# Jaeger's graphs and compatible linear partitions

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**Abstract**

A *strong matching* $C$ in a graph $G$ is a matching $C$ such that there is no edge of $E(G)$ connecting any two edges of $C$. A cubic graph $G$ is a *Jaeger's graph* if it contains a perfect matching which is a union of two disjoint strong matchings. We survey here some known results about this family and we give some new results. We define the operation of $(L, U)$-extension and we show that the family of Jaeger's graphs is generated from some small Jaeger's graphs by using this operation. A *linear forest* is a graph whose connected components are chordless paths. A *linear partition* of a graph $G$ is a partition of its edge set into linear forests and $la(G)$ is the minimum number of linear forests in a linear partition. It is well known that $la(G) = 2$ for any cubic graph $G$. For a linear partition $L = (L_B, L_R)$ of $G$ and for each vertex $v$ we define $e_L(v)$ as the edge incident to $v$ which is an end edge of a maximal path in $L_B$ or $L_R$. We shall say that two linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are *compatible* whenever $e_L(v) \neq e_{L'}(v)$ for each vertex $v$. We show that every Jaeger's graph has two compatible linear partitions and we give some sthrenthening to the conjecture: Every cubic graph has two compatible linear partitions.

*Keywords:* cubic graphs, linear partition, linear forest, strong matching

## 1. Introduction.

For any undirected graph $G$, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. A *linear forest* is a graph in which each component is a chordless path.

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The *linear arboricity* of an undirected graph $G$ is defined as the minimum number of linear forests needed to partition the set $E(G)$. The linear arboricity was introduced by Harary [9] and is denoted by $la(G)$. In this paper we consider *cubic graphs*, that is to say finite simple 3-regular graphs.

It was shown by Akiyama, Exoo and Harary [1] that $la(G) = 2$ when $G$ is a cubic graph. A partition $L$ of $E(G)$ into two linear forests $L_B$ and $L_R$ is called a *linear partition* and we shall denote this linear partition by $L = (L_B, L_R)$ . We say that every path of $L_B$ and every path of $L_R$ is an *unicoloured* path (for instance, a *Blue* path or a *Red* path). For $c \in \{B, R\}$ , we shall denote by $E(L_c)$ the set of edges of $L_c$, by $l(L_c)$ the length of a longest path in $L_c$. A linear partition $L = (L_B, L_R)$ is said to be *odd* whenever each path of $L_B \cup L_R$ has odd length and *semi-odd* whenever each path of $L_B$ (or each path of $L_R$) has odd length.
A *strong matching* $C$ in a graph $G$ is a matching $C$ such that there is no edge of $E(G)$ connecting any two edges of $C$. A perfect matching that is union of two disjoint strong matchings is said to be a *Jaeger's matching*.

**Definition 1.1.** A cubic graph $G$ is a *Jaeger's graph* if it contains a Jaeger's matching.

In his thesis [10] Jaeger called these cubic graphs *equitable* and pointed out that the above 2-colouring of their vertices leads to a *balanced colouring* as defined by Bondy [4].

**Proposition 1.2.** [10] *Every Jaeger's graph is 3-edge colourable.*

In a previous paper [7] we used some tools that we called *associated partition* and *associated linear construction* that we shall use again in the following sections.
Let $M$ be a matching transversal of the odd cycles of a cubic graph $G$. Since $G \setminus M$ is bipartite, we can colour the vertices of $G$ in two colours *Blue* and *Red* accordingly to the bipartition of $G \setminus M$. Let $M_B$ (respectively $M_R$) be the set of edges of $M$ such that their two end vertices are *Blue* vertices (respectively *Red* vertices). An edge of $G$ is said to be *mixed* when one end is *Blue* while the other is *Red*. Hence, the edges of $G \setminus M$ are mixed while $M$ is partitioned into three sets (some of them, possibly empty) $M = M_B + M_R + M'$ where $M'$ is the subset of mixed edges of $M$. Note that $M_B$ and

$M_R$ induce strong matchings in $G$. We shall say that a partition of $M$ in $M_B + M_R + M'$ is an *associated partition.*

**Lemma 1.3.** *[7] A cubic graph is* 3*-edge colourable graph if and only if there is a partition of its vertex set into two sets, Blue and Red and a perfect matching $M$ such that every edge in $G - M$ is mixed.*

**Proof** For the sake of completeness we give again the proof.
Let $G$ be a cubic 3-edge colourable graph. Any colour of a 3-edge colouring of $G$ induces a perfect matching $M$, and the two others colours induce a graph in which each component is an even cycle. Let us colour the vertices of these cycles in *Blue* and *Red* alternately. Hence every edge lying on these cycles is mixed.
Conversely, assume that $G$ has a perfect matching $M$ and a partition of its vertex set into *Blue* and *Red* such that every edge in $G - M$ is mixed. Let us consider the 2-factor of $G$ obtained by deleting $M$. Since every edge outside $M$ is mixed, this 2-factor is even, that is $G$ is 3-edge colourable. □

*Remark* 1.4. Under conditions of Lemma 1.3 we certainly have the same number of *Blue* vertices and *Red* vertices, since every edge of the 2-factor $G - M$ is mixed. When considering $M = M_B + M_R + M'$ we have $|M_B| = |M_R|$ since every mixed edge of $M$ uses a vertex in each colour.

Assume that $G$ is a 3-edge colourable cubic graph with a perfect matching $M$ given as in Lemma 1.3 by a 3-edge colouring, and let $M = M_B + M_R + M'$ an associated partition. Let us fix an arbitrary orientation to the cycles of $G \setminus M$. To each vertex $v$ of $V(G)$ we can associate an edge $o(v)$ of $E(G) \setminus M$ such that $v$ is the origin of $o(v)$ with respect to the chosen orientation of the cycle through $v$. It will be convenient to denote by $s(v)$ ( *successor* of $v$) the end of $o(v)$ in that orientation and by $p(v)$ its *predecessor.* We can colour $o(v)$ in *Blue* or *Red* accordingly to the colour of $v$. $M_B$ being coloured with *Blue* and $M_R$ with *Red*, we get hence a larger set $CL_B$ of edges coloured with *Blue* (and a set $CL_R$ of edges coloured with *Red*). It is easily seen that the $CL_B$ and $CL_R$ are linear-forests where each maximal unicoloured path has length 1 or 3. Moreover each edge of $M_B \cup M_R$ is the central edge of a path of length 3. At this point, the only edges which are not coloured are the edges of $M'$ and we do not know how we can assign a colour to these edges in order to get a linear partition of $E(G)$. We used this construction in [7] that we have called the *associated linear construction* to the partition $M = M_B + M_R + M'$.

*Remark* 1.5. The fact that a Jaeger's graph is a cubic 3-edge colourable graph is an easy consequence of Lemma 1.3

Indeed, if $G$ is a Jaeger's graph and $M$ is a perfect matching of $G$ union of two disjoint strong matchings $M_B$ and $M_R$, then let us colour with *Blue* the vertices which are end vertices of edges in $M_B$ and *Red* those which are end vertices of edges in $M_R$. Since $M_B$ and $M_R$ are strong matchings the remaining edges are mixed. Hence, by Lemma 1.3, $G$ is 3-edge colourable.

In order to give here a first illustration of the usefulness of the associated linear construction we give a proof of an old result.

**Theorem 1.6.** [2] [13] *A cubic graph $G$ has a linear partition $L = (L_B, L_R)$ such that each path has length 3 if and only if $G$ is a Jaeger's graph.*

**Proof** : Suppose that $G$ has a linear partition $L = (L_B, L_R)$ such that each path has length 3. Let $C_B$ (respectively $C_R$) be the set of the middle edges of the paths of $L_B$ (respectively $L_R$). It is an easy task to check that $C_B$ and $C_R$ are strong matchings and $|C_B| = |C_R|$. Moreover $M = C_B \cup C_R$ is a perfect matching and $G$ is a Jaeger's graph. Conversely, let us suppose that $G$ is a Jaeger's graph and let $M = M_B + M_R$ be an associated partition. Since $M'$ is empty, by using the associated linear construction above, we have coloured every edge of $G$ and each unicoloured path has length 3. □

Finally, the following result shows that Jaeger's graphs play a pivotal role in the family of cubic 3-edge colourable graphs.

**Theorem 1.7.** [8] *Let $G$ be a cubic graph. Then $G$ can be factored into two odd linear forests $L = (L_B, L_R)$ such that*

*i) Each path in $L_B$ has odd length at most 3*

*ii) Each path in $L_R$ has odd length at least 3.*

*if and only if $\chi'(G) = 3$.*

## 2. Some classes of Jaeger's graphs.

We can construct a Jaeger's graph starting from any cubic 3-edge colourable graph. Indeed, consider a perfect matching $M$ of $G$ together with a bipartition of its vertex set in *Blue* and *Red* induced by a 3-edge colouring of $G$ given by Lemma 1.3. If there are no mixed edge, we are done since $G$ itself is a Jaeger's graph. Otherwise for any mixed edge apply the transformation depicted in figure 1 on the *Blue* vertices. Every such *Blue* vertex is transformed into a new triangle containing a new *Blue* edge while the mixed edge is transformed into a *Red* edge. The resulting graph is a Jaeger's graph.

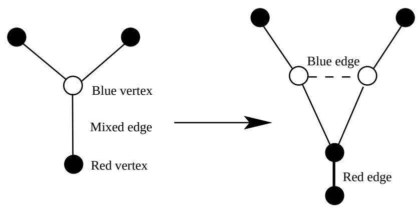


Figure 1: Triangle Extension

The *square* $G^2$ of a graph $G$ has $V(G^2) = V(G)$ with $u$, $v$ adjacent in $G^2$ whenever there exists a path of length at most 2 joining them in $G$.

**Proposition 2.1.** [2] *If $G$ is a cubic graph such that $G^2$ is 4-chromatic then $G$ is a Jaeger's graph.*

A cubic planar graph is a *multi-k-gon* [5] (with $3 \leq k \leq 5$) if all of its faces have length multiple of $k$. Since Jaeger [10] has shown that a multi-k-gon $G$ with $k = 3, 4$ has a square $G^2$ which is 4-chromatic, the following result is a consequence of Proposition 2.1.

**Proposition 2.2.** [2] *If $G$ is a multi-k-gon with $k = 3, 4$ then $G$ is a Jaeger's graph.*

When $G$ is a cubic graph having a 2-factor of $C_4$'s, we consider the auxiliary 2-regular graph $G'$ defined as follows : every $C_4$ of this 2-factor is replaced with its complementary graph (which is a $2K_2$). In a previous paper we have proved the following.

**Theorem 2.3.** [8] *Let $G$ be a connected cubic graph having a 2-factor of squares, say $\mathcal{F}$ and let $p$ be the number of cycles of the auxiliary 2-regular graph $G'$. Then there are $2^{p-1}$ Jaeger's matchings which intersect $\mathcal{F}$.*

Since by Theorem 2.3 every cubic graph having a 2-factor of squares has at least one Jaeger's matching, we have:

**Corollary 2.4.** [8] *A cubic graph having a 2-factor of squares is a Jaeger's graph.*

By Shaefer's result [12], we know that it is NP-complete to recognize a Jaeger's graph. Assume that $G$ is a Jaeger's graph, is it difficult to find a perfect matching union of two disjoint strong matchings? We do not have a general answer, but we can derive from the proof of Theorem 2.3 (see [8]) a simple linear time algorithm for finding a Jaeger's matching in a connected cubic graph which have a 2-factor of squares.

**Corollary 2.5.** [8] *If $G$ is a cubic graph with a 2-factor of squares then we can construct in $O(n)$ time a perfect matching union of two disjoint strong matchings with the same size.*

It can be noticed that every cubic graph with a perfect matching $M$ can be transformed into a Jaeger's graph by using the transformation (*square extension*) depicted in figure 2 on each edge of $M$. Indeed, the resulting graph has a 2-factor of squares and we can apply Theorem 2.3

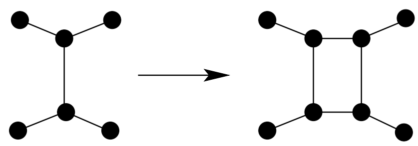

Figure 2: Square Extension

To conclude this section we note that we have studied in [6] a family of cubic graphs having a large number of perfect matchings (and containing the flower snarks). The subfamily of the 3-edge colourable graphs of this family contains some Jaegers's graphs that we have totally characterized, and we proved that such a graph on at least 12 vertices has exactly 6 Jaeger's matchings.

## 3. Compatible partitions

Let $L = (L_B, L_R)$ be a linear partition of $G$. For each vertex $v$ we can define $e_L(v)$ as the edge incident to $v$ which is an end edge of a maximal path in $L_B$ or $L_R$. We shall say that two linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are *compatible* whenever $e_L(v) \neq e_{L'}(v)$ for each vertex $v$. The qualifying adjective "compatible" refers to the notion of compatible Euler's tours introduced by Kotzig [11] (see Bondy [3] for an introduction to this question). In 1991, the first author, J.-L. Fouquet, remarked that a Jaeger's graph can be provided with two compatible linear partitions and he conjectured:

**Conjecture 3.1.** *Every cubic graph has two compatible linear partitions.*

We do not solve this conjecture but we shall give some results strengthening it.

**Definition 3.2.** Let $\alpha$ and $\beta$ be any two distinct colours of a 3-edge colouring of a cubic 3-edge colourable graph $G$. In the following $SM_G(\alpha, \beta)$ will denote a strong matching of $G$ intersecting every $\alpha\beta$-cycle (when such a strong matching exists).

Some results of the following subsections are consequences of the two following theorems.

**Theorem 3.3.** [8] *Let $G$ be a 3-edge coloured cubic graph and let $\alpha$ and $\beta$ be any two distinct colours of $E(G)$. Then there exists a strong matching $SM_G(\alpha, \beta)$ intersecting every cycle of $\Phi(\alpha, \beta)$.*

We derive from Theorem 3.3 a result on unicoloured transversals of the 2-factors induced by any 3 edge-colouring of cubic graph with chromatic index 3.

**Theorem 3.4.** [8] *Let $G$ be a cubic 3-edge colourable graph and let $\Phi$ be a 3-edge colouring of $G$. Let $\alpha$ and $\beta$ be any two distinct colours of $\Phi$ and let $\gamma$ be the third colour. Then there exists a set $F_\alpha$ of $\alpha$-edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that the set $F_\alpha$ together with the $\gamma$-edges has no cycle.*

### *3.1. Compatible partitions and associated linear construction*

For cubic 3-edge colourable graphs we obtain the following result.

**Theorem 3.5.** *Let $G$ be a cubic 3-edge colourable graph with an associated partition $M = M_B + M_R + M'$. Assume that, we can colour the edges of $M'$ in Blue and Red in* *such a way that, for each associated linear construction, the whole colouring of $E(G)$ so obtained is a linear partition. Then $G$ has two compatible linear partitions.*

**Proof** An associated linear construction is obtained in fixing an arbitrary orientation to the cycles of $G \backslash M$ . To each vertex $v$ of $V(G)$ we associate an edge $o(v)$ of $E(G) \backslash M$ such that $v$ is the origin of $o(v)$ with respect to the chosen orientation of the cycle through $v$. We colour $o(v)$ in *Blue* or *Red* accordingly to the colour of $v$.

We give a colour to each edge of $M'$, accordingly to some rule depending on the class of graphs we are studying. When the edge of $M'$ incident to some vertex $v$ is coloured in *Blue* or *Red*, $v$ is transformed into a vertex of degree 2 in one of the two colours (say *Blue*) and a vertex of degree 1 in the other (*Red*). Since the whole colouring of $E(G)$ leads to a linear partition $L = (L_B, L_R)$ , the edge incident to $v$ of this last colour is hence the edge $e_L(v)$.

On each cycle of $G \backslash M$ we can give now the opposite orientation. In the associated linear construction, the colours of the edges of the cycles incident to each vertex are exchanged. Since, with the same colouring of $M'$, we have supposed that the whole colouring of $E(G)$ leads to a linear partition $L' = (L'_B, L'_R)$, the above vertex $v$ remains a vertex of degree 2 in the *Blue* colour. Hence the *Red* edge incident to $v$ $e_{L'}(v)$ is distinct from $e_L(v)$.

Since $e_L(v) \neq e_{L'}(v)$ for each vertex $v$, the two linear partitions $L$ and $L'$ so obtained are compatible. $\square$

**Corollary 3.6.** *Let $G$ be a Jaeger's graph then $G$ has two compatible linear partitions in which every path has length* 3.

**Proof** In that case, we have a perfect matching $M = M_B + M_R$ and any associated linear construction is a linear partition in which every path has length 3. $\square$

**Theorem 3.7.** *Let $G$ be cubic 3-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that $M'$ can be partitioned into two strong matchings $M'_B$ and $M'_R$. Then there is an odd linear partition of $E(G)$ every maximal path of which has length* $1, 3, 5$ *or* 7.

**Proof** : Let $CL_B$ and $CL_R$ be the linear-forests of the associated linear construction. Recall that each maximal path of $CL_B$ (respectively $CL_R$) has length 1 or 3 and is unicoloured with *Blue* (respectively unicoloured with *Red*). Let $L_B = CL_B \cup M'_B$ and $L_R = CL_R \cup M'_R$, in addition $M = M_B + M_R + M'$ and $B$ denotes the set of *Blue* vertices of $G$ and $R$ its set of *Red* vertices.
We now prove that the components of $L_B$ and $L_R$ are odd paths of length at most 7. We only have to consider components that contain an edge of $M'$. Without loss of generality, let $C$ be a component of $L_B$ which contains an edge of $M'_B$.

**Claim 1.** *Let $br$ be an edge of $M'_B$ such that $b \in B \cap C$ and $r \in R \cap C$ and let $r' = s(b)$. Then the unique neighbour of $r'$ in $C$ is $b$.*

**Proof of Claim** Since $G \backslash M$ contains only mixed edges $r'$ is a *Red* vertex. Observe that $o(r')$ is a *Red* edge while the edge of $M$ incident to $r'$ in $G$, say $e$, cannot belong to $M'_B$ since $M'_B$ is a strong matching. Moreover, $e$ having a *Red* end must belong either to $M_R$ or to $M'_R$, consequently $e$ belongs to $L_R$. Thus, among the three edges incident to $r'$, only $o(b)$ belongs to $C$ and the result follows. □

Let $b_1r_1$ be an edge of $C \cap M'_B$ ($b_1 \in B$, $r_1 \in R$). Let us set $r_2 = s(b_1)$. We know by the above Claim that $r_2$ is a pendant vertex in $C$.
Let $b_2 = p(r_1)$ and $r_3 = p(b_2)$, obviously $b_2 \in B$, $r_3 \in R$, $b_2r_1$ is a *Blue* edge and $r_3b_2$ is a *Red* one. Consider in $G$ the edge of $M$ incident to $b_2$, say $e$. $M'_B$ being a strong matching, the edge $e$ cannot belong to $M'_B$. Moreover, the edge $e$ has a *Blue* end, namely $b_2$, and thus cannot belong to $M_R$. If $e$ belongs to $M'_R$ we are done since $e$ is a *Red* edge and the component $C$ is reduced to the path of length 3 $r_2b_1r_1b_2$.
Assume in the following that $e$ belongs to $M_B$. From now on $e$ will be denoted $b_2b_3$ ($b_3 \in B$) and $s(b_3)$ will be denoted $r_4$, we have $r_4 \in R$. Let $e'$ be the edge of $M$ incident to $r_4$ in. Since $r_4$ is a *Red* end of $e'$, $e'$ cannot belong to $M_B$. If $e'$ is a member of $M_R \cup M'_R$ we are done since $e'$ and $o(r_4)$ both belong to $L_R$ and $C$ is reduced to a path of length 5, namely $r_2b_1r_1b_2b_3r_4$.
Suppose now that $e' \in M'_B$. Let us denote $e'$ as $r_4b_4$ ($b_4 \in B$) and $s(b_4)$ as $r_5$. But now, by the above Claim $b_4$ is the unique neighbour of $r_5$ in $C$ and thus $C = \{r_2, b_1, r_1, b_2, b_3, r_4, b_4, r_5\}$ induces a path of length 7. □

As a corollary of Theorems 3.5 and 3.7 we obtain:

**Corollary 3.8.** *Let $G$ be cubic 3-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that $M'$ can be partitioned into two strong matchings $M'_B$ and $M'_R$. $G$ has two compatible linear partitions in which every path has length $1, 3, 5$ or $7$.*

**Proof** In proof of Theorem 3.7 we have coloured the edges of $M'_B$ in *Blue* and those of $M'_R$ in *Red* and we have shown that the associated linear construction together with this colouring of $M'$ leads to a linear partition with each path of length $1, 3, 5$ or 7. Applying Theorem 3.5 above leads to the result. □

*3.2. Compatible partitions in hamiltonian cubic graphs*

As a corollary of Theorem 3.3, we shall show now that cubic hamiltonian graphs satisfy Conjecture 3.1. In fact, a stronger result is obtained since the two compatible partitions are odd.

**Theorem 3.9.** *Let $G$ be a cubic hamiltonian graph, then $G$ has two compatible odd linear partitions.*

**Proof** Let $C = a_0, a_1 \dots a_{n-1}$ be a hamiltonian cycle of $G$. This hamiltonian cycle induces a 3-edge colouring $\Phi : E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us colour every edge $a_i a_{i+1}$ of $C$ with $\alpha$ when $i \equiv 0(2)$ and with $\beta$ otherwise, while the remaining perfect matching is coloured with $\gamma$. From Theorem 3.3 we know that there is a strong matching $F$ intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$. We choose $F$ minimal for the inclusion (that is $F$ intersects each cycle of $\Phi(\beta, \gamma)$ in exactly one edge). Let $M_\alpha$ be the set of $\alpha$-coloured edges. Since $F$ is a strong matching intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$, we can construct an odd linear partition $L' = (L'_B, L'_R)$ :

$$L'_R = M_\alpha \cup F$$
$$L'_B = E(G) - L'_R$$

For each vertex $v$, the edge $e_{L'}(v)$ is coloured with $\alpha$ excepted when $v$ is an end vertex of an edge of $F$. In that case $e_{L'}(v)$ coloured with $\gamma$ when $v$ is an end vertex of the edge of $F \cap C$ and with $\beta$ when $v$ is an end vertex of an edge of $F \setminus C$.

**Case 1** : *$F$ contains some edges of $C$*

Hence the edges of $F \cap C$ are coloured with $\beta$. $F \cap C$ is a strong matching intersecting the 2-factor made of the unique cycle $C$ leading to the following odd linear partition $L = (L_B, L_R)$ :

$$L_B = C - F$$
$$L_R = E(G) - L_B$$

For each vertex $v$, $e_L(v)$ is coloured with $\gamma$ excepted when $v$ is an end vertex of the edge of $F \cap C$. In that case we have $e_L(v)$ coloured with $\beta$.

We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex $v$, since the colours of these edges are distinct. Hence, the two odd partitions $L$ and $L'$ are compatible.

**Case 2** : *Each edge of $F$ is coloured with $\gamma$ and there is an edge $a_i a_{i+1}$ (i odd) of $C$ coloured with $\beta$ which is not incident to an edge of $F$.*

Let $L = (L_B, L_R)$ be the following odd linear partition:

$$L_B = C - a_i a_{i+1}$$
$$L_R = E(G) - L_B$$

For each vertex $v$, $e_L(v)$ is coloured with $\gamma$ excepted when $v$ is $a_i$ or $a_{i+1}$. In that case $e_L(v)$ is coloured with $\alpha$.

For each vertex $v$, $e_{L'}(v)$ is coloured with $\alpha$ unless when $v$ is an end vertex of an edge of $F$. In that case $e_{L'}(v)$ is coloured with $\beta$ when $v$ is an end vertex of an edge of $F$.

We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex $v$, since the colours of these edges are distinct. Hence, the two odd partitions $L$ and $L'$ are compatible.

**Case 3** : *Each edge of $F$ is coloured with $\gamma$ and each edge of $C$ coloured with $\beta$ is incident to an edge of $F$.*

Without loss of generality assume that $a_2$ is incident to $F$. Edge $a_3 a_4$ being incident to $F$, we must have $a_4$ incident to $F$. Going through $C$, we get that $a_2, a_4, \ldots a_{2k}, \ldots a_0$ must be incident to $F$. The remaining edges coloured with $\gamma$ are edges joining vertices of $C$ with odd index. It is an easy task to see that this set $K$ of edges is a strong matching. The perfect matching coloured with $\gamma$ is the union of two strong disjoint matchings $F$

and $K$ with the same size. Hence $G$ is a Jaeger's graph . Theorem 3.6 implies that we have two compatible odd linear partitions. □

*3.3. Compatible odd linear partitions*

The results of Theorem 3.9 and Theorem 3.5 leads us to a strengthening of Conjecture 3.1.

**Conjecture 3.10.** *Let $G$ be a cubic 3-edge colourable graph then we can find two compatible odd linear partitions.*

As a partial result we have:

**Theorem 3.11.** *Let $G$ be a cubic 3-edge colourable graph then we can find three odd linear partitions $L$, $L'$ and $L''$ such that for each vertex $v$*

$$|\{e_L(v), e_{L'}(v), e_{L''}(v)\}| \geq 2$$

**Proof** Let us consider a 3-edge colouring $\Phi : E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us denote by $M_\gamma$ the perfect matching consisting of the $\gamma$-coloured edges. Theorem 3.4 implies that there exists a set $F_\alpha$ of $\alpha$-coloured edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that $F_\alpha \cup M_\gamma$ is acyclic. In that way, we obtain an odd linear partition $L = (L_1, L_2)$

$$L_1 = F_\alpha \cup M_\gamma$$
$$L_2 = E(G) - L_1$$

For each vertex $v$ we have $e_L(v)$ coloured with $\gamma$ excepted for the vertices which are the end vertices of an edge of $F_\alpha$. In that case $e_L(v)$ is coloured with $\beta$.

Let us consider the perfect matching $M_\beta$, the bicoloured cycles of $\Phi(\alpha, \gamma)$ and a matching $F_\gamma$ obtained by Theorem 3.4. Hence, we get an odd linear partition $L' = (L'_1, L'_2)$ such that for every vertex $v$ the edge $e_{L'}(v)$ is coloured either with $\beta$ (when the vertex $v$ is an end vertex of an edge of $F_\gamma$) or with $\alpha$. Finally we obtain a third odd linear partition $L'' = (L''_1, L''_2)$ from the perfect matching $M_\alpha$, the bicoloured cycles of $\Phi(\beta, \gamma)$ and a matching $F_\beta$, such that the edges $e_{L''}(v)$ are coloured $\alpha$ or $\gamma$ ($\gamma$ when the considered vertices are the end vertices of an edge of $F_\beta$).

The three sets $F_\alpha$, $F_\beta$ and $F_\gamma$ being obviously pairwise disjoint, it is a routine matter to see that for each vertex $v$ two edges in $\{e_L(v), e_{L'}(v), e_{L''}(v)\}$, at least, have distinct colours. These two edges are thus distinct and we get the result.

□

## 4. Generation of Jaeger's graphs

Let $G$ be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that $a$ and $d$ are internal vertices in $L_B$ while $b$ and $c$ are internal vertices in $L_R$.
Let us consider the linear forest $L_R$ and the edges of $L_R$ incident to the vertices of $U = \{a, b, c, d\}$. We notice that, without loss of generality, there are six distinct cases (by exchanging the role of $a$ for that of $d$ or/and the role of $b$ for that of $c$). See figure 3.

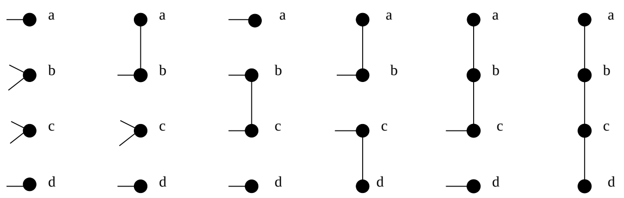


Figure 3: Six distinct cases

Analogous situations appear for the linear forest $L_B$ (by exchanging $\{a, d\}$ for $\{b, c\}$).

**Definition 4.1.** Let $G$ be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that $a$ and $d$ are internal vertices in $L_B$ while $b$ and $c$ are internal vertices in $L_R$. A $(L, U)$-*extension* of $G$ is a cubic simple graph $G'$ obtained from $G$ in the following way.

1) The set $U$ is splitted into two sets $U_R = \{a_R, b_R, c_R, d_R\}$ and $U_B = \{a_B, b_B, c_B, d_B\}$ (that is $V(G') = VG) \setminus U \cup (U_R \cup U_B))$.
2) For $x, y \in V(G) \setminus U$, if $xy \in E(G)$ then $xy \in E(G')$.
3) For $x \in V(G)\setminus U$ and $y \in U$ if $xy \in E(L_B)$ then $xy_B \in E(G')$ and if $xy \in E(L_R)$ then $xy_R \in E(G')$.

4) For $x, y \in U$ if $xy \in E(L_B)$ then $x_B y_B \in E(G')$ and if $xy \in E(L_R)$ then $x_R y_R \in E(G')$.

5) The remaining edges of $G'$ are the edges of two paths of length 3 on the sets $U_R$ and $U_B$, respectively, such that the obtained graph $G'$ is cubic (see figures 4 and 5).

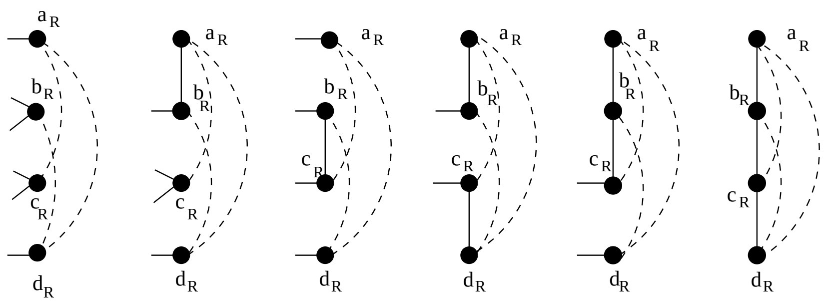


Figure 4: Addition of a path of length 3 on $U_R$

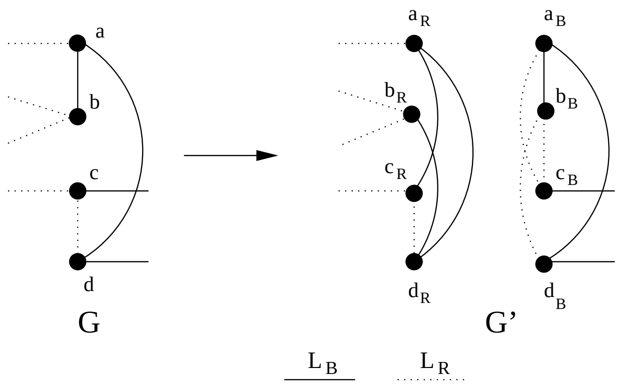


Figure 5: Example of (L, U)-extension

It is clear that the added path on $U_R$ (respectively $U_B$) can be added to $L_B$ (respectively $L_R$) in order to obtain a linear partition of $G'$ each path of which has length 3.

Note that the graph $G'$ is not uniquely defined, namely if the subgraph induced on $U$ in $L_R$ (respectively, in $L_B$) is a stable set or has exactly one edge connecting the vertices of degree 2 in $L_R$ (respectively, in $L_B$).

So, by Theorem 1.6 we have the following Proposition.

**Proposition 4.2.** *Let $G$ be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of $G$ each path of which has length 3. Let $U = \{a, b, c, d\}$ be a set of 4 vertices of $G$ such that $a$ and $d$ are internal vertices in $L_B$ while $b$ and $c$ are internal vertices in $L_R$. Then any $(L, U)-$extension of $G$ on $U$ is a Jaeger's graph.*

**Definition 4.3.** Let $G$ be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition for which every path has length 3. Assume that $P \in L_B$ and $Q \in L_R$ where $P = \{a_1, b_1, c_1, d_1\}$ and $Q = \{a_2, b_2, c_2, d_2\}$ are vertex disjoint paths in $G$. A *PQ-reduction* of $G$ on $P$ and $Q$ is a cubic simple graph $G'$ obtained from $G$ by deleting the edges of $P$ and $Q$ and identifying the internal vertices of $P$ with the end vertices of $Q$ and the internal vertices of $Q$ with the end vertices of $P$.

Note that the $PQ$-reduction of $G$ has a linear partition each path of which has length 3. Hence we have the following.

**Proposition 4.4.** *Let $G$ be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of $G$ every path of which has length 3. Assume that $P \in L_B$ and $Q \in L_R$ are vertex disjoint paths in $G$. Then the PQ-reduction of $G$ on $P$ and $Q$ is a Jaeger's graph.* □

We get immediately from Propositions 4.2 and 4.4

**Theorem 4.5.** *Every Jaeger's graph on $n \geq 20$ vertices is obtained from a Jaeger's graph on 16 vertices by a sequence of $(L, U)-$extensions.*

**Proof** Assume that $G$ is a Jaeger's graph on $n$ vertices with $n \geq 20$. Let $L = (L_B, L_R)$ be a linear partition each of whose paths have length 3. Since each path of $L$ is incident to at most 4 distinct paths, as soon as $n \geq 20$ we are sure to find a path in $P \in L_B$ and a path $Q \in L_R$ such that $V(P) \cap V(Q) = \emptyset$. By a $PQ$-reduction on these two paths we get a Jaeger's graph on $n - 4$ vertices (Proposition 4.4). The proof is complete. □


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