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# Two-dimensional unit-length vector fields of vanishing divergence

Radu Ignat\*

## Abstract

We prove the following regularity result: any two-dimensional unit-length divergence-free vector field belonging to  $H^{1/2}$  (or  $W^{1,1}$ ) is locally Lipschitz except at a locally finite number of vortices. We also prove approximation results for such vector fields: the dense sets are formed either by unit-length divergence-free vector fields that are smooth except at a finite number of points and the approximation result holds in the  $W_{loc}^{1,p}$ -topology ( $1 \leq p < 2$ ), or by everywhere smooth unit-length vector fields (not necessarily divergence-free) and the approximation result holds in a weaker topology.

*AMS classification:* 35B65, 46E35

*Keywords:* Sobolev spaces, regularity, approximation, vortices, kinetic formulation, entropy.

## 1 Introduction

Let  $\Omega \subset \mathbf{R}^2$  be an open bounded set. We will focus on measurable vector fields  $m : \Omega \rightarrow \mathbf{R}^2$  that satisfy

$$|m| = 1 \text{ a.e. in } \Omega \quad \text{and} \quad \nabla \cdot m = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (1)$$

One can equivalently consider measurable vector fields  $v : \Omega \rightarrow \mathbf{R}^2$  such that

$$|v| = 1 \text{ a.e. in } \Omega \quad \text{and} \quad \nabla \times v = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (2)$$

(The passage from (1) to (2) is done via  $v = m^\perp = (-m_2, m_1)$ .) Locally,  $m$  (resp.  $v$ ) can be written in terms of a stream function  $\psi$ , i.e.,  $m = \nabla^\perp \psi$  (resp.  $v = \nabla \psi$ ) so that we get to the eikonal equation through  $\psi$ :

$$|\nabla \psi| = 1. \quad (3)$$

Typically, one can construct such vector fields by considering stream functions of the form  $\psi = \text{dist}(\cdot, K)$  for some closed set  $K \subset \mathbf{R}^2$ ; these vector fields are called Landau states in micromagnetic jargon (see Figure 1). However, not every stream function can be written as a distance function (up to a sign  $\pm 1$  and an additive constant); for example, if  $\psi(x) = \max\{\text{dist}(x, P_1), \text{dist}(x, P_2)\}$  for two different points  $P_1, P_2 \in \mathbf{R}^2$ , then (3) holds even if  $\psi$  is not a distance function.

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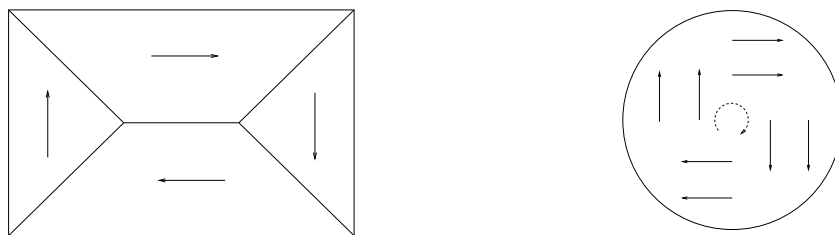


Figure 1: Landau states in a rectangle and a disk.

## 2 Main results

For  $p \geq 1$  and  $s > 0$ , we denote by

$$W_{div}^{s,p}(\Omega, S^1) = \{m \in W^{s,p}(\Omega, \mathbf{R}^2) : m \text{ satisfies (1)}\}.$$

### 2.1 Regularity results

The first goal is to prove the following regularity result:

**Theorem 1** *If  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ) then  $m$  is locally Lipschitz continuous inside  $\Omega$  except at a locally finite number of singular points. Moreover, every singular point  $P$  of  $m$  corresponds to a vortex singularity of degree 1 of  $m$ , i.e., there exists a sign  $\alpha = \pm 1$  such that*

$$m(x) = \alpha \frac{(x - P)^\perp}{|x - P|} \quad \text{for every } x \neq P \text{ in any convex neighborhood of } P \text{ in } \Omega.$$

*In particular, if  $m \in H_{div}^1(\Omega, S^1)$  then  $m$  is locally Lipschitz.*

**Remark 1** The above result was proved by Jabin-Otto-Perthame [22] in the particular case of zero-energy states of a line-energy Ginzburg-Landau model. More precisely, for  $\varepsilon > 0$ , one defines the functional  $E_\varepsilon : H^1(\Omega, \mathbf{R}^2) \rightarrow \mathbf{R}_+$  by

$$E_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_\Omega (1 - |m_\varepsilon|^2)^2 dx + \frac{1}{\varepsilon} \|\nabla \cdot m_\varepsilon\|_{H^{-1}(\Omega)}^2, \quad m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)$$

(we refer to [1, 3, 24, 13, 23, 28, 22] for the analysis of this model). A vector field  $m : \Omega \rightarrow \mathbf{R}^2$  is called zero-energy state if there exists a family  $\{m_\varepsilon \in H^1(\Omega, \mathbf{R}^2)\}_{\varepsilon \rightarrow 0}$  satisfying

$$m_\varepsilon \rightarrow m \quad \text{in } L^1(\Omega) \quad \text{and} \quad E_\varepsilon(m_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Then  $m$  satisfies (1) and shares the structure stated in Theorem 1 (see [22]).

The hypothesis  $m \in W^{1,1}$  (or  $m \in H^{1/2}$ ) in Theorem 1 is a critical regularity assumption in order to avoid line-singularities for vector fields  $m$  satisfying (1) (see Proposition 8 in Section 7). As consequence of Theorem 1, one has

$$\{m \in W_{loc}^{1,1}(\Omega, \mathbf{R}^2) : m \text{ satisfies (1)}\} = \{m \in H_{loc}^{1/2}(\Omega, \mathbf{R}^2) : m \text{ satisfies (1)}\}.$$

Let us now discuss the optimality of the result in Theorem 1: Firstly, observe that Lipschitz regularity cannot be improved.

**Proposition 1** *There exist Lipschitz vector fields  $m : \Omega \rightarrow \mathbf{R}^2$  that satisfy (1) and are not  $C^1$  in  $\Omega$ .*

In general, a vector field  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ) (without interior vortex singularities) is only locally Lipschitz, and not necessary globally Lipschitz in  $\Omega$ . This is the case of a "boundary vortex" vector field, e.g.,  $m(x) = \frac{(x-P)^\perp}{|x-P|}$  for every  $x \in \Omega$  where  $P$  is some point on  $\partial\Omega$ . If the domain  $\Omega$  has a cusp in  $P \in \partial\Omega$ , the "boundary vortex" vector field could belong even to  $H^1(\Omega, \mathbf{R}^2)$ ; moreover, there even exist convex domains  $\Omega$  and  $m \in H_{div}^1(\Omega, S^1)$  such that  $m$  is not globally Lipschitz in  $\Omega$  (see Subsection 4.2).

The geometry of  $\Omega$  influences the number of vortex singularities of  $W^{1,1}$  (or  $H^{1/2}$ )-vector fields satisfying (1). For example, if  $\Omega$  is convex, then every vector field  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ) is either a "vortex" vector field (i.e.,  $m(x) = \pm \frac{(x-P)^\perp}{|x-P|}$  for every  $x \in \Omega$  where  $P$  is some point in  $\Omega$ ), or locally Lipschitz (i.e. no interior vortex); therefore, convex domains do not allow for more than one interior vortex (see Remark 6). However, we prove that there are nonconvex domains where configurations with arbitrary number of vortices do exist.

**Proposition 2** *There exist an open simply-connected nonconvex piecewise Lipschitz domain  $\Omega$  and a vector field  $m \in W_{div}^{1,p}(\Omega, \mathbf{R}^2)$  for every  $p \in [1, 2)$  that has infinitely many vortices  $\{P_1, P_2, \dots\}$ .*

Observe that  $W_{loc}^{1,p}(\Omega, S^1) \subset H_{loc}^{1/2}(\Omega, S^1)$  for  $p > 1$ , and the embedding fails for  $p = 1$  (see Proposition 9 in Section 7). Also notice that configurations with infinitely many (interior) vortices can occur only in a non-Lipschitz domain  $\Omega$ ; indeed, if  $\partial\Omega$  is Lipschitz, then a configuration  $m \in W_{div}^{1,1}(\Omega, S^1)$  or  $m \in H_{div}^{1/2}(\Omega, S^1)$  has only a finite number of interior vortex singularities (see Proposition 7).

The main ingredient of the proof of Theorem 1 resides in the following kinetic formulation. It is a generalization to the case of  $W_{div}^{1,1}(\Omega, S^1)$  or  $H_{div}^{1/2}(\Omega, S^1)$  vector fields of the result in [22] for zero-energy states of  $E_\varepsilon$  in Remark 1:

**Proposition 3 (Kinetic formulation)** *Let  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ). For every direction  $\xi \in S^1$ , we define  $\chi(\cdot, \xi) : \Omega \rightarrow \{0, 1\}$  (resp.  $\tilde{\chi}(\cdot, \xi) : S^1 \rightarrow \{0, 1\}$ ) by*

$$\chi(x, \xi) = \tilde{\chi}(m(x), \xi) = \begin{cases} 1 & \text{for } m(x) \cdot \xi > 0, \\ 0 & \text{for } m(x) \cdot \xi \leq 0. \end{cases}$$

*Then the following kinetic equation holds for every  $\xi \in S^1$ :*

$$\xi \cdot \nabla \chi(\cdot, \xi) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (4)$$

Here,  $\chi$  corresponds to the concept of characteristic of a weak solution  $m$  satisfying (1). Indeed, if  $m$  is smooth around a point  $x \in \Omega$ , then the characteristic of  $m$  at  $x$  (by means of the eikonal equation (3) with  $m = \nabla^\perp \psi$  around  $x$ ) is given by  $\dot{X}(t, x) = m^\perp(X(t, x))$  with the initial condition  $X(0, x) = x$ ; then the orbit  $\{X(t, x)\}_t$  is a straight line (i.e.,  $X(t, x) = x + tm^\perp(x)$  for  $t$  in some interval around 0) along which  $m$  is perpendicular and constant. Therefore, in the direction  $\xi := m^\perp(x)$ , either  $\nabla \chi(\cdot, \xi)$  locally vanishes (if  $m$  is constant in a neighborhood of  $x$ ), or it

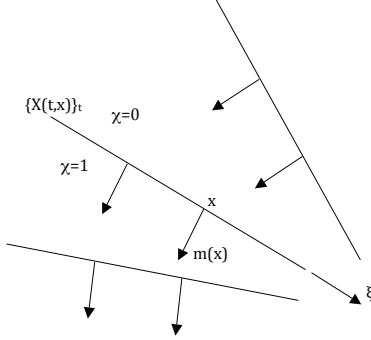


Figure 2: Characteristics of  $m$ .

concentrates on  $\{X(t, x)\}_t$  and is oriented by  $\xi^\perp$  (see Figure 2). The knowledge of  $\chi(\cdot, \xi)$  in every direction  $\xi \in S^1$  determines completely the vector field  $m$  due to the straightforward formula

$$m(x) = \frac{1}{2} \int_{S^1} \xi \chi(x, \xi) d\xi \quad \text{for a.e. } x \in \Omega. \quad (5)$$

**Remark 2** Classical kinetic averaging lemma (see e.g. Golse-Lions-Perthame-Sentis [15]) shows that a measurable vector-field  $m : \Omega \rightarrow S^1$  satisfying (4) belongs to  $H_{loc}^{1/2}$  (due to (5)). This property can be read as the inverse of Proposition 3 for the case  $m \in H^{1/2}(\Omega, S^1)$ . Moreover, such a vector field has stronger regularity since it shares the structure described in Theorem 1.

**Remark 3** The proof of Proposition 3 strongly relies on the structure of lifting of vector fields  $m \in W^{1,1}(\Omega, S^1)$  (resp.  $m \in H^{1/2}(\Omega, S^1)$ ) and an appropriate chain rule. More precisely, if  $m \in W^{1,1}(\Omega, S^1)$ , then there exists a lifting  $\Theta \in SBV(\Omega, \mathbf{R})$  such that  $m = e^{i\Theta}$  a.e. in  $\Omega$  (see e.g. [14], [7], [9], [17]). While if  $m \in H^{1/2}(\Omega, S^1)$ , then one can find a lifting  $\Theta = \Theta_1 + \Theta_2$  with  $\Theta_1 \in H^{1/2}$ ,  $\Theta_2 \in SBV$  and  $e^{i\Theta_2} \in H^{1/2} \cap W^{1,1}$  (see [6]). Recall that  $SBV(\Omega, \mathbf{R}^d)$  is the subspace of vector fields  $m \in BV(\Omega, \mathbf{R}^d)$  whose differential  $Dm$  has vanishing Cantor part  $D^c m$  (i.e.,  $D^c m \equiv 0$  as a measure in  $\Omega$ ).

We address the following open problem:

**Open Problem 1** *Is it true that every  $m \in BV(\Omega, \mathbf{R}^2)$  with (1) satisfies  $m \in SBV$ ?*

This question is related with a recent work of Bianchini-DeLellis-Robyr [5]: they show that the viscosity solution  $\psi$  of a Hamilton-Jacobi equation  $H(\nabla\psi) = 0$  in  $\Omega$  (with a uniformly convex hamiltonian  $H$ ) satisfies  $\nabla\psi \in SBV$ . Open Problem 1 asks whether for the particular case of the eikonal equation (3), the result in [5] still holds when replacing the assumption of viscosity solution with the hypothesis of a general solution  $\psi$  of (3) with  $\nabla\psi \in BV$ .

## 2.2 Density results

The second goal of the paper is to present approximation results for the class of vector fields  $W_{div}^{1,1}(\Omega, S^1)$  (or  $H_{div}^{1/2}(\Omega, S^1)$ ): our subsets are formed either by divergence-free vector fields that

are smooth except at a finite number of points and the approximation result holds in the  $W^{1,1}$  (or  $H^{1/2}$ )-topology, or by everywhere smooth vector fields (not necessarily divergence-free) and the approximation result holds in a weaker topology. We start by extending Bethuel-Zheng's density result (see [4]) for  $W^{1,1}(\Omega, S^1)$  vector fields, respectively Riviere's density result (see [27]) for  $H^{1/2}(\Omega, S^1)$  vector fields to the case of divergence-free vector fields:

**Theorem 2** *Let  $\Omega$  be a Lipschitz bounded simply-connected domain and  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ). Then  $m$  has a finite number  $k \geq 0$  of vortices  $\{P_1, \dots, P_k\}$  and  $m$  can be approximated in  $W_{loc}^{1,p}(\Omega)$  (for any  $p \in [1, 2)$ ) by divergence-free vector fields  $m_n \in C^\infty(\Omega \setminus \{P_1, \dots, P_k\}, S^1)$  that are smooth except at the  $k$  vortex points of  $m$ . In particular, if  $m \in H_{div}^1(\Omega, S^1)$ , the sequence  $\{m_n\}$  can be chosen to be smooth everywhere in  $\Omega$  and the approximation result holds in  $H_{loc}^1(\Omega)$ .*

In various applications (see e.g. Remarks 1 and 4), we need to approximate vector fields  $m$  (with the structure given in Theorem 1) by  $H^1(\Omega, S^1)$  vector fields. But  $H^1(\Omega, S^1)$ -vector fields cannot allow for vortices. Therefore, an approximation result by everywhere smooth vector fields is needed in some weak topology. What is the optimal weak topology where such a density result holds? The following result shows that  $L^1$ -topology is too strong for having density of smooth vector fields of vanishing divergence and values in  $S^1$ .

**Proposition 4** *Let  $m : B^2 \rightarrow S^1$  be the vortex configuration  $m(x) = \frac{x^\perp}{|x|}$  in the unit disk  $B^2$ . Then there exists no sequence of vector fields  $m_n \in C^\infty(\Omega, S^1)$  of vanishing divergence such that  $m_n \rightarrow m$  a.e. in  $B^2$ .*

We now generalize this property: the density result still fails if we relax the divergence-free constraint on the approximated smooth vector fields, but we impose this restriction in the limit in  $L^1$ -topology (or  $H^{-s}$  weak topology for some  $s \in [0, \frac{1}{2})$ ).

**Proposition 5** *Let  $m : B^2 \rightarrow S^1$  be the vortex configuration  $m(x) = \frac{x^\perp}{|x|}$  in  $B^2$ . Then there exists no sequence of vector fields  $m_n \in C^\infty(\Omega, S^1)$  such that  $m_n \rightarrow m$  a.e. in  $B^2$  and one of the following two conditions holds:*

- a)  $\nabla \cdot m_n \rightarrow 0$  in  $L^1(B^2)$ ;
- b)  $\nabla \cdot m_n \rightharpoonup 0$  weakly in  $H^{-s}(B^2)$  for some  $s \in [0, \frac{1}{2})$ .

Finally, we prove an approximation result in  $L^1$ -topology by smooth vector fields with values in  $S^1$  (not necessary divergence-free), but the divergence-free constraint holds in the limit in the  $H^{-1/2}$  topology. This topology is optimal due Proposition 5 b).

**Theorem 3** *Let  $\Omega$  be a Lipschitz bounded simply-connected domain and  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ). Then there exists a sequence of vector fields  $m_n \in C^\infty(\Omega, S^1)$  such that  $m_n \rightarrow m$  a.e. in  $\Omega$  and  $(\nabla \cdot m_n)\mathbf{1}_\Omega \rightarrow 0$  in  $\dot{H}^{-1/2}(\mathbf{R}^2)$ .*

**Remark 4** The motivation of Theorem 3 comes from thin-film micromagnetics. The following 2D energy (see [12]) is considered as an approximation of the full 3D micromagnetic model for thin films: for  $\varepsilon > 0$ , one defines the functional  $F_\varepsilon : H^1(\Omega, S^1) \rightarrow \mathbf{R}_+$  by

$$F_\varepsilon(m_\varepsilon) = \varepsilon \int_{\Omega} |\nabla m_\varepsilon|^2 dx + \|(\nabla \cdot m_\varepsilon)\mathbf{1}_\Omega\|_{\dot{H}^{-1/2}(\mathbf{R}^2)}^2, \quad m_\varepsilon \in H^1(\Omega, S^1).$$

This model was analyzed in [11], [21], [18]. In particular, it is proved in [18] that a vortex configuration  $m_0(x) = \frac{x^\perp}{|x|}$  in  $B^2$  is a zero-energy state, i.e., there exists a family  $\{m_\varepsilon \in H^1(B^2, S^1)\}$  such that  $m_\varepsilon \rightarrow m_0$  a.e. in  $B^2$  and  $F_\varepsilon(m_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The role of Theorem 3 is to generalize this approximation result for every vector field  $m \in W^{1,1}$  (resp.  $m \in H^{1/2}$ ) satisfying (1).

Most of the results of the paper have been announced in [16].

The outline of the paper is the following: in Section 3, we define a class of entropies that is used in the proof of Proposition 3 and Theorem 1. In Section 4, we present several examples of vector fields satisfying (1); in particular, we prove Propositions 1 and 2. Section 5 deals with the proof of non-density results in Propositions 5 and 4, while in Section 6, we present the proofs of density results in Theorems 2 and 3. We finish with Section 7 where we recall some properties of Sobolev spaces.

### 3 Entropies. Proof of Theorem 1.

The starting point consists in regarding the structure (1) of our configurations as a scalar conservation law. Indeed, writing  $m = (u, h(u))$  for the flux  $h(u) = \pm\sqrt{1-u^2}$ , the vanishing divergence condition in  $m$  turns into

$$\partial_t u + \partial_s h(u) = 0, \tag{6}$$

where  $(t, s) := (x_1, x_2)$  correspond to (time, space) variables. Let us recall some definitions from the theory of scalar conservation laws. Since the flux  $h$  is nonlinear, there is in general no smooth solution of the Cauchy problem associated to (6). Therefore, the solutions of (6) are to be understood in the sense of distributions and in general, there are infinitely many weak solutions for the Cauchy problem. The concept of entropy solution has been formulated in order to have uniqueness (see Kruřkov [25]). To introduce this notion, the pair (entropy, entropy-flux) is defined as a couple of scalar functions  $(\eta, q)$  such that  $\frac{dq}{ds} = \frac{dh}{ds} \frac{d\eta}{ds}$  and for every smooth solution  $u$  of (6), the entropy production vanishes, i.e.,

$$\partial_t[\eta(u)] + \partial_s[q(u)] = 0.$$

A solution  $u$  of (6) (in the sense of distributions) is called entropy solution if for every convex entropy  $\eta$ , the entropy production  $\partial_t[\eta(u)] + \partial_s[q(u)] \leq 0$  is a nonpositive measure. Such solutions  $u$  have the property that for every pair  $(\eta, q)$ , the entropy production concentrates on lines (corresponding to "shocks" of  $u$ ). It suggests the interest of using "global" quantities  $(\eta, q)$  to detect "local" line-singularities of  $u$ . This idea has been used by Jin-Kohn [24], Aviles-Giga [3] DeSimone-Kohn-Müller-Otto [13], Ambrosio-DeLellis-Mantegazza [1], Ignat-Merlet [20, 19].

In the sequel we will always use the following notion of entropy introduced in [13] (see also [10], [19]). It corresponds to the pair (entropy, entropy-flux) from the scalar conservation laws, but the pair is defined in terms of the couple  $(u, h(u))$ .

**Definition 1** ([13]) *We will say that  $\Phi \in C^\infty(S^1, \mathbf{R}^2)$  is an entropy if*

$$\frac{d}{d\theta}\Phi(z) \cdot z = 0, \quad \text{for every } z = e^{i\theta} = (\cos \theta, \sin \theta) \in S^1. \quad (7)$$

Here,  $\frac{d}{d\theta}\Phi(z) = \frac{d}{d\theta}[\Phi(e^{i\theta})]$  stands for the angular derivative of  $\Phi$ . The set of all entropies is denoted by *ENT*.

This notion is coherent with the property that a smooth vector field  $m$  satisfying (1) induces vanishing entropy production  $\nabla \cdot [\Phi(m)] = 0$ . In fact, it is equivalent to Definition 1 as stated in the following property:

**Proposition 6** *Let  $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ . Then  $\Phi$  is an entropy if and only if for every  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ), the following identity holds:*

$$\nabla \cdot [\Phi(m)] = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (8)$$

**Proof.** We divide the proof in several steps:

*Step 1.* If  $\Phi \in ENT$  and  $m \in W_{div}^{1,1}(\Omega, S^1)$ , then (8) holds. Indeed, let us consider an SBV lifting  $\Theta$  of  $m$  in  $\Omega$ , i.e.,  $m = e^{i\Theta}$  in  $\Omega$  (see e.g. [14], [7], [9], [17]). For the SBV-function  $\Theta$ , the measure  $D\Theta$  splits into two terms

$$D\Theta = D^a\Theta + (\Theta^+ - \Theta^-)\nu\mathcal{H}^1 \llcorner J(\Theta) \quad (9)$$

where  $D^a\Theta = \nabla^a\Theta\mathcal{H}^2$  is the absolutely continuous part of the measure  $D\Theta$  with respect to two-dimensional Lebesgue measure  $\mathcal{H}^2$  and the last term stands for the jump part concentrated on the  $\mathcal{H}^1$ -rectifiable set  $J(\Theta)$  oriented by the unit normal vector  $\nu$  and the traces of  $\Theta$  on  $J(\Theta)$  with respect to  $\nu$  are denoted by  $\Theta^\pm(x) = \lim_{\varepsilon \downarrow 0} \Theta(x \pm \varepsilon\nu(x))$  in  $L^1_{loc}(J(\Theta))$ . (Recall that a SBV function  $\Theta$  has vanishing Cantor part of the measure  $D\Theta$ .) Since  $m \in W^{1,1}$ , the chain rule applied to  $m = e^{i\Theta}$  implies that  $\Theta^+ - \Theta^- \in L^1(J(\Theta), 2\pi\mathbb{Z})$  and  $\nabla^a\Theta = m \wedge \nabla m \in L^1(\Omega)$ . Moreover, since  $\nabla \cdot m = 0$ , the chain rule also yields  $m^\perp \cdot \nabla^a\Theta = 0$  in  $L^1(\Omega)$ . Therefore, there exists a function  $\lambda := m \cdot \nabla^a\Theta \in L^1(\Omega)$  such that  $\nabla^a\Theta = \lambda m$ . Applying now the chain rule for  $\Phi(e^{i\Theta}) \in W^{1,1}$ , we deduce

$$\nabla \cdot [\Phi(m)] = \nabla \cdot [\Phi(e^{i\Theta})] = \frac{d}{d\theta}\Phi(m) \cdot \nabla^a\Theta = \frac{d}{d\theta}\Phi(m) \cdot m \lambda \stackrel{(7)}{=} 0 \quad \text{in } L^1(\Omega), \quad (10)$$

i.e., (8) holds.

*Step 2.* If  $\Phi \in ENT$  and  $m \in H_{div}^{1/2}(\Omega, S^1)$ , then (8) holds. Indeed, let  $B \subset \Omega$  be an arbitrary ball and let us consider a lifting  $\Theta = \Theta_1 + \Theta_2$  of  $m$  in  $B$  with  $\Theta_1 \in H^{1/2}(B)$ ,  $\Theta_2 \in SBV(B)$  and  $h = e^{i\Theta_2} \in W^{1,1} \cap H^{1/2}(B, S^1)$  (see Brezis-Bourgain-Mironescu [6], Theorem 5). Therefore, the corresponding decomposition (9) of  $\Theta_2$  satisfies  $\Theta_2^+ - \Theta_2^- \in L^1(J(\Theta_2), 2\pi\mathbb{Z})$  and  $\nabla^a\Theta_2 = h \wedge \nabla h \in L^1(B)$ .

**Claim.** For every  $\Psi \in C^2(S^1, \mathbf{R}^2)$ , then  $\Psi(m) \in H^{1/2}(B)$  and

$$\nabla \cdot [\Psi(m)] = \frac{d}{d\theta}\Psi(m) \cdot (\nabla\Theta_1 + \nabla^a\Theta_2) \quad \text{in } \mathbf{H}^{-1/2}(B). \quad (11)$$

Here,  $\mathbf{H}^{-1/2}(B)$  is the dual space of  $H_{00}^{1/2}(B)$  (see Section 7 for more details).



**Proof of Claim.** The fact that if  $m \in H^{1/2}(B)$  then  $\Psi(m) \in H^{1/2}(B)$  is standard and follows from

$$\int_B \int_B \frac{|\Psi(m(x)) - \Psi(m(y))|^2}{|x - y|^3} dx dy \leq \|\nabla \Psi\|_{L^\infty}^2 \int_B \int_B \frac{|m(x) - m(y)|^2}{|x - y|^3} dx dy < \infty.$$

Therefore,  $\nabla \cdot [\Psi(m)] \in \mathbf{H}^{-1/2}(B)$  (since the differential operator is continuous from  $H^{1/2}(B)$  to  $\mathbf{H}^{-1/2}(B)$ , see e.g. Lions-Magenes [26]). Before proving (11), let us observe that the RHS of (11) is a distribution (a-priori, it doesn't belong to  $\mathbf{H}^{-1/2}(B)$ ). Indeed, one has that  $\frac{d}{d\theta} \Psi(m) \in H^{1/2} \cap L^\infty(B)$  (here we use that  $\Psi \in C^2(S^1)$  so that  $\frac{d}{d\theta} \Psi \in C^1(S^1)$ ); therefore,  $\frac{d}{d\theta} \Psi(m) \cdot \nabla^a \Theta_2 \in \mathcal{D}'(B)$  as a duality product between  $L^\infty$  and  $L^1$ , while  $\frac{d}{d\theta} \Psi(m) \cdot \nabla \Theta_1 \in \mathcal{D}'(B)$  since for every test function  $\zeta \in C_c^\infty(B)$ , one has

$$\langle \frac{d}{d\theta} \Psi(m) \cdot \nabla \Theta_1, \zeta \rangle_{(\mathcal{D}'(B), C_c^\infty(B))} = \langle \nabla \Theta_1, \zeta \frac{d}{d\theta} \Psi(m) \rangle_{(\mathbf{H}^{-1/2}(B), H_{00}^{1/2}(B))}. \quad (12)$$

In order to prove (11), we consider an approximating sequence  $\Theta_{1,n} \in C^1(\bar{B})$  such that  $\Theta_{1,n} \rightarrow \Theta_1$  in  $H^{1/2}(B)$ . We set  $m_n = e^{i(\Theta_{1,n} + \Theta_2)} \in W^{1,1} \cap H^{1/2}(B, S^1)$ . Applying the chain rule as in (10) for  $\Psi(m_n) \in W^{1,1}$ , we obtain

$$\nabla \cdot [\Psi(m_n)] = \frac{d}{d\theta} \Psi(m_n) \cdot (\nabla \Theta_{1,n} + \nabla^a \Theta_2) \quad \text{in } L^1 \cap \mathbf{H}^{-1/2}(B).$$

We want to pass to the limit  $n \rightarrow \infty$  in order to get to (11). For that, we have  $e^{i\Theta_{1,n}} \rightarrow e^{i\Theta_1}$  in  $H^{1/2}(B)$  and we deduce that  $m_n \rightarrow m$  in  $H^{1/2}(B)$  by Proposition 10 (see Section 7) and also, a.e. in  $B$  (up to a subsequence). The continuity of the differential operator from  $H^{1/2}$  to  $\mathbf{H}^{-1/2}$  combined with Proposition 10 lead to  $\nabla \cdot [\Psi(m_n)] \rightarrow \nabla \cdot [\Psi(m)]$  in  $\mathbf{H}^{-1/2}(B)$ . On the other hand, the same arguments lead to  $\frac{d}{d\theta} \Psi(m_n) \rightarrow \frac{d}{d\theta} \Psi(m)$  in  $H^{1/2}(B)$  and a.e. in  $B$  (up to a subsequence); thus, by duality as in (12), one has  $\frac{d}{d\theta} \Psi(m_n) \cdot \nabla \Theta_{1,n} \rightarrow \frac{d}{d\theta} \Psi(m) \cdot \nabla \Theta_1$  in  $\mathcal{D}'(B)$  and by dominated convergence theorem, one also deduces that  $\frac{d}{d\theta} \Psi(m_n) \cdot \nabla^a \Theta_2 \rightarrow \frac{d}{d\theta} \Psi(m) \cdot \nabla^a \Theta_2$  in  $L^1(B)$ , which yields (11).  $\square$

Coming back to Step 2, (11) applied for  $\Psi(z) = z$  for  $z \in S^1$  yields  $m^\perp \cdot (\nabla \Theta_1 + \nabla^a \Theta_2) = 0$  in  $\mathbf{H}^{-1/2}(\Omega)$  since  $\nabla \cdot m = 0$ . Idem, defining  $\Psi(z) = -z^\perp$  for  $z \in S^1$ , (11) leads to  $\lambda := m \cdot (\nabla \Theta_1 + \nabla^a \Theta_2) = \nabla \cdot [\Psi(m)] \in \mathbf{H}^{-1/2}(B)$ . Therefore, one writes (as at Step 1)  $\nabla \Theta_1 + \nabla^a \Theta_2 = m \lambda$  which belongs to  $\mathcal{D}'(B)$  by duality as in (12). As in (10), we apply (11) to obtain

$$\nabla \cdot [\Phi(m)] = \frac{d}{d\theta} \Phi(m) \cdot m \lambda = 0 \quad \text{in } \mathcal{D}'(B).$$

Since  $B$  is an arbitrary ball in  $\Omega$ , using a partition of unity, we conclude that (8) holds in  $\mathcal{D}'(\Omega)$ .

*Step 3.* Conversely, let  $\Phi \in C^\infty(S^1, \mathbf{R}^2)$  such that (8) holds for every  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ). Set  $z \in S^1$ . We prove that (7) holds for  $z$  using the same argument as in [20, 19]. Up to translations, we may assume that  $\Omega$  contains the origin 0. Motivated by (10), we consider a map  $m$  given by the vortex structure centered at  $z^\perp$ , i.e.,

$$m(x) := \left( \frac{x - z^\perp}{|x - z^\perp|} \right)^\perp \quad \text{for } x \in \Omega.$$

Then  $m \in W_{div}^{1,p}(\Omega, S^1)$  for every  $p \in [1, 2)$  (in particular,  $m \in H_{div}^{1/2}(\Omega, S^1)$ ) and  $m(0) = z$ . Moreover, since  $m$  is smooth around the origin 0,  $m$  has a smooth lifting  $\Theta$  around 0 (unique up to a constant) that satisfies  $\nabla\Theta(0) = z$ . Then by (8) we know that  $\nabla \cdot [\Phi(m)](0) = 0$ . Therefore, as in (10), we obtain

$$\frac{d}{d\theta}\Phi(z) \cdot z = \frac{d}{d\theta}\Phi(m(0)) \cdot \nabla\Theta(0) = \nabla \cdot [\Phi(m)](0) = 0.$$

□

**Remark 5** A way to construct entropies is given by the following equivalent definition: if  $\Phi \in C^\infty(S^1, \mathbf{R}^2)$ , then  $\Phi \in ENT$  is an entropy if and only if there exists a (unique)  $2\pi$ -periodic  $\varphi \in C^\infty(\mathbf{R})$  such that for every  $z = e^{i\theta} \in S^1$ ,

$$\Phi(z) = \varphi(\theta)z + \frac{d\varphi}{d\theta}(\theta)z^\perp \quad (13)$$

(see details in [13, 19]).

As a consequence, we prove the kinetic formulation (4):

**Proof of Proposition 3.** For every  $\xi \in S^1$ , we define "elementary entropies"  $\Phi^\xi : S^1 \rightarrow \mathbf{R}^2$  given by

$$\Phi^\xi(z) := \xi \tilde{\chi}(z, \xi) = \begin{cases} \xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \leq 0. \end{cases}$$

Although  $\Phi^\xi$  is not a smooth entropy (in fact,  $\Phi^\xi$  has a jump at the points  $\pm\xi^\perp \in S^1$ ), the equality (7) trivially holds in  $\mathcal{D}'(S^1)$ . That's why  $\Phi^\xi$  is a generalized entropy. Moreover, as shown in [13], there exists a sequence of smooth entropies  $\{\Phi_k\} \subset ENT$  such that  $\{\Phi_k\}$  is uniformly bounded and  $\lim_k \Phi_k(z) = \Phi^\xi(z)$  for every  $z \in S^1$ . Indeed, this smoothing result follows by (13): if one writes  $\xi = e^{i\theta_0}$  with  $\theta_0 \in (-\pi, \pi]$ , then the unique  $2\pi$ -periodic function  $\varphi \in C(\mathbf{R})$  satisfying (13) for  $\Phi^\xi$  is given by:

$$\varphi(\theta) = \xi \cdot z \mathbf{1}_{\{z \cdot \xi > 0\}} = \cos(\theta - \theta_0) \mathbf{1}_{\{\theta - \theta_0 \in (-\pi/2, \pi/2)\}} \quad \text{for } z = e^{i\theta}, \theta \in (-\pi + \theta_0, \pi + \theta_0).$$

By (13) for  $\Phi^\xi$ , the choice of  $\varphi'$  is fixed at the jump points  $\pm\xi^\perp \in S^1$ :

$$\varphi'(\theta) = -\sin(\theta - \theta_0) \mathbf{1}_{\{\theta - \theta_0 \in (-\pi/2, \pi/2)\}} \quad \text{for } \theta \in (-\pi + \theta_0, \pi + \theta_0).$$

Now, one regularizes  $\varphi$  by  $2\pi$ -periodic functions  $\varphi_k \in C^\infty(\mathbf{R})$  that are uniformly bounded in  $W^{1,\infty}(\mathbf{R})$  and  $\lim_k \varphi_k(\theta) = \varphi(\theta)$  as well as  $\lim_k \varphi_k'(\theta) = \varphi'(\theta)$  for every  $\theta \in \mathbf{R}$ . Thus, the desired (smooth) approximating entropies  $\Phi_k$  are given by  $\varphi_k$  via (13). Therefore, Proposition 6 implies that for every  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ), one has  $\int_\Omega \Phi_k(m) \cdot \nabla\zeta \, dx = 0$  for every  $\zeta \in C_c^\infty(\Omega)$  and by the dominated convergence theorem, one concludes that

$$0 = \nabla \cdot [\Phi^\xi(m)] = \nabla \cdot [\xi \tilde{\chi}(m, \xi)] = \nabla \cdot [\xi \chi(\cdot, \xi)] = \xi \cdot \nabla \chi(\cdot, \xi).$$

□

**Proof of Theorem 1.** It is a consequence of Proposition 3 combined with the strategy of Jabin-Otto-Perthame (see Theorem 1.3 in [22]). For completeness of the writing, let us recall the main steps of that argument: let  $m : \Omega \rightarrow S^1$  be a measurable function that satisfies (4) for every  $\xi \in S^1$ . Notice that the divergence-free condition is automatically satisfied (in  $\mathcal{D}'(\Omega)$ ) because of (5). The first step consists in defining a  $L^\infty$ -trace of  $m$  on each segment  $\Sigma \subset \Omega$ . More precisely, if  $\Sigma := \{0\} \times [-1, 1] \subset \Omega$ , then there exists a trace  $\tilde{m} \in L^\infty(\Sigma, S^1)$  such that

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_{-r}^r \int_{-1}^1 |m(x_1, x_2) - \tilde{m}(x_2)| dx_2 dx_1 = 0$$

and for each Lebesgue point  $(0, x_2) \in \Sigma$  of  $m$ , one has  $m(0, x_2) = \tilde{m}(x_2)$ . Observe that this step is straightforward in the case of  $m \in W_{div}^{1,1}(\Omega, S^1)$ ; however, it is essential for the case of  $m \in H_{div}^{1/2}(\Omega, S^1)$ . The second step is to prove that the trace  $\tilde{m}$  of  $m$  on  $\Sigma$  is almost everywhere orthogonal at  $\Sigma$  (which coincides with the classical principle of characteristics for smooth vector fields  $m$ ). The key point for that resides in a relation of order of characteristics of  $m$ , i.e., for every two Lebesgue points  $x, y \in \Omega$  of  $m$  with the segment  $[x, y] \subset \Omega$ , the following implication holds:

$$m(x) \cdot (y - x) > 0 \Rightarrow m(y) \cdot (y - x) > 0.$$

The final step consists in proving that on any open convex subset  $\omega \subset \Omega$  with  $d = \text{dist}(\omega, \partial\Omega) > 0$ , only two situations may occur: either two characteristics of  $m$  intersect at  $P \in \omega$  with  $\text{dist}(P, \omega) < d$  and  $m(x) = \pm \frac{(x-P)^\perp}{|x-P|}$  for  $x \in \omega \setminus \{P\}$ , or  $m$  is  $1/d$ -Lipschitz in  $\omega$ , i.e.,

$$|m(x) - m(y)| \leq \frac{1}{d}|x - y|, \quad \text{for every } x, y \in \omega$$

(in this case, every two characteristics passing through  $\omega$  may intersect only at distances  $\geq d$  outside  $\omega$ ). Notice that  $m$  may have infinitely many vortex points  $P_k$  and any vortex point has degree one, but the orientation  $\alpha_k$  of the vortex  $P_k$  could change or not in  $\Omega$  (see Section 4).  $\square$

## 4 Several examples

### 4.1 Lipschitz vector fields (1) that are not $C^1$

**Proof of Proposition 1.** Let  $\Theta : (0, 1) \rightarrow (\frac{\pi}{4}, \frac{\pi}{3})$  be a Lipschitz function that is not in  $C^1(0, 1)$ . On the "space" axis  $s$ , we define  $m$  as given by

$$m(s, 0) = (\cos \Theta(s), \sin \Theta(s)) = e^{i\Theta(s)} \in S^1 \quad \text{for every } s \in (0, 1).$$

Then  $m$  has a unique Lipschitz extension satisfying (1): The initial value (at "time"  $t = 0$ ) of  $m$  determines the characteristics along which  $m$  remains constant. More precisely, we define the flow of characteristics  $F : \text{Dom}(F) = (0, 1) \times (-\frac{1}{3\|\frac{d}{ds}\Theta\|_{L^\infty}}, \frac{1}{3\|\frac{d}{ds}\Theta\|_{L^\infty}}) \rightarrow \mathbf{R}^2$  as

$$F(s, t) = (s, 0) + tm(s, 0)^\perp = s + ite^{i\Theta(s)} \in \mathbf{C} \quad \text{for every } (s, t) \in \text{Dom}(F).$$

The choice of "time" range is done in order that  $F$  is a bi-Lipschitz homeomorphism onto its open range, denoted  $\Omega$  (which implies that characteristics of  $m$  do not intersect in the domain  $\Omega$ ).

Indeed, one computes that

$$\nabla F(s, t) = \left( \frac{\partial}{\partial s} F \quad \frac{\partial}{\partial t} F \right) = \begin{pmatrix} 1 - t \cos \Theta(s) \frac{d}{ds} \Theta(s) & -\sin \Theta(s) \\ -t \sin \Theta(s) \frac{d}{ds} \Theta(s) & \cos \Theta(s) \end{pmatrix}$$

with

$$\det \nabla F(s, t) = \cos \Theta(s) - t \frac{d}{ds} \Theta(s) \geq \frac{1}{2} - \left| t \frac{d}{ds} \Theta(s) \right| \geq \frac{1}{6} \quad \text{in } \text{Dom}(F).$$

Here, we used that

$$|t| \leq \frac{1}{3 \left\| \frac{d}{ds} \Theta \right\|_{L^\infty}} \quad \text{and} \quad \cos \Theta(s) \geq 1/2. \quad (14)$$

Therefore,  $F$  is a local bi-Lipschitz homeomorphism. In order to be a global homeomorphism, we show that  $F$  is injective on  $\text{Dom}(F)$ . Assume by contradiction that there exist two points  $(s, t) \neq (\tilde{s}, \tilde{t})$  in  $\text{Dom}(F)$  so that  $F(s, t) = F(\tilde{s}, \tilde{t})$ . Then  $t, \tilde{t} \neq 0$  and  $\frac{\tilde{t}}{t} = \frac{\cos \Theta(s)}{\cos \Theta(\tilde{s})}$ . It follows that

$$\begin{aligned} |s - \tilde{s}| &= \left| t(\sin \Theta(s) - \frac{\tilde{t}}{t} \sin \Theta(\tilde{s})) \right| = \left| \frac{t}{\cos \Theta(\tilde{s})} \sin(\Theta(s) - \Theta(\tilde{s})) \right| \\ &\leq \frac{2}{3 \left\| \frac{d}{ds} \Theta \right\|_{L^\infty}} |\Theta(s) - \Theta(\tilde{s})| \leq \frac{2}{3} |s - \tilde{s}| \end{aligned}$$

which would mean that  $s = \tilde{s}$  and then,  $t = \tilde{t}$  which is a contradiction. (In the above inequalities, we used again (14).) Denoting by  $G = (G_1, G_2) : \Omega \rightarrow \text{Dom}(F)$  the inverse of  $F$ , we have that

$$\nabla G(F(s, t)) = \frac{1}{\cos \Theta(s) - t \frac{d}{ds} \Theta(s)} \begin{pmatrix} \cos \Theta(s) & \sin \Theta(s) \\ t \sin \Theta(s) \frac{d}{ds} \Theta(s) & 1 - t \cos \Theta(s) \frac{d}{ds} \Theta(s) \end{pmatrix}. \quad (15)$$

We now define  $m : \Omega \rightarrow S^1$  by

$$m(F(s, t)) = m(s, 0) \quad \text{for every } (s, t) \in \text{Dom}(F).$$

Obviously,  $m$  is a Lipschitz function in  $\Omega$  with values in  $S^1$ . Since the open segment  $(0, 1) \times \{0\} \subset \Omega$  and  $\Theta$  is not  $C^1$  in  $(0, 1)$ , then  $m$  is not a  $C^1$  map in  $\Omega$ . Finally, we check that  $\nabla \cdot m = 0$  in  $\Omega$ . Indeed, if  $(x_1, x_2) = F(s, t) \in \Omega$ , then  $m(x) = m(G_1(x), 0) = e^{i\Theta(G_1(x))}$  and

$$\begin{aligned} \nabla \cdot m(x) &= \frac{\partial}{\partial x_1} [\cos \Theta(G_1(x))] + \frac{\partial}{\partial x_2} [\sin \Theta(G_1(x))] \\ &= \left( -\sin \Theta(s) \frac{\partial}{\partial x_1} G_1(F(s, t)) + \cos \Theta(s) \frac{\partial}{\partial x_2} G_1(F(s, t)) \right) \frac{d}{ds} \Theta(s) \stackrel{(15)}{=} 0. \end{aligned}$$

□

## 4.2 $H^1$ vector fields (1) that are not globally Lipschitz

**The case of non-convex domains  $\Omega$ .** We consider the kink domain

$$\Omega = \{r e^{i\theta} : r \in (0, 1), |\theta| < r\}$$

and the boundary vortex configuration in the origin:  $m(x) = \frac{x^\perp}{|x|}$  for  $x \in \Omega$ . Then  $m \in H^1(\Omega, S^1)$  and  $m$  is not globally Lipschitz in  $\Omega$  (but only locally Lipschitz).

**The case of convex domains  $\Omega$ .** There exist a convex domain  $\Omega$  and a vector field  $m \in H_{div}^1(\Omega, S^1)$ , such that  $m$  is not globally Lipschitz in  $\Omega$ . Indeed, we use the construction in the proof of Proposition 1. Let  $\Theta : (0, 1/10) \rightarrow (-\infty, 0)$  be given by  $\Theta(s) = -\sqrt{s}$  for  $s \in (0, 1/10)$ . On the "space" axis  $(0, 1) \times \{0\}$ , we define  $m$  by

$$m(s, 0) = (\cos \Theta(s), \sin \Theta(s)) = e^{i\Theta(s)} \in S^1 \quad \text{for every } s \in (0, 1/10).$$

The flow of characteristics is defined by

$$F : \text{Dom}(F) = \{(s, t) : s \in (0, 1/10), 0 < t < \sqrt{s}/10\} \rightarrow \mathbf{R}^2$$

as

$$F(s, t) = (s, 0) + tm(s, 0)^\perp = s + ite^{i\Theta(s)} \in \mathbf{C} \quad \text{for every } (s, t) \in \text{Dom}(F).$$

Then  $F$  is locally a bi-Lipschitz homeomorphism since

$$\det \nabla F(s, t) = \cos \Theta(s) - t \frac{d}{ds} \Theta(s) \geq \cos \Theta(1/10) > 0 \quad \text{in } \text{Dom}(F).$$

We show that no characteristics intersect inside  $\text{Dom}(F)$  (i.e.,  $F$  is injective in  $\text{Dom}(F)$ ): Assume by contradiction that there exist two points  $(s, t) \neq (\tilde{s}, \tilde{t})$  in  $\text{Dom}(F)$  so that  $F(s, t) = F(\tilde{s}, \tilde{t})$ . Then  $t, \tilde{t} \neq 0$  and  $\frac{\tilde{t}}{t} = \frac{\cos \Theta(s)}{\cos \Theta(\tilde{s})}$  which would imply

$$|s - \tilde{s}| = |t(\sin \Theta(s) - \frac{\tilde{t}}{t} \sin \Theta(\tilde{s}))| = \left| \frac{t}{\cos \sqrt{\tilde{s}}} \sin(\sqrt{s} - \sqrt{\tilde{s}}) \right| \leq \frac{\sqrt{s}}{5} \frac{|s - \tilde{s}|}{\sqrt{s} + \sqrt{\tilde{s}}} \leq \frac{1}{5} |s - \tilde{s}|$$

and would mean that  $s = \tilde{s}$  and then  $t = \tilde{t}$  which is a contradiction. Here, we used that  $0 < t < \frac{\sqrt{s}}{10}$  and  $\cos \Theta(s) \geq 1/2$  if  $s \in (0, 1/10)$ . Let  $\Omega$  be any convex domain included in  $F(\text{Dom}(F))$  such that  $\tilde{\Omega}$  contains the segment  $[0, 1/10] \times \{0\}$  and  $G : \Omega \rightarrow F^{-1}(\Omega)$  be the inverse of  $F$ . Then (15) holds and we set

$$m(F(s, t)) = m(s, 0) \quad \text{for every } (s, t) \in G(\Omega).$$

We compute that

$$\nabla m(F(s, t)) = \left( \frac{dm}{ds}(s, 0) \quad \mathbf{0}_{\mathbf{R}^2} \right) \cdot \nabla G(F(s, t))$$

so that (15) implies

$$|\nabla m|(F(s, t)) = \frac{1}{\det \nabla F(s, t)} \left| \frac{dm}{ds}(s) \right| = \frac{1}{\cos(\sqrt{s}) + \frac{t}{2\sqrt{s}}} \left| \frac{d\Theta}{ds}(s) \right|.$$

Then

$$\begin{aligned} \int_{\Omega} |\nabla m(x)|^2 dx &= \int_{G(\Omega)} |\nabla m|^2(F(s, t)) \det \nabla F(s, t) ds dt \\ &\leq \int_0^{1/10} \left| \frac{d\Theta}{ds}(s) \right| ds \int_0^{\sqrt{s}/10} \frac{1}{\cos(\sqrt{s}) - t \frac{d\Theta}{ds}(s)} \left| \frac{d\Theta}{ds}(s) \right| dt \\ &\leq \int_0^{1/10} \frac{ds}{2\sqrt{s}} \left( \ln(\cos(\sqrt{s}) + 1/20) - \ln(\cos(\sqrt{s})) \right) < \infty. \end{aligned}$$

Therefore,  $m \in H_{div}^1(\Omega, S^1)$  and  $m$  is not globally Lipschitz in  $\Omega$  since  $\frac{d\Theta}{ds}$  blows up at  $s = 0$ .  $\square$

### 4.3 Vector fields (1) with arbitrary many vortices

The geometry of the domain  $\Omega$  determines the number of vortices of a vector field (1).

**Remark 6** (i) If  $\Omega$  is a convex domain and  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ), then  $m$  has either no (interior) vortices, or one interior vortex in  $\Omega$ . It is a consequence of the final step in the proof of Theorem 1 (see [22] for more details).

(ii) If  $\Omega \neq \mathbf{R}^2$  is a smooth simply connected domain and additionally we impose the boundary condition  $m \cdot n = 0$  on  $\partial\Omega$ , then either  $\Omega$  is a disk and  $m$  has a vortex placed in the center of the disk, or  $\Omega$  is a strip and  $m$  is constant (see [22]).

**Proof of Proposition 2.** First of all, we construct domains  $\Omega_n$  and  $W^{1,p}$  vector fields (1) defined on  $\Omega_n$  with  $n$  vortices for every  $n \geq 1$ . For  $n = 1$ , we choose  $\Omega_1 := B(0, 2)$  to be the ball centered at the origin of radius 2 and  $m_1(x) = \frac{x^\perp}{|x|}$  in  $\Omega_1$ . Then  $m_1 \in W_{div}^{1,p}(\Omega_1, S^1)$  and  $\|\nabla m_1\|_{L^p(\Omega_1)}^p = 2\pi \frac{2^{2-p}}{2-p}$ , for any  $p \in [1, 2)$ . For  $n = 2$ , the construction is the following: Set  $f, g : \mathbf{R} \rightarrow \mathbf{R}$  with  $f(t) = |t|$  and

$$g(t) = \begin{cases} 2(t+1) & \text{for } t \leq -1 \\ -\frac{1}{2}(t+1) & \text{for } t \geq -1 \end{cases}$$

and define the curves

$$\gamma^+ = \{(x_1, f(x_1)) : x_1 \in [-2, 1]\} \quad \text{and} \quad \gamma^- = \{(x_1, g(x_1)) : x_1 \in [-2, 1]\}.$$

Fixing the vortex points  $P_1 := (-2, 0)$  and  $P_2 := (1, 0)$ , we define the domain

$$\begin{aligned} \Omega_2 := & \left( B(P_1, 2) \cap \{x_1 \leq -2\} \right) \cup \left\{ (x_1, x_2) : x_1 \in [-2, 1], g(x_1) < x_2 < f(x_1) \right\} \\ & \cup \left( B(P_2, 1) \cap \{x_1 \geq 1\} \right) \end{aligned}$$

(see Figure 3). We define  $m_2 : \Omega_2 \rightarrow S^1$  as follows:

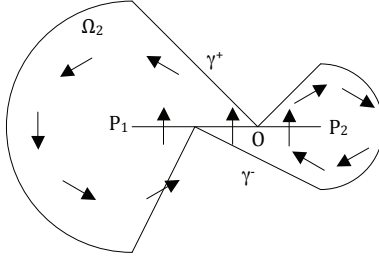


Figure 3: Two vortices of different orientation.

$$m_2(x) = \begin{cases} \frac{(x-P_1)^\perp}{|x-P_1|} & \text{in } \left\{ x \in \Omega_2 : x_1 \leq -1 \text{ or } \left( x_1 \in (-1, 0) \text{ and } x_2 > 0 \right) \right\}, \\ -\frac{(x-P_2)^\perp}{|x-P_2|} & \text{in } \left\{ x \in \Omega_2 : x_1 \geq 0 \text{ or } \left( x_1 \in (-1, 0) \text{ and } x_2 < 0 \right) \right\}. \end{cases}$$

Then  $m_2 \in W_{div}^{1,p}(\Omega, S^1)$  has two vortices in  $P_1$  and  $P_2$  of degree 1, but different orientation. One easily checks that  $\|\nabla m_2\|_{L^p(\Omega_1)}^p \lesssim \frac{1+2^{2-p}}{2-p}$ ,  $p \in [1, 2)$ .

For arbitrary  $n$ , the above construction is to be repeated by an inductive argument: for every positive integer  $n$ , we construct a non-convex Lipschitz simply-connected domain  $\Omega_n$  and a vector field  $m_n \in W_{div}^{1,p}(\Omega_n, S^1)$  having  $n$  vortex singularities and such that there exists a vortex point  $P_n \in \Omega_n$  where  $(P_n + \mathbf{R}_+ \times \mathbf{R}) \cap \Omega_n$  is a half disk of radius  $2^{-n+2}$  and  $m_n(x) = (-1)^{n-1} \frac{(x-P_n)^\perp}{|x-P_n|}$  for every  $x \in (P_n + \mathbf{R}_+ \times \mathbf{R}) \cap \Omega_n$  and

$$\|\nabla m_n\|_{L^p(\Omega_1)}^p \lesssim \frac{2^{2-p} + \sum_{k=0}^{n-2} 2^{-k(2-p)}}{2-p} \leq C(p), \quad p \in [1, 2),$$

where the constant  $C(p) > 0$  is independent of  $n$ .

Letting  $n \rightarrow \infty$ , one gets

$$\Omega = \{x \in \mathbf{R}^2 : \exists n(x) \geq 1, \forall n \geq n(x), x \in \Omega_n\} \quad (16)$$

which is a non-convex piecewise Lipschitz simply-connected domain. We define  $m : \Omega \rightarrow S^1$  as follows: for every  $x \in \Omega$ , set  $m(x) = m_n(x)$ , for  $n \geq n(x)$  (given in (16)). Then  $m \in W_{div}^{1,p}(\Omega, S^1)$  for every  $p \in [1, 2)$  and has infinitely many vortices  $\{P_1, P_2, \dots\}$ . (Here, the sequence of vortices  $\{P_k\}$  accumulates on some point  $P \in \partial\Omega$ , therefore  $\partial\Omega \setminus \{P\}$  is a Lipschitz curve, but not  $\partial\Omega$ ).  $\square$

**Remark 7** In the above construction, the vortex singularities have alternative orientations. However, one can construct domains where the vortex singularities have the same orientation. Here is the example of two vortex configuration in  $P_1 = (-3, 0)$  and  $P_2 = (1, 0)$ :  $\omega = \omega_1 \cup \omega_2 \cup \omega_3$  with  $\omega_1$  be a union of a square and a rectangle

$$\omega_1 := \left( (-5, -1) \times (-2, 2) \right) \cup \left( (-5, -3) \times (-4, -2) \right)$$

and  $\omega_2 := \left( (0, 2) \times (-1, 1) \right) \cup \left( (1, 2) \times (-4, -1) \right)$  and  $\omega_3 = (-3, 1) \times (-4, -3)$ . Then choose

$$m(x) = \begin{cases} \frac{(x-P_1)^\perp}{|x-P_1|} & \text{in } \omega_1, \\ \frac{(x-P_2)^\perp}{|x-P_2|} & \text{in } \omega_2, \\ (1, 0) & \text{in } \omega_3. \end{cases}$$

Then  $m \in W_{div}^{1,p}(\omega, S^1)$  for every  $p \in [1, 2)$  (see Figure 4).

Let us explain why in general a domain  $\Omega$  satisfying the properties in Proposition 2 (i.e. admitting configurations with infinitely many vortices) is not Lipschitz.

**Proposition 7** *If  $\Omega$  is a Lipschitz domain and  $m \in W_{div}^{1,1}(\Omega, S^1)$  (or  $m \in H_{div}^{1/2}(\Omega, S^1)$ ), then  $m$  has only a finite number of interior vortex singularities.*

**Proof.** Assume by contradiction that  $m$  has infinitely many interior vortex singularities  $\{P_1, P_2, \dots\}$ . Obviously, discarding a subsequence, we can assume that the points  $P_k$  converge to a point  $P \in \bar{\Omega}$ .

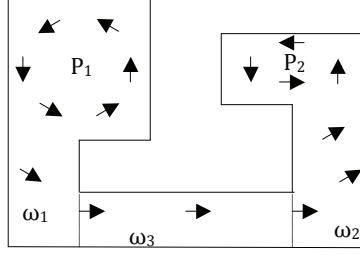


Figure 4: Two vortices with the same orientation.

**Claim:** There exist two cones  $\mathcal{C}_{k_1} \subset \Omega$  and  $\mathcal{C}_{k_2} \subset \Omega$  centered in  $P_{k_1}$  and  $P_{k_2}$  (of some positive height and positive angle) such that  $\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}$  has nonempty interior.

**Proof of Claim:** The case of a limit point  $P$  belonging to the interior of  $\Omega$  is obvious. Let us suppose that  $P \in \partial\Omega$ . Since  $\partial\Omega$  is Lipschitz, we may assume that there exists a ball  $B$  centered at  $P$  such that  $\partial\Omega \cap B$  is the Lipschitz graph  $\{(x_1, \gamma(x_1)) : x_1 \in (-\delta, \delta)\}$  with  $\gamma(0) = \delta$ ,  $P = (0, \gamma(0))$  and

$$U = \left\{ (x_1, x_2) : 0 \leq x_2 < \gamma(x_1), x_1 \in (-\delta, \delta) \right\} \subset \Omega \cap B.$$

Therefore, for every  $P_k = (x_{1,k}, x_{2,k}) \in U$ , the vertical segment  $S_k$  between  $(x_{1,k}, 0)$  and  $P_k$  belongs to  $U$ . Since  $\gamma$  is a Lipschitz function, there exists an angle

$$\alpha = \alpha(\gamma) := \pi/2 - \arctan\left(\left\| \frac{d\gamma}{dx_1} \right\|_{L^\infty(-\delta, \delta)}\right) > 0$$

such that the cone  $\mathcal{C}_k$  centered at  $P_k$  of angle  $\alpha$  and having  $S_k$  as height is included in  $\Omega$ . Due to the fact that  $P_k$  converges to  $P$ , it follows that there exists  $k_0 \geq 1$  such that two cones  $\mathcal{C}_{k_1}$  and  $\mathcal{C}_{k_2}$  have nonempty interior intersection for any  $k_1, k_2 \geq k_0$ . This finishes the proof of Claim.

The contradiction comes from the structure proved in Theorem 1 since  $m(x) = \pm \frac{(x - P_{k_j})^\perp}{|x - P_{k_j}|}$  in  $\mathcal{C}_{k_1} \cap \mathcal{C}_{k_2}$  for both  $j = 1, 2$  which is absurd.  $\square$

**Remark 8** For a Lipschitz domain  $\Omega$ , we say that a point  $P \in \partial\Omega$  is a *boundary vortex* of  $m$  if there exists a cone  $\mathcal{C} \subset \bar{\Omega}$  (of some height  $> 0$  and angle  $\beta > 0$ ) centered at  $P$  such that  $m(x) = \pm \frac{(x - P)^\perp}{|x - P|}$  for  $x \in \mathcal{C} \setminus \{P\}$ . By Jabin-Otto-Perthame strategy, a boundary vortex  $P$  of  $m$  is always assigned to a maximal cone  $\mathcal{C}$ , in the sense that for every point  $x \in \Omega \setminus \mathcal{C}$  outside the maximal cone, the characteristic of  $m$  passing through  $x$  stays outside the cone  $\mathcal{C}$ . For  $p < 2$ , one can construct  $W^{1,p}$  vector fields (1) in a Lipschitz simply connected domain with infinitely many boundary vortices (that accumulate on the boundary).



## 5 Non-density results. Proof of Propositions 4 and 5

**Proof of Proposition 5.** Assume by contradiction that such a sequence exists. By dominated convergence theorem, it would mean in particular that  $m_n \rightarrow m$  in  $L^1(B^2)$ , i.e.,

$$\int_0^1 \int_{\partial B_r} |m_n - m| d\mathcal{H}^1 dr \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Up to a subsequence, there exists  $0 < r < 1$  such that

$$\int_{\partial B_r} |m_n - m| d\mathcal{H}^1 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (17)$$

The key point of the proof consists in a dynamical system argument related to the topology of the flow of  $m_n^\perp$ : Using the technique in [11, 21, 18], we will consider the autonomous system

$$\dot{X} = m_n^\perp(X). \quad (18)$$

First of all, (18) has no critical point and no cycle (i.e., no closed loop): Since  $|m_n^\perp| = 1$  in  $B^2$  and  $m_n^\perp$  is smooth, the degree of  $m_n^\perp$  on a closed curve in  $B^2$  is zero and therefore, an orbit of (18) cannot be closed in  $B^2$ . Now set  $X_n$  be the orbit of (18) passing by 0 (see Figure 5), i.e.,

$$\begin{cases} \dot{X}_n(t) = m_n^\perp(X_n(t)), \\ X_n(0) = 0. \end{cases}$$

Then either the orbit  $X_n$  reaches the boundary  $\partial B^2$  in finite time, or the limit points of  $X_n$  (see

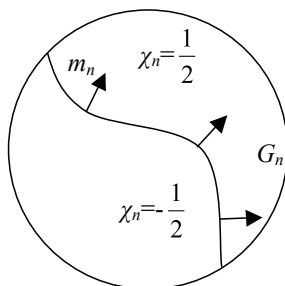


Figure 5: The orbit  $X_n$  of the vector field  $m_n^\perp$  passing by 0 in the ball  $B^2$ .

[8], Chapter 16) belong to the boundary  $\partial B^2$ : Suppose that this is not the case, i.e., there is a limit point inside the ball  $B^2$ . Since (18) has no critical point, Poincaré-Bendixson's theorem (see [8], Theorem 2.1) implies that the limit set of  $X_n$  should contain a periodic orbit which is a contradiction with the nonexistence of cycles for (18). Hence, the orbit  $X_n$  separates the ball  $B_r$  into a right side  $G_n$  (where  $m_n$  is the inner normal vector to  $\partial G_n$ ) and a left side  $B_r \setminus G_n$  (see Figure 5). We define

$$\chi_n = \begin{cases} \frac{1}{2} & \text{in } G_n, \\ -\frac{1}{2} & \text{in } B_r \setminus G_n, \\ 0 & \text{in } B^2 \setminus B_r. \end{cases}$$

Then  $\chi_n \in BV(B^2)$  with

$$\begin{aligned} D\chi_n &= m_n \mathcal{H}^1 \llcorner (\{X_n\} \cap B_r) + \chi_n^- m^\perp \mathcal{H}^1 \llcorner \partial B_r \\ &= m_n |D\chi_n| \llcorner B_r + \chi_n^- m^\perp \mathcal{H}^1 \llcorner \partial B_r \quad \text{in } \mathcal{D}'(B^2), \end{aligned} \quad (19)$$

where  $\chi_n^-$  is the interior trace of  $\chi_n$  on  $\partial B_r$  with respect to the normal outer vector  $\frac{x}{|x|} = -m^\perp(x)$ . Notice that  $|\chi_n^-| = \frac{1}{2}$  on  $\partial B_r$ . Moreover, we have that

$$\int_{B_r} |D\chi_n| = \mathcal{H}^1(\{X_n \in B_r\}) \geq 2r \quad \text{and} \quad \int_{B^2} |D\chi_n| = \int_{B_r} |D\chi_n| + \pi r. \quad (20)$$

Integration by parts leads to

$$\begin{aligned} 2r &\stackrel{(20)}{\leq} \int_{B_r} |D\chi_n| \stackrel{(19)}{=} \int_{B_r} m_n \cdot D\chi_n \\ &= - \int_{\partial B_r} m_n \cdot m^\perp \chi_n^- d\mathcal{H}^1 - \int_{B_r} \nabla \cdot m_n \chi_n dx \\ &\stackrel{(17)}{=} o(1) - \int_{B_r} \nabla \cdot m_n \chi_n dx \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (21)$$

If *a*) holds, then the contradiction follows immediately from (21): since  $|\chi_n| = 1/2$  in  $B_r$ , then  $\int_{B_r} \nabla \cdot m_n \chi_n dx = o(1)$  which would mean that  $o(1) \geq r > 0$  that is absurd. If *b*) holds, we prove first that  $\{\chi_n\}$  is uniformly bounded in  $BV(B^2)$ . Indeed, since  $\{\nabla \cdot m_n\}$  is bounded in  $H^{-s}(B^2)$  (here  $s \in [0, 1/2)$ ), by Gagliardo-Nirenberg's inequality (see Proposition 8 in Section 7) we have

$$\begin{aligned} \left| \int_{B_r} \nabla \cdot m_n \chi_n dx \right| &= \left| \int_{B^2} \nabla \cdot m_n \chi_n dx \right| \leq \|\nabla \cdot m_n\|_{H^{-s}(B^2)} \|\chi_n\|_{H^s(B^2)} \\ &\stackrel{\text{Proposition 8}}{\leq} C \|\nabla \cdot m_n\|_{H^{-s}(B^2)} \left( \|\chi_n\|_{L^\infty(B^2)}^{1/2} \|\chi_n\|_{BV(B^2)}^{1/2} + \|\chi_n\|_{L^\infty(B^2)} \right) \\ &\stackrel{(20)}{\leq} C \left( 1 + \left( \int_{B_r} |D\chi_n| \right)^{1/2} \right). \end{aligned}$$

By (21), we deduce that  $\{\chi_n\}$  is uniformly bounded in  $BV(B_r)$  and also in  $BV(B^2)$  due to (20). Therefore,  $\{\chi_n\}$  is relatively compact in  $H^s(B^2)$  (since  $s \in [0, 1/2)$ , see Proposition 8 in Section 7) and we conclude by assumption *b*) that

$$\int_{B_r} \nabla \cdot m_n \chi_n dx = \int_{B^2} \nabla \cdot m_n \chi_n dx = o(1)$$

which is a contradiction with (21).  $\square$

**Proof of Proposition 4.** Obviously, the statement of Proposition 4 is a direct consequence of Proposition 5 a). However, there is an easier proof in this particular case. More precisely, we assume by contradiction that there exists a sequence of divergence-free vector fields  $m_n \in C^\infty(\Omega, S^1)$  satisfying  $m_n \rightarrow m$  a.e. in  $B^2$ . In particular, dominated convergence theorem leads to  $m_n \rightarrow m$  in  $L^3(B^2)$ . First of all, Poincaré's lemma yields the existence of smooth stream functions  $\varphi_n \in C^\infty(\Omega)$  such that  $m_n = \nabla^\perp \varphi_n$  in  $B^2$ . Since  $|m_n| = 1$ ,  $\varphi_n$  is a 1-Lipschitz function. Observe that  $m = \nabla^\perp \varphi$

with  $\varphi(x) = |x|$  in  $B^2$ . Subtracting eventually a constant, we can assume that  $\int_{B^2} \varphi_n dx = \int_{B^2} \varphi dx$  so that  $\varphi_n \rightarrow \varphi$  uniformly in  $B^2$ : indeed, for every  $x \in B^2$ , one has

$$\begin{aligned} |\varphi_n(x) - \varphi(x)| &= \left| \int_{B^2} [(\varphi_n - \varphi)(x) - (\varphi_n - \varphi)(y)] dy \right| \\ &\leq \int_{B^2} \int_0^1 |x - y| |\nabla(\varphi_n - \varphi)|(x + t(y - x)) dy dt \\ &\leq C \int_0^1 \left( \int_{B^2} |\nabla(\varphi_n - \varphi)|^3(x + t(y - x)) dy \right)^{1/3} dt \\ &\leq C \|m_n - m\|_{L^3(B^2)} \int_0^1 \frac{dt}{t^{2/3}} \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

Since 0 is a strict minimum of  $\varphi$ , it would imply that  $\varphi_n$  has a minimum inside  $B^2$ , i.e.,  $\nabla\varphi_n$  vanishes somewhere inside  $B^2$  which is a contradiction with the fact that  $|\nabla\varphi_n| = 1$  in  $B^2$ .  $\square$

## 6 Dense subsets: Proof of Theorems 2 and 3

We start by proving Theorem 2:

**Proof of Theorem 2.** By Proposition 7, we know that  $m$  has only a finite number of (interior) vortex singularities in  $\Omega$  (call this set  $\mathcal{A}$ ).

*Case 1. There are no interior vortices, i.e.,  $\mathcal{A} = \emptyset$ .* By Theorem 1 we know that  $m$  is locally Lipschitz in  $\Omega$ . For  $d > 0$  small, we choose a smooth simply connected subdomain  $\Omega^d$  of  $\Omega$  that is close to  $\Omega$  in the sense that  $\text{dist}(x, \partial\Omega) \in (d, 3d/2)$  for every  $x \in \partial\Omega^d$ . Then, by Theorem 1, we have that  $m$  is globally Lipschitz in  $\Omega^d$ . For each characteristic of  $m$  passing through a point  $x \in \Omega^d$  we call *extremal points* in  $\bar{\Omega}^d$  the two intersection points  $P(x)$  and  $Q(x)$  of the characteristic with  $\partial\Omega^d$ . By the Jabin-Otto-Perthame procedure (as recalled in the proof of Theorem 1), any two characteristics of  $m$  passing through two points  $x, y \in \Omega^d$  intersect outside  $\Omega^d$  at a distance larger than  $d$  with respect to their extremal points. As a consequence,  $|m(x) - m(y)| \leq \frac{1}{d}|x - y|$  for every  $x, y \in \Omega^d$ , i.e.,  $m$  is globally Lipschitz in  $\Omega^d$  with the Lipschitz constant  $\leq 1/d$  (see [22]).

A standard geometry argument shows the existence of a countable set of segments  $\gamma_j : [0, 1] \rightarrow \bar{\Omega}^d$ ,  $j \in J \subset \mathbb{N}$  such that

- 1) every two (open) segments  $\gamma_j((0, 1))$  and  $\gamma_k((0, 1))$  are disjoint,  $j \neq k$ ;
- 2) the characteristics of  $m$  passing through  $\gamma_j(t)$  for every  $t \in [0, 1]$  and  $j \in J$  cover the whole domain  $\bar{\Omega}^d$ , i.e.,  $\bar{\Omega}^d = \cup_{j \in J, t \in [0, 1]} [P(t)Q(t)]$  where  $P(t) \in \partial\Omega^d$  and  $Q(t) \in \partial\Omega^d$  are the extremal points in  $\bar{\Omega}^d$  of the characteristic passing through the point  $\gamma_j(t)$ .
- 3) for every  $t \in (0, 1)$  and  $j \in J$ , the characteristic  $[P(t)Q(t)]$  passing through  $\gamma_j(t)$  intersects the set of segments  $\cup_{k \in J} \{\gamma_k([0, 1])\}$  only at  $\{\gamma_j(t)\}$ .

One could have two end points  $P_j$  and  $P_k$  of segments  $\gamma_j$  and  $\gamma_k$  that coincide, or the characteristic passing through  $P_j$  may intersect  $\gamma_k$  at the end point  $P_k$ . (There exist smooth vector fields  $m$  with (1) on  $C^2$  simply connected domain such that any set of curves  $\{\gamma_j\}$  satisfying 1), 2), 3) are necessary infinitely countable, i.e.,  $J$  is not finite; one could think of a local boundary given by the graph  $\{(x_1, \gamma(x_1))\}$  with  $\gamma : x_1 \mapsto x_1^4 \sin \frac{1}{x_1}$  around 0 and  $m$  be a small perturbation of the constant vector field  $\mathbf{e}_2$ .)

We consider the lifting  $\Theta$  (unique up to a constant) of  $m$  in  $\Omega^d$ , so  $\Theta \in W^{1,\infty}(\Omega^d)$ . The smoothing procedure is the following: for each segment  $\gamma_j([0,1])$ , we approximate the lifting  $\Theta|_{\gamma_j}$  in  $H^1(\{\gamma_j\})$  (or any  $W^{1,q}$  with  $q < \infty$ ) by a sequence of  $C^\infty$  liftings  $\Theta_n|_{\gamma_j}$  such that  $\Theta_n|_{\partial\gamma_j} = \Theta|_{\partial\gamma_j}$  and the Lipschitz constant of  $\Theta_n$  is less than  $1/(d-1/n)$ . Then we set  $m_n := e^{i\Theta_n}|_{\gamma_j}$  on the segment  $\gamma_j$ ; after that,  $m_n$  is uniquely smoothly extended along the characteristics starting from any point  $\gamma_j(t)$  due to the initial value  $m_n(\gamma_j(t))$  since these characteristics (passing through points  $\gamma_j(t)$ ) could intersect only outside  $\Omega^d$  at a distance  $\geq d - \frac{1}{n}$  with respect to their extremal points in  $\Omega^d$ . The new vector fields  $m_n$  satisfy (1) in  $\Omega^d$  and approximate  $m$  in  $H^1(\Omega^d)$ ; they are smooth on the domain covered by the characteristics passing through  $\gamma_j((0,1))$ . However, globally in  $\Omega^d$ , they could be only Lipschitz (and not smooth) at the end points  $\gamma_j(\{0,1\})$ . Let us call these corresponding characteristics as "bad" characteristics of  $m_n$ . In order to smooth everywhere  $m_n$ , we will proceed as follows: we will restrict to a subdomain  $\tilde{\Omega}^d$  of  $\Omega^d$  such that  $0 < \text{dist}(x, \partial\Omega^d) < d/2$  for every  $x \in \partial\tilde{\Omega}^d$ . Then any "bad" characteristic  $S$  has the following property:  $S \cap \tilde{\Omega}^d$  splits  $\tilde{\Omega}^d$  into two open subdomains  $\omega_n$  and  $\tilde{\omega}_n$  so that  $m_n$  is locally smooth around  $S$  in  $\omega_n$  respectively in  $\tilde{\omega}_n$ . Then one considers a small segment  $\tilde{S} = [A\tilde{A}]$  orthogonal to  $S$  such that the middle point  $M := (A + \tilde{A})/2$  of  $\tilde{S}$  belongs to the "bad" characteristic  $S$  and  $m_n$  is smooth on  $(AM) \subset \omega_n$  respectively  $(\tilde{A}M) \subset \tilde{\omega}_n$ . Considering the lifting  $\Theta_n$  of  $m_n$  on  $\tilde{S}$ , one repeats the same smoothing argument as above, but asking that the new vector field  $\tilde{m}_n$  coincides with  $m_n$  in a neighborhood of  $\partial\tilde{S}$ . Therefore, we conclude that  $m$  can be regularized in  $H^1(\tilde{\Omega}^d)$  by smooth vector fields with (1) defined in  $\tilde{\Omega}^d$ . Letting  $d \rightarrow 0$ , the conclusion follows in Case 1 (the approximation holding in  $H_{loc}^1(\Omega)$ ).

*Case 2. There are interior vortices, i.e.,  $\mathcal{A} \neq \emptyset$ .* We recall that  $\mathcal{A}$  is finite. As before, for  $d > 0$  small, we choose a smooth simply connected subdomain  $\Omega^d$  of  $\Omega$  that is close to  $\Omega$  in the sense that  $\text{dist}(x, \partial\Omega) \in (d, 3d/2)$  for every  $x \in \partial\Omega^d$  and  $\mathcal{A} \subset \Omega^d$  and  $\text{dist}(\mathcal{A}, \partial\Omega^d) \geq d$ . Therefore, all the characteristics passing through  $\Omega^d$  intersect either in a vortex point of  $\mathcal{A}$ , or at a distance larger than  $d$  outside  $\Omega^d$ . We decompose the domain in a partition  $\Omega^d := \Omega_1(m) \cup \Omega_2(m)$  where

$$\Omega_1(m) = \{x \in \Omega^d : \text{there exist } r > 0, P \in \mathcal{A}, m(z) = \pm \frac{(z-P)^\perp}{|z-P|} \text{ for all } z \in B(x, r) \subset \Omega^d\}.$$

Then  $\Omega_1(m)$  is an open subset of  $\Omega^d$ ,  $\bar{\Omega}_1(m)$  contains all the (interior) vortex singularities of  $m$  (i.e.  $\mathcal{A} \subset \bar{\Omega}_1(m)$ ) and  $m \in C^\infty(\Omega_1(m) \setminus \mathcal{A})$ .

To construct the desired  $m_n$  in  $\Omega^d$ , the idea is the following: Set first  $m_n := m$  on  $\Omega_1(m)$  so that  $m_n$  has the same (interior) vortices  $\mathcal{A}$  as  $m$  and  $m_n \in C^\infty(\Omega_1(m) \setminus \mathcal{A})$ . It remains to smooth  $m$  on  $\Omega_2(m)$ ; since  $\mathcal{A}$  is not included in  $\Omega_2(m)$  (in fact,  $\mathcal{A}$  is placed at distance larger than  $d$  outside  $\partial\Omega_2(m)$ ), it means that any two characteristics of  $m$  passing through  $\Omega_2(m)$  intersect at a distance larger than  $d$  outside  $\Omega_2(m)$ , i.e.,  $m$  is globally Lipschitz on  $\Omega_2(m)$  with a Lipschitz constant less than  $1/d$ . We will construct  $m_n$  on  $\Omega_2(m)$  as at Case 1: we find a countable set of segments  $\gamma_j : [0,1] \rightarrow \bar{\Omega}_2(m)$ ,  $j \in J \subset \mathbb{N}$  such that

- 1) every two (open) lines  $\gamma_j((0,1))$  and  $\gamma_k((0,1))$  are disjoint,  $j \neq k$ ;
- 2) the characteristics of  $m$  passing through  $\gamma_j(t)$  for every  $t \in [0,1]$  and  $j \in J$  cover the whole domain  $\bar{\Omega}_2(m)$ , i.e.,  $\bar{\Omega}_2(m) = \cup_{j \in J, t \in [0,1]} P(t)Q(t)$  where  $P(t) \in \partial\Omega_2(m)$  and  $Q(t) \in \partial\Omega_2(m)$  are the extremal points of the characteristic passing through  $\gamma_j(t)$  in  $\Omega_2(m)$ .

3) for every  $t \in (0, 1)$  and  $j \in J$ , the characteristic  $P(t)Q(t)$  passing through  $\gamma_j(t)$  intersects the set of curves  $\cup_{k \in J} \{\gamma_k\}$  only at  $\{\gamma_j(t)\}$ .

It could happen that some lines  $\gamma_j$  have their end points in  $\partial\Omega_1(m)$  (i.e.,  $\gamma_j(\{0, 1\}) \subset \partial\Omega_1(m)$ ). Since  $\Omega_1(m)$  is open, it means that one can extend a little bit  $\gamma_j$  by the line  $\gamma_j \cup \tilde{\gamma}_j$  in the interior of  $\Omega_1(m)$  (i.e.,  $\tilde{\gamma}_j \subset \Omega_1(m)$ ); for such lines, the smoothing process of Case 1 is to be repeated by adding the constraint that the approximation  $m_n$  coincides with  $m$  on  $\tilde{\gamma}_j$ . On the other lines  $\gamma_j$  (not intersecting  $\partial\Omega_1(m)$ ), the smoothing process in Case 1 is to be repeated. As before,  $m_n$  is not smooth around "bad" characteristics. Eventually by considering a subdomain  $\tilde{\Omega}^d \subset \Omega^d$ , one can also smooth  $m_n$  around the "bad" characteristics. Therefore, by letting  $d \rightarrow 0$ , the conclusion follows also in Case 2 (the approximation holding in  $W_{loc}^{1,p}(\Omega)$  for any  $p < 2$ ) and the approximating smooth vector fields  $m_n$  have the same vortices  $\mathcal{A}$  as  $m$ . Moreover, since  $m_n = m$  locally around  $\mathcal{A}$ , one has that for any compact  $K \subset \Omega$  and  $q < \infty$ ,  $\|m_n - m\|_{W^{1,q}(K)} \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**Proof of Theorem 3.** Let  $\mathcal{A} = \{P_k : k \in K\}$  be the set of interior vortices of  $m$  with  $K$  be a finite set. For each interior vortex  $P_k$  of  $m$ , we can find a cone  $\mathcal{C}_k \subset \Omega$  (of center  $A_k$  small angle) such that  $P_k \in \text{int}(\mathcal{C}_k)$  and  $\Omega \setminus \mathcal{C}_k$  is a Lipschitz simply-connected domain. Set  $d_k = \text{dist}(P_k, \partial\mathcal{C}_k) > 0$  and  $\tilde{\Omega} := \Omega \setminus \cup_{k \in K} \mathcal{C}_k$ . The smoothing process is the following: in a first step, we smooth  $m$  inside each cone  $\mathcal{C}_k$  by nonvanishing divergence vector fields, and in a second step we smooth  $m$  by divergence-free vector fields inside  $\tilde{\Omega}$ .

*Step 1. Smoothing inside the cones.* We will use the strategy in [18]. For simplicity of the writing, we suppose that  $P_k = O(0, 0)$  is the origin,  $m(x) = \frac{x^\perp}{|x|}$  in  $\mathcal{C}_k$  and for every small  $\varepsilon$ , denoting

$$\lambda = \lambda(\varepsilon) = \frac{1}{|\ln \varepsilon|^2},$$

we will assume that the cone  $\mathcal{C}_k$  contains the sets

$$\begin{aligned} \omega_1 &= \omega_1(\varepsilon) = \{x \in \mathbf{R}^2 : \lambda < |x| := \sqrt{x_1^2 + x_2^2} < 1, x_2 \geq 0, |x_1| \leq \lambda\} \\ \text{and } \omega_2 &= \omega_2(\varepsilon) = \{x \in \mathbf{R}^2 : |x| < \lambda\} \end{aligned}$$

and  $\partial\Omega \cap \{x_2 > 0, |x_1| \leq \lambda\}$  is contained in  $\bar{\omega}_1$  (see Figure 6). We construct a family  $\{m_\varepsilon \in H^1(\mathcal{C}_k, S^1)\}$  such that  $m_\varepsilon \rightarrow m$  a.e. in  $\mathcal{C}_k$  and  $(\nabla \cdot m_\varepsilon)\mathbf{1}_{\mathcal{C}_k} \rightarrow 0$  in  $\dot{H}^{-1/2}(\mathbf{R}^2)$  as  $\varepsilon \rightarrow 0$ .

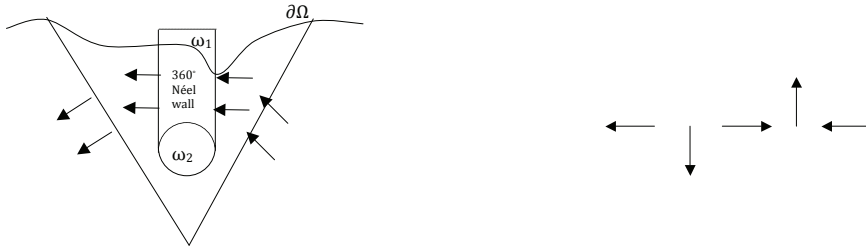


Figure 6: Smooth vortex approximation inside a cone using a 360° Néel wall transition (described in the right picture).

*Definition of  $m_\varepsilon$ .* We will denote the phase of  $m_\varepsilon$  by  $\Theta_\varepsilon$ , i.e.  $m_\varepsilon = e^{i\Theta_\varepsilon}$ . In the region  $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$ ,  $m_\varepsilon$  will coincide with the vortex  $m$  (in particular,  $m_\varepsilon$  is a smooth vector field (1) in  $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$ ). More precisely, in polar coordinates, the phase is given by

$$\Theta_\varepsilon(r, \theta) = \theta + \pi/2 \quad \text{in} \quad \mathcal{C}_k \setminus (\omega_1 \cup \omega_2).$$

In the region  $\omega_1$ ,  $m_\varepsilon$  will turn clockwise as a transition layer of degree  $-1$  (known in micromagnetic jargon as  $360^\circ$ -Néel wall of initial angle 0, see [18]) and in the region  $\omega_2$  (standing for the core of the vortex), we apply some linear cut-off in the radius for the phase of  $m_\varepsilon$ .

*Transition layer of degree  $-1$ .* Let

$$\delta = \delta(\varepsilon) = \varepsilon |\ln \varepsilon|^3.$$

In  $\omega_1$ , we first denote by  $(\tilde{u}_\delta, \tilde{v}_\delta) = e^{i\tilde{\varphi}_\delta} : \mathbf{R} \rightarrow S^1$ , the following approximation of the  $360^\circ$ -Néel wall of initial angle 0 (magnetization turning in clockwise direction, i.e., of topological degree  $-1$ ):

$$\tilde{u}_\delta(t) = \begin{cases} 1 - \frac{2}{|\ln \delta|} \ln \frac{1}{\sqrt{t^2 + \delta^2}} & \text{if } |t| \leq \sqrt{\frac{1}{4} - \delta^2} \\ \cos \tilde{\theta}_\delta(|t|) & \text{if } \sqrt{\frac{1}{4} - \delta^2} \leq |t| \leq 1 \\ 1 & \text{if } |t| \geq 1, \end{cases} \quad \text{and} \quad \tilde{v}_\delta(t) = \begin{cases} -\sqrt{1 - \tilde{u}_\delta^2(t)} & \text{if } t < 0, \\ \sqrt{1 - \tilde{u}_\delta^2(t)} & \text{if } t > 0, \end{cases} \quad (22)$$

where  $\tilde{\theta}_\delta : [\sqrt{1/4 - \delta^2}, 1] \rightarrow [0, \frac{\pi}{2}]$  is defined by

$$\tilde{\theta}_\delta := \text{linear function with } \tilde{\theta}_\delta(\sqrt{1/4 - \delta^2}) = \arccos \frac{|\ln 4\delta|}{|\ln \delta|} \quad \text{and} \quad \tilde{\theta}_\delta(1) = 0 \quad (23)$$

(see Figure 7). In view of (22), we may assume that  $\tilde{\varphi}_\delta(0) = -\pi$ . We then have  $\tilde{\varphi}_\delta(-\infty) = 0$  and

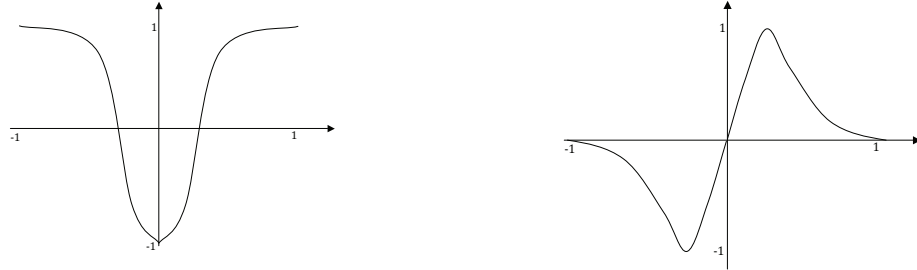


Figure 7: The components  $\tilde{u}_\delta$  and  $\tilde{v}_\delta$  of a  $360^\circ$  Néel wall transition of degree  $-1$ .

$\tilde{\varphi}_\delta(+\infty) = -2\pi$  and since  $\tilde{\varphi}_\delta + \pi$  is antisymmetric in  $t$ , we also get

$$\tilde{\varphi}_\delta(-t) + \tilde{\varphi}_\delta(t) = -2\pi \quad \text{for every } t \in \mathbf{R}. \quad (24)$$

We rescale the transition layer  $(\tilde{u}_\delta, \tilde{v}_\delta)$  so that it is contained in  $\omega_1$ : In polar coordinates, for each arc of radius  $r \in (\lambda, 1)$  fixed in  $\omega_1$  and where the angle  $\theta$  varies inside the interval  $\theta \in (\frac{\pi}{2} \pm \arcsin \frac{\lambda}{r})$ , we define the rescaled transition layer  $(u_\varepsilon, v_\varepsilon)$  with phase  $\varphi_\varepsilon$  by

$$(u_\varepsilon, v_\varepsilon)(\theta) = e^{i\varphi_\varepsilon(\theta)} := (\tilde{u}_\delta, \tilde{v}_\delta) \left( \frac{\theta - \frac{\pi}{2}}{\arcsin \lambda} \right),$$

or equivalently, the rescaled phase is given by

$$\varphi_\varepsilon(\theta) = \tilde{\varphi}_\delta \left( \frac{\theta - \frac{\pi}{2}}{\arcsin \lambda} \right). \quad (25)$$

The profile  $m_\varepsilon$  is defined in terms of its components in radial direction  $\vec{r}$  and angular direction  $\vec{\theta}$  for rotation around arcs with fixed radius,

$$m_\varepsilon(r, \theta) = e^{i\Theta_\varepsilon(r, \theta)} = u_\varepsilon(\theta)\vec{\theta} - v_\varepsilon(\theta)\vec{r} \quad \text{in } \omega_1.$$

Notice that in  $\omega_1$  (as well as in  $\mathcal{C}_k \setminus (\omega_1 \cup \omega_2)$ ), the profile  $m_\varepsilon$  (together with its phase  $\Theta_\varepsilon$ ) are invariant in  $r$ . Also, we have the following relation between  $\varphi_\varepsilon$  and  $\Theta_\varepsilon$ :

$$\Theta_\varepsilon(r, \theta) = \theta + \varphi_\varepsilon(\theta) + \frac{\pi}{2} \quad \text{in } \omega_1 \quad (26)$$

and the phase  $\Theta_\varepsilon$  is continuous in  $\mathcal{C}_k \setminus \omega_2$ . Finally, in the core region  $\omega_2$ , we define the profile  $m_\varepsilon(r, \theta) = e^{i\Theta_\varepsilon(r, \theta)}$  in polar coordinates by

$$\Theta_\varepsilon(r, \theta) = \frac{r}{\lambda} \Theta_\varepsilon(\lambda, \theta) \quad \text{in } \omega_2, \quad (27)$$

where we recall by (26) that  $\Theta_\varepsilon(\lambda, \theta) = \theta + \frac{\pi}{2} + \tilde{\varphi}_\delta(\frac{\theta - \frac{\pi}{2}}{\arcsin \lambda})$  for every  $\theta \in (0, 2\pi)$ .

*Energy estimates.* We refer to [18] for the following estimates:

$$\int_{\mathcal{C}_k} |\nabla m_\varepsilon|^2 dx = o\left(\frac{1}{\varepsilon |\ln \varepsilon|}\right) \quad \text{and} \quad \|(\nabla \cdot m_\varepsilon)\mathbf{1}_{\mathcal{C}_k}\|_{\dot{H}^{1/2}(\mathbf{R}^2)}^2 = O\left(\frac{1}{|\ln \varepsilon|}\right). \quad (28)$$

Notice that  $m_\varepsilon$  is continuous, but not smooth in  $\mathcal{C}_k$ . Obviously, one can approximate  $m_\varepsilon$  by  $\tilde{m}_{\varepsilon, \eta} \in C^\infty(\mathcal{C}_k, S^1)$  in  $H^1(\mathcal{C}_k)$  as  $\eta \rightarrow 0$ , so that by a diagonal argument,  $m$  is approximated by smooth vector fields  $m_n \in C^\infty(\mathcal{C}_k, S^1)$  in  $L^1$  such that  $(\nabla \cdot m_n)\mathbf{1}_{\mathcal{C}_k} \rightarrow 0$  in  $H^{-1/2}(\mathbf{R}^2)$ .

*Step 2. Smoothing outside the cones.* Observe that  $m$  doesn't have any interior vortices in  $\tilde{\Omega}$  that is Lipschitz simply connected. By the proof of Theorem 2 our vector field  $m$  can be approximated in  $W_{loc}^{1,q}(\tilde{\Omega})$  for  $q < \infty$  (in particular, a.e. in  $\tilde{\Omega}$ ) by smooth vector fields  $m_n \in C^\infty(\tilde{\Omega}, S^1)$  of vanishing divergence. Moreover,  $m_n = m$  in  $\mathcal{C}_k$  around  $\partial\mathcal{C}_k \cap \partial\tilde{\Omega}$  and the characteristics of  $m$  passing through  $\tilde{\Omega}$  around  $\mathcal{C}_k$  intersect at a distance larger than  $d_k$  outside  $\tilde{\Omega}$  so that the proof of Theorem 2 enables  $m_n$  to be chosen  $C^\infty$  around the boundary of  $\partial\tilde{\Omega} \cap \partial\mathcal{C}_k$ .

Therefore, we have constructed smooth vector fields  $m_n \in C^\infty(\Omega, S^1)$  converging to  $m$  a.e. that are divergence-free in  $\tilde{\Omega}$ . It remains to show that  $(\nabla \cdot m_n)\mathbf{1}_\Omega \rightarrow 0$  in  $\dot{H}^{-1/2}(\Omega)$ . Observe that  $(\nabla \cdot m_n)\mathbf{1}_\Omega = (\nabla \cdot m_n)\mathbf{1}_{\cup_{k \in K} \mathcal{C}_k} \in L^2(\mathbf{R}^2)$  (since  $m_n \in H^1(\mathcal{C}_k)$  for each  $k \in K$ ). Therefore, by (28), it follows

$$\|(\nabla \cdot m_n)\mathbf{1}_\Omega\|_{\dot{H}^{-1/2}(\mathbf{R}^2)} \rightarrow 0.$$

which concludes our statement.  $\square$

## 7 Appendix

For  $s \in (0, \frac{1}{2}]$  and  $\Omega \subset \mathbf{R}^d$  a Lipschitz domain, the Sobolev space  $H^s(\Omega)$  is a Hilbert space defined as

$$H^s(\Omega) = \{f \in L^2(\Omega) : \|f\|_{H^s(\Omega)}^2 := \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} dx dy < \infty\}$$

and endowed by the norm  $\|f\|_{\dot{H}^s(\Omega)}^2 := \|f\|_{L^2(\Omega)}^2 + \|f\|_{\dot{H}^s(\Omega)}^2$  for every  $f \in H^s(\Omega)$ . (Convention:  $H^0(\Omega) := L^2(\Omega)$ .) It is known that

$$H^s(\Omega) = H_0^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$$

where  $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ . The dual space  $H^{-s}(\Omega)$  of  $H^s(\Omega)$  with respect  $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$  is defined as

$$H^{-s}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \|u\|_{H^{-s}(\Omega)} := \sup_{\substack{f \in \mathcal{D}(\Omega) \\ \|f\|_{H^s(\Omega)} \leq 1}} \langle u, f \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} < \infty\}$$

(for more details, see [29], Chapter 4)

We introduce a closed subspace of  $H^{1/2}(\Omega)$ :

$$H_{00}^{1/2}(\Omega) = \{f \in H^{1/2}(\Omega) : |f|_{00}^2 = \|f\|_{H^{1/2}(\Omega)}^2 + \int_{\Omega} \frac{|f(x)|^2}{\rho(x)} dx < \infty\}$$

where  $\rho(x) = \text{dist}(x, \partial\Omega)$ . In fact,  $H_{00}^{1/2}(\Omega)$  can be seen as the closure of  $C_c^\infty(\Omega)$  in  $H^{1/2}(\mathbf{R}^d)$ . Recall that the map  $u \mapsto \zeta u$  is continuous from  $H^{1/2}(\Omega)$  to  $H_{00}^{1/2}(\Omega)$  for any  $\zeta \in C_c^\infty(\Omega)$ ; indeed, this is due to the inequality

$$\|\zeta u\|_{\dot{H}^{1/2}} \leq \|\zeta\|_{L^\infty} \|u\|_{\dot{H}^{1/2}} + C(\Omega) \|\nabla \zeta\|_{L^\infty} \|u\|_{L^2}.$$

We denote by  $\mathbf{H}^{-1/2}(\Omega)$  the dual of  $(H_{00}^{1/2}(\Omega), |\cdot|_{00})$  with respect to  $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$ . So,  $H^{-1/2}(\Omega)$  is a closed subspace of  $\mathbf{H}^{-1/2}(\Omega)$ . Recall that  $u \mapsto \nabla u$  is continuous in  $H^{1/2}(\Omega)$  to  $\mathbf{H}^{-1/2}(\Omega)$  (see details in [26]).

The space  $BV(\Omega)$  is the set of functions  $f \in L^1(\Omega)$  such that the derivative of  $f$  (in the sense of the distributions) is a finite Radon measure, i.e.,

$$\int_{\Omega} |\nabla f| = \sup \left\{ \int_{\Omega} f \nabla \cdot \zeta dx : \zeta \in C_c^1(\Omega, \mathbf{R}^d), |\zeta(x)| \leq 1, \forall x \in \Omega \right\} < \infty,$$

where  $|\cdot|$  is the Euclidean norm in  $\mathbf{R}^d$ . We denote the  $BV$ -norm as follows:

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + \int_{\Omega} |\nabla f|.$$

One has that  $W^{1,1}(\Omega) = \overline{C^\infty(\Omega)}^{BV(\Omega)}$ . It is known that the zero extension operator  $T : BV(\Omega) \rightarrow BV(\mathbf{R}^d)$  defined as  $Tf(x) = f(x)$  for  $x \in \Omega$  and  $Tf(x) = 0$  for  $x \in \mathbf{R}^d \setminus \Omega$  is a linear continuous operator and we denote  $\|T\|_{BV(\Omega)}$  the norm of operator  $T$  (more details in [2]).

**Proposition 8 (Gagliardo-Nirenberg's inequality)** *Let  $\Omega \subset \mathbf{R}^d$  be a bounded Lipschitz domain and  $s \in [0, \frac{1}{2})$ . Then the embedding  $BV \cap L^\infty(\Omega) \subset H^s(\Omega)$  is compact and one has*

$$\|f\|_{\dot{H}^s(\Omega)}^2 \leq C \|T\|_{BV(\Omega)} \frac{\text{diam}(\Omega)^{1-2s}}{1-2s} \|f\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla f| \quad \text{for every } f \in BV \cap L^\infty(\Omega),$$

where  $C > 0$  is universal positive constant.



**Proof.** If  $f \in C^\infty(\Omega)$ , one has

$$\begin{aligned}
\int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|Tf(x) - Tf(y)|^2}{|x - y|^{d+2s}} dx dy \\
&\leq 2\|f\|_{L^\infty(\Omega)} \int_{|h| \leq \text{diam}(\Omega)} \int_{\mathbf{R}^n} \frac{|Tf(x+h) - Tf(x)|}{|h|^{d+2s}} dh dx \\
&\leq 2\|f\|_{L^\infty(\Omega)} \int_{|h| \leq \text{diam}(\Omega)} \frac{dh}{|h|^{d+2s-1}} \int_{\mathbf{R}^n} \int_0^1 |\nabla(Tf)(x+th)| dt dx \\
&\leq C\|T\|_{BV(\Omega)} \frac{\text{diam}(\Omega)^{1-2s}}{1-2s} \|f\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla f|.
\end{aligned}$$

The inequality holds true for general  $f \in BV \cap L^\infty(\Omega)$  due to the density of smooth functions in  $BV$  endowed with the topology induced by the strict convergence, i.e., there exists  $f_n \in C^\infty \cap BV(\Omega)$  such that  $|f_n| \leq \|f\|_{L^\infty}$  and  $f_n \rightarrow f$  a.e. in  $\Omega$  and  $\|f_n\|_{BV(\Omega)} \rightarrow \|f\|_{BV(\Omega)}$  as  $n \rightarrow \infty$  (see Remark 3.22 in AFP). Obviously,  $\|f\|_{L^2(\Omega)} \leq C(\Omega)\|f\|_{L^\infty(\Omega)}$ . Therefore, one obtains the embedding  $BV \cap L^\infty(\Omega) \subset H^s(\Omega)$ . Moreover, this embedding is compact since for some  $s' \in (s, \frac{1}{2})$ , one has  $BV \cap L^\infty(\Omega) \subset H^{s'}(\Omega)$  and  $H^{s'}(\Omega) \subset H^s(\Omega)$  is compact (see Proposition 4.4. in [29]).  $\square$

**Proposition 9** For every open set  $\Omega \subset \mathbf{R}^d$ ,  $d \geq 1$ , one has  $W^{1,1}(\Omega, S^1) \setminus H^{1/2}(\Omega, S^1) \neq \emptyset$ .

**Proof.** The idea is the following (as explained in [11]): it is known that  $BV \cap L^\infty \not\subset H^{1/2}$  in any dimension and the counter-example is given by any jump function, e.g.,  $\varphi_0 = \mathbf{1}_{(0,1)}$  in  $\Omega = (-1, 1)$ . Moreover, if one regularizes  $\varphi_0$  by  $\varphi_n = \varphi_0$  in  $\Omega \setminus (0, 1/n)$  and  $\varphi_n(x) = nx$  for  $x \in (0, 1/n)$ , then we have

$$\int_{\Omega} |\varphi_n'| = 1, \quad \|\varphi_n\|_{L^\infty} = 1 \quad \text{and} \quad \|\varphi_n\|_{H^{1/2}} \rightarrow \infty.$$

Based on this idea, one can construct a function  $\varphi \in W^{1,1}(\Omega) \setminus H^{1/2}(\Omega)$  with  $\Omega = (-1, 1)$  and set  $m : \Omega \rightarrow \mathbf{R}$  with  $m = e^{i\varphi}$ . Then  $m$  satisfies the desired properties. In higher dimensions, the same example (depending on a single variable) holds.  $\square$

**Proposition 10** a) (Stability by composition in  $H^{1/2}$ ) Let  $\Omega \subset \mathbf{R}^d$  be a bounded open set,  $h_n, h \in H^{1/2}(\Omega, \mathbf{R}^k)$  ( $k = 1, 2$ ) and  $\Psi : \mathbf{R}^k \rightarrow \mathbf{R}$  be a Lipschitz function. If  $h_n \rightarrow h$  in  $H^{1/2}$  then  $\Psi(h_n) \rightarrow \Psi(h)$  in  $H^{1/2}$ .

b) (Stability by complex multiplication in  $H^{1/2}$ ) Let  $h, m_n, m \in H^{1/2}(\Omega, S^1)$  and  $m_n \rightarrow m$  in  $H^{1/2}$  then  $h m_n \rightarrow h m$  in  $H^{1/2}$  (the last terms are to be interpreted as product of complex numbers).

**Proof.** a) Let

$$R(h, x, y) := \frac{h(x) - h(y)}{|x - y|^{(d+1)/2}}, \quad x \neq y \in \Omega.$$

Observe that  $h_n \rightarrow h$  in  $H^{1/2}$  yields  $h_n \rightarrow h$  in  $L^2(\Omega)$  and  $R(h_n, \cdot, \cdot) \rightarrow R(h, \cdot, \cdot)$  in  $L^2(\Omega \times \Omega)$ . Discarding eventually a subsequence, we may assume that  $h_n \rightarrow h$  a.e. in  $\Omega$  and  $|R(h_n, \cdot, \cdot)| \leq T$  for a.e.  $x, y \in \Omega$  for some  $T \in L^2(\Omega \times \Omega, \mathbf{R}_+)$ . Therefore,  $|R(\Psi(h_n), \cdot, \cdot)| \leq \|\Psi\|_{Lip} T$  and  $R(\Psi(h_n), \cdot, \cdot) \rightarrow R(\Psi(h), \cdot, \cdot)$  a.e. in  $\Omega \times \Omega$ . By dominated convergence theorem, we get that  $R(\Psi(h_n), \cdot, \cdot) \rightarrow R(\Psi(h), \cdot, \cdot)$  in  $L^2(\Omega \times \Omega)$ . Since  $|\Psi(h_n) - \Psi(h)| \leq \|\Psi\|_{Lip} |h_n - h|$  a.e. in  $\Omega \times \Omega$ ,

we conclude that  $\Psi(h_n) \rightarrow \Psi(h)$  in  $H^{1/2}$ . (Since the limit is unique, the above argument holds for the whole sequence  $n$  in the metric space  $H^{1/2}$ .)

b) Discarding eventually a subsequence, we may assume as before that  $m_n \rightarrow m$  a.e. in  $\Omega$ . Let  $v_n := m_n \bar{m}$  where  $\bar{m}$  is the complex conjugate of  $m$ . Then  $v_n \rightarrow 1$  in  $H^{1/2}$ . Indeed, we first have that  $|v_n| = 1$  and  $v_n \rightarrow 1$  a.e. in  $\Omega$ . Then  $R(v_n, \cdot, \cdot) \rightarrow 0$  a.e. in  $\Omega \times \Omega$  and  $|R(v_n, \cdot, \cdot)| \leq |R(m_n, \cdot, \cdot)| + |R(m, \cdot, \cdot)| \in L^2(\Omega \times \Omega)$ . The dominated convergence theorem yields  $v_n \rightarrow 1$  in  $H^{1/2}$ . Finally, we conclude

$$\begin{aligned} \|hm_n - hm\|_{\dot{H}^{1/2}}^2 &= \|hm(v_n - 1)\|_{\dot{H}^{1/2}}^2 \\ &\leq 2\|v_n\|_{\dot{H}^{1/2}}^2 + 2 \int_{\Omega} \int_{\Omega} |v_n(y) - 1|^2 |R(mh, x, y)|^2 dx dy \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

by dominated convergence theorem (since  $mh \in H^{1/2}$ ). □

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