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Some Properties of Inclusions of Multisets and Strictly Increasing Boolean Functions

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Abstract

Consider the following curious puzzle: call an n -tuple $\bar{X} = (X_1, \dots, X_n)$ of sets smaller than another n -tuple \bar{Y} if it has fewer *unordered sections*. We show that equivalence classes for this preorder are very easy to describe and characterize the preorder in terms of the simpler pointwise inclusion and the existence of a special strictly increasing boolean function $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$. We also show that contrary to plain boolean functions or increasing boolean functions, strictly increasing boolean functions are not finitely generated, which might explain why this preorder is not easy to describe concretely.

Keywords: multiset, system of representative, boolean functions

2000 MSC: 06A06, 06E30, 94C10

Introduction: a puzzle

Choose a (fixed) set N and a (fixed) natural number n . We can consider the following partial order on $\mathcal{P}(N)^n$, the collection of n -tuples of *nonempty* subsets of N :

$$\bar{X} \subseteq \bar{Y} \stackrel{\text{def}}{=} \prod_{1 \leq i \leq n} X_i \subseteq \prod_{1 \leq i \leq n} Y_i .$$

Because we restrict to nonempty subsets, this preorder is just pointwise inclusion:

$$(X_1, \dots, X_n) \subseteq (Y_1, \dots, Y_n) \iff \forall 1 \leq i \leq n, X_i \subseteq Y_i .$$

Consider now a commutative version of the cartesian product where instead of the usual ordered n -tuples, we take “unordered n -tuples”.

Definition 1. If $\bar{X} = (X_1, \dots, X_n)$ is an n -tuple of nonempty subsets of N , define $\mathcal{S}(\bar{X})$, the set of *unordered sections* of \bar{X} , as

$$\mathcal{S}(\bar{X}) \stackrel{\text{def}}{=} \left(\prod_{1 \leq i \leq n} X_i \right) / S_n , \tag{1}$$

where $_ / S_n$ denotes quotienting by the action of the symmetric group S_n .

Note that strictly speaking, S_n does not really act on $\prod_i X_i$ but on N^n . The definition does work, as the orbit of an element of $\prod_i X_i$ is well defined even if not all the points of the orbits are themselves in $\prod_i X_i$. From now on, we will drop the adjective “unordered” and refer to an element of $\mathcal{S}(\bar{X})$ simply as a section of \bar{X} . We now define the preorder \sqsubseteq on $\mathcal{P}(N)^n$:

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Definition 2. If \bar{X} and \bar{Y} are nonempty n -tuples of subsets of N , we define $\bar{X} \sqsubseteq \bar{Y}$ to mean $\mathcal{S}(\bar{X}) \subseteq \mathcal{S}(\bar{Y})$. We write $\bar{X} \approx \bar{Y}$ for “ $\bar{X} \sqsubseteq \bar{Y}$ and $\bar{Y} \sqsubseteq \bar{X}$ ”, that is, for $\mathcal{S}(\bar{X}) = \mathcal{S}(\bar{Y})$.

Note that the relation \sqsubseteq is not antisymmetric: $(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \approx (X_1, \dots, X_n)$ for any permutation σ .

The aim of this note is to answer the following questions:

1. When do we have $\bar{X} \approx \bar{Y}$?
2. What is the relation between $\bar{X} \sqsubseteq \bar{Y}$ and $\bar{X} \subseteq \bar{Y}$?

The problem is subtler than it appears and the first question makes for an interesting puzzle: while elementary, the proof is more complex than what most people initially think. Readers are thus encouraged to spend a couple of minutes playing with the problem before reading on.

Related notions. The notion of *system of representatives* was introduced by P. Hall in 1935 [1]. A system of representatives for the n -tuple of sets \bar{X} is simply an n -tuple \bar{x} such that $x_i \in X_{\sigma(i)}$ for each $i = 1, \dots, n$, for some permutation σ . Equivalence classes of those under permutations are exactly the unordered sections of \bar{X} of Definition 1. A lot of attention has been devoted to systems of *distinct* representatives, also called *transversal*, where the components of \bar{x} are pairwise distinct [2]. Rather than looking at them individually, we look here at the collection of all possible systems of representatives. This shift of focus seems to be new in itself, as is the notion of strictly increasing boolean function that appears later. Relating the two will give a concise answer to the second question.

The fact that all the results contained in this note are elementary makes it all the more interesting, but also more likely that I missed some related notions in the existing literature, or maybe just an exercise in a classical textbook.¹

Notation. To make formulas less verbose, we will abuse the vector notation by lifting “ \in ” pointwise: just as $\bar{X} \subseteq \bar{Y}$ means “ $\forall 1 \leq i \leq n, X_i \subseteq Y_i$ ”, the notation $\bar{a} \in \bar{X}$ is a synonym for “ $a_i \in X_i$ for all $1 \leq i \leq n$ ”. The action of S_n on n -tuples is written with a dot and is defined as $\sigma \cdot \bar{a} \stackrel{\text{def}}{=} (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)})$. When talking about n -multisets (orbits for the action of S_n), we identify an n -tuple with its orbit. In particular, $\bar{a} \in \mathcal{S}(\bar{X})$ means that $\sigma \cdot \bar{a} \in \bar{X}$ for some permutation σ , i.e., that $a_i \in X_{\sigma(i)}$ for all $1 \leq i \leq n$.

1. The equivalence relation

The first question has a simple answer: equivalence is just equality up to a permutation of the sets. In other words, the failure of antisymmetry is captured by the remark coming after Definition 2.

Proposition 1. *Given any \bar{X} and \bar{Y} in $\mathcal{P}(N)^n$, we have*

$$\bar{X} \approx \bar{Y} \iff \exists \sigma \in S_n, \sigma \cdot \bar{X} = \bar{Y}. \quad (2)$$

This proposition is slightly surprising because the left side is definitionally equal to

$$\forall \bar{a} \in \bar{X}, \exists \sigma \in S_n, \sigma \cdot \bar{a} \in \bar{Y} \quad \text{and} \quad \forall \bar{a} \in \bar{Y}, \exists \sigma' \in S_n, \sigma' \cdot \bar{a} \in \bar{X}, \quad (3)$$

while the right side is definitionally equal to

$$\exists \sigma \in S_n, \left(\forall \bar{a} \in \bar{X}, \sigma \cdot \bar{a} \in \bar{Y} \text{ and } \forall \bar{a} \in \bar{Y}, \sigma^{-1} \cdot \bar{a} \in \bar{X} \right).$$

That the latter implies the former is trivial. Proposition 1 asserts the converse: in (3) above, we can choose the permutation uniformly for all the $\bar{a} \in \bar{X}$ and $\bar{a} \in \bar{Y}$!

¹This problem comes from a very different area: denotational models of linear logic. In [3], the relation $\mathcal{S}(\bar{X}) \subseteq T$ played an important role, where the set T was an arbitrary collection of n -multisets. Thinking about the preorder “ \sqsubseteq ” was a natural step from there.

Lemma 1. *We have*

1. *If $\overline{X} \sqsubseteq \overline{Y}$ then, for all $1 \leq j \leq n$, there is some $1 \leq i \leq n$ s.t. $X_i \subseteq Y_j$.*
2. *If $\overline{X} \approx \overline{Y}$ then $X_{i_0} = Y_{j_0}$ for some pair i_0, j_0 .*

PROOF. For the first point, suppose that there is some j_0 satisfying $X_i \not\subseteq Y_{j_0}$ for all $1 \leq i \leq n$. This means that there is an $\bar{a} \in \overline{X}$ s.t. $a_i \notin Y_{j_0}$ for all i . This \bar{a} cannot be a section of \overline{Y} . Contradiction!

The second point follows easily: starting from Y_1 and repeatedly using the first point, we can construct an infinite chain of reverse inclusions:

$$Y_1 \supseteq X_{i_1} \supseteq Y_{j_2} \supseteq X_{i_2} \supseteq Y_{j_2} \supseteq \dots$$

Because there are finitely many X_i 's and Y_j 's, the chain must contain cycles. This gives an equality between some X_i and Y_j . \square

Thus, if $\overline{X} \approx \overline{Y}$, one of the sets appears on both sides and we can start the construction of σ in (2), Proposition 1. To finish the proof of Proposition 1 by induction on n , we need to show the following implication:

$$(Z, X_2, \dots, X_n) \approx (Z, Y_2, \dots, Y_n) \Rightarrow (X_2, \dots, X_n) \approx (Y_2, \dots, Y_n). \quad (4)$$

If unordered sections are seen as a “commutative cartesian product”, the next definition would be the corresponding “division”:

Definition 3. Let T be a collection of n -multisets and $Z \in \mathcal{P}(N)$ we put

$$T \div Z \stackrel{\text{def}}{=} \left\{ (a_2, \dots, a_n) \mid \forall a \in Z, (a, a_2, \dots, a_n) \in T \right\}.$$

Implication (4) follows from the following lemma:

Lemma 2. *We have:*

$$\mathcal{S}(X_1, X_2, \dots, X_n) \div X_1 = \mathcal{S}(X_2, \dots, X_n). \quad (5)$$

PROOF. The “ \supseteq ” inclusion follows from the definition.

For the “ \subseteq ” inclusion, suppose $(a_2, \dots, a_n) \in \mathcal{S}(\overline{X}) \div X_1$. Choose $b \in X_1$; we necessarily have that $(b, a_2, \dots, a_n) \in \mathcal{S}(\overline{X})$, i.e., there is a permutation τ s.t. $(b, a_2, \dots, a_n) \in (X_{\tau(1)}, \dots, X_{\tau(n)})$.

If $\tau(1) = 1$, we have $(a_2, \dots, a_n) \in (X_{\tau(2)}, \dots, X_{\tau(n)})$ and we can conclude directly.

If $\tau(1) \neq 1$, up to permuting the sets X_2, \dots, X_n and choosing an appropriate element in the orbit of (a_2, \dots, a_n) , we can assume that $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(i) = i$ when $2 < i \leq n$, or in other words, that $b \in X_2$, $a_2 \in X_1$, and $a_i \in X_i$ whenever $2 < i \leq n$.

Put $a_1 \stackrel{\text{def}}{=} a_2$. Because $a_1 = a_2 \in X_1$, we have $(a_1, a_2, \dots, a_n) \in \mathcal{S}(\overline{X})$ by hypothesis, that is, $\sigma \cdot \bar{a} \in \overline{X}$ for some permutation σ . Note that since $a_1 = a_2$, we can interchange the values $\sigma(1)$ and $\sigma(2)$ and still have $\sigma \cdot \bar{a} \in \overline{X}$.

Let $k \stackrel{\text{def}}{=} \min \{i \mid i > 0, \sigma^i(1) \in \{1, 2\}\}$. Up to changing the values of $\sigma(1)$ and $\sigma(2)$, we can assume that $\sigma^k(1) = 1$ and that the set $I \stackrel{\text{def}}{=} \{1, \sigma(1), \dots, \sigma^{k-1}(1)\}$ is the cycle containing 1. We define the set I^c as $\{1, \dots, n\} \setminus I$.

Rearrange the columns of

$$\begin{array}{cccccc} a_1 & a_2 & \dots & a_i & \dots & a_n \\ \bullet & \bullet & & \bullet & & \bullet \\ \downarrow \in & \downarrow \in & & \downarrow \in & & \downarrow \in \\ X_{\sigma(1)} & X_{\sigma(2)} & \dots & X_{\sigma(i)} & \dots & X_{\sigma(n)} \end{array}$$

into two parts:

$$\underbrace{\begin{array}{cccc} a_1 & a_{\sigma(1)} & \dots & a_{\sigma^{k-1}(1)} \\ \vdots \in & \vdots \in & & \vdots \in \\ X_{\sigma(1)} & X_{\sigma^2(1)} & \dots & X_{\sigma^k(1)} = X_1 \end{array}}_I \quad \underbrace{\begin{array}{cccc} a_2 & \dots & a_i & \dots \\ \vdots \in & & \vdots \in & \\ X_{\sigma(2)} & \dots & X_{\sigma(i)} & \dots \end{array}}_{I^c} .$$

The indices of \bar{X} on the left are exactly those in I , and so are the indices of \bar{a} . Thus, the indices of \bar{X} and \bar{a} on the right are exactly those in I^c . This shows that $(a_i)_{i \in I^c}$ is a section of $(X_i)_{i \in I^c}$. Also, because each of $\sigma(1), \dots, \sigma^{k-1}(1)$ is strictly more than 2 (by the definition of k), we have $a_{\sigma^i(1)} \in X_{\sigma^i(1)}$ for all $1 \leq i \leq k-1$ by a previous hypothesis. This shows that the permutation

$$\rho : \{2, \dots, n\} \rightarrow \{2, \dots, n\}, \quad \rho(i) \stackrel{\text{def}}{=} \begin{cases} i & \text{if } i \in \{\sigma(1), \dots, \sigma^{k-1}(1)\} \\ \sigma(i) & \text{otherwise} \end{cases}$$

satisfies $\rho \cdot (a_2, \dots, a_n) \in (X_2, \dots, X_n)$. This finishes the proof that (a_2, \dots, a_n) is indeed a section of (X_2, \dots, X_n) . \square

2. The preorder

The initial question was not very formal and read as: “*What is the relation between $\bar{X} \sqsubseteq \bar{Y}$ and $\bar{X} \subseteq \bar{Y}$?*” It is obvious that $\bar{X} \subseteq \bar{Y}$ implies $\bar{X} \sqsubseteq \bar{Y}$, but unfortunately, the converse does not hold, even if we consider n -tuples of sets up to permutations. For example, we have

$$\bar{X} := (\{3\}, \{1, 2, 3\}) \quad \sqsubseteq \quad \bar{Y} := (\{1, 3\}, \{2, 3\})$$

because the sections of \bar{X} are all sections of \bar{Y} :

$$\mathcal{S}(\bar{X}) = \{[3, 1], [3, 2], [3, 3]\} \quad \subset \quad \mathcal{S}(\bar{Y}) = \{[1, 2], [1, 3], [3, 2], [3, 3]\} .$$

However, one of the sets on the left side is strictly bigger than all the sets on the right side: $\{1, 2, 3\} \supsetneq \{2, 3\}$ and $\{1, 2, 3\} \supsetneq \{1, 3\}$!

More generally, $(Y_1 \cap Y_2, Y_1 \cup Y_2) \sqsubseteq (Y_1, Y_2)$ and any operator F on n -tuples of sets obtained by composing functions $(Y_i, Y_j) \mapsto (Y_i \cap Y_j, Y_i \cup Y_j)$ on any pairs of coordinates,² will satisfy $F(\bar{Y}) \sqsubseteq \bar{Y}$. For example,

$$\begin{aligned} F(Y_1, Y_2, Y_3) & \stackrel{\text{def}}{=} (Y_1 \cap Y_3, Y_2 \cap (Y_1 \cup Y_3), Y_2 \cup (Y_1 \cup Y_3)) \\ & \sqsubseteq (Y_1 \cap Y_3, Y_2, Y_1 \cup Y_3) \\ & \sqsubseteq (Y_1, Y_2, Y_3) . \end{aligned}$$

We will characterize which F s on $\mathcal{P}(N)^n$ satisfy $F(\bar{Y}) \sqsubseteq \bar{Y}$ by looking at functions on \mathbf{B}^n .

Definition 4. Let $\mathbf{B} \stackrel{\text{def}}{=} \{0, 1\}$ equipped with the order $0 \leq 1$. This is a complete lattice with operations written \vee and \wedge . The lattice structure is lifted pointwise to \mathbf{B}^n .

If $u \in \mathbf{B}^n$, the *weight* of u is the number of 1s in u . It is written $|u|$.

We write elements of \mathbf{B}^n as words: for example, “011101 $\in \mathbf{B}^6$ ”.

Definition 5. If $a \in N$ and $\bar{X} \in \mathcal{P}(N)^n$, the *characteristic function of a along \bar{X}* , written $\chi_{\bar{X}}(a) \in \mathbf{B}^n$ is defined by $(\chi_{\bar{X}}(a))_i = 1$ iff $a \in X_i$.

²provided the intersection isn't empty to agree with Definition 1

Thus, $\chi_{\overline{X}}(a)$ describes in which components of \overline{X} the element a appears, and $|\chi_{\overline{X}}(a)|$ is the number of those components of \overline{X} which contain a . There is a necessary condition for $\overline{X} \sqsubseteq \overline{Y}$: “for all $a \in N$, if a appears in k components of \overline{Y} , then it appears in at most k components of \overline{X} ”. Concisely, this condition can be written as “ $|\chi_{\overline{X}}(a)| \leq |\chi_{\overline{Y}}(a)|$ for all $a \in N$ ”. This condition is reminiscent of the condition appearing in Hall’s celebrated “marriage theorem” [1]. Like in the marriage theorem, this condition is also sufficient in the appropriate setting:

Proposition 2. *Given \overline{X} and \overline{Y} two n -tuples of non-empty subsets of N that satisfy*

1. *the function $a \mapsto \chi_{\overline{Y}}(a)$ is bijective from N to $\mathbf{B}^n \setminus \{0 \cdots 0\}$,*
2. *the function $f : \chi_{\overline{Y}}(a) \mapsto \chi_{\overline{X}}(a)$ is increasing;*

we have $\overline{X} \sqsubseteq \overline{Y}$ if and only if $|\chi_{\overline{X}}(a)| \leq |\chi_{\overline{Y}}(a)|$ for all $a \in N$.

Looking at the example $(\{3\}, \{1, 2, 3\}) \sqsubseteq (\{2, 3\}, \{1, 3\})$ might help to understand the conditions of the proposition. The first condition means that there is exactly one element that belongs only to Y_1 (“2”), exactly one element that belongs only to Y_2 (“1”) and exactly one element that belongs to both (“3”). When the first condition is satisfied, the second condition amounts to “when a appears in more sets than b on the right side, then a appears in more sets than b on the left side”. The graph of the resulting function f can be read below

$a :$	1	2	3	
$\chi_{\overline{Y}}(a) :$	01	10	11	
$\chi_{\overline{X}}(a) :$	01	01	11	.

Proposition 2 is a corollary to the following more general lemma:

Lemma 3. *We have $\overline{X} \sqsubseteq \overline{Y}$ iff $f(u) \stackrel{\text{def}}{=} \bigvee_{\chi_{\overline{Y}}(a) \leq u} \chi_{\overline{X}}(a)$ satisfies $|f(u)| \leq |u|$ for all u .*

The function f is the least (for the extensional order) increasing function satisfying $f(\chi_{\overline{Y}}(a)) \geq \chi_{\overline{X}}(a)$.

PROOF. For the “ \Leftarrow ” implication, suppose that $\overline{a} \in \overline{X}$. We want to show that $\sigma \cdot \overline{a} \in \overline{Y}$ for some permutation σ . If we look at the bipartite graph $G_{\overline{a}, \overline{Y}}$

$$G_{\overline{a}, \overline{Y}} \stackrel{\text{def}}{=} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ Y_1 & Y_2 & \cdots & Y_n \end{array},$$

with an edge between a_i and Y_j when $a_i \in Y_j$, finding a σ s.t. $\sigma \cdot \overline{a} \in \overline{Y}$ is equivalent to finding a perfect matching in the bipartite graph. By Hall’s marriage theorem, this is equivalent to “every subset of $\{a_1, \dots, a_n\}$ of cardinality c has at least c neighbors”.

Take some subset $U \subseteq \{a_1, \dots, a_n\}$ of cardinality c . Because \overline{a} is a section of \overline{X} , by the marriage theorem, this set has at least c neighbors in the corresponding $G_{\overline{a}, \overline{X}}$ bipartite graph. By construction, we have

$$c \leq \underbrace{\left| \bigvee_{a \in U} \chi_{\overline{X}}(a) \right|}_{\substack{\# \text{ of neighbors} \\ \text{of } U \text{ in } G_{\overline{a}, \overline{X}}}} \leq \left| f \left(\bigvee_{a \in U} \chi_{\overline{Y}}(a) \right) \right| \leq \underbrace{\left| \bigvee_{a \in U} \chi_{\overline{Y}}(a) \right|}_{\substack{\# \text{ of neighbors} \\ \text{of } U \text{ in } G_{\overline{a}, \overline{Y}}}}$$

where the second inequality follows from the definition of f , and the third inequality follows from the hypothesis. This concludes the “ \Leftarrow ” implication.

For the “ \Rightarrow ” implication, let $\overline{X} \sqsubseteq \overline{Y}$ and $c = |u| < |f(u)|$. We can find a set $\{a_1, \dots, a_k\} \subseteq N$ which satisfies $\bigvee_{i \leq k} \chi_{\overline{X}}(a_i) \leq u$ and $|\bigvee_{i \leq k} \chi_{\overline{X}}(a_i)| > c$. In particular, we have

$$\left| \bigvee_{1 \leq i \leq k} \chi_{\overline{Y}}(a_i) \right| \leq c < \left| \bigvee_{1 \leq i \leq k} \chi_{\overline{X}}(a_i) \right|.$$

For each 1 in $\bigvee_{i \leq k} \chi_{\overline{X}}(a_i)$ take one element of $\{a_1, \dots, a_k\}$ that accounts for this 1. Call the resulting tuple \overline{a} . Note that this tuple has length strictly greater than c and may contain repetitions. This is a partial section of \overline{X} : take the sets X_j s.t. $\bigvee_{i \leq k} \chi_{\overline{X}}(a_i)$ has a 1 on the j -th coordinate. To complete this tuple into a section of the whole \overline{X} , add one element from each of the remaining sets to obtain a section $(\overline{a}, \overline{a}')$ of \overline{X} . This is also a section of \overline{Y} and in particular, each element of \overline{a} needs to fit in one component of \overline{Y} . This is impossible because there are at most c sets Y_j that can contain the elements of the tuple \overline{a} . Contradiction! \square

Lemma 3 does characterize the \sqsubseteq preorder but still looks a little ad-hoc. We will now give a more concise answer using the notion of *strictly increasing boolean functions*.

3. Interlude: strictly increasing boolean functions

The next notion is quite natural, but the literature is scarce on the subject. While they are a natural specialization of increasing boolean functions, neither the big book [4] nor the survey [5] seem to mention them. Note that in computer science, a boolean function is implicitly a function from some \mathbf{B}^n to \mathbf{B} . What we are interested in here are functions from some \mathbf{B}^n to some \mathbf{B}^m , that is, boolean functions with several outputs.

Strictly increasing boolean functions are not necessarily injective, but we have the following result:

Lemma 4 (Very Simple Lemma). *If $f : \mathbf{B}^n \rightarrow \mathbf{B}^m$ is strictly increasing, then $n \leq m$.*

PROOF. Just note that chains in \mathbf{B}^n have length at most n and that a strictly increasing function preserves the cardinality of chains. \square

Because of this, it is not possible to decompose a strictly increasing function $\mathbf{B}^n \rightarrow \mathbf{B}^m$ into an m -tuple of strictly increasing functions $\mathbf{B}^n \rightarrow \mathbf{B}$. In other words, the outputs of a strictly increasing boolean function cannot be treated independently. This might be the reason why the notion does not appear much in the literature.

Lemma 5 (Simple Lemma). *If $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ is increasing, then it is strictly increasing iff it is weight preserving, that is if $|f(u)| = |u|$ for all u .*

In particular, the values $f(0 \cdots 0)$ and $f(1 \cdots 1)$ are fixed to $0 \cdots 0$ and $1 \cdots 1$. Examples of such functions are the action of permutations: they are exactly the invertible strictly increasing boolean functions:

Lemma 6. *Let $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ be a strictly increasing boolean function; the following are equivalent:*

- f is bijective,
- f is injective on words of weight 1,
- f is the action of a permutation $u \mapsto \sigma \cdot u$ for some $\sigma \in S_n$.

A less trivial and more interesting example of strictly increasing boolean function is the “and/or” function: $(b_1, b_2) \mapsto (b_1 \wedge b_2, b_1 \vee b_2)$. Its full graph is

$$\left\{ \begin{array}{lll} 00 & \mapsto & 00 \\ 10 & \mapsto & 01 \\ 01 & \mapsto & 01 \\ 11 & \mapsto & 11 \end{array} \right. .$$

Applying this “and/or” function repeatedly yields many more examples, including the function pushing all the 1s to the right:

$$u \quad \mapsto \quad 00 \cdots 00 \underbrace{11 \cdots 11}_{|u|} .$$

There are however strictly increasing boolean functions that cannot be obtained by composing permutations and $(b_i, b_j) \mapsto (b_i \wedge b_j, b_i \vee b_j)$ functions. Here is a counter example:

$$\left\{ \begin{array}{ll} & 0000 & \mapsto & 0000 \\ 0001, 0010, 0100, 1000 & \mapsto & 0001 \\ & 1100, 0110, 0011 & \mapsto & 0011 \\ & 1010, 0101, 1001 & \mapsto & 0101 \\ 1110, 1101, 1011, 0111 & \mapsto & 0111 \\ & 1111 & \mapsto & 1111 \end{array} \right. .$$

We will see in Section 5 that strictly increasing boolean functions are not finitely generated.

Because of global constraints, not every partial weight-preserving increasing function can be promoted to a total one. For example,

$$\left\{ \begin{array}{ll} 1100 & \mapsto & 1100 \\ 1010 & \mapsto & 0011 \end{array} \right.$$

cannot appear in the graph of a strictly increasing function as it would imply $1110 \mapsto 1111$, which contradicts the “simple lemma” (Lemma 5). In general, to know if a partial weight-preserving function p can be extended to a total one, compute the function p^\uparrow with $p^\uparrow(u) \stackrel{\text{def}}{=} \bigvee_{v \leq u} p(v)$, the least increasing total function agreeing with p .³ For p to be part of a total strictly increasing function, it is necessary (by the “very simple lemma”) to have $|p^\uparrow(u)| \leq |u|$ for all $u \in \mathbf{B}^n$. The next lemma shows that this is also sufficient.

Lemma 7. *An increasing (total) function $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ is extensionally smaller than a strictly increasing function if and only if $|f(u)| \leq |u|$ for all u .*

PROOF. The implication “ \Rightarrow ” follows directly from the “simple lemma”. For the converse, we define a new function f^\nearrow from f in such a way that:

- $f \leq f^\nearrow$ for the extensional order,
- f^\nearrow is increasing and weight preserving, i.e., is strictly increasing.

The function f^\nearrow is defined inductively by looking at tuples of decreasing weights. Whenever $|f(u)| = |u|$, we put $f^\nearrow(u) \stackrel{\text{def}}{=} f(u)$. Otherwise, we look at $\bigwedge_{u < v} f^\nearrow(v)$. Note that because we start with the “heavy” tuples, this is well defined. In particular, we get $f^\nearrow(1 \cdots 1) = 1 \cdots 1$. The two important cases are

1. $|\bigwedge_{u < v} f^\nearrow(v)| = |u|$, in this case we use $\bigwedge_{u < v} f^\nearrow(v)$ for the value of $f^\nearrow(u)$,
2. $|\bigwedge_{u < v} f^\nearrow(v)| = 1 + |u|$, in that case, the left-side is strictly greater than $|f(u)|$ by hypothesis, so we choose a 1 in $\bigwedge_{u < v} f^\nearrow(v)$ that is not in $f(u)$ and replace it with a 0.

Provided no other case appears, this defines a weight-preserving increasing function greater than f .

Claim. *During the construction, we always have*

$$|u| \leq \left| \bigwedge_{u < v} f^\nearrow(v) \right| \leq 1 + |u| .$$

³Because p is partial, the supremum is computed on the set of defined values.

Note that the infimum can be obtained by looking at the successors⁴ of u , whose weight is always $1 + |u|$. Because f^\nearrow is weight-preserving, we necessarily have $|\bigwedge_{u < v} f^\nearrow(v)| \leq 1 + |u|$.

For the other inequality, suppose by contradiction that $|\bigwedge_{u < v} f^\nearrow(v)| < |u|$. We can find some w_1 and w_2 satisfying:

- w_1 and w_2 are successors of v ,
- $|f^\nearrow(w_1) \wedge f^\nearrow(w_2)| < |u|$.

Those boolean tuples can be constructed as follows: enumerate the successors v_1, \dots, v_k of u and expand the infimum as

$$\bigwedge_{v < u} f^\nearrow(v) = \underbrace{f^\nearrow(v_1) \wedge \dots \wedge f^\nearrow(v_i)}_{|\dots| \leq |u|} \wedge \dots \wedge \underbrace{f^\nearrow(v_j)}_{|\dots| < |u|} \wedge \dots$$

with minimal i and j . We can put “ $w_1 \stackrel{\text{def}}{=} v_i$ and $w_2 \stackrel{\text{def}}{=} v_j$ ”. We obtain a contradiction because

$$\begin{aligned} 2 + |u| &= |f^\nearrow(w_1 \vee w_2)| \\ &\geq |f^\nearrow(w_1) \vee f^\nearrow(w_2)| \\ &= |f^\nearrow(w_1)| + |f^\nearrow(w_2)| - |f^\nearrow(w_1) \wedge f^\nearrow(w_2)| \\ &> 1 + |u| + 1 + |u| - |u| \\ &= 2 + |u| \quad . \end{aligned}$$

This finishes the proof of the claim, and thus of Lemma 7. \square

4. Back to the preorder

We can now give a more elegant answer to the original question. First note that we can lift any function $f : \mathbf{B}^n \rightarrow \mathbf{B}^m$ to a function $\mathcal{P}(N)^n \rightarrow \mathcal{P}(N)^m$:

Definition 6. Suppose $f : \mathbf{B}^n \rightarrow \mathbf{B}^m$, define $\hat{f} : \mathcal{P}(N)^n \rightarrow \mathcal{P}(N)^m$ as

$$\hat{f}(\bar{Y}) \stackrel{\text{def}}{=} \bar{X} \quad \text{with } a \in X_i \text{ iff } f(\chi_{\bar{Y}}(a)) \text{ has a 1 at coordinate } i .$$

This transformation is, in a precise categorical sense, *natural*. It amounts to lifting the boolean operations \wedge and \vee to their set theoretic versions \cap and \cup in a way that is compatible with function composition. For example, with the “and/or” function $(b_1, b_2) \mapsto (b_1 \wedge b_2, b_1 \vee b_2)$ we obtain the “intersection/union” function sending (Y_1, Y_2) to $(Y_1 \cap Y_2, Y_1 \cup Y_2)$. Putting Lemma 3 and Lemma 7 together, we can now answer the initial question:

Proposition 3. For any \bar{X} and \bar{Y} , we have

$$\bar{X} \sqsubseteq \bar{Y} \iff \bar{X} \subseteq \hat{g}(\bar{Y}) \quad \text{for some strictly increasing } g : \mathbf{B}^n \rightarrow \mathbf{B}^n .$$

PROOF. We know that $\bar{X} \sqsubseteq \bar{Y}$ is equivalent to having $|f(u)| \leq |u|$ for all u in \mathbf{B}^n , where f is defined as in Lemma 3. By Lemma 7, this is equivalent to saying that there is a strictly increasing g s.t. $f \leq g$. The two following points are thus equivalent:

- $\bar{X} \sqsubseteq \bar{Y}$,
- there is a strictly increasing g on \mathbf{B}^n s.t. $\chi_{\bar{Y}}(a) \leq u$ implies $\chi_{\bar{X}}(a) \leq g(u)$.

The second point implies that $\bar{X} \subseteq \hat{g}(\bar{Y})$: use $u \stackrel{\text{def}}{=} \chi_{\bar{Y}}(a)$ to check that $a \in X_i$ is an element of the i -th set of $\hat{g}(\bar{Y})$. For the converse, suppose g is strictly increasing with $\bar{X} \subseteq \hat{g}(\bar{Y})$ and let $\chi_{\bar{Y}}(a) \leq u$. Suppose that $\chi_{\bar{X}}(a)$ contains a 1 in position i . This means that $a \in X_i$ and thus a is in the i -th set of $\hat{g}(\bar{Y})$. We can conclude that $g(\chi_{\bar{Y}}(a))$ contains a 1 in position i . This implies that $g(u)$ also contains a 1 in position i . \square

⁴ v is a successors of u if $v > u$ and $|v| = |u| + 1$.

5. Strictly increasing functions are not finitely generated

If one had a simple representation of strictly increasing boolean functions from \mathbf{B}^n to itself, then Proposition 2 would give a simple representation of the \sqsubseteq preorder. It is well known that all boolean functions $\mathbf{B}^n \rightarrow \mathbf{B}^m$ with n inputs and m outputs can be represented by a boolean circuit using only “and”, “or” together with “not” cells. Strictly speaking, we also need constant values and need a way to forget or duplicate inputs. The complete set of cells is depicted in Figure 1, where the last cells are:

- constants 1 and 0 (zero input, one output),
- drop (one input, zero output),
- duplicate (one input, two outputs),
- crossing (two inputs, two outputs).

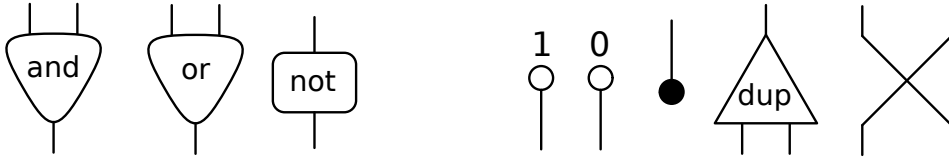


Figure 1: Boolean cells

These cells, together with a finite set of relations expressing properties of the operations (associativity, etc.), give a finite presentation of the monoidal category of boolean functions.⁵ We can generate the subcategory of increasing functions by removing the “not” cell from the generators. Unfortunately, no such thing is possible for *strictly increasing* boolean functions.

Proposition 4. *Strictly increasing boolean functions are not finitely generated.*

In other words, any finite set of cells will either miss some strictly increasing boolean function, or generate some non strictly increasing boolean function.

PROOF. First note that because of the “ultra simple lemma”, we cannot use cells with more inputs than outputs as they would not be strictly increasing on their own. In particular, any strictly increasing boolean function from \mathbf{B}^n to itself must be represented using cells with as many inputs as outputs. An example of such cell could be $(b_1, b_2) \mapsto (b_1 \wedge b_2, b_1 \vee b_2)$, with two inputs and two outputs.

We will prove Proposition 3 by contradiction: suppose there is a finite set of cells that generates all strictly increasing boolean functions, and write m for the maximal arity of the cells in this set.

Any non-invertible function has a representation as in Figure 2 where

- the topmost rectangle contains only crossing (or invertible cells),
- the cell C is not invertible and has arity $c \leq m$,
- and the lowermost rectangle contains the rest of the circuit.

By Lemma 6, we know that the cell C is not injective on inputs of weight 1. It means there are two input wires i_1 and i_2 s.t. C gives the same value on the two elements of \mathbf{B}^n consisting of 0s and a single 1 in position i_1 or i_2 . Because this is independent of the inputs \bar{v} on the $n - m$ remaining wires, we obtain:

⁵All of this has a precise algebraic meaning, see [6] for details.

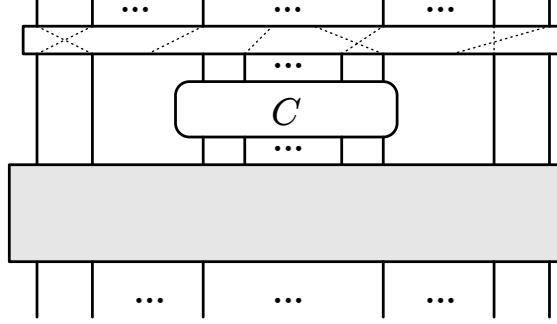


Figure 2: Circuits

Claim. *Supposing strictly increasing boolean functions were finitely generated with a cells of arity less than m , then for any non-invertible strictly-increasing boolean function $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$, with $n \geq m$, we have:*

$$\exists \sigma \in S_n \quad \forall \bar{v} \in \mathbf{B}^{n-m} \quad f(\sigma(01.\bar{0}.\bar{v})) = f(\sigma(10.\bar{0}.\bar{v}))$$

The permutation σ is used to simplify the notation: it reorders the wires to put i_1 and i_2 in positions 1 and 2, and the remaining input wires for C in positions $3, \dots, c$.

For any maximal arity m , we will construct a (large) n together with a function $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ that contradicts this fact: whenever we choose input wires i_1 and i_2 and put any $m - 2$ other input wires to 0, we can complete the remaining input wires in such a way that putting $i_1 \stackrel{\text{def}}{=} 0$ and $i_2 \stackrel{\text{def}}{=} 1$, or putting $i_2 \stackrel{\text{def}}{=} 0$ and $i_1 \stackrel{\text{def}}{=} 1$ makes a difference in the output of the function. Thus, this function will not be representable using the given set of cells.

Given a (large) n , define $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ as:

$$f(u) \stackrel{\text{def}}{=} \begin{cases} 0^n & \text{if } |u| = 0 & (1) \\ 1 \ 0^{n-1} & \text{if } |u| = 1 & (2) \\ 1^k \ 0^{n-k} & \text{if } |u| = k \text{ is even} & (3) \\ 1101 \ 0^{n-4} & \text{if } u = 0 \dots 0 \ 110^l 1 \ 0 \dots 0, \text{ with } l > 0 & (4) \\ 1110 \ 0^{n-4} & \text{if } |u| = 3 \text{ but } u \text{ not of the previous shape} & (5) \\ 1^{2^k} 01 \ 0^{n-2^k-2} & \text{if } u = 0 \dots 0 \ 1^{2^k} 0^{2^k} 1 \ 0 \dots 0, \text{ with } k > 1 & (6) \\ 1^{2^k} 01 \ 0^{n-2^k-2} & \text{if } u = 0 \dots 0 \ 10^{2^k} 1^{2^k} \ 0 \dots 0, \text{ with } k > 1 & (7) \\ 1^{2^k} 10 \ 0^{n-2^k-2} & \text{in all the remaining cases.} & (8) \end{cases}$$

This function is strictly increasing because whenever v is a successor of u , we have $f(v) > f(u)$:

- $f(u) = 1^{2^k} 0 \dots$ when $|u| = 2k$
- $f(u) = 1^{2^k} 10 \ 0 \dots$ or $f(u) = 1^{2^k} 01 \ 0 \dots$ when $|u| = 2k + 1$.

Suppose input wires k_1, \dots, k_{m-2} are fixed to 0 and we want to differentiate between input wires i_1 and i_2 , with $i_1 < i_2$. By putting some 1s in the appropriate remaining wires, we can make f give different results when “ $i_1 \stackrel{\text{def}}{=} 0, i_2 \stackrel{\text{def}}{=} 1$ ” and “ $i_1 \stackrel{\text{def}}{=} 1, i_2 \stackrel{\text{def}}{=} 0$ ”.

- If there are two consecutive wires between i_1 and i_2 (but not touching i_2) which are not among k_1, \dots, k_{m-2} , we put those two wires to 1 and all the other wires to 0. By lines (4) and (5) from the definition of f , we will get two different results.
- If not, the wires i_1 and i_2 cannot be too far apart. (There are at most $2m - 2$ wires between them...) If we can find a sequence of 2^k consecutive wires at distance 2^k to the left of i_1 , or a sequence of 2^k consecutive wires at distance 2^k to the right of i_2 , we can put those wires to 1 and the rest to 0. By lines (6) and (8) or (7) and (8) of the definition of f , we will also get different results.

For this to work, we have to make sure n is big enough. At worst, the wires k_1, \dots, k_{m-2} can prevent us from finding an appropriate sequence $m-2$ times. In particular, if i_1 is big enough (bigger than 2^{m+1}), such a sequence is bound to happen. The same is true when i_2 is small enough compared to n . In the end, choosing n bigger than, say, 2^{2m+2} plus an additional ε will guarantee that we can differentiate any i_1 and i_2 among any set of m wires. A more careful analysis shows that it is in fact enough to take $n \stackrel{\text{def}}{=} 2^{m+1} + 4$. This concludes the proof. \square

∞ Further questions

There are more questions one can ask about this preorder, or more generally about strictly increasing boolean functions. Readers will probably come up with more examples but the most immediate ones are:

- Strictly increasing boolean functions are not finitely generated, but how many cells are needed to generate those functions that are in $\mathbf{B}^n \rightarrow \mathbf{B}^n$? (Asymptotic bounds etc.) Can we give canonical minimal generating families? If so, what can be said about the complexity of strictly increasing boolean functions expressed with those cells?
- How many strictly increasing boolean functions from \mathbf{B}^n to itself are there? The first four values of this sequence are “1, 4, 66, 7128” or “1, 2, 11, 297” if one divides each number by $n!$ to take symmetries on the outputs into account. Neither of them appears in Sloane’s encyclopedia [7]. This question is a variant of *Dedekind’s problem* from [8] of counting the number of increasing boolean functions from \mathbf{B}^n to \mathbf{B} . Besides some asymptotic bounds, only the first eight Dedekind numbers are known [9]. The global nature of *strict* monotonicity makes it unlikely that the answers to these two questions are related.

- [1] P. Hall, On representatives of subsets, *Journal of the London Mathematical Society* 10 (1935) 26–30.
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- [4] I. Wegener, *The complexity of Boolean functions*, Wiley-Teubner, 1987.
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URL <http://stacks.iop.org/0036-0279/58/i=5/a=R02>
- [6] Y. Lafont, Towards an algebraic theory of boolean circuits, *Journal of Pure and Applied Algebra* 184 (2003) 2003.
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- [8] R. Dedekind, Über Zerlegungen von Zahlen durch Ihre grössten gemeinsame Theiler, Unknown, 1897, reprinted in [10], pp. 103147.
- [9] D. Wiedemann, A computation of the eighth Dedekind number, *A Journal on The Theory of Ordered Sets and Its Applications* 8 (1991) 5–6. doi:10.1007/BF00385808.
- [10] R. Dedekind, H. v. R. Fricke, E. Noether, Ø. Ore, *Gesammelte mathematische werke*, Vol. 2 of *Gesammelte mathematische werke*, Friedr. Vieweg & Sohn, 1930, <http://gdz.sub.uni-goettingen.de/en/dms/load/img/?PPN=PPN235693928>.
- [11] D. E. Knuth, *Combinatorial Algorithms*, Vol. 4A of *The Art of Computer Programming*, Addison-Wesley Professional, 2011.

None of the references in this bibliography is directly concerned with either the pre-order “ \sqsubseteq ” or strictly increasing boolean functions. Any reference to papers or books mentioning them would be most welcome. Moreover, besides [1] or any book covering Hall’s theorem, none of them is actually a prerequisite to this note.

An algorithm

Proposition 2 is more elegant but Lemma 3 has an interesting byproduct: it gives a concrete algorithm to check if $\overline{X} \sqsubseteq \overline{Y}$. For that, construct the function f from Lemma 3 and check that it satisfies the condition. Just as a proof of concept, here is the main part of the algorithm, in the Python programming language. Minor alterations have been made to make it more readable. The most difficult (and fun) part was to write

the function combinations” that generates all the vectors of length n and weight w using one of the subtle algorithms from [11]⁶

```
def check(N,n,X,Y):
    # N is a set, n is an integer, X / Y are tuples of sets.
    def combinations(w):
        # generates all vectors of weight w
        # omitted (see Knuth, or use you favorite method)
    def sup(u,v):
        # computes the pointwise "or" on n-tuples
        # omitted (simple)
    def weight(u):
        # computes the weight of an n-tuple
        # omitted (simple)
    def chi(a,Z):
        # complexity: O(n log(z)) (z is cardinality of Z)
        for i in range(n):
            # we use Python builtin "set" type
            if a in Z[i]:
                # so that "a in Z[i]" mean "a belongs to Z[i]"
                u[i] = 1
        return u

F = {}
# F is a finite map with at most 2^n elements,
# access is logarithmic: O(log(2^n)) = O(n)
for a in N:
    chiX = chi(a,X)
    chiY = chi(a,Y)
    F[chiY] = sup(F[chiY], chiX)
for w in range(n+1):
    # generating all tuples
    # complexity : about 2^n *
    for u in combinations(w):
        # complexity : about 2^n *
        v = F[u]
        for i in range(n):
            # complexity : about 2^n *
            if u[i] == 1:
                # complexity : about 2^n *
                u[i] = 0
                v = sup(v,F[u])
                # complexity : about 2^n *
                u[i] = 1
        F[u] = v
        if weight(v) > w:
            # complexity : about 2^n *
            return False
return True
# if we reached this far, the condition is satisfied
```

If N has cardinality c and the components of \bar{X} and \bar{Y} have cardinalities at most x and y ; and if we suppose that the standard operations on sets and finite functions have logarithmic complexity, the hints in the comments give a total complexity of roughly $O(cn(\log(x) + \log(y)) + 2^n n^3)$. Because both $x = O(c)$ and $y = O(c)$, we get a complexity of $O(nc \log(c) + 2^n n^3)$. If c is fixed, this is $O(n^3 2^n)$; if n is fixed, this is $O(c \log(c))$. In almost all cases, this is better (and much easier to write) than the naive approach that checks if each $\bar{a} \in \bar{X}$ is a section of \bar{Y} , even if we are allowed to use an oracle to guess the permutations.

⁶The file is available from <http://lama.univ-savoie.fr/~hyvernat/research.php>