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► **To cite this version:**

Jean-Luc Fouquet, Henri Thuillier, Jean-Marie Vanherpe, Adam Pawel Wojda. Jaeger's graphs and compatible linear partitions. 2011. <hal-00601554>

HAL Id: hal-00601554

<https://hal.science/hal-00601554v1>

Preprint submitted on 19 Jun 2011

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Jaeger's graphs and compatible linear partitions

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Abstract

A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C . A cubic graph G is a *Jaeger's graph* if it contains a perfect matching which is a union of two disjoint strong matchings. We survey here some known results about this family and we give some new results. We define the operation of (L, U) -extension and we show that the family of Jaeger's graphs is generated from some small Jaeger's graphs by using this operation. A *linear forest* is a graph whose connected components are chordless paths. A *linear partition* of a graph G is a partition of its edge set into linear forests and $la(G)$ is the minimum number of linear forests in a linear partition. It is well known that $la(G) = 2$ for any cubic graph G . For a linear partition $L = (L_B, L_R)$ of G and for each vertex v we define $e_L(v)$ as the edge incident to v which is an end edge of a maximal path in L_B or L_R . We shall say that two linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are *compatible* whenever $e_L(v) \neq e_{L'}(v)$ for each vertex v . We show that every Jaeger's graph has two compatible linear partitions and we give some strengthening to the conjecture: Every cubic graph has two compatible linear partitions.

Keywords: cubic graphs, linear partition, linear forest, strong matching

1. Introduction.

For any undirected graph G , we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. A *linear forest* is a graph in which each component is a chordless path.

¹The research of APW was partially sponsored by polish Ministry of Science and Higher Education.
Preprint submitted to Elsevier *June 18, 2011*

The *linear arboricity* of an undirected graph G is defined as the minimum number of linear forests needed to partition the set $E(G)$. The linear arboricity was introduced by Harary [9] and is denoted by $la(G)$. In this paper we consider *cubic graphs*, that is to say finite simple 3-regular graphs.

It was shown by Akiyama, Exoo and Harary [1] that $la(G) = 2$ when G is a cubic graph. A partition L of $E(G)$ into two linear forests L_B and L_R is called a *linear partition* and we shall denote this linear partition by $L = (L_B, L_R)$. We say that every path of L_B and every path of L_R is an *unicoloured* path (for instance, a *Blue* path or a *Red* path). For $c \in \{B, R\}$, we shall denote by $E(L_c)$ the set of edges of L_c , by $l(L_c)$ the length of a longest path in L_c . A linear partition $L = (L_B, L_R)$ is said to be *odd* whenever each path of $L_B \cup L_R$ has odd length and *semi-odd* whenever each path of L_B (or each path of L_R) has odd length.

A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C . A perfect matching that is union of two disjoint strong matchings is said to be a *Jaeger's matching*.

Definition 1.1. A cubic graph G is a *Jaeger's graph* if it contains a Jaeger's matching.

In his thesis [10] Jaeger called these cubic graphs *equitable* and pointed out that the above 2-colouring of their vertices leads to a *balanced colouring* as defined by Bondy [4].

Proposition 1.2. [10] *Every Jaeger's graph is 3-edge colourable.*

In a previous paper [7] we used some tools that we called *associated partition* and *associated linear construction* that we shall use again in the following sections.

Let M be a matching transversal of the odd cycles of a cubic graph G . Since $G \setminus M$ is bipartite, we can colour the vertices of G in two colours *Blue* and *Red* accordingly to the bipartition of $G \setminus M$. Let M_B (respectively M_R) be the set of edges of M such that their two end vertices are *Blue* vertices (respectively *Red* vertices). An edge of G is said to be *mixed* when one end is *Blue* while the other is *Red*. Hence, the edges of $G \setminus M$ are mixed while M is partitioned into three sets (some of them, possibly empty) $M = M_B + M_R + M'$ where M' is the subset of mixed edges of M . Note that M_B and

M_R induce strong matchings in G . We shall say that a partition of M in $M_B + M_R + M'$ is an *associated partition*.

Lemma 1.3. [7] *A cubic graph is 3-edge colourable graph if and only if there is a partition of its vertex set into two sets, Blue and Red and a perfect matching M such that every edge in $G - M$ is mixed.*

Proof For the sake of completeness we give again the proof.

Let G be a cubic 3-edge colourable graph. Any colour of a 3-edge colouring of G induces a perfect matching M , and the two others colours induce a graph in which each component is an even cycle. Let us colour the vertices of these cycles in *Blue* and *Red* alternately. Hence every edge lying on these cycles is mixed.

Conversely, assume that G has a perfect matching M and a partition of its vertex set into *Blue* and *Red* such that every edge in $G - M$ is mixed. Let us consider the 2-factor of G obtained by deleting M . Since every edge outside M is mixed, this 2-factor is even, that is G is 3-edge colourable. \square

Remark 1.4. Under conditions of Lemma 1.3 we certainly have the same number of *Blue* vertices and *Red* vertices, since every edge of the 2-factor $G - M$ is mixed. When considering $M = M_B + M_R + M'$ we have $|M_B| = |M_R|$ since every mixed edge of M uses a vertex in each colour.

Assume that G is a 3-edge colourable cubic graph with a perfect matching M given as in Lemma 1.3 by a 3-edge colouring, and let $M = M_B + M_R + M'$ an associated partition. Let us fix an arbitrary orientation to the cycles of $G \setminus M$. To each vertex v of $V(G)$ we can associate an edge $o(v)$ of $E(G) \setminus M$ such that v is the origin of $o(v)$ with respect to the chosen orientation of the cycle through v . It will be convenient to denote by $s(v)$ (*successor* of v) the end of $o(v)$ in that orientation and by $p(v)$ its *predecessor*. We can colour $o(v)$ in *Blue* or *Red* accordingly to the colour of v . M_B being coloured with *Blue* and M_R with *Red*, we get hence a larger set CL_B of edges coloured with *Blue* (and a set CL_R of edges coloured with *Red*). It is easily seen that the CL_B and CL_R are linear-forests where each maximal unicoloured path has length 1 or 3. Moreover each edge of $M_B \cup M_R$ is the central edge of a path of length 3. At this point, the only edges which are

not coloured are the edges of M' and we do not know how we can assign a colour to these edges in order to get a linear partition of $E(G)$. We used this construction in [7] that we have called the *associated linear construction* to the partition $M = M_B + M_R + M'$.

Remark 1.5. The fact that a Jaeger's graph is a cubic 3-edge colourable graph is an easy consequence of Lemma 1.3

Indeed, if G is a Jaeger's graph and M is a perfect matching of G union of two disjoint strong matchings M_B and M_R , then let us colour with *Blue* the vertices which are end vertices of edges in M_B and *Red* those which are end vertices of edges in M_R . Since M_B and M_R are strong matchings the remaining edges are mixed. Hence, by Lemma 1.3, G is 3-edge colourable.

In order to give here a first illustration of the usefulness of the associated linear construction we give a proof of an old result.

Theorem 1.6. [2] [13] *A cubic graph G has a linear partition $L = (L_B, L_R)$ such that each path has length 3 if and only if G is a Jaeger's graph.*

Proof : Suppose that G has a linear partition $L = (L_B, L_R)$ such that each path has length 3. Let C_B (respectively C_R) be the set of the middle edges of the paths of L_B (respectively L_R). It is an easy task to check that C_B and C_R are strong matchings and $|C_B| = |C_R|$. Moreover $M = C_B \cup C_R$ is a perfect matching and G is a Jaeger's graph. Conversely, let us suppose that G is a Jaeger's graph and let $M = M_B + M_R$ be an associated partition. Since M' is empty, by using the associated linear construction above, we have coloured every edge of G and each unicoloured path has length 3. \square

Finally, the following result shows that Jaeger's graphs play a pivotal role in the family of cubic 3-edge colourable graphs.

Theorem 1.7. [8] *Let G be a cubic graph. Then G can be factored into two odd linear forests $L = (L_B, L_R)$ such that*

- i) Each path in L_B has odd length at most 3*
- ii) Each path in L_R has odd length at least 3.*

if and only if $\chi'(G) = 3$.

2. Some classes of Jaeger's graphs.

We can construct a Jaeger's graph starting from any cubic 3-edge colourable graph. Indeed, consider a perfect matching M of G together with a bipartition of its vertex set in *Blue* and *Red* induced by a 3-edge colouring of G given by Lemma 1.3. If there are no mixed edge, we are done since G itself is a Jaeger's graph. Otherwise for any mixed edge apply the transformation depicted in figure 1 on the *Blue* vertices. Every such *Blue* vertex is transformed into a new triangle containing a new *Blue* edge while the mixed edge is transformed into a *Red* edge. The resulting graph is a Jaeger's graph.

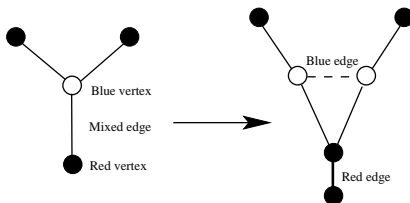


Figure 1: Triangle Extension

The *square* G^2 of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever there exists a path of length at most 2 joining them in G .

Proposition 2.1. [2] *If G is a cubic graph such that G^2 is 4-chromatic then G is a Jaeger's graph.*

A cubic planar graph is a *multi- k -gon* [5] (with $3 \leq k \leq 5$) if all of its faces have length multiple of k . Since Jaeger [10] has shown that a multi- k -gon G with $k = 3, 4$ has a square G^2 which is 4-chromatic, the following result is a consequence of Proposition 2.1.

Proposition 2.2. [2] *If G is a multi- k -gon with $k = 3, 4$ then G is a Jaeger's graph.*

When G is a cubic graph having a 2-factor of C_4 's, we consider the auxiliary 2-regular graph G' defined as follows : every C_4 of this 2-factor is replaced with its complementary graph (which is a $2K_2$). In a previous paper we have proved the following.

Theorem 2.3. [8] *Let G be a connected cubic graph having a 2-factor of squares, say \mathcal{F} and let p be the number of cycles of the auxiliary 2-regular graph G' . Then there are 2^{p-1} Jaeger's matchings which intersect \mathcal{F} .*

Since by Theorem 2.3 every cubic graph having a 2-factor of squares has at least one Jaeger's matching, we have:

Corollary 2.4. [8] *A cubic graph having a 2-factor of squares is a Jaeger's graph.*

By Shaefer's result [12], we know that it is NP-complete to recognize a Jaeger's graph. Assume that G is a Jaeger's graph, is it difficult to find a perfect matching union of two disjoint strong matchings? We do not have a general answer, but we can derive from the proof of Theorem 2.3 (see [8]) a simple linear time algorithm for finding a Jaeger's matching in a connected cubic graph which have a 2-factor of squares.

Corollary 2.5. [8] *If G is a cubic graph with a 2-factor of squares then we can construct in $O(n)$ time a perfect matching union of two disjoint strong matchings with the same size.*

It can be noticed that every cubic graph with a perfect matching M can be transformed into a Jaeger's graph by using the transformation (*square extension*) depicted in figure 2 on each edge of M . Indeed, the resulting graph has a 2-factor of squares and we can apply Theorem 2.3

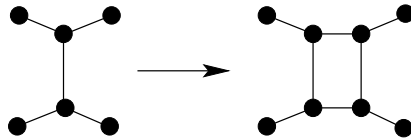


Figure 2: Square Extension

To conclude this section we note that we have studied in [6] a family of cubic graphs having a large number of perfect matchings (and containing the flower snarks). The sub-family of the 3-edge colourable graphs of this family contains some Jaeger's graphs that we have totally characterized, and we proved that such a graph on at least 12 vertices has exactly 6 Jaeger's matchings.

3. Compatible partitions

Let $L = (L_B, L_R)$ be a linear partition of G . For each vertex v we can define $e_L(v)$ as the edge incident to v which is an end edge of a maximal path in L_B or L_R . We shall say that two linear partitions $L = (L_B, L_R)$ and $L' = (L'_B, L'_R)$ are *compatible* whenever $e_L(v) \neq e_{L'}(v)$ for each vertex v . The qualifying adjective "compatible" refers to the notion of compatible Euler's tours introduced by Kotzig [11] (see Bondy [3] for an introduction to this question). In 1991, the first author, J.-L. Fouquet, remarked that a Jaeger's graph can be provided with two compatible linear partitions and he conjectured:

Conjecture 3.1. *Every cubic graph has two compatible linear partitions.*

We do not solve this conjecture but we shall give some results strengthening it.

Definition 3.2. Let α and β be any two distinct colours of a 3-edge colouring of a cubic 3-edge colourable graph G . In the following $SM_G(\alpha, \beta)$ will denote a strong matching of G intersecting every $\alpha\beta$ -cycle (when such a strong matching exists).

Some results of the following subsections are consequences of the two following theorems.

Theorem 3.3. [8] *Let G be a 3-edge coloured cubic graph and let α and β be any two distinct colours of $E(G)$. Then there exists a strong matching $SM_G(\alpha, \beta)$ intersecting every cycle of $\Phi(\alpha, \beta)$.*

We derive from Theorem 3.3 a result on unicoloured transversals of the 2-factors induced by any 3 edge-colouring of cubic graph with chromatic index 3.

Theorem 3.4. [8] *Let G be a cubic 3-edge colourable graph and let Φ be a 3-edge colouring of G . Let α and β be any two distinct colours of Φ and let γ be the third colour. Then there exists a set F_α of α -edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that the set F_α together with the γ -edges has no cycle.*

3.1. Compatible partitions and associated linear construction

For cubic 3-edge colourable graphs we obtain the following result.

Theorem 3.5. *Let G be a cubic 3-edge colourable graph with an associated partition $M = M_B + M_R + M'$. Assume that, we can colour the edges of M' in Blue and Red in*

such a way that, for each associated linear construction, the whole colouring of $E(G)$ so obtained is a linear partition. Then G has two compatible linear partitions.

Proof An associated linear construction is obtained in fixing an arbitrary orientation to the cycles of $G \setminus M$. To each vertex v of $V(G)$ we associate an edge $o(v)$ of $E(G) \setminus M$ such that v is the origin of $o(v)$ with respect to the chosen orientation of the cycle through v . We colour $o(v)$ in *Blue* or *Red* accordingly to the colour of v .

We give a colour to each edge of M' , accordingly to some rule depending on the class of graphs we are studying. When the edge of M' incident to some vertex v is coloured in *Blue* or *Red*, v is transformed into a vertex of degree 2 in one of the two colours (say *Blue*) and a vertex of degree 1 in the other (*Red*). Since the whole colouring of $E(G)$ leads to a linear partition $L = (L_B, L_R)$, the edge incident to v of this last colour is hence the edge $e_L(v)$.

On each cycle of $G \setminus M$ we can give now the opposite orientation. In the associated linear construction, the colours of the edges of the cycles incident to each vertex are exchanged. Since, with the same colouring of M' , we have supposed that the whole colouring of $E(G)$ leads to a linear partition $L' = (L'_B, L'_R)$, the above vertex v remains a vertex of degree 2 in the *Blue* colour. Hence the *Red* edge incident to v $e_{L'}(v)$ is distinct from $e_L(v)$.

Since $e_L(v) \neq e_{L'}(v)$ for each vertex v , the two linear partitions L and L' so obtained are compatible. \square

Corollary 3.6. *Let G be a Jaeger's graph then G has two compatible linear partitions in which every path has length 3.*

Proof In that case, we have a perfect matching $M = M_B + M_R$ and any associated linear construction is a linear partition in which every path has length 3. \square

Theorem 3.7. *Let G be cubic 3-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that M' can be partitioned into two strong matchings M'_B and M'_R . Then there is an odd linear partition of $E(G)$ every maximal path of which has length 1, 3, 5 or 7.*

Proof : Let CL_B and CL_R be the linear-forests of the associated linear construction. Recall that each maximal path of CL_B (respectively CL_R) has length 1 or 3 and is unicoloured with *Blue* (respectively unicoloured with *Red*). Let $L_B = CL_B \cup M'_B$ and $L_R = CL_R \cup M'_R$, in addition $M = M_B + M_R + M'$ and B denotes the set of *Blue* vertices of G and R its set of *Red* vertices.

We now prove that the components of L_B and L_R are odd paths of length at most 7. We only have to consider components that contain an edge of M' . Without loss of generality, let C be a component of L_B which contains an edge of M'_B .

Claim 1. *Let br be an edge of M'_B such that $b \in B \cap C$ and $r \in R \cap C$ and let $r' = s(b)$. Then the unique neighbour of r' in C is b .*

Proof of Claim Since $G \setminus M$ contains only mixed edges r' is a *Red* vertex. Observe that $o(r')$ is a *Red* edge while the edge of M incident to r' in G , say e , cannot belong to M'_B since M'_B is a strong matching. Moreover, e having a *Red* end must belong either to M_R or to M'_R , consequently e belongs to L_R . Thus, among the three edges incident to r' , only $o(b)$ belongs to C and the result follows. \square

Let b_1r_1 be an edge of $C \cap M'_B$ ($b_1 \in B$, $r_1 \in R$). Let us set $r_2 = s(b_1)$. We know by the above Claim that r_2 is a pendant vertex in C .

Let $b_2 = p(r_1)$ and $r_3 = p(b_2)$, obviously $b_2 \in B$, $r_3 \in R$, b_2r_1 is a *Blue* edge and r_3b_2 is a *Red* one. Consider in G the edge of M incident to b_2 , say e . M'_B being a strong matching, the edge e cannot belong to M'_B . Moreover, the edge e has a *Blue* end, namely b_2 , and thus cannot belong to M_R . If e belongs to M'_R we are done since e is a *Red* edge and the component C is reduced to the path of length 3 $r_2b_1r_1b_2$.

Assume in the following that e belongs to M_B . From now on e will be denoted b_2b_3 ($b_3 \in B$) and $s(b_3)$ will be denoted r_4 , we have $r_4 \in R$. Let e' be the edge of M incident to r_4 in. Since r_4 is a *Red* end of e' , e' cannot belong to M_B . If e' is a member of $M_R \cup M'_R$ we are done since e' and $o(r_4)$ both belong to L_R and C is reduced to a path of length 5, namely $r_2b_1r_1b_2b_3r_4$.

Suppose now that $e' \in M'_B$. Let us denote e' as r_4b_4 ($b_4 \in B$) and $s(b_4)$ as r_5 . But now, by the above Claim b_4 is the unique neighbour of r_5 in C and thus $C = \{r_2, b_1, r_1, b_2, b_3, r_4, b_4, r_5\}$ induces a path of length 7. \square

As a corollary of Theorems 3.5 and 3.7 we obtain:

Corollary 3.8. *Let G be cubic 3-edge colourable graph and an associated partition $M_B + M_R + M'$. Assume that M' can be partitioned into two strong matchings M'_B and M'_R . G has two compatible linear partitions in which every path has length 1, 3, 5 or 7.*

Proof In proof of Theorem 3.7 we have coloured the edges of M'_B in *Blue* and those of M'_R in *Red* and we have shown that the associated linear construction together with this colouring of M' leads to a linear partition with each path of length 1, 3, 5 or 7. Applying Theorem 3.5 above leads to the result. \square

3.2. Compatible partitions in hamiltonian cubic graphs

As a corollary of Theorem 3.3, we shall show now that cubic hamiltonian graphs satisfy Conjecture 3.1. In fact, a stronger result is obtained since the two compatible partitions are odd.

Theorem 3.9. *Let G be a cubic hamiltonian graph, then G has two compatible odd linear partitions.*

Proof Let $C = a_0, a_1 \dots a_{n-1}$ be a hamiltonian cycle of G . This hamiltonian cycle induces a 3-edge colouring $\Phi : E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us colour every edge $a_i a_{i+1}$ of C with α when $i \equiv 0(2)$ and with β otherwise, while the remaining perfect matching is coloured with γ . From Theorem 3.3 we know that there is a strong matching F intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$. We choose F minimal for the inclusion (that is F intersects each cycle of $\Phi(\beta, \gamma)$ in exactly one edge). Let M_α be the set of α -coloured edges. Since F is a strong matching intersecting every bicoloured cycle of $\Phi(\beta, \gamma)$, we can construct an odd linear partition $L' = (L'_B, L'_R)$:

$$\begin{aligned} L'_R &= M_\alpha \cup F \\ L'_B &= E(G) - L'_R \end{aligned}$$

For each vertex v , the edge $e_{L'}(v)$ is coloured with α excepted when v is an end vertex of an edge of F . In that case $e_{L'}(v)$ coloured with γ when v is an end vertex of the edge of $F \cap C$ and with β when v is an end vertex of an edge of $F \setminus C$.

Case 1 : *F contains some edges of C*

Hence the edges of $F \cap C$ are coloured with β . $F \cap C$ is a strong matching intersecting the 2-factor made of the unique cycle C leading to the following odd linear partition $L = (L_B, L_R)$:

$$\begin{aligned} L_B &= C - F \\ L_R &= E(G) - L_B \end{aligned}$$

For each vertex v , $e_L(v)$ is coloured with γ excepted when v is an end vertex of the edge of $F \cap C$. In that case we have $e_L(v)$ coloured with β .

We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex v , since the colours of these edges are distinct. Hence, the two odd partitions L and L' are compatible.

Case 2 : *Each edge of F is coloured with γ and there is an edge $a_i a_{i+1}$ (i odd) of C coloured with β which is not incident to an edge of F.*

Let $L = (L_B, L_R)$ be the following odd linear partition:

$$\begin{aligned} L_B &= C - a_i a_{i+1} \\ L_R &= E(G) - L_B \end{aligned}$$

For each vertex v , $e_L(v)$ is coloured with γ excepted when v is a_i or a_{i+1} . In that case $e_L(v)$ is coloured with α .

For each vertex v , $e_{L'}(v)$ is coloured with α unless when v is an end vertex of an edge of F . In that case $e_{L'}(v)$ is coloured with β when v is an end vertex of an edge of F .

We can check that $e_L(v) \neq e_{L'}(v)$ for each vertex v , since the colours of these edges are distinct. Hence, the two odd partitions L and L' are compatible.

Case 3 : *Each edge of F is coloured with γ and each edge of C coloured with β is incident to an edge of F.*

Without loss of generality assume that a_2 is incident to F . Edge $a_3 a_4$ being incident to F , we must have a_4 incident to F . Going through C , we get that $a_2, a_4, \dots, a_{2k}, \dots, a_0$ must be incident to F . The remaining edges coloured with γ are edges joining vertices of C with odd index. It is an easy task to see that this set K of edges is a strong matching. The perfect matching coloured with γ is the union of two strong disjoint matchings F

and K with the same size. Hence G is a Jaeger's graph . Theorem 3.6 implies that we have two compatible odd linear partitions. \square

3.3. Compatible odd linear partitions

The results of Theorem 3.9 and Theorem 3.5 leads us to a strengthening of Conjecture 3.1.

Conjecture 3.10. *Let G be a cubic 3-edge colourable graph then we can find two compatible odd linear partitions.*

As a partial result we have:

Theorem 3.11. *Let G be a cubic 3-edge colourable graph then we can find three odd linear partitions L, L' and L'' such that for each vertex v*

$$|\{e_L(v), e_{L'}(v), e_{L''}(v)\}| \geq 2$$

Proof Let us consider a 3-edge colouring $\Phi : E(G) \longrightarrow \{\alpha, \beta, \gamma\}$. Let us denote by M_γ the perfect matching consisting of the γ -coloured edges. Theorem 3.4 implies that there exists a set F_α of α -coloured edges intersecting every cycle of $\Phi(\alpha, \beta)$ such that $F_\alpha \cup M_\gamma$ is acyclic. In that way, we obtain an odd linear partition $L = (L_1, L_2)$

$$L_1 = F_\alpha \cup M_\gamma$$

$$L_2 = E(G) - L_1$$

For each vertex v we have $e_L(v)$ coloured with γ excepted for the vertices which are the end vertices of an edge of F_α . In that case $e_L(v)$ is coloured with β .

Let us consider the perfect matching M_β , the bicoloured cycles of $\Phi(\alpha, \gamma)$ and a matching F_γ obtained by Theorem 3.4. Hence, we get an odd linear partition $L' = (L'_1, L'_2)$ such that for every vertex v the edge $e_{L'}(v)$ is coloured either with β (when the vertex v is an end vertex of an edge of F_γ) or with α . Finally we obtain a third odd linear partition $L'' = (L''_1, L''_2)$ from the perfect matching M_α , the bicoloured cycles of $\Phi(\beta, \gamma)$ and a matching F_β , such that the edges $e_{L''}(v)$ are coloured α or γ (γ when the considered vertices are the end vertices of an edge of F_β).

The three sets F_α , F_β and F_γ being obviously pairwise disjoint, it is a routine matter to see that for each vertex v two edges in $\{e_L(v), e_{L'}(v), e_{L''}(v)\}$, at least, have distinct colours. These two edges are thus distinct and we get the result. □

4. Generation of Jaeger's graphs

Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that a and d are internal vertices in L_B while b and c are internal vertices in L_R .

Let us consider the linear forest L_R and the edges of L_R incident to the vertices of $U = \{a, b, c, d\}$. We notice that, without loss of generality, there are six distinct cases (by exchanging the role of a for that of d or/and the role of b for that of c). See figure 3.

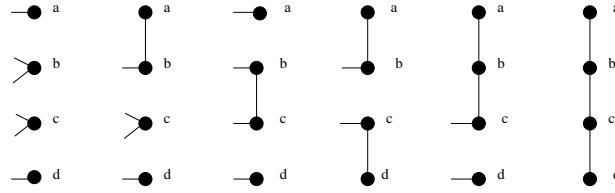


Figure 3: Six distinct cases

Analogous situations appear for the linear forest L_B (by exchanging $\{a, d\}$ for $\{b, c\}$).

Definition 4.1. Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition every path of which has length 3. Assume that $U = \{a, b, c, d\}$ is a set of 4 vertices such that a and d are internal vertices in L_B while b and c are internal vertices in L_R . A (L, U) -extension of G is a cubic simple graph G' obtained from G in the following way.

- 1) The set U is splitted into two sets $U_R = \{a_R, b_R, c_R, d_R\}$ and $U_B = \{a_B, b_B, c_B, d_B\}$ (that is $V(G') = VG \setminus U \cup (U_R \cup U_B)$).
- 2) For $x, y \in V(G) \setminus U$, if $xy \in E(G)$ then $xy \in E(G')$.
- 3) For $x \in V(G) \setminus U$ and $y \in U$ if $xy \in E(L_B)$ then $xy_B \in E(G')$ and if $xy \in E(L_R)$ then $xy_R \in E(G')$.

- 4) For $x, y \in U$ if $xy \in E(L_B)$ then $x_B y_B \in E(G')$ and if $xy \in E(L_R)$ then $x_R y_R \in E(G')$.
- 5) The remaining edges of G' are the edges of two paths of length 3 on the sets U_R and U_B , respectively, such that the obtained graph G' is cubic (see figures 4 and 5).

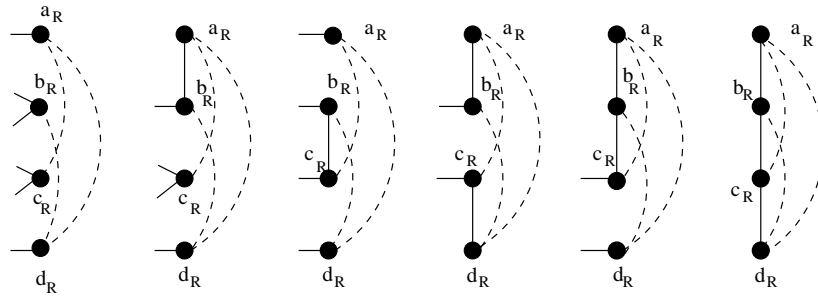


Figure 4: Addition of a path of length 3 on U_R

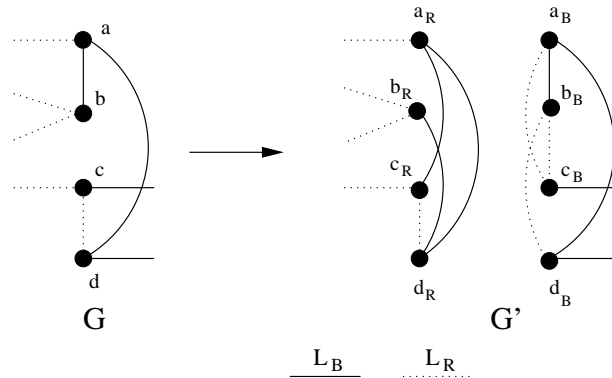


Figure 5: Example of (L, U)-extension

It is clear that the added path on U_R (respectively U_B) can be added to L_B (respectively L_R) in order to obtain a linear partition of G' each path of which has length 3.

Note that the graph G' is not uniquely defined, namely if the subgraph induced on U in L_R (respectively, in L_B) is a stable set or has exactly one edge connecting the vertices of degree 2 in L_R (respectively, in L_B).

So, by Theorem 1.6 we have the following Proposition.

Proposition 4.2. *Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of G each path of which has length 3. Let $U = \{a, b, c, d\}$ be a set of 4 vertices of G such that a and d are internal vertices in L_B while b and c are internal vertices in L_R . Then any (L, U) -extension of G on U is a Jaeger's graph.*

Definition 4.3. Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition for which every path has length 3. Assume that $P \in L_B$ and $Q \in L_R$ where $P = \{a_1, b_1, c_1, d_1\}$ and $Q = \{a_2, b_2, c_2, d_2\}$ are vertex disjoint paths in G . A PQ -reduction of G on P and Q is a cubic simple graph G' obtained from G by deleting the edges of P and Q and identifying the internal vertices of P with the end vertices of Q and the internal vertices of Q with the end vertices of P .

Note that the PQ -reduction of G has a linear partition each path of which has length 3. Hence we have the following.

Proposition 4.4. *Let G be a Jaeger's graph and let $L = (L_B, L_R)$ be a linear partition of G every path of which has length 3. Assume that $P \in L_B$ and $Q \in L_R$ are vertex disjoint paths in G . Then the PQ -reduction of G on P and Q is a Jaeger's graph. \square*

We get immediately from Propositions 4.2 and 4.4

Theorem 4.5. *Every Jaeger's graph on $n \geq 20$ vertices is obtained from a Jaeger's graph on 16 vertices by a sequence of (L, U) -extensions.*

Proof Assume that G is a Jaeger's graph on n vertices with $n \geq 20$. Let $L = (L_B, L_R)$ be a linear partition each of whose paths have length 3. Since each path of L is incident to at most 4 distinct paths, as soon as $n \geq 20$ we are sure to find a path in $P \in L_B$ and a path $Q \in L_R$ such that $V(P) \cap V(Q) = \emptyset$. By a PQ -reduction on these two paths we get a Jaeger's graph on $n - 4$ vertices (Proposition 4.4). The proof is complete. \square

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