the AMM model.

Conclusion

In this work we present a possible use of agent-based systems in social sciences and in particular in economics. Building on the standard urban economics model (AMM model), we run simulations of a simple model with agents interacting in an urban area. The dynamics of the model consist mainly in agents moving and bidding on housing to represent a competition on the urban housing market. This pushes our system in the direction of the equilibrium. This equilibrium corresponds to a discrete version of the analytical equilibrium of the AMM model. A comparison shows the very good agreement of the analytical and the agent-based monocentric models with two income groups.

Then we study the evolution of this equilibrium when the monocentric hypothesis is abandoned to explore polycentric cities. Our results present economic and environmental outcomes of simple polycentric forms within the agent-based model.

The introduction of several centers, when compared to the monocentric city model, has a positive impact on agents' welfare, as transport expenses and competition on the housing market decrease. Commuting distances are reduced, which gives a positive environmental outcome of the polycentric city in this model. However, the increase of housing surfaces may counterbalance this decrease of greenhouse gases emissions. Although the global effect of a reduction of competition for land between agents is clear, its impact on the different variables of this simple urban model and on different income groups is not obvious, as the results show.

The use of agent-based systems on calibrated urban models could test the effect of different urban policies, and provide a global view of their influence on the urban system. In this goal a calibration of a version of this model where housing construction is endogenous is an interesting perspective of research.

Acknowledgements

This research has been partly funded by a PhD grant awarded to Rémi Lemoy by the French Ministry of Research, under the supervision of Charles Raux and Pablo Jensen, and a grant from the transport research program of the French Ministry for Sustainable Development (PREDIT / MEDTL). The authors thank gratefully Florence Goffette-Nagot (GATE, CNRS and University of Lyon) for her valuable comments.

Appendix A. Reproducibility of the results

In order to confirm that the equilibrium reached by the agent-based model is unique, we perform the same simulation 15 times. In spite of the stochasticity of the dynamics of the model, each run converges to the same equilibrium, in a sense which is defined more precisely here.

The simulations are stopped only once the two conditions ensuring that the equilibrium is reached, described in section 2, are verified : the homogeneity of utility ΔU_{\max} is smaller than 10^{-6} (section 2.2) and the share of "empty" surface $S_{\text{empty}}/S_{\text{tot}}$ is smaller than 0.5% (section 2.3).

The results of these simulations are given in table A.2 for two models presented in this work: the first part corresponds to model 1, the reference monocentric model with two income groups. The second corresponds to model 2 with $d = 9$ and $m = 0.2$. The equilibrium values of the variables characterizing the models have only very small variations across different simulations. The maximal variation observed, computed for variable X as $(X_{\max} - X_{\min})/X_{\min}$, is of approximately 0.1% under the two previous conditions.

Model 1	U_r	U_p	$U_r - U_p$	D_{mean}^r	D_{mean}^p	D_{tot}	R_{tot}	p_{mean}	S_{tot}
Variations (in %)	0.009	0.001	0.02	0.09	0.06	0.08	0.02	0.11	0.08
Model 2	U_p	U_0	U_1	D_{mean}^0	D_{mean}^1	D_{tot}	R_{tot}	p_{mean}	S_{tot}
Variations (in %)	0.008	0.01	0.02	0.09	0.05	0.06	0.02	0.09	0.08

Table A.2: Reproducibility of the results: maximal relative variations of the variables characterizing the models across 15 runs of the same simulation.

Appendix B. Parameters of the agent-based model

In this section, we give an example of stationary configuration⁶ of the agent-based model when the parameters specific to the agent-based model, ϵ and T_p , are not chosen so as to minimize the inhomogeneity of utility ΔU_{\max} and the share of "empty" surface $S_{\text{empty}}/S_{\text{tot}}$. As a consequence, the system does not reach an equilibrium which corresponds to the analytical one.

We keep the values of parameters given in table 1, except T_p , which we take as $T_p = 3000$. The results of this simulation are shown on figure B.8. They should be compared with the results of figure 1. Because of this much higher value of T_p the price of vacant cells decreases very slowly. It even decreases too slowly to manage to compensate price increases due to agents' bids, which prevents the system from reaching an equilibrium corresponding to the analytical one. Indeed, as can be seen on the left panel of figure B.8, as the price of cells decreases too slowly, some cells, even close to the CBD, are left vacant after their price has increased too much. The bid mechanism still manages to bring the system to a state with homogeneous utility, where $\Delta U_{\max} < 10^{-6}$. However, a lot of space is not optimally used, which is indicated by the value of the share of "empty" surface $S_{\text{empty}}/S_{\text{tot}} \simeq 130\%$ (in the other simulations presented in this work, this variable is smaller than 0.5% – see section 2.3). Numerous cells where no agents live have a price which is higher than the agricultural rent, a situation which cannot be observed if space is optimally used.

⁶This configuration corresponds actually to a state of the system where the evolution is very slow, so that the configuration seems stationary. We do not study this configuration more precisely here and present it as an illustration of a simulation not converging to an equilibrium corresponding to the analytical one.

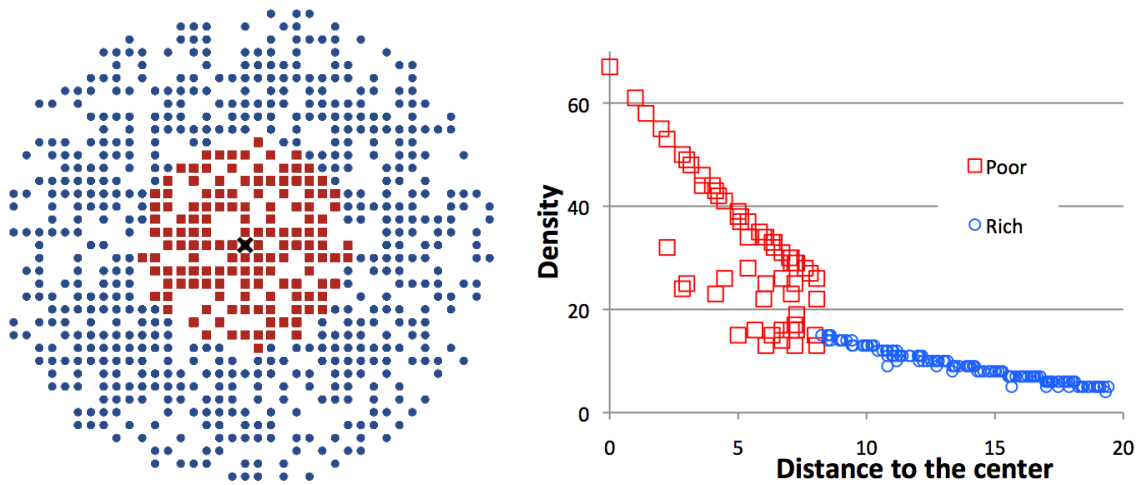


Figure B.8: Monocentric city with two income groups, with $T_p = 3000$. Left panel: shape of the city. Right panel: density as a function of the distance to the center. Same symbols as figure 1.

References

- Alonso, W., 1964. Location and land use. Harvard University Press.
- Anas, A., Arnott, R., Small, K.A., 1998. Urban spatial structure. *Journal of Economic Literature* 36, 1426.
- Asami, Y., Fujita, M., Smith, T.E., 1991. On the foundations of land use theory: Discrete versus continuous populations. *Regional Science and Urban Economics* 20, 473–508.
- Berliant, M., Sabarwal, T., 2008. When worlds collide: Different comparative static predictions of continuous and discrete agent models with land. *Regional Science and Urban Economics* 38, 438–444.
- Brownstone, D., Golob, T.F., 2009. The impact of residential density on vehicle usage and energy consumption. *Journal of Urban Economics* 65, 91–98.
- Brueckner, J., Thisse, J., Zenou, Y., 1999. Why is central Paris rich and downtown Detroit poor? An amenity-based theory. *European Economic Review* 43, 91–107.
- Brueckner, J.K., 1987. The structure of urban equilibria: A unified treatment of the Muth-Mills model, in: Mills, E.S. (Ed.), *Urban Economics*. Elsevier, North Holland. volume 2 of *Handbook of Regional and Urban Economics*. chapter 20, pp. 821–845.
- Caruso, G., Peeters, D., Cavailhes, J., Rounsevell, M., 2007. Spatial configurations in a periurban city. A cellular automata-based microeconomic model. *Regional Science and Urban Economics* 37, 542–567.
- Ewing, R., 1997. Is Los-Angeles-style sprawl desirable? *Journal of the American Planning Association* 63, 107–126.
- Fujita, M., 1985. Existence and uniqueness of equilibrium and optimal land use: Boundary rent curve approach. *Regional Science and Urban Economics* 15, 295–324.
- Fujita, M., 1989. *Urban Economic Theory*. Cambridge University Press.

- Fujita, M., Ogawa, H., 1982. Multiple equilibria and structural transition of non-monocentric urban configurations. *Regional Science and Urban Economics* 12, 161–196.
- Fujita, M., Smith, T.E., 1987. Existence of continuous residential land-use equilibria. *Regional Science and Urban Economics* 17, 549–594.
- Fujita, M., Thisse, J., 2003. *Economie des villes et de la localisation*. De Boeck.
- Giuliano, G., Small, K.A., 1991. Subcenters in the Los Angeles region. *Regional Science and Urban Economics* 21, 163–182.
- Glaeser, E.L., Kahn, M.E., 2010. The greenness of cities: Carbon dioxide emissions and urban development. *Journal of Urban Economics* 67, 404–418.
- Gordon, P., Richardson, H.W., 1997. Are Compact Cities a Desirable Planning Goal? *Journal of the American Planning Association* 63, 95–106.
- Hartwick, P.G., Hartwick, J.M., 1974. Efficient resource allocation in a multinucleated city with intermediate goods. *The Quarterly Journal of Economics* 88, 340–52.
- Hotchkiss, D., White, M.J., 1993. A simulation model of a decentralized metropolitan area with two-worker, traditional, and female-headed households. *Journal of Urban Economics* 34, 159–185.
- Kohlhase, J.E., 1986. Labor supply and housing demand for one- and two-earner households. *The Review of Economics and Statistics* 68, 48–57.
- Lemoy, R., Bertin, E., Jensen, P., 2011. Socio-economic utility and chemical potential. *EPL (Europhysics Letters)* 93, 38002.
- Lemoy, R., Raux, C., Jensen, P., 2010. An agent-based model of residential patterns and social structure in urban areas. *Cybergeo: European Journal of Geography* .
- Madden, J.F., White, M.J., 1980. Spatial implications of increases in the female labor force: A theoretical and empirical synthesis. *Land Economics* 56, 432–446.
- Mills, E.S., 1967. An aggregative model of resource allocation in a metropolitan area. *American Economic Review* 57, 197–210.
- Mills, E.S., 1972. *Urban economics*. Scott, Foresman, Glenview, Illinois.
- Mills, E.S., 2000. A thematic history of urban economic analysis. *Brookings-Wharton Papers on Urban Affairs* , 1–52.
- Muth, R.F., 1969. *Cities and Housing*. University of Chicago Press.
- Seto, K.C., Fragkias, M., Güneralp, B., Reilly, M.K., 2011. A meta-analysis of global urban land expansion. *PLoS ONE* 6, e23777.
- Spivey, C., 2008. The Mills-Muth model of urban spatial structure: Surviving the test of time? *Urban Studies* 45, 295.
- Straszheim, M., 1987. Straszheim, in: Mills, E.S. (Ed.), *Urban Economics*. Elsevier, North Holland. volume 2 of *Handbook of Regional and Urban Economics*, pp. 717–757.

- Sullivan, A.M., 1986. A general equilibrium model with agglomerative economies and decentralized employment. *Journal of Urban Economics* 20, 55–74.
- Wheaton, W.C., 1977. Income and urban residence: An analysis of consumer demand for location. *American Economic Review* 67, 620–31.
- Wheaton, W.C., 2004. Commuting, congestion, and employment dispersal in cities with mixed land use. *Journal of Urban Economics* 55, 417–438.
- White, M.J., 1976. Firm suburbanization and urban subcenters. *Journal of Urban Economics* 3, 323–343.
- White, M.J., 1988. Location choice and commuting behavior in cities with decentralized employment. *Journal of Urban Economics* 24, 129–152.
- Wieand, K.F., 1987. An extension of the monocentric urban spatial equilibrium model to a multicenter setting: The case of the two-center city. *Journal of Urban Economics* 21, 259–271.
- Yinger, J., 1992. City and suburb: Urban models with more than one employment center. *Journal of Urban Economics* 31, 181–205.