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ON HOMOGENEOUS HERMITE-LORENTZ SPACES

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ABSTRACT. We define naturally Hermite-Lorentz metrics on almost-complex manifolds as special case of pseudo-Riemannian metrics compatible with the almost complex structure. We study their isometry groups.

1. INTRODUCTION

Let us call a quadratic form q on a complex space of dimension $n + 1$ of **Hermite-Lorentz** type if it is $\mathbb{C}$-equivalent to the standard form $q_0 = -|z_0|^2 + |z_2|^2 + \ldots + |z_n|^2$ on $\mathbb{C}^{n+1}$. In other words, q is Hermitian, and as a real form, it has a signature $- - + \ldots +$. Here, Lorentz refers to the occurrence of exactly one negative sign (in the complex presentation). Classically, this one negative sign distinguishes, roughly, between time and space components. (A "complex-Lorentz" form could perhaps be an equally informative terminology?)

One can then define Hermite-Lorentz metrics on almost complex manifolds. If (M, J) is an almost complex manifold, then g is a Hermite-Lorentz metric if g is a tensor such that for any $x \in M$, $(T_x M, J_x, g_x)$ is a Hermite-Lorentz linear space.

Hermite-Lorentz metrics generalize (definite) Hermitian metrics, and they are the nearest from them in the sense that they have the minimal (non-trivial) signature.

Hermite-Lorentz metrics are to compare, on one hand with (definite) Hermitian metrics in complex geometry, and with Lorentz metrics in (real) differential geometry.

H-structure. Let $\mathsf{U}(1, \mathsf{n}) \subset \mathsf{GL}_{\mathsf{n}+1}(\mathbb{C})$ be the unitary group of q_0. Then, a Hermite-Lorentz structure on a manifold M of real dimension $2n + 2$ is a reduction of the structural group of TM to $\mathsf{U}(1, \mathsf{n})$. They are different from "complex Riemannian" metrics which are reduction to $\mathsf{O}(n + 1, \mathbb{C})$.

Kähler-Lorentz spaces. As in the positive definite case, on defines a Kähler form ω by $\omega(u, v) = g(u, Jv)$. It is a J-invariant 2-differential form. A Kähler-Lorentz metric corresponds to the case where J is integrable and ω is closed. A **Kähler-Lorentz** manifold is in particular symplectic.

Conversely, from the symplectic point of view, a symplectic manifold (M, ω) is Kähler-Lorentz if ω can be calibrated with a special complex structure J. Let us generalize the notion of calibration by letting it to mean that J is such that $g(u, v) = \omega(u, Jv)$ is non-degenerate, i.e. g is a pseudo-Riemannian metric. Now, the classical Kähler case means that g is Riemannian and in addition J is integrable.

So Kähler-Lorentz means that g is "post-Riemannian" in the sense that it has a Hermite-Lorentz signature.

Terminology. Hermite-Lorentz metrics are a particular case of Hermitian pseudo-Riemannian metrics. They are however many terminologies for them in the literature: indefinite-Hermitian, pseudo-Hermitian, pseudo-Kählerian... Also, the word "pseudo-Hermitian" is sometimes used to mean "Hermitian sub-Riemannian" (as in the situation of CR-structures)!

Holomorphic sectional curvature. Differential geometry can be developed for Hermite-Lorentz metrics exactly as in the usual Hermitian as well as the usual Lorentz cases [15]. In particular, there is a Levi-Civita connection and a Riemann curvature tensor R. The **holomorphic sectional curvature** $K(u)$ is the sectional curvature of the real 2-plane $\mathbb{C}u$; $K(u) = \frac{g(R(u,Ju)u,Ju)}{g(u,u)g(v,v)-g(u,v)^2}$ (this requires u to be non isotropic $g(u,u) \neq 0$, in order to divide by the volume volume $u \wedge Ju$). So, K is a real function on an open set of the projectivization bunble of TM (which fibers over M with fiber type $\mathbb{P}^n(\mathbb{C})$).

In fact, K determines the full Riemann tensor in the Kähler case [12, 4] (but not in the general Hermitian case). In particular, the case K constant in the (usual) Kähler case corresponds to the most central homogeneous spaces: $\mathbb{C}^n$, $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{H}^n(\mathbb{C})$ (the complex hyperbolic space). Kähler-Lorentz spaces of constant curvature are introduced below.

1.1. **Examples.** We are going to give examples of homogeneous spaces $M = G/H$, where the natural (generally unique) G-invariant geometric structure is a Kähler-Lorentz metric.

1.1.1. *Universal Kähler-Lorentz spaces of constant holomorphic curvature.* If a Kähler-Lorentz metric has constant holomorphic sectional curvature, then it is locally isometric to one of the following spaces:

   (1) The universal (flat Hermite-Lorentz) complex Minkowski space $\mathsf{Mink}_n(\mathbb{C})$ (or $\mathbb{C}^{1,n-1}$), that is $\mathbb{C}^n$ endowed with $q_0 = -|z_1|^2 + |z_2|^2 + \ldots + |z_{n-1}|^2$.

   (2) The complex de Sitter space $\mathsf{dS}_n(\mathbb{C}) = \mathsf{SU}(1,n)/\mathsf{U}(1,\mathsf{n}-1)$[1] It has a positive constant holomorphic sectional curvature.

   (3) The complex anti de Sitter space $\mathsf{AdS}_n(\mathbb{C}) = \mathsf{SU}(2,n-1)/\mathsf{U}(1,\mathsf{n}-1)$. It has negative curvature.

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1. Here $\mathsf{U}(1,\mathsf{n}-1)$ as a subgroup of $\mathsf{SU}(1,n)$ stands for matrices of the form

$$\begin{pmatrix} \lambda A & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, |\lambda| = 1, A \in \mathsf{SU}(1,n-1).$$

In general $\mathsf{U}(1,\mathsf{n}-1)$ designs a group isomorphic to a product $\mathsf{U}(1) \times \mathsf{SU}(1,\mathsf{n}-1)$, where the embedding $U(1)$ depends on the context.

• As said above, Hermite-Lorentz metrics are generalizations of both Hermitain metrics (from the definite to the indefinite) and Lorentz metrics (from the real to the complex). Let us draw up in the following table the analogous of our previous spaces in both Hermitian and Lorentzian settings.

Kähler-Lorentz spaces of constant curvature	Hermitian (positive definite) counterpart	(Real) Lorentz counterpart
$\mathsf{Mink}_n(\mathbb{C})$	$\mathbb{C}^n$	$\mathsf{Mink}_n(\mathbb{R})$
$\mathsf{dS}_n(\mathbb{C}) =$ $\mathsf{SU}(1, \text{n})/\mathsf{U}(1, \text{n-1})$	$\mathbb{P}^n(\mathbb{C}) =$ $\mathsf{SU}(1+n)/\mathsf{U}(n)$	$\mathsf{dS}_n(\mathbb{R}) =$ $\mathsf{SO}(1,n)/\mathsf{SO}(1,n-1)$
$\mathsf{AdS}_n(\mathbb{C}) =$ $\mathsf{SU}(2,n-1)/\mathsf{U}(1,\text{n}-1)$	$\mathbb{H}^n(\mathbb{C}) =$ $\mathsf{SU}(1,n)/\mathsf{U}(n)$	$\mathsf{AdS}_n(\mathbb{R}) =$ $\mathsf{SO}(2,n-1)/\mathsf{SO}(1,n-1)$

(Of course, we also have as Riemannian counterparts of constant sectional curvature, respectively, the Euclidean, spherical and hyperbolic spaces, $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$).

• The Kähler-Lorentz spaces of constant holomorphic sectional curvature are pseudo-Riemannian symmetric spaces (see below for further discussion). They are also holomorphic symmetric domains in $\mathbb{C}^n$. Indeed, $\mathsf{dS}_n(\mathbb{C})$ is the exterior of a ball in the projective space $\mathbb{P}^n(\mathbb{C})$. It is strictly pseudo-concave. The ball of $\mathbb{P}^n(\mathbb{C})$ is identified with the hyperbolic space $\mathbb{H}^n(\mathbb{C})$, and then $\mathsf{dS}_n(\mathbb{C})$ is the space of geodesic complex hypersurfaces of $\mathbb{H}^n(\mathbb{C})$.

As for $\mathsf{AdS}_n(\mathbb{C})$, it can be represented as the open set $q < 0$ of $\mathbb{P}^n(\mathbb{C})$, where $q = -|z_1|^2 - |z_2|^2 + |z_3|^2 + \ldots + |z_n|^2$. It is pseudo-convex, but not strictly, as its boundary is ruled.

1.1.2. *Irreducible Kähler-Lorentz symmetric spaces.* Let $M = G/H$ be a homogeneous space. Call p the base point $1.H$. The isotropy representation at p is identified with the adjoint representation $\rho : H \to \mathsf{GL}(\mathfrak{g}/\mathfrak{h})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the respective Lie algebras of G and H. The homogeneous space is of Hermite-Lorentz type if ρ is conjugate to a representation in $\mathsf{U}(1, \text{n})$ (where the real dimension of G/H is $2n + 2$).

The space G/H is symmetric if $-Id_{T_pM}$ belongs to the image of ρ. This applies in particular to the two following spaces:

$$\mathbb{C}\mathsf{dS}_n = \mathsf{SO}(1, n+1)/\mathsf{SO}(1, n-1) \times \mathsf{SO}(2)$$

$$\mathbb{C}\mathsf{AdS}_n = \mathsf{SO}(3, n-1)/\mathsf{SO}(2) \times \mathsf{SO}(1, n-1)$$

Complexification. The isotropy representation of these spaces is the complexification of the $\mathsf{SO}(1, n-1)$ standard representation in $\mathbb{R}^n$, i.e. its diagonal action on $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$; augmented with the complex multiplication by $\mathsf{U}(1) \cong \mathsf{SO}(2)$.

If one agrees that a complexification of a homogeneous space X is a homogeneous space $\mathbb{C}X$ whose isotropy is the complexification of that of X, then $\mathbb{C}\mathsf{dS}_n$ and $\mathbb{C}\mathsf{AdS}_n$ appear naturally as complexification of $\mathsf{dS}_n(\mathbb{R})$ and $\mathsf{AdS}_n(\mathbb{R})$ respectively. In contrast, $\mathsf{dS}_n(\mathbb{C})$ and $\mathsf{AdS}_n(\mathbb{C})$ are the set of complex points of the same algebraic object as $\mathsf{dS}_n(\mathbb{R})$ and $\mathsf{AdS}_n(\mathbb{R})$. As another example, the

complexification of $\mathbb{S}^n$ is not $\mathbb{P}^n(\mathbb{C})$ but rather the Kähler Grassmanian space $\mathsf{SO}(n+2)/\mathsf{SO}(n) \times \mathsf{SO}(2)$?!

1.1.3. *List.* There are lists of pseudo-Riemannian **irreducible** symmetric spaces [5]. (Here irreducibility concerns isotropy, but for symmetric spaces, besides the flat case, the holonomy and isotropy groups coincide. In particular, holonomy irreducible symmetric spaces are isotropy irreducible) . It turns out that the five previous spaces are all the Kähler-Lorentz (or equivalently Hermite-Lorentz) ones. Our theorem 1.1 below will give a non list-checking proof of this classification.

1.2. **Results.**

1.2.1. *Objective.* Our aim here is the study of isometry groups $\mathsf{Iso}(M, J, g)$ of Hermite-Lorentz manifolds. They are Lie groups acting holomorphically on M. If g were (positive definite) Hermitian, then $\mathsf{Iso}(M, J, g)$ acts properly on M, and is in particular compact if M is compact.

This is no longer true for g indefinite.

In the real case, that is without the almost complex structure, there have been many works tending to understand how and why the isometry group of a Lorentz manifold can act non-properly. The Lorentz case is the simplest among all the pseudo-Riemannian cases, since, with its one negative sign, it lies as the nearest to the Riemannian case. For instance, the situation of signature $- - + \ldots +$ presents more formidable difficulties. With this respect, the Hermite-Lorentz case seems as an intermediate situation, which besides mixes in a beautiful way pseudo-Riemannian and complex geometries.

1.2.2. *Homogeneous vs Symmetric.* We are going to prove facts characterizing these Kähler-Lorentz symmetric spaces by means of a homogeneity hypothesis.

In pseudo-Riemannian geometry, it is admitted that, among homogeneous spaces, the most beautiful are those of constant sectional curvature, and then the symmetric ones, and so on... This also applies to pseudo-Kähler spaces, where the sectional curvature is replaced by the holomorphic sectional curvature.

In general, being (just) homogeneous is so weaker than being symmetric which in turn is weaker than having constant (sectional or holomorphic sectional) curvature.

For instance, Berger spheres are homogeneous Riemannian metrics on the 3-sphere that have non constant sectional curvature and are not symmetric. On the other hand different Grassmann spaces are irreducible symmetric Riemannian (or Hermitian) spaces but do not have constant sectional (or holomorphic) curvature.

Our first theorem says that in the framework of Hermite-Lorentz spaces, being homogeneous implies essentially symmetric!

Theorem 1.1. *Let (M, J, g) be a Hermite-Lorentz almost complex space, homogeneous under the action of a Lie group G. Suppose that the isotropy group G_p of some point p acts $\mathbb{C}$-irreducibly on T_pM, and $\dim_{\mathbb{C}} M > 3$. Then M is a global Kähler-Lorentz symmetric space, and it is isometric, up to a cyclic cover, to $\mathsf{Mink}_n(\mathbb{C})$, $\mathsf{dS}_n(\mathbb{C})$, $\mathsf{AdS}_n(\mathbb{C})$, $\mathbb{C}\mathsf{dS}_n$ or $\mathbb{C}\mathsf{AdS}_n$.*

• The content of the theorem is:

1. Irreducible isotropy $\Longrightarrow$ symmetric,

2. The list of Hermite-Lorentz symmetric spaces with irreducible isotropy are the five mentioned ones. This fact may be extracted from Berger's classification of pseudo-Riemannian irreducible symmetric spaces. Here, we provide a direct proof.

• In the (real) Lorentz case, there is a stronger version, which states that an isotropy irreducible homogeneous space has constant sectional curvature [6] (the fact that irreducible and symmetric implies constancy of the curvature was firstly observed in [7] by consulting Berger's list).

• The theorem is not true in the Riemannian case. As an example of a compact irreducible isotropy non-symmetric space, we have $M = G/K$, where $G = \mathsf{SO}(\frac{n(n-1)}{2})$, and K is the image of the representation of $\mathsf{SO}(n)$ in the space of trace free symmetric 2-tensors on $\mathbb{R}^n$ (see [5] Chap 7).

1.2.3. *Actions of semi-simple Lie groups.* Let now (M, J, g) be an almost Hermite-Lorentz manifold and G a Lie group acting (not necessarily transitively) on M by preserving its structure. We can not naturally make a hypothesis on the isotropy in this case, since it can be trivial (at least for generic points). It is however more natural to require dynamical properties on the action. As discussed in many places, non-properness of the G-action is a reasonable condition allowing interplay between dynamics and the geometry of the action. For instance, without it everything is possible; a Lie group G acting by left translation on itself can be equipped by any type of tensors by prescribing it on the Lie algebra.

The literature contains many investigations on non-proper actions preserving Lorentz metrics [1, 2, 3, 9] and specially [13]. We are going here to ask similar questions on the Hermite-Lorentz case. We restrict ourselves here to transitive actions, since the general idea, within this geometric framework, is that a G-non-proper action must have non-proper G-orbits, i.e. orbits with non-precompact stabilizer. The natural starting point is thus the study of non-proper transitive actions.

Theorem 1.2. *Let G be a non-compact simple (real) Lie group of finite center not locally isomorphic to $\mathsf{SL}_2(\mathbb{R})$, $\mathsf{SL}_2(\mathbb{C})$ or $\mathsf{SL}_3(\mathbb{R})$. Let G act non-properly transitively holomorphically and isometrically on an almost complex Hermite-Lorentz space (M, J, g), with $\dim M > 3$. Then, M is a global Kähler-Lorentz irreducible symmetric space, and is isometric, up to a cyclic cover, to $\mathsf{dS}_n(\mathbb{C})$, $\mathsf{AdS}_n(\mathbb{C})$, $\mathbb{C}\mathsf{dS}_n$ or $\mathbb{C}\mathsf{AdS}_n$.*

1.2.4. *Comments.*

Integrabilities. Observe that we do not assume a priori neither that J is integrable, nor g is Kähler.

The exceptional cases. The hypotheses $\dim M > 3$ and G different form $\mathsf{SL}_2(\mathbb{R})$, $\mathsf{SL}_2(\mathbb{C})$ and $\mathsf{SL}_3(\mathbb{R})$ are due on the one hand to "algebraic" technical difficulties in proofs and on the other hand to that statements become complicated in this cases.

As an example, $\mathsf{SL}_2(\mathbb{C})$ with its complex structure admits a left invariant Hermite-Lorentz metric g which is moreover invariant by the right action of $\mathsf{SL}_2(\mathbb{R})$. So, its

isometry group $G = \mathsf{SL}_2(\mathbb{C}) \times \mathsf{SL}_2(\mathbb{R})$ and its isotropy is $\mathsf{SL}_2(\mathbb{R})$ acting by conjugacy. On the Lie algebra g is defined as: $\langle a, b \rangle = \mathsf{tr}(a\bar{b})$, where $a, b \in sl_2(\mathbb{C}) \subset \mathsf{Mat}_2(\mathbb{C})$. This metric is not Kähler, neither symmetric, although the isotropy is $\mathbb{C}$-irreducible, and so Theorem 1.1 does not apply in this case.

In the case of $\mathsf{SL}_3(\mathbb{R})$ one can construct an example of a left invariant Hermite-Lorentz structure (J, g), with J non-integrable, invariant under the action by conjugacy of a one parameter group, and therefore Theorem 1.2 does not apply to the $\mathsf{SL}_3(\mathbb{R})$-case. Notice on the other hand that, although $\mathsf{SL}_3(\mathbb{R})$ is a not a complex Lie group, it admits left invariant complex structures. This can be seen for instance by observing that its natural action on $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ has an open orbit on which it acts freely. We hope to come back to this discussion elsewhere.

About the proof. The tangent space at a base point of M is identified to $\mathbb{C}^{n+1}$, and the isotropy H to a subgroup of $\mathsf{U}(1, n)$. A classification of such subgroups into amenable and (essentially) simple is yielded as a fundamental tool for proofs. Regarding Theorem 1.1, since it acts irreducibly, the possibilities given for H (more precisely its Zariski closure) are $\mathsf{U}(1, n)$, $\mathsf{SU}(1, n)$, $\mathsf{U}(1) \times \mathsf{SO}(1, n)$, and $\mathsf{SO}(1, n)$ (the last act $\mathbb{C}$-irreducibly but not $\mathbb{R}$-irreducibly). Geometric and algebraic manipulations yield the theorem...

As for Theorem 1.2, the idea is to apply Theorem 1.1 by showing that H is irreducible (assuming it non-precompact and G simple). One starts proving that H is big enough. – If H is simple, irreducibility consists in excluding the intermediate cases $\mathsf{SO}(1, k) \subset H \subset \mathsf{U}(1, k)$, for $k < n$. – The most delicate situation to exclude is the non-reductive one: H amenable. Observe however that, in general, homogeneous pseudo-Riemannian manifolds with a semi-simple Lie group may have a non-reductive isotropy. For example, $\mathsf{SL}_n(\mathbb{R})$ acts diagonally on $\mathbb{R}^n \times \mathbb{R}^{n*}$ by preserving the pairing $\langle u, v^* \rangle = v^*(u)$ which determines a pseudo-product of signature (n, n). Then $\mathsf{SL}_n(\mathbb{R})$ has an open orbit and non-reductive isotropy, for $n > 2$.

CONTENTS

2. SOME PREPARATORY FACTS

$\mathbb{C}^{n+1}$ is endowed with the standard Hermite-Lorentz form $q_0 = -|z_0|^2 + |z_2|^2 + \ldots + |z_n|^2$. The Hermitian product is denoted $\langle, \rangle$.

Recall that u is lightlike (or isotropic) if $q(u) = 0$. A $\mathbb{C}$-hyperplane is lighlike if it equals the orthogonal $\mathbb{C}u^\perp$ of a lightlike vector u.

It is also sometimes useful to consider the equivalent form $q_1 = z_0 \bar{z}_n + |z_2|^2 + \ldots + |z_{n-1}|^2$.

As usually, $\mathsf{U}(q_0)$ is denoted $\mathsf{U}(1, n)$, and $\mathsf{SU}(1, n)$ its special subgroup.

The Lorentz group $\mathsf{SO}(1, n)$ is a subgroup of $\mathsf{SU}(1, n)$; it acts diagonally on $\mathbb{C}^{n+1} = \mathbb{R}^{n+1} + i\mathbb{R}^{n+1}$, by $A(x + iy) = A(x) + iA(y)$.

2.1. **Some $\mathsf{SO}(1, n)$-invariant theory.**

Levi form.

FACT **2.1.** *1. For $n > 1$, there is no non-vanishing $\mathsf{SO}(1, n)$-invariant anti-symmetric form $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$.*

2. Let $n > 1$ and $b : (\mathbb{R}^{n+1} + i\mathbb{R}^{n+1}) \times (\mathbb{R}^{n+1} + i\mathbb{R}^{n+1}) \to \mathbb{R}$ be a $\mathsf{SO}(1, n)$-invariant anti-symmetric bilinear form. Then, up to a constant, $b(u + iv, u' + iv') = \langle u, v' \rangle - \langle v, u' \rangle$. (That is, up to a constant, b concides with the Kähler form $i(-dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n)$).

Proof.

1. Let $b : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be such a form, and u a timelike vector, that is $\langle u, u \rangle < 0$. Thus, the metric on $\mathbb{R}u^\perp$ is positive and the action of the stablizer (in $\mathsf{SO}(1, n)$) of u on it is equivalent to the usual action of $\mathsf{SO}(n)$ on $\mathbb{R}^n$. The linear form $v \in \mathbb{R}u^\perp \to b(u, v) \in \mathbb{R}$ is $\mathsf{SO}(n)$-invariant, and hence vanishes (since its kernel is invariant, but the $\mathsf{SO}(n)$-action is irreducible). Thus u belongs to the kernel of b, and so is any timelike vector, and therefore $b = 0$

2. Let now $b : (\mathbb{R}^{n+1} + i\mathbb{R}^{n+1}) \times (\mathbb{R}^{n+1} + i\mathbb{R}^{n+1}) \to \mathbb{R}$. From the previous point $b(u + i0, v + +i0) = b(0 + iu, 0 + iv) = 0$. It remains to consider $b(u, iv)$. It can be written $b(u, iv) = \langle u, Av \rangle$, for some $A \in \mathsf{End}(\mathbb{R}^n)$, commuting with $\mathsf{SO}(1, n)$. By the (absolute) irreducibility of $\mathsf{SO}(1, n)$, A is scalar. (Indeed, by irreducibility, A has exactly one eigenvalue λ with eigenspace the whole $\mathbb{R}^{n+1}$. If λ is pure imaginary, then $\mathsf{SO}(1, n)$ preserves a complex structure, but this is impossible (for instance hyperbolic elements of $\mathsf{SO}(1, n)$ have simple real eigenvalues). Thus A is a real scalar).

The rest of the proof follows. $\qquad\square$

Kähler form.

FACT **2.2.** *For $n > 2$, there is no non-vanishing (real) exterior 3-form α on $\mathbb{C}^{n+1} (= \mathbb{R}^{n+1} + i\mathbb{R}^{n+1})$ invariant under the $\mathsf{SO}(1, n)$-action.*

Proof. Let α be such a form. Let $e \in \mathbb{R}^{n+1}$ be spacelike: $\langle e, e \rangle > 0$, and consider $\alpha_e = i_e \alpha$. First, $\alpha(e, ie, z)$ is a linear form on $\mathbb{C}e^\perp$ invariant under a group conjugate to $\mathsf{SO}(1, n-1)$, and hence vanishes. On $\mathbb{C}e^\perp$, α_e is a 2-form as in the fact above. It then follows that for any $u \in \mathbb{C}e$, and $v, w \in \mathbb{C}u^\perp$, $\alpha(u, v, w) = \phi(u)\omega(v, w)$, where ω is the Kähler form on $\mathbb{C}e^\perp$, $\phi : \mathbb{C}e \to \mathbb{R}$ is a function, necessarily linear. There is $u \in \mathbb{C}e$ such that $\phi(u) = 0$, and hence $u \in \ker \alpha$. This kernel is a $\mathsf{SO}(1, n)$-invariant space. If it is not trivial, then it has the form $\{au + biu, u \in \mathbb{R}^{n+1}\}$, where a and b are constant. But α induces a form on the quotient $\mathbb{C}^{n+1}/\ker \alpha$ which vanishes for same reasons. Hence $\alpha = 0$.

□

Nijenhuis tensor.

FACT **2.3.** *For $n > 2$, there is no non-trivial anti-symmetric bilinear form $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \to \mathbb{C}^{1+n}$, equivariant under $\mathsf{SO}(1,n)$.*

Proof. Let $b : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a $\mathsf{SO}(1,k)$-invariant anti-symmetric form.

Let us first consider the restriction of b to $\mathbb{R}^{n+1}$. Let $u, v \in \mathbb{R}^{n+1}$ two linearly independent lightlike vectors and w in the orthogonal space $\mathsf{Span}_{\mathbb{C}}(u,v)^{\perp} \cap \mathbb{R}^{n+1}$. Consider H the subgroup of $A \in \mathsf{SO}(1,n)$ such that there exists $\lambda \in \mathbb{R}$, $A(u) = \lambda u$, $A(v) = \lambda^{-1}v$, and $A(w) = w$. By equivariance $b(u,v)$ is fixed by H. But H is too big; its fixed point set is $\mathbb{C}w$. Indeed its action on $\mathsf{Span}_{\mathbb{C}}(u,v,w)^{\perp}$ is equivalent to the action of $\mathsf{SO}(n-2)$ on $\mathbb{R}^{n-2}$. Therefore, $b(u,v) \in \mathbb{C}w$. But, since $n \geq 3$, we have freedom to choose w in $\mathsf{Span}_{\mathbb{C}}(u,v)^{\perp} \cap \mathbb{R}^{n+1}$. Hence, $b(u,v) = 0$. Last, observe that $\mathbb{R}^{n+1}$ is generated by lightlike vectors and hence $b = 0$ on $\mathbb{R}^{n+1}$.

One can prove in the same meaner that $b(u, iv) = 0$, for $u, v \in \mathbb{R}^{n+1}$. It then follows that $b = 0$.

□

Remark 2.4. *[Dimension 3] For $n = 2$, the vector product $\mathbb{R}^{2+1} \times \mathbb{R}^{2+1} \to \mathbb{R}^{2+1}$ is anti-symmetric and $\mathsf{SO}(1,2)$-equivariant. One can equally define a vector product on $\mathbb{C}^{2+1}$ equivariant under $\mathsf{SU}(1,2)$. For given u,v, $u \wedge v$ is such that $\det(w,u,v) = \langle w, u \wedge v \rangle$ (here $\langle, \rangle$ is the Hermitian product on $\mathbb{C}^{2+1}$). Observe nevertheless that this vector product is not equivariant under $\mathsf{U}(1,2)$.*

2.2. **Parabolic subgroups.** By definition, a maximal parabolic subgroup of $\mathsf{SU}(1,n)$ is the stabilizer of a lightlike direction. It is convenient here to consider the form $q_1 = z_0\bar{z}_n + |z_2|^2 + \ldots + |z_{n-1}|^2$. Thus, e_0 is lightlike and the stabilizer P of $\mathbb{C}e_0$ is a semi-direct product $(\mathbb{C}^* \times \mathsf{SU}(n-1)) \ltimes \mathsf{Heis}$. The elements of $\mathbb{C}^* \times \mathsf{SU}(n-1)$ have the form:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix}$$

where, $a \in \mathbb{C}^*$, $A \in \mathsf{SU}(n-1)$

The Heisenberg group is the unipotent radical of P and consists of:

$$\begin{pmatrix} 1 & t & -\frac{\|t\|^2}{2} \\ 0 & 1 & -\bar{t} \\ 0 & 0 & 1 \end{pmatrix}$$

where $t \in \mathbb{C}^{n-1}$.

We see in particular that P is **amenable**.

2.3. **Lightlike geodesic hypersurfaces.** We will meet (especially in §7.1) special complex hypersurfaces $L \subset M$. We say that L is **lightlike** if for any $y \in L$, T_yL is a lightlike complex hyperplane of (T_yM, g_y). The kernel of (T_yL, g_y) defines a complex line sub-bundle N of TL (not necessarily holomorphic). The metric on TL/N is positive.

We say that L is (totally) **geodesic** if for any $u \in TL$, the geodesic γ_u tangent to u, is locally contained in L (there exists ϵ, such that $\gamma_u(]-\epsilon, +\epsilon[) \subset L$). This is equivalent to invariance of TL by the Levi-Civita connection; if X and Y are vector fields defined in a neighborhood of L, and are tangent to L (i.e. $X(y), Y(y) \in TL$, for $y \in L$), then $\nabla_X Y(y) \in TL$, for $y \in L$.

Let us prove in this case that N is parallel along L and thus it is in particular integrable. For this, consider three vector fields X, Y and Z tangent to L, with X tangent to N. We have $\langle X, Z \rangle = 0$, and thus $0 = Y \langle X, Z \rangle = \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle = \langle \nabla_Y X, Z \rangle$ (since X is tangent to N). This is true for any Z, and therefore $\nabla_Y X$ is tangent to N, which means that N is a parallel 2-plane field.

Denote by $\mathcal{N}$ the so defined foliation of L. The leaves are complex curves. Transversally, $\mathcal{N}$ is a **Riemannian foliation**, that is, there is a well defined projected Riemmannian metric on the leaf (local) quotient space $Q = L/\mathcal{N}$. Equivalently, the Lie derivative $L_X h = 0$, where h is the metric restricted to L and X is tangent to N. This is turn is equivalent to that, for any Y invariant under the X-flow, i.e. $[X, Y] = 0$, the product $\langle Y, Y \rangle$ is X-invariant. To check this, observe that $X \langle Y, Y \rangle = \langle \nabla_X Y, Y \rangle = \langle \nabla_Y X, Y \rangle = 0$, since as we have just proved, N is parallel (that is $\nabla_Y X$ is tangent to N).

Corollary 2.5. *Let f be an isometry of M preserving L and fixing a point $x \in L$. Assume $D_x f \in \mathsf{GL}(T_x M)$ is unipotent (i.e. $D_x f - Id$ is nilpotent). Then f preserves (individually) each leaf of $\mathcal{N}$.*

Proof. f acts as an isometry $\hat{f}$ of the (local) quotient space $L/\mathcal{N}$ endowed with its projected Riemannian metric. The derivative $D_{\hat{x}} \hat{f}$ at the projection of x is unipotent. But the orthogonal group $O(n)$ contains no non-trivial unipotent elements. Therefore, $D_{\hat{x}} \hat{f} = Id_{T_{\hat{x}} Q}$, and hence as a Riemannian isometry, $\hat{f} = Id_Q$ (of course, we are tacitly assuming everything connected). $\square$

3. SUBGROUPS OF $\mathsf{U}(1, n)$

The following proposition says roughly that, up to compact objects, a subgroup of $\mathsf{SU}(1, n)$ is either contained in a parabolic group, or conjugate to one of the standard subgroups $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$.

Proposition 3.1. *Let H be a **non-precompact connected** Lie subgroup of $\mathsf{SU}(1, n)$. Then:*

(1) *H is amenable iff it preserves a lightlike hyperplane (that is, by definition, H is contained in a maximal parabolic subgroup).*

(2) *In opposite, if H acts $\mathbb{C}$-irreducibly on $\mathbb{C}^{n+1}$ (there is no non-trivial complex invariant subspace), then H equals $\mathsf{SO}(1, n)$ or $\mathsf{SU}(1, n)$.*

(3) *In the general (intermediate) case, when H is not amenable, it acts $\mathbb{R}$-irreducibly on some non-trivial subspace E, such that:*

    (a) *Either E is totally real, and up to a conjugacy in $SU(1, n)$, $E = \mathbb{R}^{k+1} \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+1} + i\mathbb{R}^{n+1} = \mathbb{C}^{n+1}$, and up to finite index,*

H is a product $C \times SO(1,k)$, for C a pre-compact subgroup acting trivially on E.

(b) *or E is a complex subspace, and up to conjugacy in $\mathsf{SU}(1,n)$, $E = \mathbb{C}^{k+1}$, and H is $C \times \mathsf{SU}(1,k)$, where C is as previously.*

Remark 3.2. *As it will be seen from its proof, this classification naturally generalizes to connected subgroups of all simple Lie groups of rank 1. The proof uses essentially one standard result from simple Lie groups theory, attributed in this form to Mostow [14]. It states that a given Cartan decomposition of a Lie subgroup extends to a Cartan decomposition of the ambient simple Lie group. An essentially geometric (algebraic free) approach is also available in the case of $\mathsf{SO}(1,n)$, see [6, 10].*

– Observe finally that we do not assume H to be closed.

Proof. Let $H \subset \mathsf{SU}(1,n)$ be as in the proposition.

3.0.1. *Hyperbolicity.* Let $G = \mathsf{SU}(1,n)$, $K = \mathsf{U}(1,\mathsf{n}-1)$ and consider $X = G/K = \mathbb{H}^n(\mathbb{C})$ the associated Riemannian symmetric space. We let $\mathsf{SU}(1,n)$ act on the (visual Hadamard) boundary $\partial_\infty X$, which is identified to the space of complex lightlike directions of $\mathbb{C}^{1+n}$. (See [11] to learn about the geometry of $\mathbb{H}^n(\mathbb{C})$).

By definition, a maximal parabolic subgroup P is the stabilizer of a lightlike direction, or equivalently a point of $\partial_\infty X$. From §2.2, P is amenable (the fact that maximal parabolic groups are amenable characterizes rank 1 groups). Therefore, any group fixing a point at $\partial_\infty X$ is amenable.

We have to prove conversely that a non-precompact amenable group fixes some point at $\partial_\infty X$. In general, elements of $\mathsf{SU}(1,n)$ are classified into elliptic, parabolic or hyperbolic. An isometry is elliptic if it fixes some point in X, equivalently it lies in a compact subgroup of $\mathsf{SU}(1,n)$.

A non-elliptic element f has (exactly) one or two fixed points in $\partial_\infty X$. In this case, the centralizer $Cent(f)$ preserves this set of one or two points, and up to index 2, $Cent(f)$ fixes individually these points. One can adapt this argument to check that any solvable group R having a non-elliptic element fixes some point at $\partial_\infty X$ (up to index 2). More generally, if a group L contains such a R as a normal subgroup, then L has the same fixed point (up to finite index).

Any amenable Lie group H is, up to finite index, a semi-direct product of a semi-simple compact group by a solvable one R. If H is non-precompact, then so is R, and hence H fixes a point in $\partial_\infty X$. This completes the proof of (1) in the proposition.

3.0.2. *Non-amenable case.* H is a semi-direct product $(S \times C) \ltimes R$ (up to finite index) where S is semi-simple with no compact factor, C is compact semi-simple, and R is the (solvable) radical. From above, R must be compact, since otherwise H would be contained in a maximal parabolic group (which is amenable). This implies that the semi-direct product is in fact direct (up to finite index). Let us say that H is a product $S \times C'$ where C' is precompact. We now investigate S and come back later on to $S \times C'$. Observe first that S is simple. Indeed, if $S = S_1 \times S_2$,

then any non-elliptic $f \in S_2$, will centralize S_1, which implies S_1 has a fixed point at $\partial_\infty X$ and hence amenable.

3.0.3. *Simple Lie subgroups.* In order to understand the geometry of S, we investigate the symmetric space X (rather than its boundary as in the previous case). Let p be a base point, say that fixed by the maximal compact K. We have a Cartan decomposition of the Lie algebra of G: $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is the Lie subalgebra of K, and $\mathfrak{p}$ is (the unique) K-invariant supplementary space of $\mathfrak{k}$ in $\mathfrak{g}$. Geometrically, $\mathfrak{p}$ is identified with $T_p M$, and if $u \in p$, the orbit $\exp(tu).p$ is the geodesic of X determined by u.

More other properties are: K acts irreducibly on $\mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

If $S \subset G$ is a simple Lie subgroup, then Mostow's theorem [14] states, up to a conjugacy in G (or equivalently a modification of the base point), we get a Cartan decomposition by taking intersection: $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{p} \oplus \mathfrak{s} \cap \mathfrak{k}$.

Observe that $\mathfrak{s} \cap \mathfrak{p}$ determines $\mathfrak{s}$, since $\mathfrak{s} \cap \mathfrak{k} = [\mathfrak{s} \cap \mathfrak{p}, \mathfrak{s} \cap \mathfrak{p}]$.

In our case, $\mathfrak{p} = T_p X$ is identified to $\mathbb{C}^n$. The subspace $T = \mathfrak{s} \cap \mathfrak{p}$ is either complex or totally real, since $T \cap iT$ is $S \cap K$-invariant, and this last group acts irreducibly on E. Now, $K = \mathsf{U}(1, \mathsf{n} - 1)$ acts transitively on the set of totally real (resp. complex) planes of a given dimension k. Thus, up to conjugacy, $\mathfrak{s} \cap \mathfrak{p}$ is the canonical $\mathbb{R}^k$ or $\mathbb{C}^k$ in $\mathbb{C}^n$. Since $\mathfrak{s} \cap \mathfrak{p}$ determines completely $\mathfrak{s}$, it follows that S is conjugate to one of the standard subgroups $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$.

End. We have thus proved (2) and (3) of the proposition at the group level: H is conjugate in $\mathsf{SU}(1, n)$ to $S \times C$, with $S = \mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$. Since the precompact factor C commutes with the non-compact S, it is contained in $\mathsf{SO}(n - k)$ or $\mathsf{SU}(n - k)$, respectively. In particular H preserves $\mathbb{R}^{k+1}$ or $\mathbb{C}^{k+1}$. This completes the proof of the proposition. $\square$

Corollary 3.3. *Let L be a subgroup of $\mathsf{SU}(1, n)$ acting $\mathbb{C}$-irreducibly on $\mathbb{C}^{n+1}$. Then, the identity component of its Zariski closure equals $\mathsf{SU}(1, n)$ or $\mathsf{SO}(1, n)$. If the identity component L^0 is not trivial, L itself equals $\mathsf{SU}(1, n)$ or $\mathsf{SO}(1, n)$.*

We have more:

Corollary 3.4. *Let L be a subgroup of $\mathsf{SU}(1, n)$. If L is non-amenable, then its Zariski closure contains a copy of $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$ for some $k > 0$. If furthermore L^0 is non- pre-compact, then L itself contains $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$. (Of course $\mathsf{SO}(1, k) \subset \mathsf{SU}(1, k)$, but we prefer our formulation here for a later use).*

Proof. The only one point to justify is that if L^0 is non-precompact, then it is non-amenable (assuming L non-amenable). If not the Zariski closure L^{Zar} will normalize L^0. But $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$ normalize no amenable non-compact connected subgroup of $\mathsf{SU}(1, n)$. $\square$

3.1. **Subgroups of** $\mathsf{U}(1, \mathsf{n})$**.** Consider now L a subgroup of $\mathsf{U}(1, \mathsf{n}) = \mathsf{U}(1) \times \mathsf{SU}(1, \mathsf{n})$, and denote by π the projection $\mathsf{U}(1, \mathsf{n}) \to \mathsf{U}(1)$ having $\mathsf{SU}(1, n)$ as kernel. We have an exact sequence $1 \to (L \cap \mathsf{SU}(1, n)) \to L \to \pi(L) \to 1$.

If $L \cap \mathsf{SU}(1, n) = 1$, then π sends injectively L in $U(1)$ and hence L is abelian, and can not act irreducibly. One can in fact prove that L acts irreducibly on $\mathbb{C}^{n+1}$ iff $L \cap \mathsf{SU}(1, n)$ do.

Corollary 3.5. *Let L be a subgroup of $\mathsf{U}(1,\mathsf{n})$ acting irreducibly on $\mathbb{C}^{n+1}$. Then, its Zariski closure contains $\mathsf{SO}(1,n)$ or $\mathsf{SU}(1,n)$.*

If furthermore $L^0 \neq 1$, and its Zariski closure does not contain $\mathsf{SU}(1,n)$, then, either, L^0 equals $\mathsf{SO}(1,n)$, or $L^0 \supset \mathsf{U}(1)$.

Proof. The first part is obvious from the discussion above, let us prove the second one. In this case L is a subgroup of $\mathsf{U}(1) \times \mathsf{SO}(1,\mathsf{n})$.

– If $L^0 \cap \mathsf{SO}(1,n) \neq 1$, then its equals $\mathsf{SO}(1,n)$ by irreducibility of $L \cap \mathsf{SO}(1,n)$, in particular $L \supset \mathsf{SO}(1,n)$. Furthermore, if a product ab, $a \in \mathsf{U}(1)$, $b \in \mathsf{SO}(1,n)$ belongs to L, then $b \in L$, that is $\pi(L) = L \cap \mathsf{U}(1)$. This last group is either $\mathsf{U}(1)$ (in which case $L = \mathsf{U}(1) \times \mathsf{SO}(1,\mathsf{n})$), or totally discontinuous, in which case $L^0 = \mathsf{SO}(1,n)$.

– Assume now that $L^0 \cap \mathsf{SO}(1,n) = 1$. Let $l^t = a^t b^t$ be a one parameter group in L^0, and $c \in L \cap \mathsf{SO}(1,n)$. The commutator $[c, a^t b^t]$ equals $[c, b^t]$. This is a one parameter group in $L \cap \mathsf{SO}(1,n)$, and hence must be trivial. But since we can choose c in a Zariski dense set in $\mathsf{SO}(1,n)$, the one parameter group b^t must be trivial. This means that $l^t \in \mathsf{U}(1)$, and hence $L^0 \supset \mathsf{U}(1)$. $\qquad\square$

4. Proof of Theorem 1.1

Let (M, J, g) be an almost complex Hermite-Lorentz space on which a group G acts transitively with $\mathbb{C}$-irreducible isotropy.

Let p be a base point of M, and call H its isotropy group in G. The tangent space $T_p M$ is identified to $\mathbb{C}^{1+n}$ and H to a subgroup of $\mathsf{U}(1,\mathsf{n})$. By hypothesis H acts $\mathbb{C}$-irreducibly on $\mathbb{C}^{1+n}$.

The first part of Theorem 1.1, that is J is integrable and g is Kähler will be proved quickly. Indeed, by Corollary 3.5 the Zariski closure of H (in $\mathsf{U}(1,\mathsf{n})$) contains $\mathsf{SO}(1,n)$.

Kähler Character. Let ω be the Kähler form of g. Its differential at p, $\alpha = d\omega_p$ is an H-invariant 3-form on $\mathbb{C}^{n+1}$. By Corollary 3.5, α is $\mathsf{SO}(1,n)$-invariant. By Fact 2.2, $\alpha = 0$, that is, M is Kähler.

Integrability of the complex structure. The (Nijenhuis, obstruction to) integrability tensor at p is an $\mathbb{R}$-anti- symmetric bilinear vectorial form $\mathbb{C}^{1+n} \times \mathbb{C}^{1+n} \to \mathbb{C}^{1+n}$. The same argument, using Fact 2.3 yields its vanishing, that is J is integrable.

Remark 4.1. *Observe that we need $\dim M > 3$ in order to apply Facts 2.2 and 2.3.*

4.0.1. *Classification.* The rest of this section is devoted to the identification of M as one of theses spaces: $\mathsf{Mink}_{n+1}(\mathbb{C})$, $\mathsf{dS}_{n+1}(\mathbb{C})$, $\mathsf{AdS}_{n+1}(\mathbb{C})$, $\mathbb{C}\mathsf{dS}_{n+1}$ or $\mathbb{C}\mathsf{AdS}_{n+1}$ (up to a central cyclic cover in some cases).

4.0.2. *The identity component H^0.* Let us prove that $H^0 \neq 1$. If not G is a covering of M, in particular $T_p M \cong \mathbb{C}^{n+1}$ is identified to the Lie algebra $\mathfrak{g}$, and H acts by conjugacy. The bracket is an $\mathbb{R}$-bilinear form like the integrability tensor,

and hence vanishes, that is $\mathfrak{g}$ is abelian. This contradicts the fact that H acts non-trivially by conjugacy. Therefore H^0 is non-trivial. Applying 3.1, we get three possibilities:

1. The Zariski closure of H contains $\mathsf{SU}(1, n)$
2. $H^0 = \mathsf{SO}(1, n)$
3. H^0 contains $\mathsf{U}(1)$.

4.1. **Case 1: the Zariski closure of H contains $\mathsf{SU}(1, n)$.** The holomorphic sectional curvature at p is an H-invariant function on the open subset in $\mathbb{P}^n(\mathbb{C})$ of non-lightlike $\mathbb{C}$-lines of $\mathbb{C}^{n+1}$. But $\mathsf{SU}(1, n)$ acts transitively on this set. It follows that this holomorphic sectional curvature is constant. Therefore M is a Kähler-Lorentz manifold of constant holomorphic sectional curvature, and thus M is locally isometric to one the universal spaces $\mathsf{Mink}_{n+1}(\mathbb{C})$, $\mathsf{dS}_{n+1}(\mathbb{C})$ or $\mathsf{AdS}_{n+1}(\mathbb{C})$ [12, 4, 15]. We will see below (§4.4) that M is (globally) isometric to $\mathsf{Mink}_{n+1}(\mathbb{C})$, $\mathsf{dS}_{n+1}(\mathbb{C})$ or to a cover of $\mathsf{AdS}_{n+1}(\mathbb{C})$.

4.2. **Case 2: $H^0 = \mathsf{SO}(1, n)$.** The final goal here is to show that M is $\mathsf{Mink}_{n+1}(\mathbb{C})$. First we replace $M = G/H$ by G/H^0 which enjoys all the properties of the initial M. In other words, we can assume $H = H^0 = \mathsf{SO}(1, n)$.

Invariant distributions. $\mathsf{SO}(1, n)$ acts $\mathbb{C}$-irreducibly but not $\mathbb{R}$-irreducibly. We set a G-invariant distribution S on M as follows. Define S to be equal to $\mathbb{R}^{n+1}$ at p. For $x = gp$, define $S_x = D_p g(S_p)$. This does not depend on the choice of g since S_p is H-invariant.

The orthogonal distribution $S^\perp$ is in fact determined similarly by means of the H-invariant space $i\mathbb{R}^{n+1}$.

Integrability of distributions. The obstruction to the integrability of S is encoded in the anti–symmetric **Levi** form $II : S \times S \to S^\perp$, where $II(X, Y)$ equals the projection on $S^\perp$ of $[X, Y]$, for X and Y sections of S. At p, we get an anti-symmetric bilinear form $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, equivariant under the $\mathsf{SO}(1, n)$-action. By Fact 2.3, this must vanish and hence S and analogously $S^\perp$ are integrable.

We denote by $\mathcal{S}$ and $\mathcal{S}^\perp$ the so defined foliations.

Observe that since G preserves these foliations, then each leaf of them is homogeneous. If F is such a leaf, $x, y \in F$, and $g \in G$ is such that $y = gx$, then g sends the distribution at x to that at y, and hence, $gF = F$.

Leaves of $\mathcal{S}$ or $\mathcal{S}^\perp$ are (real) homogeneous Lorentz manifolds with (maximal) isotropy $\mathsf{SO}(1, n)$. They are easy to handle du to the following fact, the proof of which is standard:

FACT **4.2.** *Let $F = A/B$ be a homogeneous Lorentz manifold of dimension $n + 1$ such that the action of B on the quotient $\mathfrak{a}/\mathfrak{b}$ of Lie algebras is equivalent to the standard action of $\mathsf{SO}(1, n)$ on $\mathbb{R}^{n+1}$. Then M has constant sectional curvature. If M is flat, then $M = \mathsf{Mink}_{n+1}$ and $A = \mathsf{SO}(1, n) \ltimes \mathbb{R}^{n+1}$. If M has positive curvature then it equals dS_{n+1} and $A = \mathsf{SO}(1, n + 1)$. Finally, in the negative curvature case, M is a cover of AdS_{n+1}, and A covers $\mathsf{SO}(2, n)$.*

Now if leaves of $\mathcal{S}$ are not flat, then L is $\mathsf{SO}(1, n + 1)$ in case of positive curvature, and $L = \mathsf{SO}(2, n)$ in the negative curvature case. Consider the (local)

quotient space $Q = M/\mathcal{S}$, it has dimension $n + 1$. The group L acts by fixing F, seen as a point of Q. But L ($= \mathsf{SO}(1, n+1)$ or $\mathsf{SO}(2, n)$) has no linear representation of dimension $n + 1$. Therefore, it acts trivially on the tangent space $T_F Q$. But this tangent space is identified to $S_p^\perp$. There, $\mathsf{SO}(1, k)$, as an isotropy subgroup, acts non-trivially. This contradiction implies that the leaves of $\mathcal{S}$ and analogously $\mathcal{S}^\perp$ are flat.

Now, we need to study further the geometry of our foliations. We claim that their leaves are in fact totally geodesic. Indeed, there is a symmetric Levi form measuring the obstruction of geodesibility. More exactly, it is given by $II^*(X, Y) = $ the orthogonal projection of the covariant derivative $\nabla_X Y$. From 2.3, since equivariant symmetric bilinear forms do not exist, the foliations $\mathcal{S}$ and $\mathcal{S}^\perp$ are geodesic. It is classical that the existence of a couple of orthogonal geodesic foliations implies a metric splitting of the space (see for example [12] about the proof of the de Rham decomposition Theorem). That is, at least locally, M is isometric to the product $\mathcal{S}_p \times \mathcal{S}_p^\perp$. In particular, M is a flat Hermite-Lorentz manifold, that is M is locally isometric to $\mathsf{Mink}_{1+n}(\mathbb{C})$.

One can moreover prove that G is a semi-direct product $\mathsf{SO}(1, n) \ltimes \mathbb{R}^{n+1}$ and $M = \mathsf{Mink}_{n+1}(\mathbb{C})$ (see §4.4 below for details in a similar situation).

4.3. **Case 3:** $\mathsf{U}(1) \subset \mathsf{H} \subset \mathsf{U}(1) \times \mathsf{SO}(1, n)$.

The goal here is to prove that M is flat or isomorphic to one of the two spaces $\mathbb{C}\mathsf{dS}_n = \mathsf{SO}(1, n+1)/\mathsf{SO}(1, n-1) \times \mathsf{SO}(2)$ or $\mathbb{C}\mathsf{AdS}_n = \mathsf{SO}(3, n-1)/\mathsf{SO}(2) \times \mathsf{SO}(1, n-1)$.

The crucial observation is that M is a (pseudo-Riemannian) symmetric space, that is there exists $f \in G$, such that $D_p f = -Id_{T_p M}$. Indeed, $-Id \in \mathsf{U}(1)$.

There is a de Rham decomposition of M into a product of a flat factor and irreducible symmetric spaces. In our case, there exists a subgroup of the isotropy that acts irreducibly. It follows that M is either flat, or irreducible. There is nothing to prove in the first case, we will therefore assume that M is irreducible. We can also assume that G is the full isometry group of M. It is known that isotropy groups of symmetric spaces have finitely many connected components. Thus, up to a finite cover, H must be $\mathsf{U}(1) \times \mathsf{SO}(1, n)$.

Consider a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$, where $\mathfrak{p}$ is identified with $\mathbb{C}^{n+1}$. Consider the bracket $[,] : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{h} = so(1, n) + u(1)$.

Its second component is a $\mathsf{SO}(1, n)$-invariant anti-symmetric scalar bilinear form $\alpha : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to u(1) = \mathbb{R}$. By Fact 2.1, α vanishes on $\mathbb{R}^{n+1}$, that is if $X, Y \in \mathbb{R}^{n+1}$, then $[X, Y] \in so(1, n)$. On the other hand, $SO(1, n)$ preserves $\mathbb{R}^{n+1}$, and hence if $T \in so(1, n)$ and $Z \in \mathbb{R}^{n+1}$, then $[T, Z] \in \mathbb{R}^{n+1}$.

Summarizing, if $X, Y, Z \in \mathbb{R}^{n+1}$, then $[[X, Y], Z] \in \mathbb{R}^{n+1}$. It is known that this implies that $\mathbb{R}^{n+1}$ determines a totally geodesic submanifold, say F. It has dimension $n + 1$ and isotropy $\mathsf{SO}(1, n)$. From Fact 4.2, F is a Lorentz space of constant curvature. It can not be flat since in that case, the bracket $[,]$ vanishes on $\mathbb{R}^{n+1}$, but this implies it vanishes on the whole of $\mathbb{C}^{n+1}$. So M is the de Sitter or the anti de Sitter space.

The two cases are treated identically, let us assume $F = \mathsf{dS}_{n+1}$. Its isometry group $\mathsf{SO}(1, n + 1)$ is thus contained in G.

The goal now is to show that $G = \mathsf{SO}(1, n+2)$. For this, we consider the homogeneous space $N = G/\mathsf{SO}(1, n+1)$. Since we know the dimensions of $G/\mathsf{U}(1) \times \mathsf{SO}(1, \mathsf{n})$ and $\mathsf{SO}(1, n+1)/\mathsf{SO}(1, n)$, we can compute that of $G/\mathsf{SO}(1, n+1)$, and find it equals $n + 2$.

Thus $\mathsf{SO}(1, n + 1)$ has an isotropy representation ρ in the $(n + 2)$-dimensional space E, the tangent space at the base point of N. In a direct way, we prove that this is the standard representation of $\mathsf{SO}(1, n+1)$ in $\mathbb{R}^{n+2}$. For this, we essentially use that ρ restricted to $\mathsf{SO}(1, n)$ is already known.

From Fact 4.2, G is $\mathsf{SO}(1, n+2)$ or $\mathsf{SO}(2, n+1)$. Again, in a standard way, we exclude the case $G = \mathsf{SO}(2, n + 1)$ (just because it does not contain the isotropy $\mathsf{U}(1) \times \mathsf{SO}(1, \mathsf{n})$). We have thus proved that $M = \mathsf{SO}(1, n+2)/\mathsf{SO}(1, n) \times \mathsf{SO}(2)$.

4.4. **Global symmetry.** It was proved along the investigation of cases (2) and (3) that M is (globally) symmetric. It remains to consider the first case, that is when the Zariski closure of H contains $\mathsf{SU}(1, d)$. Exactly as in the other (weaker) cases we have $\mathsf{SU}(1, n) \subset H$, or $\mathsf{U}(1) \subset \mathsf{H}$. The last case is globally symmetric, let us focus on the first one.

M is locally isometric to a universal space X of constant holomorphic sectional curvature. We let the universal cover $\tilde{G}$ act on X. Since the isotropy $\mathsf{SU}(1, n)$ of M has codimension 1 in the isotropy $\mathsf{U}(1, \mathsf{n})$ of X, $\tilde{G}$ has codimension 1 in $\mathsf{Iso}(X)$. However, if X is not flat, $\mathsf{Iso}(X)$ is a simple Lie group with no codimension 1 subgroup, since it is not locally isomorphic to $\mathsf{SL}_2(\mathbb{R})$. Therefore, $\dim G = \dim(\mathsf{Iso}(X))$, and in particular the isotropy of M is $\mathsf{U}(1, \mathsf{n})$, in particular M is (globally) symmetric.

Let us now consider the case of $X = \mathsf{Mink}_{n+1}(\mathbb{C})$. Thus $\mathsf{Iso}(X) = \mathsf{U}(1, \mathsf{n}) \ltimes \mathbb{C}^{\mathsf{n}+1}$. Since $\tilde{G}$ acts (locally) transitively, it must contain some translation, that is $A = \tilde{G} \cap \mathbb{C}^{n+1} \neq 1$. The subgroup A is $\mathsf{SU}(1, n)$-invariant. By irreducibility, $A = \mathbb{C}^{n+1}$, and thus $\tilde{G} = \mathsf{SU}(1, n) \ltimes \mathbb{R}^{n+1}$. The group G is a quotient of $\tilde{G}$ by a discrete central subgroup. But $\tilde{G}$ has no such a subgroup. It then follows that $M = \mathsf{SU}(1, n) \ltimes \mathbb{C}^{n+1}/\mathsf{SU}(1, n)$, and hence $M = \mathsf{Mink}_{n+1}(\mathbb{C})$.

This finishes the proof of Theorem 1.1. □

5. PROOF OF THEOREM 1.2: PRELIMINARIES

Let M be a Hermite-Lorentz space homogeneous under the holomorphic isometric action of a semi-simple Lie group G of finite center.

For x in M, we denote by G_x its stabilzer in G, $\mathfrak{g}$ the Lie subalgebra of G, and $\mathfrak{g}_x$ the Lie subalgebra of G_x. The goal in this section is to show that $\mathfrak{g}_x$ is big; it contains nilpotent elements.

5.1. **Stable subalgebras, actions on surfaces.**

5.1.1. *Notations.*

An element X in the Lie algebra $\mathfrak{g}$ is $\mathbb{R}$-**split** (or **hyperbolic**) if ad_X is diagonalizable with real eigenvalues. Thus $\mathfrak{g} = \Sigma_\alpha \mathfrak{g}^\alpha$, where α runs over the set of eigenvalues of Ad_X. Let

$$W_X^s = \Sigma_{\alpha(X)<0}\mathfrak{g}^\alpha, \ W_X^u = \Sigma_{\alpha(X)>0}\mathfrak{g}^\alpha \text{ and } W_X^{s0} = \Sigma_{\alpha(X)\leq0}\mathfrak{g}^\alpha$$

be respectively, the **stable**, unstable and **weakly-stable** sub-algebras of X. We have in particular $\mathfrak{g} = W_X^{s0} \oplus W_X^u$

The stable and unstable subalgebras are nilpotent in the sense that, for $Y \in W_X^s$ (or W_X^u), Ad_Y is a nilpotent element of $\mathsf{Mat}(\mathfrak{g})$, equivalently, $\exp Y$ is a unipotent element of $\mathsf{GL}(\mathfrak{g})$. It then follows that if $\mathfrak{h}$ is an Ad_Y-invariant subspace, then $\exp Y$ determines a unipotent element of $\mathsf{GL}(\mathfrak{g}/\mathfrak{h})$.

It is known that W_X^s and W_X^u are isomorphic; an adapted Cartan involution sends one onto the other. In particular the codimension of W_X^{s0} in G equals the dimension of W_X^s. Assuming (to simplify) that G is simply connected, it acts on G/L, where L is the Lie subgroup determined by W_X^{s0}. Then, $\dim(G/L) = \dim W_X^s$; summarizing:

FACT **5.1.** *If for some* X, $\dim W_X^s = 2$, *then* G *acts on a surface, that is there exists a* G *homogeneous space of (real) dimension 2.*

Semi-simple Lie groups satisfying the fact can be understood:

FACT **5.2.** *A semi-simple Lie group* G *acting (faithfully) on a surface is locally isomorphic to* $\mathsf{SL}_2(\mathbb{R}), \mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R}), \mathsf{SL}_2(\mathbb{C})$ *or* $\mathsf{SL}_3(\mathbb{R})$. *(it is well known that acting on dimension 1 implies being locally isomorphic to* $\mathsf{SL}_2(\mathbb{R})$*).*

Proof. This can be derived from the classification theory of simple Lie groups. One starts observing that the problem can be complexified, that is complexified groups act on complex surfaces; algebraically, they possess codimension 2 complex subalgebras in their complexified algebras. Let the isotropy group have a Levi decomposition $S \ltimes R$. Since S has a faithful 2-dimensional representation, it is locally isomorphic to $\mathsf{SL}_2(\mathbb{C})$. If S' is the affine subgroup of $\mathsf{SL}_2(\mathbb{C})$, then $S' \ltimes R$ is solvable and has codimension 3 in G. Therefore, a Borel group of G has codimenion ≤ 3. This implies that the cardinality of the set of positive roots is ≤ 3 (for any associated root system). With this restriction, one observes that the (complex) rank is ≤ 2, and consult a list of root systems to get our mentioned groups. □

Example 5.3. *These actions on surfaces are in fact classified (up to covers). We have the projective action of* $\mathsf{SL}_2(\mathbb{R})$ *(resp.* $\mathsf{SL}_3(\mathbb{R})$*) on the real projective space* $\mathbb{P}^1(\mathbb{R})$ *(resp.* $\mathbb{P}^2(\mathbb{R})$*). There is also the action of* $\mathsf{SL}_2(\mathbb{C})$ *on the Riemann sphere* $\mathbb{P}^1(\mathbb{C})$*, and the product action of* $\mathsf{SL}_2(\mathbb{R})^2$ *on* $\mathbb{P}^1(\mathbb{R})^2$*. Finally, the hyperbolic, de Sitter and (the punctured) affine planes are obtained as quotients of* $\mathsf{SL}_2(\mathbb{R})$ *by suitable one parameter groups. It is finally possible, in some cases, to take covers or quotients by discrete (cyclic) groups of the previous examples.*

5.2. **Non-precompactness.**

FACT **5.4.** *Let* $M = G/H$ *be a homogeneous space where* G *is semi-simple of finite center and acts non-properly (and faithfully) on* M. *Then* H *seen as the isotropy group of a base point, say* p, *in not precompact in* $\mathsf{GL}(T_pM)$.

Proof. By contradiction, if H is precompact then it preserves a Euclidean scalar product on T_pM, and hence G preserves a Riemannian metric on G/H (of course H is closed in G since it equals the isotropy of p). Let us show that H is compact. Indeed, let L be the isometry group of the Riemannian homogeneous space $X =$

G/H, and K its isotropy group in L, which is compact since the homogeneous space is of Riemannian type. Now, $H = K \cap G$. It is known that a semi-simple Lie group of finite center is closed in any Lie group where it lies. Therefore, H is a closed subgroup of K, and hence compact. $\qquad\square$

5.3. **Dynamics vs Isotropy.**

For V be a subspace (in general a subalgebra) of $\mathfrak{g}$, its **evaluation** at x is the tangent subspace $V(x) = \{v(x) \in T_x M, v \in V\}$ (here v is seen as a vector field on M).

FACT **5.5.** *(Kowalsky* [13]*) There exists $X \in \mathfrak{g}$ (depending on x), an $\mathbb{R}$-split element, such that $W_X^s(x)$ is isotropic.*

In the sprit of Kowalsky's proof, we have the following precise statement.

FACT **5.6.** *If the stabilizer algebra $\mathfrak{g}_x$ contains a nilpotent Y, then any $\mathbb{R}$-split element X of an sl_2-triplet $\{X, Y, Z\}$ (i.e. $[X, Y] = -Y, [X, Z] = +Z$, and $[Y, Z] = X$) satisfies that $(\mathbb{R}X \oplus W_X^s)(x)$ is isotropic.*

Proof. Let L be the subgroup of G determined by $\{X, Y, Z\}$. It is isomorphic up to a finite cover to $\mathsf{SL}_2(\mathbb{R})$. A Cartan KAK decomposition yields $\exp(tY) = L_t \exp(s(t)X)R_t$, where L_t and R_t belong to the compact $\mathsf{SO}(2)$.

Write $X_t = Ad(R_t)(X)$ (for t fixed), it generates the one parameter group $s \to \exp sX_t = (R_t)^{-1} \exp sX R_t$. Thus, $\exp(tY) = D_t \exp s(t)X_t$, where $D_t = L_t R_t \in \mathsf{SO}(2)$.

Let u_t^α and v_t^β be the eigenvectors for ad_{X_t} (acting on $\mathfrak{g}$) associated to two roots α and β. Since $\exp tY$ preserves $\langle, \rangle$, we have

$$\langle u_t^\alpha, v_t^\beta \rangle = \langle \exp tY u_t^\alpha, \exp tY v_t^\beta \rangle = e^{s(t)(\alpha+\beta)} \langle D_t u_t^\alpha, D_t v_t^\beta \rangle$$

• The point now is that X_t converges to X, when $t \to \infty$. This follows from a direct computation of the KAK decomposition in $\mathsf{SL}_2(\mathbb{R})$. It follows for the eigenvectors of ad_X that $\langle u^\alpha, v^\beta \rangle$ is dominated by a function of the form $e^{s(\alpha+\beta)}$. Thus $\langle u^\alpha, u^\beta \rangle = 0$, whence $\alpha < 0$, and $\beta \leq 0$. In particular $W_X^s(x)$ is isotropic and orthogonal to $X(x)$.

• It remains to verify that $X(x)$ is isotropic. For this, consider M' the $\mathsf{SL}_2(\mathbb{R})$-orbit (of the base point of M). If the isotropy of the $\mathsf{SL}_2(\mathbb{R})$-action on M' is exactly generated by $\exp tY$, then M' is the affine punctured plane $\mathbb{R}^2 - \{0\}$. The unique $\mathsf{SL}_2(\mathbb{R})$-invariant (degenerate) metric is 0 or a multiple of $d\theta^2$ in polar coordinates (θ, r), and therefore $X(x)$ is isotropic since it coincides with $\frac{\partial}{\partial r}$. In the case where the isotropy is bigger, the metric on M' must vanish.

$\qquad\square$

FACT **5.7.** *$\mathfrak{g}_x$ contains a nilpotent element unless for any X as in Fact 5.5, $\dim W_X^s \leq 2$. In particular, if G does not act (locally) on surfaces, then $\mathfrak{g}_x \cap W_X^s \neq 0$, and $\mathfrak{g}_x$ contains nilpotent elements.*

Proof. Consider the evaluation map $V \in W_X^s \to V(x) \in T_x M$. Its image is isotropic, and thus has at most dimension 2 (**since the metric is Hermite-Lorentz**). Its kernel $\mathfrak{I}_x = \mathfrak{g}_x \cap W_X^s$ consists of nilpotent elements and satisfies

$\dim(\mathfrak{I}_x) \geq \dim(W^s_X) - 2$, which is positive if for some X, $\dim W^s_X > 2$, in particular if G does not act (locally) on surfaces by Fact 5.1.

$\square$

6. Proof of Theorem 1.2: non-amenable isotropy case

Let (M, J, g) and G be as in Theorem 1.2, that is G is simple, not locally isomorphic to $\mathsf{SL}_2(\mathbb{R}), \mathsf{SL}_2(\mathbb{C})$ or $\mathsf{SL}_3(\mathbb{R})$, and acts non-properly by preserving the almost complex and Hermite-Lorentz structures on M.

Theorem 1.2 states that M is exactly as in Theorem 1.1, that is M is a global symmetric Kähler-Lorentz space. It is thus natural to prove Theorem 1.2 by showing that its hypotheses imply those of Theorem 1.1, i.e. if the acting group is simple, and the action in non-proper, then the isotropy is irreducible.

As previously, this isotropy H is a subgroup of $\mathsf{U}(1, \mathsf{n})$. Let us assume by contradiction that H does not act irreducibly on $\mathbb{C}^{n+1}$.

By Fact 5.7, the identity component H^0 is non-precompact, which allows us using Proposition 3.1.

The goal of the present section is to get a contradiction assuming H is non-irreducible and non-amenable. The amenable case will be treated in the next section.

By Proposition 3.1, up to conjugacy, H preserves $\mathbb{C}^{k+1}$, for some $1 < k < n$, and its non-compact semi-simple part is $\mathsf{SO}(1, k)$ or $\mathsf{SU}(1, k)$. Let us assume here that it is $\mathsf{SO}(1, k)$, since the situation with $\mathsf{SU}(1, k)$ is even more rigid!

Integrability of distributions. As in §4.2 during the proof of Theorem 1.1, we define a G-invariant distribution S on M, by declaring $S_p = \mathbb{C}^{k+1}$.

We first show that the distribution $S^{\perp}$ is integrable. The obstruction to its integrability is encoded in the anti–symmetric Levi form $II : S^{\perp} \times S^{\perp} \to S$, where $II(X, Y)$ equals the projection on S of $[X, Y]$, for X and Y sections of $S^{\perp}$.

At p, we get a skew-symmetric form $II : \mathbb{C}^{n-k} \times \mathbb{C}^{n-k} \to \mathbb{C}^{k+1}$, equivariant under the actions of $\mathsf{SO}(1, k)$ on $\mathbb{C}^{n-k}$ and $\mathbb{C}^{k+1}$ respectively. Observe however that $\mathsf{SO}(1, k)$ acts trivially on $\mathbb{C}^{n-k}$. Therefore, the image of II in $\mathbb{C}^{k+1}$ consists of fixed points, which is impossible since $\mathsf{SO}(1, k)$ has no such points (in $\mathbb{C}^{k+1}$).

• We denote by $\mathcal{S}^{\perp}$ the so defined foliation. Before going further, let us notice that $\mathsf{SO}(1, k)$ acts trivially on the leaf $\mathcal{S}^{\perp}_p$. Indeed, it preserves the induced (positive definite) Hermitian metric on $\mathcal{S}^{\perp}_p$. But, the derivative action of $\mathsf{SO}(1, k)$ on $T_p\mathcal{S}^{\perp}_p = S^{\perp}_p$ is trivial, and hence $\mathsf{SO}(1, k)$ acts trivially on $\mathcal{S}^{\perp}_p$.

• Let us now study S itself from the point of view of integrability. We consider a similar Levi form. This time, we get an equivariant form $\mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \to \mathbb{C}^{n-k}$. Since $\mathsf{SU}(1, k)$ acts trivially on $\mathbb{C}^{n-k}$, this form is $\mathsf{SO}(1, k)$-invariant. However, up to a constant, the Kähler form ω is the unique scalar $\mathsf{SO}(1, k)$-invariant form (Fact 2.1). It follows that there exists $v \in \mathbb{C}^{n-k}$, such that $II = \omega v$. This determines a vector field V on M such that $V(p) = v$, and a distribution $S' = S \oplus \mathbb{R}V$.

Of course, it may happen that $V = 0$, in which case S is integrable.

We claim that S' is integrable. Indeed, by construction, the bracket $[X, Y]$ of two sections of S belongs to S'. It remains to consider a bracket of the form $[V, X]$. As previously, consideration of an associated Levi form leads us to the following

linear algebraic fact: an $\mathsf{SO}(1,k)$-invariant bilinear form $\mathbb{C}^{k+1} \times \mathbb{R} \to \mathbb{R} \times \mathbb{C}^{n-k-1}$ must vanish. Its proof is straightforward.

Contradiction. Now, we have two foliations $\mathcal{S}^\perp$ and $\mathcal{S}'$. The group G acts by preserving each of them. It also acts on Q, the (local) quotient space of $\mathcal{S}'$, i.e. the space of its leaves. However, $\mathsf{SO}(1,k)$ acts trivially on Q. Indeed, as we have seen, $\mathsf{SO}(1,k)$ acts trivially on $\mathcal{S}_p^\perp$, and this is a kind of cross section of the quotient space Q; say, $\mathcal{S}_p^\perp$ meets an open set of leaves of $\mathcal{S}$. On this open set, $\mathsf{SO}(1,k)$ acts trivially. By analyticity, $\mathsf{SO}(1,k)$ acts trivially on Q.

Thus the G-action on Q has a non-trivial connected Kernel, and is therefore trivial since G is a simple Lie group. This means Q is reduced to one point, that is $k = n$, which contradicts our hypothesis that H is not irreducible.

7. Proof of Theorem 1.2 in the amenable case

We continue the proof of Theorem 1.2 started in the previous section, with here the hypothesis (by contradiction) that the isotropy H is amenable. The idea of the proof is as follows. To any x we associate, in a G-equivariant meaner, F_x, the **asymptotic leaf** of the isotropy group G_x at x. It is a (complex) codimension 1 lightlike geodesic hypersurface in M. This is got by widely general considerations (see for instance [8, 16, 17]). Next, the point is to check that $x \to F_x$ is a foliation: $F_x \cap F_y \neq \emptyset \implies F_x = F_y$. Its (local) quotient space would be a (real) surface with a G-action, which is impossible by hypotheses of Theorem 1.1; leading to that H can not be amenable.

7.0.1. *Notation and Dimension.* For x in M, we denote by G_x its stabilzer in G, $\mathfrak{g}_x$ its Lie sub-algebra, and $\mathfrak{I}_x = \mathfrak{g}_x \cap W_X^s$, where X is a fixed $\mathbb{R}$-split element as in Fact 5.6 (associated to x).

FACT **7.1.** *1)* $X(x) \neq 0$.
  2) $\dim(\mathfrak{I}_x) \geq 2$.

Proof. 1) By contradiction, if $X \in \mathfrak{g}_x$, then one first proves directly (an easy case of Fact 5.5) that also the unstable W_X^u is isotropic at x and for the same reasons $W_X^u \cap \mathfrak{g}_x \neq 0$, say $Z \in W_X^u \cap \mathfrak{g}_x$, and also $Y \in W_X^s \cap \mathfrak{g}_x$.

Thus $\{X, Y, Z\} \subset \mathfrak{g}_x$. Now, the isotropy G_x embeds in the unitary group $\mathsf{U}(T_xM, \langle, \rangle_x)$ identified with $\mathsf{U}(1, n)$. The element $\exp tX$ is an $\mathbb{R}$-split one parameter group that acts on $\mathfrak{g}_x$ with both a contracting and an expanding eigenvectors. But, by the amenability hypothesis on G_x, it is contained in a maximal parabolic subgroup P of $\mathsf{U}(1, n)$. However, $P = \mathbb{C}^* \times \mathsf{SU}(1, n) \ltimes \mathsf{Heis}$ (see §2.2) has no such elements.

2) Since $\mathfrak{I}_x \neq 0$, we apply Fact 5.6 to modify X if necessary and get that $(\mathbb{R}X \oplus W_X^s)(x)$ is isotropic. Now the kernel $\mathfrak{I}_x'$ of the evaluation $\mathbb{R}X \oplus W_X^s \to T_xM$ has dimension at least $(1 + \dim W_X^s) - 2 \geq 4 - 2 = 2$, since we assumed that G can not act on surfaces, and hence $\dim W_X^s \geq 3$.

It remains to check that $\mathfrak{I}_x'$ is contained in W_X^s to conclude that $\mathfrak{I}_x' = \mathfrak{I}_x$, and obtain the desired estimation. For this assume by contradiction that $X' = X + u \in \mathfrak{I}_x$, with $u \in W_X^s$. It is known that any such X' is conjugate to X in $\mathbb{R}X \oplus W_X^s$

(this is the Lie algebra of a semi-direct product of $\mathbb{R}$ by $\mathbb{R}^k$, with $\mathbb{R}$ acting on $\mathbb{R}^k$ by contraction. For $k = 1$, we get the affine group of $\mathbb{R}$). Therefore, we are led to the situation $X(x) = 0$, which we have just excluded.

$\square$

7.1. **Asymptotic leaf.** (see [8, 16, 17] for a similar situation).

Endow $M \times M$ with the metric $(+g) \oplus (-g)$. Let $f : M \to M$ be a diffeomorphism and $Graph(f) = \{(x, f(x)), x \in M\}$. By definition, f is isometric iff $Graph(f)$ is isotropic for $g \oplus (-g)$. Furthermore, in this case, $Graph(f)$ is a (totally) geodesic submanifold.

Let f_n be a diverging sequence in G_x, i.e. no sub-sequence of it converges in G_x. Consider the sequence of graphs $Graph(f_n)$. In order to avoid global complications, let us localize things by taking E_n the connected component of (x, x) in a (small) convex neighborhood $(O \times O) \cap Graph(f_n)$, where O is a convex neighborhood of x, that is, two points of it can be joined within it by a unique geodesic.

Let $V_n = Graph(D_x f_n) \subset T_x M \times T_x M$. Then, E_n is the image by the exponential map $\exp_{(x,x)}$ of an open neighborhood of 0 in V_n.

If V_n converge to V in the Grassmanian space of planes of $T_x M \times T_x M$, then E_n converge in a natural way to a geodesic submanifold E in $M \times M$. Let V^1 be the projection on the first factor $T_x M$. It is no longer a graph, since otherwise it would correspond to the graph of an element of G_x which is a limit of a subsequence of (f_n) (in fact the map $f \in G_x \to Graph(D_x f)$ is a homeomorphism onto its image in the Grasmann space).

Since a sequence of isometries converge iff the sequence of inverse isometries converge, V intersects both $T_x M \times 0$ and $0 \times T_x M$ non-trivially.

Since V_n is a complex (resp. isotropic) subspace, also is $V \cap (T_x M \times 0)$. Hence, because the metric on M is Hermite-Lorentz, $V \cap (T_x M \times 0)$ is a complex line. Furthermore, since V is isotropic, the projection V^1 is a lightlike complex hyperplane, with orthogonal direction $V \cap (T_x M \times 0)$.

Define similarly E^1 to be the projection of E on M. It equals the image by $\exp_x$ of an open subset of V^1. It is a lightlike geodesic complex hypersurface (see §2.3).

Finally, without assuming that V_n converge, we consider all the limits obtained by means of sub-sequences of (f_n). Any so obtained space V^1 (resp. E^1) is called **asymptotic** space (resp. leaf) of (f_n) at x. (Observe that different limits V may have a same projection V^1).

FACT **7.2.** *Let H be a non-precompact amenable subgroup of $\mathsf{U}(1, \mathsf{n})$. There is exactly one or two degenerate complex hyperplanes which are asymptotic spaces for any sequence of H, and are furthermore invariant under H. Similarly, there exist one or two asymptotic leaves for G_x (assuming it amenable and non-compact).*

Proof. H is contained in a maximal parabolic group P. By definition P is the stabilizer of a lightlike direction $u \in \mathbb{C}^{1+n}$. One then observes that $(\mathbb{C}u)^\perp$ is a common asymptotic space for all diverging sequences in P, and hence for H.

Assume now that H preserves two other different degenerate complex hyperplanes $(\mathbb{C}v)^\perp$ and $(\mathbb{C}w)^\perp$. Let us prove that H is pre-compact in this case. Indeed

H preserves the complex 3-space $W = \mathsf{Span}_{\mathbb{C}}(\mathsf{u}, \mathsf{v}, \mathsf{w})$ and 3-directions inside it, and also $W^\perp$. Since $W^\perp$ is spacelike (the metric on it is positive), it suffices to consider the case $W^\perp = 0$. So the statement reduces to the compactness of the subgroup of $\mathsf{U}(1, 2)$ preserving 3 different $\mathbb{C}$-lines. This is a classical fact related to the definition of the cross ratio. $\square$

7.1.1. *Varying x.*

At our fixed x, we have one or two asymptotic leaves. By homogeneity, we have the same property, one or two asymptotic leaves, for any $y \in M$. If there are two, we arrange to choose an asymptotic leaf denoted F_y, in order to insure continuity (at least) in a neighborhood of x.

FACT **7.3.** *Assume y and z near x. If $G_y \cap G_z$ is non-compact, then $F_y = F_z$.*

Proof. Let (f_n) is a diverging sequence in $G_y \cap G_z$. The fixed point set of each f_n is a geodesic submanifold containing y and z. The intersection of all of them when n varies is a geodesic submanifold S containing y and z, fixed by all the f_n. The graph of f_n above S is the diagonal of $S \times S$. Hence, S is contained in the projection of any limit of $Graph(f_n)$. Therefore, any asymptotic leaf at y is also asymptotic at z. $\square$

7.2. **Geometry.**

FACT **7.4.** *(Remember the notation $\mathfrak{I}_x = \mathfrak{g}_x \cap W_X^s$). If X can be chosen such that $\dim \mathfrak{I}_x \geq 3$, then for any $y \in F_x$, $\mathfrak{g}_y \cap \mathfrak{I}_x \neq 0$. In particular $F_y = F_x$.*

Proof. By Corollary 2.5, the group generated by $\mathfrak{I}_x$ preserves each leaf of the characteristic foliation $\mathcal{N}$ of F_x. Equivalently, the evaluation $\mathfrak{I}_x(y)$ is contained in the tangent space of the characteristic leaf $\mathcal{N}_y$, for any $y \in F_x$. Since $\dim \mathfrak{I}_x \geq 3$, and $\dim \mathcal{N}_y = 2$, we conclude that $\mathfrak{I}_x \cap \mathfrak{g}_y \neq 0$, and hence $F_x = F_y$ by Fact 7.3. $\square$

7.2.1. *Same conclusion in the other case.* Assume now that for any choose of X, $\dim \mathfrak{I}_x = 2$, say $\mathfrak{I}_x = \mathsf{Span}\{\mathsf{a}, \mathsf{b}\}$, and let $A = \exp \mathsf{a}$ and $B = \exp \mathsf{b}$. The set of fixed points of a nilpotent element, say a (or equivalently a unipotent element A) is a geodesic submanifold $Fix(a)$ of complex codimension 2 (one cheeks this for elements of $\mathsf{U}(1, \mathsf{n})$).

Choose $y \in Fix(a)$, then by Fact 7.3, $F_y = F_x$. Since $a \in \mathfrak{g}_y$, we can apply Facts 5.6 and 5.7 and get for the same X that $\mathfrak{I}_y = \mathfrak{g}_y \cap W_X^s$ has dimension ≥ 2.

Let $\mathfrak{I}$ be the subalgebra of W_X^s generated by $\mathfrak{I}_x$ and $\mathfrak{I}_y$. It is nilpotent since contained in W_X^s. Assume furthermore that $y \in Fix(a) - Fix(b)$, then $\mathfrak{I}_x \neq \mathfrak{I}_y$, and hence $\dim \mathfrak{I} \geq 3$.

Since it is generated by $\mathfrak{I}_x$ and $\mathfrak{I}_y$, $\mathfrak{I}$ preserves individually the leaves of the characteristic foliation $\mathcal{N}$ on F_x. As above, by the inequality on dimensions, any $z \in F_x$ is fixed by a non-trivial element of $\mathfrak{I}$. Therefore, by Fact 7.3, $F_z = F_x$.

7.3. **End, Contradiction.** The previous conclusion means that two asymptotic leaves are disjoint or equal, that is they define a foliation of M, of (real) codimension 2. This foliation is G invariant. Therefore, G acts on the (local) quotient space of the foliation. This contradicts our hypothesis that G does not act (locally) on surfaces.

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