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Local E_{11} and the gauging of the trombone symmetry

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Abstract

In any dimension, the positive level generators of the very-extended Kac-Moody algebra E_{11} with completely antisymmetric spacetime indices are associated to the form fields of the corresponding maximal supergravity. We consider the local E_{11} algebra, that is the algebra obtained enlarging these generators of E_{11} in such a way that the global E_{11} symmetries are promoted to gauge symmetries. These are the gauge symmetries of the corresponding massless maximal supergravity. We show the existence of a new type of deformation of the local E_{11} algebra, which corresponds to the gauging of the symmetry under rescaling of the fields. In particular, we show how the gauged IIA theory of Howe, Lambert and West is obtained from an eleven-dimensional group element that only depends on the eleventh coordinate via a linear rescaling. We then show how this results in ten dimensions in a deformed local E_{11} algebra of a new type.

1 Introduction

Given a supergravity theory with a global internal symmetry group and abelian vectors transforming in a representation of this group, the gauging of a subgroup thereof consists in deforming this theory turning on a gauge coupling, and collecting a subset of the vectors in the adjoint representation of the gauge subgroup, compatibly with gauge invariance with respect to the gauge subgroup and with supersymmetry. In this paper we will only be interested in theories with maximal supersymmetry. The first, and probably one of the best known examples of a gauged theory with maximal supersymmetry is the four dimensional $\mathcal{N} = 8$ theory of [1], that is a deformation of the massless maximal supergravity of [2] where an $SO(8)$ subgroup of the internal, or Cremmer-Julia, symmetry group $E_{7(7)}$ is gauged (we refer to the internal symmetry $E_{11-D(11-D)}$ of the massless maximal supergravity in D dimensions as Cremmer-Julia [3] symmetry). Gauged supersymmetric theories are sometimes called massive theories because supersymmetry typically relates coupling constants with mass terms.

A method of obtaining a lower dimensional gauged supergravity theory starting from a massless higher dimensional one is due to Scherk and Schwarz [4]. If the higher dimensional theory possesses an internal symmetry, one can perform a dimensional reduction with the fields depending on the internal coordinate via a linear internal symmetry transformation proportional to a mass parameter m . Because of the symmetry of the higher dimensional theory, this procedure is bound to give a consistent lower dimensional theory, in the sense that in the lower dimension there is no dependence on the internal coordinate. This resulting theory is a massive theory, with masses proportional to the parameter m .

As an example, we can consider the Scherk-Schwarz reduction of the IIB theory to nine dimensions [5, 6]. The IIB theory possesses an $SL(2, \mathbb{R})$ symmetry with generators R^i , $i = 1, 2, 3$. One thus performs a generalised dimensional reduction to nine dimensions, in which the fields transform under $SL(2, \mathbb{R})$ linearly in the internal coordinate and proportionally to the mass parameter m_i in the triplet of $SL(2, \mathbb{R})$. This gives rise to a massive maximal supergravity in nine dimensions, with mass m_i .

There are gauged supergravities that are not of the type discussed so far in this introduction. These arise from the gauging of the global scaling symmetry that leaves the field equations invariant, but rescales the action. This symmetry is not a symmetry of the Cremmer-Julia type, and it is referred to as “trombone” symmetry (it is important to

observe, though, that the trombone symmetry plays a crucial role in understanding the occurrence of the Cremmer-Julia symmetries in the lower dimensional theories [7]). The first example of such a theory is the gauged IIA theory of Howe, Lambert and West [8]. The massless IIA theory [9] has an internal symmetry \mathbb{R}^+ under shifts of the dilaton, and one can consider a combination of this symmetry and the scaling symmetry that leaves the vector invariant. This combined symmetry can thus be gauged, resulting in a Higgs mechanism in which the dilaton field is eaten by the vector, which becomes massive. The fact that the scaling symmetry is not a symmetry of the lagrangian implies that this theory does not admit a lagrangian formulation, but only field equations. It is probably unnecessary to stress that this theory is different from the massive IIA theory of Romans [10], corresponding to a deformation of the massless IIA in which the vector is eaten by means of a Higgs mechanism in which the 2-form becomes massive.

In [6] the gauged IIA theory was shown to arise from a generalised Scherk-Schwarz dimensional reduction from eleven dimensional supergravity. This corresponds to performing a dimensional reduction from eleven to ten dimensions in which the fields depend on the internal coordinate in terms of a linear rescaling. Given that the eleven-dimensional scaling symmetry is a symmetry of the field equations, the ten dimensional equations do not depend on the internal coordinate and as such the truncation to ten dimensions is consistent from this point of view. The lagrangian, though, has an overall scaling symmetry which is linear in the internal coordinate, and thus the truncation to ten dimensions is not consistent at the level of the lagrangian. This is another way of seeing that the theory does not have a lagrangian formulation.

Maximal supergravity theories have a very elegant and natural classification in terms of the very-extended infinite-dimensional Kac-Moody algebra E_{11} [11]. This algebra was first conjectured in [11] to be a symmetry of M-theory. The maximal supergravity theory in D dimensions corresponds to decomposing E_{11} in terms of $GL(D, \mathbb{R}) \otimes E_{11-D}$, and thus the occurrence of the internal symmetry E_{11-D} appears natural from this perspective. In particular, the IIA theory naturally has from the E_{11} viewpoint an \mathbb{R}^+ symmetry that corresponds to the shift of the dilaton. Decomposing the adjoint representation of E_{11} with respect to the subalgebra associated to the IIA theory one obtains generators that are associated to the IIA fields and their duals [11]. In this IIA decomposition of the E_{11} algebra there is a generator with nine antisymmetric ten-dimensional spacetime indices, which is associated to a 9-form in the IIA theory [12]. This 9-form has a 10-form field

strength, which can be thought as the dual of the mass parameter of Romans. Therefore the Romans massive IIA is naturally encoded in E_{11} [13].

More generally, decomposing the E_{11} algebra in a given dimension and considering only the level zero generators (that is the generators of $GL(D, \mathbb{R}) \otimes E_{11-D}$, that are associated to the graviton and the scalars) and the positive level generators with completely antisymmetric indices, that are associated to forms, one finds in all cases the field content of the D -dimensional supergravity theory, in a democratic formulation in which all fields appear together with their magnetic duals [12, 14]. One also finds generators associated to $D - 1$ forms, that are not propagating fields. Remarkably, these generators are in one to one correspondence with a constant scalar quantity, the so called embedding tensor, that parametrises all possible gaugings of subgroups of the internal symmetry E_{11-D} in any dimension, and can be thought of as belonging to a representation of E_{11-D} [15, 16, 17, 18, 19, 20, 21], which indeed is the same representation as the one to which the $D - 1$ form generators belong [14, 22]. Exactly like in the case of Romans, one thinks of the $D - 1$ form fields as being dual to the embedding tensor, obtaining in this way a classification of all possible maximal gauged supergravities in terms of E_{11} .

In the non-linear realisation, the action of positive level E_{11} generators with completely antisymmetric spacetime indices corresponds to gauge transformations for the associated form fields that are linear in the spacetime coordinates, and one wants to enlarge the algebra so that it includes arbitrary gauge transformations. This was done in [23], and the corresponding algebra includes the non-negative level generators as well as momentum and an infinite set of additional generators, that were called Ogievetsky, or Og generators, that correspond to an expansion in the spacetime coordinates of the gauge parameters. This extension is dimension-dependent, and it was called $E_{11,D}^{local}$ in [23]. From the non-linear realisation of the $E_{11,D}^{local}$ algebra with as local subalgebra the D dimensional Lorentz algebra times the maximal compact subalgebra of E_{11-D} one computes all the field strengths of the massless maximal supergravity in D dimensions.

Given the local E_{11} algebra in D dimensions, one can consider its possible massive deformations. In [24] the deformations that do not involve the $GL(D, \mathbb{R})$ generators were studied, and the consistency of the deformed algebra implies that all possible deformations are parametrised by a constant quantity that turns out to be the embedding tensor. All the possible deformations are thus in one to one correspondence with all the possible gauged supergravities resulting from the gauging of a subgroup of E_{11-D} . The non-linear

realisation then provides an extremely simple and powerful method to compute the field strengths and gauge transformations of the fields.

If the gauged supergravity theory arises from a dimensional reduction, this can be seen from the E_{11} point of view in terms of the fact that the deformed generators arise from a redefinition involving the undeformed E_{11} and Og generators in the higher dimension. This was shown in detail in [23] for the case of the Scherk-Schwarz reduction of IIB to nine dimensions. Taking the local E_{11} group element associated to the ten-dimensional IIB theory, the Scherk-Schwarz reduction corresponds to transforming this group element by an $SL(2, \mathbb{R})$ transformation that is linear in the internal coordinate and in the mass parameter m_i , with the rest of the group element only depending on the nine-dimensional coordinates. From the nine-dimensional viewpoint, this results in an algebra that is deformed by the mass parameter m_i with respect to the algebra associated to the massless nine-dimensional theory.

In this paper we show that the construction of [23] admits additional deformations that are associated to the gauging of the trombone symmetry. In particular we show that the gauged IIA theory of [8] naturally arises as a deformation of the local E_{11} algebra of a new type. We show this by considering an eleven-dimensional group element that only depends on the eleventh coordinate by a linear scaling, while the fields are taken to only depend on the ten-dimensional coordinates. This exactly reproduces the generalised Scherk-Schwarz construction of [6]. The fact that the symmetry that one is gauging is not a symmetry of the lagrangian corresponds from this point of view to the fact that the Maurer-Cartan form has an explicit dependence on the eleventh coordinate. Still, there is a very natural way of interpreting the results in ten dimensions, as will be explained in the paper. The resulting ten-dimensional algebra is the algebra corresponding to the IIA theory of [8], and the new feature is that the deformation involves not only the generator of the internal symmetry, that is the scalar generator associated to the dilaton, but also the scaling generator that is the trace of the $GL(10, \mathbb{R})$ generators. Recently a complete classification of this type of maximal gauged supergravities in any dimension was performed in [25] using the embedding tensor formalism. The analysis of the corresponding deformations of the local E_{11} algebra will be presented in a separate paper [26].

It is important to observe that the local E_{11} algebra is not compatible with the full E_{11} symmetry, including the negative level generators. The approach taken in [23, 24] was therefore to include only the non-negative level generators, and from this approach E_{11}

is not a symmetry of the eleven-dimensional group element. This is the approach taken in this paper. An attempt to describe gauged supergravity theories compatibly with the full E_{11} symmetry, based on extending the momentum operator including infinitely many charge generators to form an E_{11} representation [27], was made in [28]. That approach will not be discussed in this paper.

The paper is organised as follows. Section 2 contains a review of the description of gravity as a non-linear realisation of [23], as well as some comments on its dimensional reduction. These results are useful for the main result of the paper, that is the E_{11} description of the gauged IIA theory of [8], which is contained in section 3. Finally, section 4 contains the conclusions.

2 On gravity as a non-linear realisation

In this section we first review the formulation of gravity as a non-linear realisation of [23], and we then show that deformations of this algebra correspond to field redefinitions, and we finally discuss the issue of frame dependence in the dimensional reduction. The aim of this section is to set up the framework for the main result of the paper, which is contained in the next section.

We want to describe gravity as a non-linear realisation of the algebra of diffeomorphisms with the Lorentz algebra as local subalgebra. This was originally achieved in the four-dimensional case in [29, 30], where the algebra of diffeomorphisms was realised as the closure of $IGL(4, \mathbb{R})$ with the conformal group $SO(2, 4)$. This was generalised to D dimensions in [31], where a vierbein rather than a metric was introduced (the metric indeed arises using the Lorentz group to make a particular choice of coset representative).

The more straightforward approach of [32] (see also [33]) is to consider directly the algebra of diffeomorphisms, which is the infinite dimensional algebra generated by

$$P_\mu \quad , \quad K^\mu{}_\nu \quad , \quad K^{\mu_1\mu_2}{}_\nu \quad \dots \quad K^{\mu_1\dots\mu_n}{}_\nu \dots \quad (2.1)$$

with $K^{\mu_1\dots\mu_n}{}_\nu = K^{(\mu_1\dots\mu_n)}{}_\nu$, satisfying the commutation relations

$$[K^\mu{}_\nu, P_\rho] = -\delta_\rho^\mu P_\nu \quad (2.2)$$

$$[K^{\mu_1\dots\mu_n}{}_\nu, P_\rho] = (n-1)\delta_\rho^{(\mu_1} K^{\mu_2\dots\mu_n)}{}_\nu \quad n > 1 \quad (2.3)$$

and

$$[K^{\mu_1 \dots \mu_n}_{\rho}, K^{\nu_1 \dots \nu_m}_{\sigma}] = (n + m - 1) \left[\frac{1}{n} \delta_{\rho}^{(\nu_1 |} K^{\mu_1 \dots \mu_n | \nu_2 \dots \nu_m)}_{\sigma} - \frac{1}{m} \delta_{\sigma}^{(\mu_1} K^{\mu_2 \dots \mu_n) \nu_1 \dots \nu_m}_{\rho} \right]. \quad (2.4)$$

Here the $GL(D, \mathbb{R})$ indices μ, ν, \dots go from 1 to D and an upstairs index denotes the \mathbf{D} and a downstairs index the $\overline{\mathbf{D}}$ of $GL(D, \mathbb{R})$. Note that the last equation for $n = m = 1$ is the $GL(D, \mathbb{R})$ algebra. A realisation of the algebra of eqs. (2.2), (2.3) and (2.4) can be obtained in terms of the position and derivative operators y^{μ} and $\partial_{\mu} = \partial/\partial y^{\mu}$ by the identification

$$P_{\mu} = \partial_{\mu} \quad K^{\mu_1 \mu_2 \dots \mu_n}_{\nu} = \frac{1}{n} y^{\mu_1} y^{\mu_2} \dots y^{\mu_n} \partial_{\nu} \quad . \quad (2.5)$$

One can assign a grade to the generators - that is $K^{\mu_1 \dots \mu_n}_{\nu}$ has grade $n - 1$ and P_{μ} has grade -1 - which is preserved by the algebra above. Note that the grade of a generator is its dimension when the generator is realised in terms of position and momentum operators as is eq. (2.5). The generators of grade n higher than zero, that is all the generators apart from the momentum generator P_{μ} and the $GL(D, \mathbb{R})$ were called Ogievetsky n , or Og n , generators in [23].

Given the algebra of eqs. (2.2), (2.3) and (2.4), we consider the group element written in the form

$$g = e^{x^{\mu} P_{\mu}} \dots e^{\Phi^{\nu}_{\mu_1 \dots \mu_n}(x) K^{\mu_1 \dots \mu_n}_{\nu}} \dots e^{\Phi^{\nu}_{\mu_1 \mu_2}(x) K^{\mu_1 \mu_2}_{\nu}} e^{h_{\mu}{}^{\nu}(x) K^{\mu}_{\nu}} \quad , \quad (2.6)$$

where the momentum generator is contracted with the spacetime coordinate x^{μ} , while all the other fields are functions of x^{μ} . The fields Φ contracting the Og generators are called Og fields. In particular, $\Phi^{\nu}_{\mu_1 \dots \mu_{n+1}}$ is an Og n field.

We now consider the non-linear realisation of the algebra of eqs. (2.2), (2.3) and (2.4) with as local subalgebra the D -dimensional Lorentz algebra. We want the theory to be invariant under transformations of g of the form

$$g \rightarrow g_0 g h \quad , \quad (2.7)$$

where g_0 is a constant group element and h is a local Lorentz group transformation. The fact that the group element transforms under the Lorentz group from the right means that in the exponential of $h_{\mu}{}^{\nu}$ we have to replace the column index with a Lorentz index. As it will appear natural from the Maurer-Cartan form, we identify the exponential of $h_{\mu}{}^{\nu}$ with the vierbein,

$$e_{\mu}{}^a = (e^h)_{\mu}{}^a \quad , \quad (2.8)$$

where the a ($a = 1, \dots, D$) index is a Lorentz index. This means that the vierbein converts curved, that is $GL(D, \mathbb{R})$, indices to flat, that is Lorentz, indices. We want local Lorentz transformations, that act from the right on the group element, to only rotate the vierbein, and it is for this reason that we have written the group element with $h_\mu{}^\nu$ sitting on the far right. One can show that acting as in eq. (2.7) on g one reproduces general coordinate transformations for all the Og fields, while the vierbein transforms under general coordinate transformations and under local Lorentz transformations in the usual way [32]. This notation differs from the one used in [23], where $GL(D, \mathbb{R})$ indices were denoted with Latin letters.

The Maurer-Cartan form $g^{-1}dg$ is invariant under g_0 transformations in eq. (2.7) and only transforms under h . As a consequence, the generators have to be decomposed in irreducible representations of the Lorentz algebra, and thus the indices of the generators must be converted to Lorentz indices. One gets

$$g^{-1}dg = dx^\mu(e_\mu{}^a P_a + G_{\mu,a}{}^b K^a{}_b + G_{\mu,ab}{}^c K^{ab}{}_c + \dots) \quad , \quad (2.9)$$

with

$$G_{\mu,a}{}^b = (e^{-1}\partial_\mu e)_a{}^b - \Phi_{\mu\nu}^\rho (e^{-1})_a{}^\nu e_\rho{}^b \quad (2.10)$$

and

$$G_{\mu,ab}{}^c = (\partial_\mu \Phi_{\rho\kappa}^\lambda - 2\Phi_{\mu\rho\kappa}^\lambda - \Phi_{\mu(\rho}^\tau \Phi_{\kappa)\tau}^\lambda + \frac{1}{2}\Phi_{\rho\kappa}^\tau \Phi_{\mu\tau}^\lambda)(e^{-1})_a{}^\rho (e^{-1})_b{}^\kappa e_\lambda{}^c \quad . \quad (2.11)$$

Lorentz indices can be raised and lowered using the invariant metric η_{ab} . Moreover, apart for the momentum operator, the generators belong to reducible Lorentz representations. In particular the operator K^{ab} splits in its antisymmetric part, its symmetric traceless part and its trace, and the antisymmetric part of K^{ab} is the adjoint representation of the Lorentz algebra. Note that nothing has happened to the generators as such. The generators are invariant tensors, which one can think of as constant matrices, and we have relabelled the indices of these matrices according to the fact that we have to think about them as invariant tensors of the Lorentz algebra.

Identifying as we anticipated in eq. (2.8) the vierbein with the exponential of $h_\mu{}^\nu$, one realises that the quantity $G_\mu{}^{ab}$ defined in eq. (2.10) is part of the covariant derivative of the vierbein if one further identifies $\Phi_{\mu\nu}^\rho$ with the Christoffel connection. In particular, if one imposes that the symmetric part in ab of $G_\mu{}^{ab}$ vanishes, this forces to identify $\Phi_{\mu\nu}^\rho$

with the Levi-Civita connection [33]

$$\Phi_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho} \equiv \frac{1}{2}g^{\rho\tau}(\partial_{\nu}g_{\tau\mu} + \partial_{\mu}g_{\tau\nu} - \partial_{\tau}g_{\mu\nu}) \quad , \quad (2.12)$$

and G_{μ}^{ab} becomes the spin connection ω_{μ}^{ab} [23],

$$\omega_{\mu}^{ab} = \frac{1}{2}e^{\nu a}(\partial_{\mu}e_{\nu}^b - \partial_{\nu}e_{\mu}^b) - \frac{1}{2}e^{\nu b}(\partial_{\mu}e_{\nu}^a - \partial_{\nu}e_{\mu}^a) - \frac{1}{2}e^{\nu a}e^{\rho b}(\partial_{\nu}e_{\rho}^c - \partial_{\rho}e_{\nu}^c)e_{\mu}^c \quad , \quad (2.13)$$

where we have denoted the inverse vierbein as

$$e^{\mu}_a = (e^{-1})_a^{\mu} \quad . \quad (2.14)$$

In the term contracting the Og 1 generator, that is eq. (2.11), one can covariantly solve for the Og 2 field $\phi_{\mu\nu\rho}^{\lambda}$ in terms of the Og 1 field, which is the Levi-Civita connection in such a way that eq. (2.11) becomes the Riemann tensor

$$2G_{\mu,\rho\kappa}^{\lambda} = R_{\mu\rho}^{\lambda\kappa} \equiv \partial_{\mu}\Gamma_{\rho\kappa}^{\lambda} - \partial_{\rho}\Gamma_{\mu\kappa}^{\lambda} + \Gamma_{\mu\tau}^{\lambda}\Gamma_{\rho\kappa}^{\tau} - \Gamma_{\rho\tau}^{\lambda}\Gamma_{\mu\kappa}^{\tau} \quad . \quad (2.15)$$

One can solve for the Og fields of any grade in terms of the lower grade fields, which results in the Maurer-Cartan form only containing the Riemann tensor and covariant derivatives thereof. This concludes the review of section 2 of [23].

The algebra of eqs. (2.2), (2.3) and (2.4) can be deformed compatibly with $GL(D, \mathbb{R})$. In particular, restricting our attention to the generators up to Og 2, we can write the relevant commutators as

$$\begin{aligned} [K^{\mu}_{\nu}, P_{\rho}] &= -\delta_{\rho}^{\mu}P_{\nu} + a\delta_{\nu}^{\mu}P_{\rho} \\ [K^{\mu_1\mu_2}_{\nu}, P_{\rho}] &= \delta_{\rho}^{(\mu_1}K^{\mu_2)}_{\nu} + b\delta_{\nu}^{(\mu_1}K^{\mu_2)}_{\rho} + c\delta_{\nu}^{(\mu_1}\delta_{\rho}^{\mu_2)}K \quad , \end{aligned} \quad (2.16)$$

where K is the trace of the $GL(D, \mathbb{R})$ generators,

$$K = K^{\mu}_{\mu} \quad . \quad (2.17)$$

The parameters a , b and c satisfy conditions coming from the Jacobi identities. In particular, if $Da \neq 1$, one can without loss of generality impose $b = 0$, and then determine c to be

$$c = \frac{a}{1 - Da} \quad . \quad (2.18)$$

Summarising, the deformed algebra is

$$\begin{aligned} [K^\mu{}_\nu, P_\rho] &= -\delta_\rho^\mu P_\nu + a\delta_\nu^\mu P_\rho \\ [K^{\mu_1\mu_2}{}_\nu, P_\rho] &= \delta_\rho^{(\mu_1} K^{\mu_2)}{}_\nu + \frac{a}{1-Da}\delta_\nu^{(\mu_1}\delta_\rho^{\mu_2)}K \end{aligned} \quad (2.19)$$

for any parameter a , provided that $Da \neq 1$.

We now consider the group element of eq. (2.6), and we compute the Maurer-Cartan form using the modified commutators of eq. (2.19). The result is

$$g^{-1}dg = dx^\mu(e^{-ah}(e^h)_\mu{}^a P_a + G_{\mu,a}{}^b K^a{}_b + \dots) \quad , \quad (2.20)$$

where we have defined

$$h = h_\mu{}^\mu \quad , \quad (2.21)$$

and where

$$G_{\mu,a}{}^b = (e^{-h}\partial_\mu e^h)_a{}^b - \Phi_{\mu\nu}^\rho(e^{-h})_a{}^\nu(e^h)_\rho{}^b - \frac{a}{1-Da}\Phi_{\mu\nu}^\nu\delta_a^b \quad . \quad (2.22)$$

We now interpret the matrix contracting the momentum operator as the vierbein,

$$e_\mu{}^a = e^{-ah}(e^h)_\mu{}^a \quad , \quad (2.23)$$

and inverting this relation one gets

$$(e^h)_\mu{}^a = (\det e)^{\frac{a}{1-Da}} e_\mu{}^a \quad , \quad (2.24)$$

where we have denoted the determinant of the vierbein with $\det e$ to avoid confusion as much as possible between Euler's number and the vierbein. If we plug this relation into eq. (2.22), we get

$$G_{\mu,a}{}^b = (e^{-1}\partial_\mu e)_a{}^b + \frac{a}{1-Da}(\det e)^{-1}\partial_\mu(\det e)\delta_a^b - \Phi_{\mu\nu}^\rho(e^{-1})_a{}^\nu(e)_\rho{}^b - \frac{a}{1-Da}\Phi_{\mu\nu}^\nu\delta_a^b \quad . \quad (2.25)$$

If we now impose that the symmetric part in ab of this equation vanishes, we find that eq. (2.12) is still a solution, and the δ_a^b part of eq. (2.25) cancels because eq. (2.12) gives the well-known formula

$$\Phi_{\mu\nu}^\nu = \Gamma_{\mu\nu}^\nu = (\det e)^{-1}\partial_\mu(\det e) \quad . \quad (2.26)$$

This proves that the modification of the algebra of diffeomorphisms as in eq. (2.19) is equivalent to the redefinition of the vierbein in terms of $h_\mu{}^\nu$ as in eq. (2.23).

Before we conclude this section, we want to make a comment on dimensional reduction. We consider a circle dimensional reduction from dimension $D+1$ to dimension D , we denote with μ and a the curved and flat indices in D dimensions, and we denote with y the $D+1$ -th coordinate. The $D+1$ dimensional momentum splits in P_μ and $Q = P_y$. As shown in [23], circle dimensional reduction corresponds to a truncation of the algebra in which the operator Q is projected out, and consistently one must project out all the generators that have non-trivial commutator with Q . By looking at eq. (2.2), this implies that K^y_μ must be projected out. This implies the standard ansatz for the D dimensional vierbein,

$$\begin{pmatrix} e^{\alpha\phi} e_\mu^a & e^{\beta\phi} A_\mu \\ 0 & e^{\beta\phi} \end{pmatrix} , \quad (2.27)$$

and computing the part of the Maurer-Cartan form along dx^μ , neglecting for simplicity the Og contribution, one gets

$$dx^\mu g^{-1} \partial_\mu g = e^{\alpha\phi} e_\mu^a P_a + (e^\nu{}_a \partial_\mu e_\nu{}^b + \alpha \delta_a^b \partial_\mu \phi) K^a{}_b + e^{(\beta-\alpha)\phi} \partial_\mu A_\nu e^\nu{}_a K^a{}_y + \beta \partial_\mu \phi K^y{}_y . \quad (2.28)$$

By looking at this equation, we define the D dimensional vector and scalar generators as

$$\begin{aligned} R^\mu &= K^\mu{}_y \\ R &= \alpha K + \beta K^y{}_y \quad , \end{aligned} \quad (2.29)$$

and the non-trivial commutators, apart from the commutators with the $GL(D, \mathbb{R})$ generators which are standard, are

$$\begin{aligned} [R, R^\mu] &= (\alpha - \beta) R^\mu \\ [R, P_\mu] &= -\alpha P_\mu \quad . \end{aligned} \quad (2.30)$$

From these commutators it is then easy to show that the Maurer-Cartan form of eq. (2.28) arises from the D dimensional group element

$$g = e^{x \cdot P} e^{A_\mu R^\mu} e^{h_\mu{}^\nu K^\mu{}_\nu} e^{\phi R} . \quad (2.31)$$

This concludes the analysis of this section. In the next section we will consider a (generalised) dimensional reduction of eleven dimensional supergravity to ten dimensions, and for simplicity we will work in the frame in which $\alpha = 0$ and $\beta = 1$, but all the results can easily be generalised to any frame.

3 Local E_{11} and gauged IIA

In [23] it was shown that the Scherk-Schwarz reduction of the IIB theory corresponds to a non-linear realisation based on an E_{11} group element that is entirely nine-dimensional, apart from an overall transformation with respect to the generators of the internal symmetry of the IIB theory which is linear in the compactified coordinate. The main aim of this section is to perform for the gauged IIA theory of [8] an analysis equivalent to the one performed in [23] for the Scherk-Schwarz reduction of the IIB theory. This analysis is motivated by ref. [6], where the gauged IIA theory was derived performing a generalised Scherk-Schwarz reduction of eleven dimensional supergravity in which one performs a scaling transformation of the fields which is linear in the internal coordinate. The symmetry under rescaling of the fields was called “trombone” symmetry in [6], because although it is a symmetry of the field equations, it actually gives rise to an overall scaling of the lagrangian. The fact that this symmetry is not a symmetry of the lagrangian implies that its gauging results in a theory which does not admit a lagrangian formulation.

We first review the E_{11} analysis of 11-dimensional supergravity with the inclusion of the Og generators, as was derived originally in [23]. We use a notation similar to the one of the previous section, and we thus use Greek letters to denote the $GL(D, \mathbb{R})$ indices. This notation again differs from the one used in [23]. In particular, $GL(11, \mathbb{R})$ indices are denoted by $\hat{\mu}$ ($\hat{\mu} = 1, \dots, 11$), and similarly 11-dimensional Lorentz indices are denoted by \hat{a} . We only consider the $GL(11, \mathbb{R})$ generator $K^{\hat{\mu}}_{\hat{\nu}}$ and the 3-form generator $R^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}$, which corresponds to a truncation of the E_{11} algebra to level 1 (and only considering positive level generators). The relevant E_{11} commutators are thus the commutators giving the $GL(11, \mathbb{R})$ algebra and

$$[K^{\hat{\mu}}_{\hat{\nu}}, R^{\hat{\rho}_1\hat{\rho}_2\hat{\rho}_3}] = 3\delta_{\hat{\nu}}^{[\hat{\rho}_1} R^{|\hat{\mu}|\hat{\rho}_2\hat{\rho}_3]} \quad . \quad (3.1)$$

As explained in [23], in order to promote the 3-form constant shift to a gauge transformation, we have to add an infinite set of Og generators, the first one being $K_1^{\hat{\mu},\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3}$ satisfying

$$K_1^{\hat{\mu},\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3} = K_1^{[\hat{\mu},[\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3]} \quad K_1^{[\hat{\mu},\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3]} = 0 \quad , \quad (3.2)$$

whose commutator with momentum is

$$[K_1^{\hat{\mu},\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3}, P_{\hat{\rho}}] = \delta_{\hat{\rho}}^{\hat{\mu}} R^{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3} - \delta_{\hat{\rho}}^{[\hat{\mu}} R^{\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3]} \quad . \quad (3.3)$$

If one considers the group element in the form

$$g = e^{x \cdot P} e^{\Phi_{\text{Og}} K^{\text{Og}}} e^{A_{\hat{\mu}\hat{\nu}\hat{\rho}} R^{\hat{\mu}\hat{\nu}\hat{\rho}}} e^{h_{\hat{\mu}}^{\hat{\nu}} K^{\hat{\mu}}_{\hat{\nu}}} \quad (3.4)$$

and computes the Maurer-Cartan form, one gets

$$g^{-1}dg = dx^{\hat{\mu}} \left[e_{\hat{\mu}}^{\hat{a}} P_{\hat{a}} + (\partial_{\hat{\mu}} A_{\hat{\nu}\hat{\rho}\hat{\sigma}} - \Phi_{\hat{\mu},\hat{\nu}\hat{\rho}\hat{\sigma}}) e^{\hat{\nu}}_{\hat{a}} e^{\hat{\rho}}_{\hat{b}} e^{\hat{\sigma}}_{\hat{c}} R^{\hat{a}\hat{b}\hat{c}} + \dots \right] \quad (3.5)$$

where the dots denote both the Og generators contribution, as well as the gravity sector which is as in [23] and reviewed in section 2. As explained in [23], the inverse Higgs mechanism permits to solve covariantly for the Og 1 field $\Phi_{\hat{\mu},\hat{\nu}\hat{\rho}\hat{\sigma}}$ in terms of the derivative of the 3-form potential, in such a way that only the completely antisymmetric term $\partial_{[\hat{\mu}} A_{\hat{\nu}\hat{\rho}\hat{\sigma}]}$ survives, which is the field strength of the 3-form. Similarly, by the same mechanism the Og generators are contracted with covariant derivatives of the field strength of the 3-form.

We now consider the dimensional reduction of this system to ten dimensions. For simplicity we take $\alpha = 0$ and $\beta = 1$ in the vierbein ansatz of eq. (2.27). The construction is easy to generalise to any other frame. The notation for the dimensionally reduced gravity generators is the same as in the previous section, while the 11-dimensional 3-form generator gives rise to the 3-form $R^{\mu_1\mu_2\mu_3}$ and the 2-form $R^{\mu_1\mu_2} = R^{\mu_1\mu_2 y}$, where the Greek index μ is a $GL(10, \mathbb{R})$ index and y denotes the 11-th direction.

In terms of these generators, the E_{11} algebra becomes (we only consider the non-vanishing commutators)

$$[K^{\mu}_{\nu}, R^{\rho}] = \delta_{\nu}^{\rho} R^{\mu} \quad [K^{\mu}_{\nu}, R^{\rho_1\rho_2}] = 2\delta_{\nu}^{[\rho_1} R^{|\mu|\rho_2]} \quad [K^{\mu}_{\nu}, R^{\rho_1\rho_2\rho_3}] = 3\delta_{\nu}^{[\rho_1} R^{|\mu|\rho_2\rho_3]} \quad (3.6)$$

$$[R, R^{\mu}] = -R^{\mu} \quad [R, R^{\mu\nu}] = R^{\mu\nu} \quad (3.7)$$

$$[R^{\mu}, R^{\nu\rho}] = R^{\mu\nu\rho} \quad . \quad (3.8)$$

We also have

$$[R^{\mu}, P_{\nu}] = -\delta_{\nu}^{\mu} Q \quad , \quad (3.9)$$

where as in the previous section Q denotes the momentum operator in the y direction.

The 11-dimensional Og generator $K^{\hat{\mu},\hat{\nu}_1\hat{\nu}_2\hat{\nu}_3}$ gives rise to the 10-dimensional Og generators $K^{\mu,\nu_1\nu_2\nu_3}$, $K^{\mu,\nu_1\nu_2}$, $K^{\mu_1\mu_2\mu_3}$ and $K^{\mu_1\mu_2}$, with

$$\begin{aligned} K^{\mu,\nu_1\nu_2} &= K^{\mu,\nu_1\nu_2 y} - K^{[\mu,\nu_1\nu_2]y} \\ K^{\mu_1\mu_2\mu_3} &= \frac{4}{3} K^{y,\mu_1\mu_2\mu_3} \\ K^{\mu_1\mu_2} &= K^{y,\mu_1\mu_2 y} \quad . \end{aligned} \quad (3.10)$$

The commutators of these operators with P_μ and Q are

$$\begin{aligned}
[K^{\mu,\nu_1\nu_2\nu_3}, P_\rho] &= \delta_\rho^\mu R^{\nu_1\nu_2\nu_3} - \delta_\rho^{[\mu} R^{\nu_1\nu_2\nu_3]} & [K^{\mu,\nu_1\nu_2\nu_3}, Q] &= 0 \\
[K^{\mu_1\mu_2\mu_3}, P_\nu] &= \delta_\nu^{[\mu_1} R^{\mu_2\mu_3]} & [K^{\mu_1\mu_2\mu_3}, Q] &= R^{\mu_1\mu_2\mu_3} \\
[K^{\mu,\nu_1\nu_2}, P_\rho] &= \delta_\rho^\mu R^{\nu_1\nu_2} - \delta_\rho^{[\mu} R^{\nu_1\nu_2]} & [K^{\mu,\nu_1\nu_2}, Q] &= 0 \\
[K^{\mu_1\mu_2}, P_\nu] &= 0 & [K^{\mu_1\mu_2}, Q] &= R^{\mu_1\mu_2} \quad . \quad (3.11)
\end{aligned}$$

We also consider the 10-dimensional Og generators that arise from the 11-dimensional gravity Og 1 generator $K^{\mu\nu}{}_\rho$. In particular we are only interested in the Og generators whose lower index in is the y direction, that are

$$K^{\mu\nu} = K^{\mu\nu}{}_y \quad K^\mu = 2K^{\mu y}{}_y \quad K = K^{yy}{}_y \quad , \quad (3.12)$$

and whose commutation relation with P_μ and Q are

$$\begin{aligned}
[K^{\mu\nu}, P_\rho] &= \delta_\rho^{(\mu} R^{\nu)} & [K^{\mu\nu}, Q] &= 0 \\
[K^\mu, P_\nu] &= \delta_\nu^\mu R & [K^\mu, Q] &= R^\mu \\
[K, P_\mu] &= 0 & [K, Q] &= R \quad . \quad (3.13)
\end{aligned}$$

We now consider the non-linear realisation based on this algebra. We first consider the case of standard massless dimensional reduction, which corresponds to taking the group element

$$g = e^{x \cdot P} e^{y Q} e^{\Phi_{\text{Og}} K^{\text{Og}}} e^{A_{\mu\nu\rho} R^{\mu\nu\rho}} e^{A_{\mu\nu} R^{\mu\nu}} e^{A_\mu R^\mu} e^{\phi R} e^{h_\mu{}^\nu K^\mu{}_\nu} \quad , \quad (3.14)$$

where we take all the fields not to depend on y . We then compute the Maurer-Cartan form

$$g^{-1} dg = dx^\mu g^{-1} \partial_\mu g + dy g^{-1} \partial_y g \quad . \quad (3.15)$$

We first consider the part along dx^μ . Following [23], we use the inverse Higgs mechanism to covariantly solve for the not fully antisymmetric Og fields in terms of the other fields in such a way that all the terms in the Maurer-Cartan form are completely antisymmetric. This gives

$$\begin{aligned}
dx^\mu g^{-1} \partial_\mu g &= dx^\mu [e_\mu{}^a P_a + e^\phi A_\mu Q + (\partial_\mu \phi - \Phi_\mu) R + e^\phi \partial_{[\mu} A_{\nu]} e^\nu{}_a R^a \\
&+ e^{-\phi} (\partial_{[\mu} A_{\nu\rho]} - \Phi_{\mu\nu\rho} - \Phi_{[\mu} A_{\nu\rho]}) e^\nu{}_a e^\rho{}_b R^{ab} \\
&+ (\partial_{[\mu} A_{\nu\rho\sigma]} - \partial_{[\mu} A_{\nu\rho} A_{\sigma]} + \Phi_{[\mu\nu\rho} A_{\sigma]} + \Phi_{[\mu} A_{\nu\rho} A_{\sigma]}) e^\nu{}_a e^\rho{}_b e^\sigma{}_c R^{abc} + \dots] \quad (3.16)
\end{aligned}$$

We then consider the dy term. Again following [23] we impose that the part of the Maurer-Cartan form in the dy direction vanishes apart from the Q term. This imposes that all the Og fields associated to the Og generators that do not commute with Q must vanish:

$$\Phi = 0 \quad \Phi_\mu = 0 \quad \Phi_{\mu_1\mu_2} = 0 \quad \Phi_{\mu_1\mu_2\mu_3} = 0 \quad . \quad (3.17)$$

Plugging these conditions into eq. (3.16), we then read the field strengths

$$\begin{aligned} F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} \\ F_{\mu\nu\rho} &= \partial_{[\mu}A_{\nu\rho]} \\ F_{\mu\nu\rho\sigma} &= \partial_{[\mu}A_{\nu\rho\sigma]} - \partial_{[\mu}A_{\nu\rho}A_{\sigma]} \quad , \end{aligned} \quad (3.18)$$

that are the field strengths of the gauge fields of the massless IIA theory. Acting with g_0 transformations on the group element of eq. (3.14) one also derives the corresponding gauge transformations, that are

$$\begin{aligned} \delta A_\mu &= \partial_\mu \Lambda \\ \delta A_{\mu\nu} &= \partial_{[\mu} \Lambda_{\nu]} \\ \delta A_{\mu\nu\rho} &= \partial_{[\mu} \Lambda_{\nu\rho]} + \partial_{[\mu} \Lambda A_{\nu\rho]} \quad . \end{aligned} \quad (3.19)$$

We now want to derive the field strengths and gauge transformations of the gauged IIA theory of [8, 6] in an analogous way. We take as our starting point an 11-dimensional group element that has a non-trivial y dependence, namely

$$g = e^{x \cdot P} e^{yQ} e^{my(K+R)} e^{\Phi_{\text{Og}} K^{\text{Og}}} e^{A_{\mu\nu\rho} R^{\mu\nu\rho}} e^{A_{\mu\nu} R^{\mu\nu}} e^{A_\mu R^\mu} e^{\phi R} e^{h_\mu{}^\nu K^\mu{}_\nu} \quad , \quad (3.20)$$

where K is the trace of the $GL(10, \mathbb{R})$ generators, m is a constant parameter and we take all the fields not to depend on y . Observe that this particular choice of the group element is due to the fact the trombone scaling is generated by $K^{\hat{\mu}}{}_{\hat{\mu}}$ in eleven dimensions, and $K^{\hat{\mu}}{}_{\hat{\mu}} = K + R$ in the frame in which $\alpha = 0$ and $\beta = 1$ in eq. (2.27). One can easily generalise this to an arbitrary frame.

We now compute the Maurer-Cartan form. As in the massless case, we first consider the dx^μ term, and we use the inverse Higgs mechanism to solve for the Og fields with mixed symmetry in terms of the other fields in such a way that all the terms that are left in the Maurer-Cartan form are completely antisymmetric. With respect to eq. (3.16) the P_μ and

Q terms, as well as the Og fields, acquire a y dependence due to the non-trivial form of the group element of eq. (3.20). The result is

$$\begin{aligned}
dx^\mu g^{-1} \partial_\mu g &= dx^\mu [e^{my} e_\mu^a P_a + e^{my} e^\phi A_\mu Q + (\partial_\mu \phi - e^{my} \Phi_\mu) R + e^\phi \partial_{[\mu} A_{\nu]} e^\nu{}_a R^a \\
&+ e^{-\phi} (\partial_{[\mu} A_{\nu\rho]} - e^{my} \Phi_{\mu\nu\rho} - e^{my} \Phi_{[\mu} A_{\nu\rho]}) e^\nu{}_a e^\rho{}_b R^{ab} \\
&+ (\partial_{[\mu} A_{\nu\rho\sigma]} - \partial_{[\mu} A_{\nu\rho} A_{\sigma]} + \epsilon^{my} \Phi_{[\mu\nu\rho} A_{\sigma]} + e^{my} \Phi_{[\mu} A_{\nu\rho} A_{\sigma]}) e^\nu{}_a e^\rho{}_b e^\sigma{}_c R^{abc} + \dots] \quad (3.21)
\end{aligned}$$

The fact that this term has a non-trivial y dependence is the crucial difference with respect to the Scherk-Schwarz reduction of IIB discussed in [23]. In that case, the group element was deformed by a y -dependent $SL(2, \mathbb{R})$ transformation, which commutes with momentum. Correspondingly, the dx^μ part of the Maurer-Cartan form did not contain any y dependence. This is what guarantees the consistency of the truncation to the lower dimensional theory. In this case, the dx^μ part of the Maurer-Cartan contains a y dependence, and this is the translation in this group-theoretic language of the fact that the trombone symmetry is not a symmetry of the lagrangian but only of the field equations. As emphasised in [6], having such a symmetry is actually sufficient to guarantee that also in this case the truncation to ten dimensions is consistent at the level of the field equations. We will see in the following how this notion of consistency of the truncation is translated in our language.

We now compute the dy part of the Maurer-Cartan form. We get

$$\begin{aligned}
dy g^{-1} \partial_y g &= e^\phi e^{my} Q + m(K + R) - e^{my} \Phi R + e^\phi e^{my} (-\Phi_\mu + \Phi A_\mu) e^\mu{}_a R^a \\
&+ e^{-\phi} (-e^{my} \Phi A_{\mu\nu} - e^{my} \Phi_{\mu\nu} + 3m A_{\mu\nu}) e^\mu{}_a e^\nu{}_b R^{ab} \\
&+ (-e^{my} \Phi_\mu A_{\nu\rho} + e^{my} \Phi A_{\mu\nu} A_\rho - e^{my} \Phi_{\mu\nu\rho} + e^{my} \Phi_{\mu\nu} A_\rho + 3m A_{\mu\nu\rho} \\
&- 3m A_{\mu\nu} A_\rho) e^\mu{}_a e^\nu{}_b e^\rho{}_c R^{abc} \quad . \quad (3.22)
\end{aligned}$$

Following [23], we now use the inverse Higgs mechanism to impose that all the terms in eq. (3.22) proportional to positive level generators vanish. This gives

$$\Phi = 0 \quad \Phi_\mu = 0 \quad , \quad (3.23)$$

as well as

$$e^{my} \Phi_{\mu\nu} - 3m A_{\mu\nu} = 0 \quad e^{my} \Phi_{\mu\nu\rho} - 3m A_{\mu\nu\rho} = 0 \quad . \quad (3.24)$$

Substituting these relations in eq. (3.21), we then read the field strengths

$$\begin{aligned}
F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} \\
F_{\mu\nu\rho} &= \partial_{[\mu}A_{\nu\rho]} - 3mA_{\mu\nu\rho} \\
F_{\mu\nu\rho\sigma} &= \partial_{[\mu}A_{\nu\rho\sigma]} - \partial_{[\mu}A_{\nu\rho}A_{\sigma]} + 3mA_{[\mu\nu\rho}A_{\sigma]} \quad ,
\end{aligned} \tag{3.25}$$

that are the field strengths of the gauge fields of the gauged IIA theory [8, 6]. Acting with g_0 transformations on the group element of eq. (3.14) one also derives the gauge transformations

$$\begin{aligned}
A_\mu &\rightarrow A_\mu + \partial_\mu\Lambda \\
A_{\mu\nu} &\rightarrow e^{3m\Lambda}A_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]} \\
A_{\mu\nu\rho} &\rightarrow e^{3m\Lambda}A_{\mu\nu\rho} + \partial_{[\mu}\Lambda_{\nu\rho]} + \partial_{[\mu}\Lambda A_{\nu\rho]} \quad ,
\end{aligned} \tag{3.26}$$

which transform covariantly the field strengths of eq. (3.25), that is

$$\begin{aligned}
F_{\mu\nu} &\rightarrow F_{\mu\nu} \\
F_{\mu\nu\rho} &\rightarrow e^{3m\Lambda}F_{\mu\nu\rho} \\
F_{\mu\nu\rho\sigma} &\rightarrow e^{3m\Lambda}F_{\mu\nu\rho\sigma} \quad .
\end{aligned} \tag{3.27}$$

We now perform an analysis of the deformed algebra that parallels the one performed in [23] for the case of the Scherk-Schwarz reduction of IIB to nine dimensions. We start observing that eq. (3.24) relates the Og fields to the E_{11} fields times the deformation parameter. Iterating this one obtains for any n an Og n field identified with an E_{11} field times the n th power of the mass parameter. This generalises to all the fields in the theory whose corresponding operators have non-vanishing commutator with the operator $K + R$. Putting these solutions into the original group element of eq. (3.20) we find that it takes the form

$$g = e^{x\cdot P} e^{yQ} e^{y(K+R)} e^{\Phi_{\text{Og}}\tilde{K}^{\text{Og}}} e^{A_{\mu\nu\rho}\tilde{R}^{\mu\nu\rho}} e^{A_{\mu\nu}\tilde{R}^{\mu\nu}} e^{A_\mu R^\mu} e^{\phi R} e^{h_\mu{}^\nu K^\mu{}_\nu} \quad , \tag{3.28}$$

where

$$\begin{aligned}
\tilde{R}^{\mu\nu} &= R^{\mu\nu} + 3me^{-my}K^{\mu\nu} + \dots \\
\tilde{R}^{\mu\nu\rho} &= R^{\mu\nu\rho} + 3me^{-my}K^{\mu\nu\rho} + \dots \quad ,
\end{aligned} \tag{3.29}$$

where the dots correspond to higher powers in m multiplying higher grade Og generators, and \tilde{K} denotes deformed Og generators associated with ten-dimensional gauge transformations. We also define, as suggested by the Maurer-Cartan form of eq. (3.21), the deformed 10-dimensional momentum operator as

$$\tilde{P}_\mu = e^{my} P_\mu \quad . \quad (3.30)$$

We therefore get the commutator

$$[\tilde{R}^{\mu_1\mu_2\mu_3}, \tilde{P}_\nu] = 3m\delta_\nu^{[\mu_1} \tilde{R}^{\mu_2\mu_3]} \quad , \quad (3.31)$$

while the commutator of $\tilde{R}^{\mu_1\mu_2}$ with \tilde{P}_μ vanishes.

We think of the deformed generators constructed this way as constituting a deformed local E_{11} algebra. This deformed algebra has an algebraic classification as the set of generators that commute with the operator

$$\tilde{Q} = e^{my} Q + m(K + R) \quad . \quad (3.32)$$

This operator can be read from eq. (3.22), which indeed becomes, once one imposes the conditions of eqs. (3.23) and (3.24),

$$dye^{-\phi R} \tilde{Q} e^{\phi R} \quad . \quad (3.33)$$

In terms of the operator \tilde{Q} the commutator between R^μ and \tilde{P}_μ reads

$$[R^\mu, \tilde{P}_\nu] = -\delta_\nu^\mu \tilde{Q} + m\delta_\nu^\mu (K + R) \quad . \quad (3.34)$$

We then consider the scalar sector of eq. (3.21), that is

$$e^\phi e^{my} A_\mu Q + \partial_\mu \phi R = A_\mu e^{-\phi R} \tilde{Q} e^{\phi R} + (\partial_\mu \phi - mA_\mu) R - mA_\mu K \quad . \quad (3.35)$$

The R term in this equation gives the covariant derivative for the scalar,

$$D_\mu \phi = \partial_\mu \phi - mA_\mu \quad , \quad (3.36)$$

which is invariant under

$$\delta\phi = m\Lambda \quad \delta A_\mu = \partial_\mu \Lambda \quad . \quad (3.37)$$

Finally, we consider the gravity sector. This is again different with respect to the Scherk-Schwarz reduction discussed in [23]. Indeed, in that case the deformation of the group

element was due to an internal symmetry generator, which commutes with the gravity generators, and thus the analysis of the dimensionally reduced gravity sector was trivial. In this case the deformation involves the generator K , which is the trace of the $GL(10, \mathbb{R})$ generators, and as such this has a non-trivial effect in the gravity sector. Specifically, taking into account the K term in the Maurer-Cartan form, the K^{ab} term becomes

$$[(e^{-1}\partial_\mu e)_{ab} - \Phi_{\mu\nu}^\rho e_a^\nu e_{\rho b} - mA_\mu \eta_{ab}] K^{ab} \quad . \quad (3.38)$$

Observing that the vierbein transforms under Λ as

$$e_\mu^a \rightarrow e^{m\Lambda} e_\mu^a \quad , \quad (3.39)$$

we write the term contracting K^{ab} in eq. (3.38) as

$$(e^{-1}D_\mu e)_{ab} - \Phi_{\mu\nu}^\rho e_a^\nu e_{\rho b} \quad , \quad (3.40)$$

where D_μ is the derivative covariantised with respect to the transformation of eq. (3.39), that is $D_\mu = \partial_\mu - mA_\mu$. Applying the same arguments of [23], which are reviewed in section 2, we obtain that imposing that the symmetric part in ab of eq. (3.40) vanishes gives for the antisymmetric part the spin connection as in eq. (2.13), but with the derivative ∂_μ substituted by the covariant derivative D_μ . This is [25]

$$\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} - 2me_\mu^{[a} e^{|\nu|b]} A_\nu \quad . \quad (3.41)$$

If one plugs this into the Maurer-Cartan form and applies the inverse Higgs mechanism at the level of the next gravity Og field, one obtains that the term contracting K^{ab}_c is the covariantised Riemann tensor

$$\tilde{R}_{\mu\nu}{}^{ab} = 2\partial_{[\mu}\tilde{\omega}_{\nu]}{}^{ab} + 2\tilde{\omega}_{[\mu}{}^{ac}\tilde{\omega}_{\nu]}{}^b{}_c \quad . \quad (3.42)$$

Therefore this reproduces exactly the field theory analysis of [25] in the gravity sector.

The question we now want to address is in what sense one can truncate the algebra in such a way that the resulting theory is purely ten-dimensional. What we want to do is to project out of the algebra the operator \tilde{Q} , and consider the group element as a purely ten-dimensional one with commutation relations deformed with respect to the massless case. From eq. (3.34) we consider as a starting point for the ten-dimensional deformed algebra the commutator

$$[R^\mu, P_\nu] = m\delta_\nu^\mu(K + R) \quad , \quad (3.43)$$

where now we have for simplicity dropped the tilde from the deformed generators. We want to determine the rest of the algebra by requiring the closure of the Jacobi identities. This is exactly the method explained in [23] and applied in [24] to derive the deformed algebra associated to any gauged maximal supergravity in any dimension.

The Jacobi identity involving $K + R$, R^μ and P_μ gives

$$[R, P_\mu] = P_\mu \quad , \quad (3.44)$$

which implies

$$[K + R, P_\mu] = 0 \quad . \quad (3.45)$$

We then get

$$[R^{\mu\nu}, P_\rho] = 0 \quad [R^{\mu\nu\rho}, P_\sigma] = 3m\delta_\sigma^{[\mu} R^{\nu\rho]} \quad . \quad (3.46)$$

We thus recover the commutation relation of eq. (3.31) from a purely ten-dimensional perspective. If we then consider the ten-dimensional group element

$$g = e^{x \cdot P} e^{\Phi_{\text{Og}} K^{\text{Og}}} e^{A_{\mu\nu\rho} R^{\mu\nu\rho}} e^{A_{\mu\nu} R^{\mu\nu}} e^{A_\mu R^\mu} e^{\phi R} e^{h_\mu{}^\nu K^\mu{}_\nu} \quad , \quad (3.47)$$

the corresponding Maurer-Cartan form gives, once the inverse Higgs mechanism is applied, the field strengths of eq. (3.25), as well as the covariantised spin connection of eq. (3.41) and the covariantised Riemann tensor of eq. (3.42).

As a final comment, we discuss the overlap of this deformation, corresponding to the gauged IIA theory, with the deformation associated with Romans massive IIA theory. Denoting with m_R the mass parameter associated to Romans theory, the deformation corresponds to a non-vanishing commutator between the 2-form generator and momentum [13]

$$[R^{\mu\nu}, P_\rho] = m_R \delta_\rho^{[\mu} R^{\nu]} \quad . \quad (3.48)$$

A simple computation shows that using this commutator together with the one of eq. (3.43) the Jacobi identity involving $R^{\mu\nu}$, P_ρ and P_σ closes only if the quadratic constraint

$$mm_R = 0 \quad (3.49)$$

holds. This means that it is not consistent to turn on both deformations together. This result is perfectly consistent with the field theoretic analysis. Indeed turning on the Romans mass breaks the trombone symmetry also at the level of the field equations, and thus it is not consistent to perform the gauging of the trombone symmetry when the Romans mass parameter is non-vanishing.

4 Conclusions

In this paper we have shown that the local E_{11} algebra corresponding to the IIA theory admits a deformation which is associated to the gauged IIA theory of [8, 6]. This deformation is shown to arise from considering the Maurer-Cartan form that results from taking the eleven-dimensional group element as in eq. (3.20), and solving for the Og fields using the inverse Higgs mechanism. The deformed algebra can also be obtained directly in ten dimensions starting from the commutator of eq. (3.43) and imposing the closure of the Jacobi identities, which also imply that this deformation can not be turned on together with the Romans deformation. Given that the commutator of eq. (3.43) involves the trace of the $GL(10, \mathbb{R})$ generators, this deformation has a non-trivial effect in the gravity sector, as shown in eqs. (3.41) and (3.42).

The deformed algebra can naturally be extended to include higher rank form generators, and we expect the field equations to arise as duality relations between the corresponding field strengths. It is important to observe, though, that the only 9-form generator that is present in the IIA decomposition of E_{11} is associated to the Romans mass. This can be seen explicitly by observing that the field strength of the IIA 9-form that one obtains from the deformed E_{11} algebra associated to the Romans theory [23] coincides up to field redefinitions with the 9-form that one obtains imposing the closure of the supersymmetry algebra [34, 35], which also imposes the duality of its field strength with the Romans mass. Therefore, there is no dual form in the spectrum associated to the trombone deformation. In [25] it was observed that in any dimension D the E_{11} spectrum contains generators with $D - 1$ spacetime indices in the $(D - 2, 1)$ mixed symmetry irreducible representation of $GL(D, \mathbb{R})$ with $D - 2$ antisymmetric indices, that could be associated to the trombone deformations. In this IIA case, this would be a generator in the $(8, 1)$ representation of $GL(10, \mathbb{R})$, which is indeed present. Actually, the occurrence of these generators is completely general, as already shown in [36]. Indeed these are the first of an infinite chain of so called “dual” vector generators in the $GL(D, \mathbb{R})$ representations $(D-2, D-2, \dots, D-2, 1)$, and their presence is crucial for the universal structure of E_{11} reproducing the gauge algebra of all the form fields in all dimensions. In [37, 38] it was observed that in the case of the internal gaugings one can consider a lagrangian formulation in which the $D - 1$ forms are Lagrange multipliers for the embedding tensor (so that their field equation implies the constancy of the embedding tensor). The fact that these forms are present in the

gauge algebra is thus intrinsically related to the fact that one expects such a lagrangian formulation to be possible. In [34] this lagrangian formulation was originally derived for the IIA case, thus describing simultaneously the massless and the Romans case. The IIA theory considered in this paper does not admit a lagrangian formulation, and thus we consider the fact that there is no form generator associated to this deformation as completely consistent, and we do not expect any E_{11} generator associated to a non-propagating field to play a role in triggering this deformation.

As mentioned in the introduction, in [25] it was shown that all possible gauged maximal supergravities of the trombone type in any dimension D can be classified in terms of a new embedding tensor in the representation of E_{11-D} which is conjugate to the one to which the vectors belong. The consistency of the gauge algebra imposes quadratic constraints, which the authors of [25] also analyse in the case in which this trombone gauging is considered together with the embedding tensor associated to the internal gauging. In [26] these results are reproduced imposing the closure of the Jacobi identities of the deformed local E_{11} algebra with deformations also involving the trace of the $GL(D, \mathbb{R})$ generators, and the gauge transformations and the field strengths of the fields are computed in all cases.

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