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On the connection between Hamilton and Lagrange formalism in Quantum Field Theory

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The connection between the Hamilton and the standard Lagrange formalism is established for a generic Quantum Field Theory with vanishing vacuum expectation values of the fundamental fields. The Effective Actions in both formalisms are the same if and only if the fundamental fields and the momentum fields are related by the stationarity condition. These momentum fields in general differ from the canonical fields as defined via the Effective Action. By means of functional methods a systematic procedure is presented to identify the full correlation functions, which depend on the momentum fields, as functionals of those usually appearing in the standard Lagrange formalism. Whereas Lagrange correlation functions can be decomposed into tree diagrams the decomposition of Hamilton correlation functions involves loop corrections similar to those arising in n -particle effective actions. To demonstrate the method we derive for theories with linearized interactions the propagators of composite auxiliary fields and the ones of the fundamental degrees of freedom. The formalism is then utilized in the case of Coulomb gauge Yang-Mills theory for which the relations between the two-point correlation functions of the transversal and longitudinal components of the conjugate momentum to the ones of the gauge field are given.

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I. INTRODUCTION

The path integral method within the Lagrange formalism is a fundamental tool to formulate Quantum Field Theory. Certainly, the treatment of the same theory within the canonical operator formalism is somewhat more cumbersome. In fact, additionally to the operator-ordering problems of gauge theories [1, 2], the formal invariance properties of a generic theory admit additional transformation laws which are not present in the Lagrange formulation. The “momentum fields” in such a case are not related to the fundamental fields by means of the canonical equations, see, *e.g.*, ref. [3], but constitute separate degrees of freedom with independent transformation properties and thus symmetries. As a consequence the number of variables in the path integral increases in comparison to the Lagrange formalism. This introduces for most theories considerable additional complications in perturbative as well as non-perturbative calculations [4–7].

On the other hand, some recent studies indicate that the first order formalism proves to be a successful tool to study the required complete cancellation of the energy divergences [8] that emerge in a perturbative treatment of Coulomb gauge Yang-Mills theory within the standard Lagrange formalism [7, 9]. Coulomb gauge Yang-Mills theory has attracted attention since one possible solution to the confinement problem in QCD is provided by the “Gribov-Zwanziger” scenario [8, 10] (*cf.* also ref. [11]). However, the problem of renormalizing Coulomb gauge Yang-Mills theory is still unsolved since these energy divergences cannot be regularized using any of the standard procedures. Recently it was illustrated how these energy divergences cancel at each order in perturbation theory [12]. In order to perform explicit calculations a number of methods have been applied as *e.g.* introducing a novel method to regularize Feynman integrals in non-covariant gauges [13, 14], employing algebraic renormalizability [8], and trying to recover Coulomb gauge Yang-Mills theory as a limit of an interpolating gauge [15]. On the other hand, there are several indications that the canonical or first order formalism is better suited for studying Coulomb gauge Yang-Mills theory [4–8, 15–18]. Yet, even if such an approach would be successful, due to the dramatically complicated form of the functional equations in the first order formalism, an explicit non-perturbative study is far too involved to be computationally feasible. Therefore, an analysis in the second order formalism would be highly desirable. To this end we provide general connections between the Greens functions in the two formulations that should help to perform the renormalization in the Lagrange framework according to the insight in the renormalization procedure obtained in the Hamilton framework.

In order to establish general relations between dressed correlation functions in the different formulations we exploit the following properties. At vanishing sources associated to the momentum fields the Effective Action (*i.e.* the Generating Functional of one-particle-irreducible (1PI) Green’s functions) of the Hamilton formalism reduces to the one of the Lagrangean approach. This allows to reduce the set of Dyson-Schwinger equations (DSEs) [19, 20] in the first order formalism to the corresponding set derived from the standard path integral representation. An important special

case is given if the Hamiltonian is quadratic in the momentum fields. For a corresponding Generating Functional the integration over the momentum fields can be performed and any m -point function involving this field as the average of a polylocal function of the quantum canonical momentum fields can be determined. This means that the full correlation functions involving these canonical variables can be found as a functional of those that usually appear in the standard path integral representation. The procedure to find these connections is closely related to the functional method used in the derivation of the DSEs (see *e.g.* ref. [21]). In this paper we give the explicit form of these relations for a general four-dimensional renormalizable theory. This includes the case that the interaction terms involve the time derivative of the fields. In a final step we resolve the relations between the proper two-point functions in both frames by considering the inverse of the matrix-valued propagators in the Hamiltonian approach.

Moreover, we show that similar connections arise for theories where not the kinetic but the interaction part is linearized. Such a bosonization procedure is an important technique used in many parts of physics ranging from hadronic physics to condensed matter systems. Our results explicitly verify, in agreement with other approaches, that there is no double counting in bosonized theories, but instead a given correlation function in the underlying theory is exactly given by the sum of all possible contributions involving both the fundamental and the composite degrees of freedom in the bosonized theory.

This paper is organized as follows: In Sect. II the functional equations for DSEs and a Symmetry-Related identity (like Ward-Green-Takahashi, resp. Slavnov-Taylor, identities (STIs) of a gauge theory [22–26]) in the phase space formulation are given. In Sect. III we present the first order DSEs of theories which are quadratic in the momentum fields. In addition, we derive a method to determine the correlation functions and the STIs including momentum fields from the respective quantities that usually appear in the standard Lagrange representation. Diagrammatic rules and the general decomposition of the full proper functions in the Lagrange formalism are given in Sect. IV, while in Sect. V and Sect. VI we detail the explicit decomposition of the connected and proper two-point functions in the Hamilton framework, respectively. In Sect. VII we show that the formalism can also be utilized in the case composite of auxiliary fields, and last but not least in Sect. VIII we apply the formalism to the case of Coulomb gauge Yang-Mills theory. In the last section we conclude while essential steps of many calculations have been deferred to several appendices.

II. FUNCTIONAL EQUATIONS

We start our analysis by considering a generic Quantum Field Theory formulated within the first order formalism. The Hamiltonian density $\mathcal{H}(q_m(x), p_m(x))$ depends on the fundamental fields $q_m(x)$ and the conjugate momentum fields $p_m(x)$, respectively. As will be discussed in Sect. IV.D, for fermionic fields the relation between the two formalisms is trivial. Therefore we will treat only the case of bosonic fields in this and the next section.

In the case of gauge invariant theories we will suppose that the Hamiltonian \mathcal{H} includes the additional terms that arise when the constraints associated to such theories, like *e.g.* Gauss' law in Quantum Electrodynamics, are localized. In this context each Lagrange multiplier necessary to impose the constraints will be treated as a fundamental field. Certainly the presence of these terms leads to the main differences between gauge theories and the more conventional Hamiltonian systems. In addition, in a fixed gauge, ghost fields appear. The usual path integral representation of these Grassmannian variables is formulated within the first order formalism (see also Sect. IV.C). Thus, in the case of a gauge theory they will be analysed in an independent way (for details see Sect. VIII). In this section, to set up the problem, we will disregard for the moment these potential complications.

Let us suppose, in addition, that the fields $p_m(x)$ and $q_m(x)$ are coupled to a set of classical sources given by $J_m^p(x)$ and $J_m^q(x)$, respectively. Under such conditions the source dependent vacuum-to-vacuum transition amplitude between the asymptotic states $|\text{Vac, in}\rangle$ and $|\text{Vac, out}\rangle$ looks like

$$\begin{aligned} \mathcal{Z}[J] &= \langle \text{Vac, out} | \text{Vac, in} \rangle_J \\ &= \int \mathcal{D}[q] \mathcal{D}[p] \exp \left\{ \frac{i}{\hbar} (I[q, p, J] + i\epsilon\text{-terms}) \right\} \end{aligned} \quad (1)$$

where $[q]$ denotes the collection of all fundamental fields, whereas $[p]$ the corresponding momentum fields. For more details see section 9.2 of the Ref. [3]. Here the argument in the exponential has the structure

$$I[q, p, J] = I_0[q, p] + \int d^4x J(x) \cdot \phi(x)$$

where

$$\phi(x) \equiv \begin{pmatrix} p_m(x) \\ q_m(x) \end{pmatrix} \quad \text{and} \quad J(x) \equiv \begin{pmatrix} J_m^p(x) \\ J_m^q(x) \end{pmatrix},$$

include both momentum and fundamental fields and sources. The components of the sources in this notation are distinguished by upper labels, whereas

$$I_0[q, p] = \int_{-\infty}^{\infty} d\tau \left(\int d^3\mathbf{x} \left(p_m(x) \dot{q}_m(x) - \mathcal{H}(q(x), p(x)) \right) \right) \quad (2)$$

looks like the classical expression for the Hamiltonian action. Note that the part concerning to the ϵ terms in Eq. (1) have the function to produce the necessary $i\epsilon$'s in the denominators of all propagators such that the correct boundary conditions of the fields at asymptotic times, $q_m(\mathbf{x}, \pm\infty)$, are implemented (see again Ref. [3]).

The fact that $I_0[q, p]$ looks like the action expressed in terms of canonical variables is somewhat misleading since the momentum fields $p_m(x)$ are independent variables and therefore **not yet** related to the fundamental fields $q_m(x)$ or their derivatives. In particular, since the path integral is not saturated by its saddle point(s) they are not constrained to obey the equations of motion of classical Hamiltonian dynamics $\delta I_0/\delta\phi = 0$ where

$$\frac{\delta I_0}{\delta\phi} \equiv \begin{pmatrix} \frac{\delta I_0}{\delta p_m} \\ \frac{\delta I_0}{\delta q_m} \end{pmatrix} = \begin{pmatrix} \dot{q}_m - \frac{\partial \mathcal{H}}{\partial p_m} \\ \dot{p}_m - \nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla q_m)} + \frac{\partial \mathcal{H}}{\partial q_m} \end{pmatrix}. \quad (3)$$

The path integrals, however, contain the information about these equations of motion. Indeed, let us consider the operator version of Eq. (3). Its vacuum expectation values in the presence of the external classical sources can be expressed as

$$\frac{\overline{\delta I_0}}{\delta\phi} = \frac{\int \mathcal{D}[q] \mathcal{D}[p] \frac{\hbar}{i} \frac{\delta}{\delta\phi} \exp \left\{ \frac{i}{\hbar} I[q, p, J] \right\}}{\langle 0_{\text{out}} | 0_{\text{in}} \rangle_J} - J. \quad (4)$$

Assuming the absence of any boundary terms, the integral of such a functional derivative vanishes. Then by substituting each of the elementary objects present in Eq. (3) by the derivative with regard to the respective classical sources, the following functional differential equation arises

$$\left. \frac{\delta I_0}{\delta\phi} \right|_{\phi(x) \rightarrow \frac{\hbar}{i} \frac{\delta}{\delta J(x)}} \mathcal{Z}[J] = -J(x) \mathcal{Z}[J]. \quad (5)$$

It contains all equations of motions fulfilled by all Green's functions.

Introducing the Generating Functional of connected Green's functions, $\mathcal{W}^H[J] = -i \ln(\mathcal{Z}[J])$, allows us to rewrite Eq. (5) as

$$\left. \frac{\delta I_0}{\delta\phi} \right|_{\phi(x) \rightarrow \frac{\delta \mathcal{W}^H}{\delta J(x)} + \frac{\hbar}{i} \frac{\delta}{\delta J(x)}} = -J(x), \quad (6)$$

where in the last step we made use of the following identity

$$\mathfrak{F} \left(\frac{\delta}{\delta J} \right) \exp(\mathfrak{G}(J)) = \exp(\mathfrak{G}(J)) \mathfrak{F} \left(\frac{\delta \mathfrak{G}(J)}{\delta J} + \frac{\delta}{\delta J} \right) \quad (7)$$

for arbitrary functionals \mathfrak{F} and \mathfrak{G} . This set of equations constitutes the DSEs for the connected Green's functions.

In the next step we introduce the vacuum expectation values of the momentum and the fundamental fields in the presence of sources

$$\bar{\phi}[J] = \frac{1}{i} \frac{\delta \mathcal{W}^H}{\delta J(x)} \equiv \begin{pmatrix} \bar{p}[J] \\ \bar{q}[J] \end{pmatrix} = \begin{pmatrix} \frac{1}{i} \frac{\delta \mathcal{W}^H}{\delta J_m^p} \\ \frac{1}{i} \frac{\delta \mathcal{W}^H}{\delta J_m^q} \end{pmatrix}. \quad (8)$$

We assume that it is possible to invert these relations such that the sources are expressed as functionals of the vacuum expectation values of $\hat{\Pi}(x)$ and $\hat{Q}(x)$. Hereby the replacements of variables in Eq. (6) becomes

$$\phi(x) \rightarrow \bar{\phi}[J](x) + \frac{\hbar}{i} \int d^4x' \mathbb{D}[J](x, x') \frac{\delta}{\delta \bar{\phi}(x')} \quad (9)$$

where

$$\mathbb{D}[J](x, x') \equiv \begin{pmatrix} \Delta_{ml}^{pp}[J](x, x') & \Delta_{ml}^{pq}[J](x, x') \\ \Delta_{ml}^{qp}[J](x, x') & \Delta_{ml}^{qq}[J](x, x') \end{pmatrix} \quad (10)$$

are the source-dependent two-point Green's function where

$$\Delta^{pq} = \frac{1}{i} \frac{\delta \mathcal{W}^H[J]}{\delta \tilde{J}^p(x) \delta \tilde{J}^q(x')} = \frac{\delta \tilde{q}_l[J](x')}{\delta \tilde{J}^p(x)} \quad (11)$$

and $\Delta^{pp}[J]$, $\Delta^{qq}[J]$ and $\Delta^{qp}[J]$ are analogously defined.

The Effective Action in the first order formalism $\Gamma^H[\bar{q}_m, \bar{p}_m]$ can be defined by the Legendre transform of $\mathcal{W}^H[J]$ with respect to the associated ‘‘averaged’’ fields of the theory

$$\Gamma^H[\bar{\phi}] \stackrel{def}{=} \mathcal{W}^H[J] - \int d^4x J(x) \cdot \bar{\phi}(x). \quad (12)$$

Remember that the compact notation implies a twofold Legendre transformation in the two independent components q and p . By considering the functional derivative of $\Gamma^H[\bar{q}, \bar{p}]$ with respect to $\bar{p}_i(x)$ and $\bar{q}_i(x)$ we obtain as expected

$$\frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = -J(x). \quad (13)$$

Substituting Eq. (13) into Eq. (6), however, we obtain with the replacement Eq. (9) the desired functional differential equations for $\Gamma^H[\bar{q}, \bar{p}]$,

$$\frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = \frac{\delta I_0}{\delta \phi(x)} \Big|_{\phi(x) \rightarrow \bar{\phi}[J](x) + \frac{i}{\hbar} \int d^4x' \mathbb{D}[J](x, x') \frac{\delta}{\delta \bar{\phi}(x')}}. \quad (14)$$

This relation generates all DSEs for the 1PI Green's functions of the first order formalism. The method to obtain such a functional relation will be employed several times in the following.

The derivation of the proper propagators is straightforward when keeping in mind that they are embedded in a matrix. In fact, taking a derivative with regard $\phi(x')$ in Eq. (14) we obtain

$$\mathbb{G}[\phi](x'', x') \equiv \begin{pmatrix} \frac{\delta^2 \Gamma^H}{\delta \bar{p}_m(x'') \delta \bar{p}_n(x')} & \frac{\delta^2 \Gamma^H}{\delta \bar{p}_m(x'') \delta \bar{q}_n(x')} \\ \frac{\delta^2 \Gamma^H}{\delta \bar{q}_m(x'') \delta \bar{p}_n(x')} & \frac{\delta^2 \Gamma^H}{\delta \bar{q}_m(x'') \delta \bar{q}_n(x')} \end{pmatrix}. \quad (15)$$

This expression and Eq. (10) are related via the identity

$$\int d^4x'' \mathbb{D}[J](x, x'') \mathbb{G}[\phi](x'', x') = -\mathbb{I}, \quad (16)$$

which can be obtained by taking a derivative with respect to the source J in Eq. (13). Hereby $\mathbb{I} = \delta_{mn} \delta^4(x-x') \otimes 1_{2 \times 2}$ denotes the identity in the considered space. Any other connected or proper Green's function can be derived by considering higher order derivatives with respect to J and ϕ in Eq. (10) and Eq. (15), respectively.

In general, these functions are constrained by the symmetry properties of the initial ‘‘action’’. In the case of a gauge theory a corresponding derivation leads to the STIs. For Coulomb gauge Yang-Mills theory it is presented in Sect. VIII. Here we will illustrate the potential complications arising in the first order formalism. For this it sufficient to assume that I_0 is invariant under the simultaneous infinitesimal transformations

$$\delta \phi[\phi] \equiv \epsilon \cdot \begin{pmatrix} \mathfrak{G}_m^q[p, q] \\ \mathfrak{F}_m^p[p, q] \end{pmatrix} \quad (17)$$

and also assume that they leave the path integration measure invariant. Performing this substitution in Eq. (1) we can obtain the following ‘‘Symmetry-Related Functional Identity’’

$$0 = \int \mathcal{D}[q] \mathcal{D}[p] \int d^4x J(x) \cdot \delta \phi(x) \exp \left[\frac{i}{\hbar} I[\phi, J] \right]. \quad (18)$$

In the next step we substitute the fundamental and momentum fields by the respective derivatives with respect to the classical source. Following a procedure similar to the derivation of the DSEs we rewrite the symmetry-related identity Eq. (18) and express it in terms of the full Generating Functional $\mathcal{Z}[J]$:

$$0 = \int d^4x J(x) \cdot \delta\phi \left[\frac{\hbar}{i} \frac{\delta}{\delta J} \right] \mathcal{Z}[J]. \quad (19)$$

Analogously, the above relation can be rewritten for connected Green's functions

$$0 = \int d^4x J(x) \cdot \delta\phi[\phi] \Big|_{\phi \rightarrow \frac{\delta W^H}{\delta J} + \frac{\hbar}{i} \frac{\delta}{\delta J}}. \quad (20)$$

As before, Eq. (18) can be reformulated in terms of the Effective Action

$$0 = \int d^4x \frac{\delta \Gamma^H}{\delta \phi} \cdot \delta\phi[\phi] \Big|_{\phi \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \int d^4y \mathbb{D}[J](x,y) \frac{\delta}{\delta \phi(y)}}. \quad (21)$$

This identity verifies that the Effective Action in first order formalism preserves the continuous symmetries of the initial canonical quantum action.

At this point the DSEs and the symmetry-related identity Eq. (18) of a QFT seem to be more cumbersome than the usual one which appear in the standard path integral formulation. However, in the next section we shall show that under certain conditions the above system can be reduced to the latter one.

III. THE CONNECTION BETWEEN LAGRANGE AND HAMILTON FORMALISM

From here on we will denote all the variables a field depends on by a single latin index. For instance in case of a gauge field such indices will indicate the space-time point x , the vectorial index μ , and the adjoint gauge group index a . Repeated indices are summed and integrated over for discrete and continuous variables, respectively. Also all fundamental fields will be denoted by the same letter q . Similarly, p will represent all momentum fields.

Let us now consider the path integral representation of a n -point function involving only momentum fields

$$\Delta_{i_1 \dots i_n}^{p \dots p} = \frac{\langle 0_{\text{out}} | T \{ \hat{\Pi}_{i_1} \dots \hat{\Pi}_{i_n} \} | 0_{\text{in}} \rangle_J}{i^{1-n} \langle 0_{\text{out}} | 0_{\text{in}} \rangle_J} = \frac{\int \mathcal{D}[q] \mathcal{D}[p] p_{i_1} \dots p_{i_n} \exp \left[\frac{i}{\hbar} I[q, p, J] \right]}{i^{1-n} \int \mathcal{D}[q] \mathcal{D}[p] \exp \left[\frac{i}{\hbar} I[q, p, J] \right]}.$$

Here T is the time ordering operator, and $\hat{\Pi}_i \equiv \hat{\Pi}_m(\mathbf{x}, \tau)$ are the Hermitian momentum field operators in the Heisenberg picture. The expression (22) can be written via functional derivatives with respect to J_i^p as

$$\Delta_{i_1 \dots i_n}^{p \dots p} = \frac{\int \mathcal{D}[q] \mathcal{D}[p] \frac{\hbar}{i} \frac{\delta}{\delta J_{i_1}^p} \dots \frac{\hbar}{i} \frac{\delta}{\delta J_{i_n}^p} \exp \left[\frac{i}{\hbar} \{ I[q, p, J] \} \right]}{i^{1-n} \int \mathcal{D}[q] \mathcal{D}[p] \exp \left[\frac{i}{\hbar} \{ I[q, p, J] \} \right]}. \quad (22)$$

In the following we will consider only the important class of Hamiltonians which are at most quadratic in the momentum fields. Note that this case is realized for all renormalizable and most non-renormalizable theories. For later use we write the Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2} p_i \mathcal{A}_{ij}[q] p_j + \mathcal{B}_i[q] p_i + \mathcal{C}[q], \quad (23)$$

which defines a real, symmetric, positive and non-singular matrix \mathcal{A} as well as the real functionals of q , $\mathcal{B}_i[q]$ and $\mathcal{C}[q]$. For the considered class of Hamiltonians the Gaussian integration over p can be performed analytically. As shown in appendix A it yields an expression of the form

$$\Delta_{i_1 \dots i_n}^{p \dots p} = \frac{\int \mathcal{D}[q] \frac{\hbar}{i} \frac{\delta}{\delta J_{i_1}^p} \dots \frac{\hbar}{i} \frac{\delta}{\delta J_{i_n}^p} \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q, J] \right]}{i^{1-n} \int \mathcal{D}[q] \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q, J] \right]}. \quad (24)$$

The general result for the new action arising in the exponential is given in Eq. (A5).

For simplicity we will discuss first the standard Hamiltonian with $\mathcal{A}_{ij}[q] = \delta_{ij}$. The general case where \mathcal{A} is non-trivial is realized *e.g.* in the case of Coulomb gauge QCD which will be discussed in Sect. VIII. (For the moment we

only note that then the inverse matrix, $\mathcal{A}^{-1}[q]$, appears in the following expressions as a prefactor of J_p making in general the action S_0 non-local.) In the considered simpler standard case the new action possesses the structure

$$\tilde{\mathcal{S}}[q, J] = \mathcal{S}_0 + \frac{1}{2} J_i^p J_i^p + J_i^p \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} + J_i^q q_i \quad (25)$$

where \mathcal{S}_0 is the standard action

$$\mathcal{S}_0[q] = \frac{1}{2} \dot{q}^i \dot{q}^i - \dot{q}^i \mathcal{B}^i[q] + \frac{1}{2} \mathcal{B}^i[q] \mathcal{B}^i[q] - \mathcal{C}[q]. \quad (26)$$

The application of Eq. (7) on the integrand present in Eq. (24) makes it possible to write

$$\Delta_{i_1 \dots i_n}^{p \dots p} = \frac{\int \mathcal{D}[q] \left\{ \hat{\mathcal{O}}_{i_1 \dots i_{n-1}}^{p \dots p} \left(J_{i_n}^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_n}} \right) \right\} \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}} \right]}{i^{1-n} \int \mathcal{D}[q] \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}} \right]}. \quad (27)$$

where the operator

$$\hat{\mathcal{O}}_{i_1 \dots i_{n-1}}^{p \dots p} = \prod_{l=1}^{n-1} \left(J_{i_l}^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_l}} + \frac{\hbar}{i} \frac{\delta}{\delta J_{i_l}^p} \right) \quad (28)$$

only acts on the function inside the curly brackets. Note that the integrand in the numerator is a functional depending on the fields q as well as the sources J_q and J_p .

A. Alternative form of the DSEs

Analogous to Eq. (6) we can write the partially integrated expression Eq. (27) in terms of the Generating Functional of connected Green's functions. Let us specialize to the case of one-point Green's functions, *i.e.*, the vacuum expectation values of the momentum fields

$$\bar{p}_i[J] = \left. \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \right|_{q \rightarrow \frac{\delta \mathcal{W}^H}{\delta J^q} + \frac{\hbar}{i} \frac{\delta}{\delta J^q}} + J_i^p = \frac{\delta \mathcal{W}^H}{\delta J_i^p(\mathbf{x}, \tau)}. \quad (29)$$

Eq. (13) allows to express Eq. (29) as

$$\frac{\delta \Gamma^H}{\delta \bar{p}_i} = -\bar{p}_i + \left. \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \right|_{q \rightarrow \bar{q}[J] + \frac{\hbar}{i} \Delta^{qq}[J] \frac{\delta}{\delta \bar{q}}}. \quad (30)$$

Here the dependence on the average momentum fields is made explicit and the action in the Lagrange formalism appears which depends only on the q -fields. From a functional point of view it is more convenient to express this via Eq. (29) as

$$\frac{\delta \Gamma^H}{\delta \bar{p}_i} = - \left[p_i - \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \right]_{\phi \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \Delta^{\phi\phi}[J] \frac{\delta}{\delta \phi}}. \quad (31)$$

Now, employing the identity

$$0 = \int \mathcal{D}[q] \frac{\delta}{\delta q_m} \int \mathcal{D}[p] \exp \left[\frac{i}{\hbar} I[q, p, J] \right] \quad (32)$$

we perform the p -integration, so that the above expression reads

$$0 = \int \mathcal{D}[q] \left(\frac{\delta \mathcal{S}_0}{\delta q_i} + J_m^p \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta q_m} + J_i^q \right) \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q, J] \right], \quad (33)$$

which makes it possible to obtain an analogous form of the field equations

$$\frac{\delta \Gamma^H}{\delta \bar{q}_i} = \left(\frac{\delta \mathcal{S}_0}{\delta q_i} - \frac{\delta \Gamma^H}{\delta \bar{p}_m} \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta q_m} \right) \Big|_{\phi \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \Delta^{\phi\phi}[J] \frac{\delta}{\delta \phi}}. \quad (34)$$

Eqs. (31) and (34) represent alternative forms of the first order DSEs (14).

This alternative form of the DSEs allows us to find corresponding equations for the fundamental proper Green's functions by taking derivatives with respect to the momentum and/or fundamental fields. This way the proper momentum propagator can be written as

$$\frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{p}_j} = -\delta_{ij} + \frac{\delta}{\delta \bar{p}_i} \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \Big|_{q \rightarrow \bar{q}[J] + \frac{\hbar}{i} \Delta^{qq}[J] \frac{\delta}{\delta \bar{q}}} \right), \quad (35)$$

whereas the corresponding mixed Green's function is

$$\frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{p}_j} = \frac{\delta}{\delta \bar{q}_i} \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \Big|_{q \rightarrow \bar{q}[J] + \frac{\hbar}{i} \Delta^{qq}[J] \frac{\delta}{\delta \bar{q}}} \right). \quad (36)$$

In a similar way, the other mixed propagator can be written as

$$\frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{q}_j} = \frac{\delta}{\delta \bar{p}_i} \left(\frac{\delta \mathcal{S}_0}{\delta q_j} \Big|_{q \rightarrow \bar{q}[J] + \frac{\hbar}{i} \Delta^{qq}[J] \frac{\delta}{\delta \bar{q}}} \right) - \frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{p}_m} \frac{\delta^2 \mathcal{S}_0}{\delta q_j \delta \dot{q}_m} \Big|_{\phi \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \Delta^{\phi\phi}[J] \frac{\delta}{\delta \phi}}, \quad (37)$$

whereas the fundamental field propagator is given by

$$\frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{q}_j} = \frac{\delta}{\delta \bar{q}_i} \left(\frac{\delta \mathcal{S}_0}{\delta q_j} \Big|_{q \rightarrow \bar{q}[J] + \frac{\hbar}{i} \Delta^{qq}[J] \frac{\delta}{\delta \bar{q}}} \right) - \frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{p}_m} \frac{\delta^2 \mathcal{S}_0}{\delta q_j \delta \dot{q}_m} \Big|_{\phi_j \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \Delta^{\phi\phi}[J] \frac{\delta}{\delta \phi}}. \quad (38)$$

Hereby, in the derivation of Eqs. (37) and (38) we have discarded those terms that vanish when the sources are set to zero. In addition, note that the proliferated occurrence of Green's functions involving the p -field arises from functional derivatives on $\Delta_{ij}^{qq}[J]$. As will become clear below, despite the different appearance, the structure of Eqs. (37) and (38) is related to the one of Eqs. (35) and (36).

B. Pure and mixed momentum correlation functions

In this subsection we derive expressions for first order correlation functions entirely in terms of second order correlation functions. To this end, we start from Eq. (27). Subsequently, we again follow a procedure similar to that used in the derivation of Eq. (6) thus arriving at a general expression for arbitrary momentum correlation functions,

$$\Delta_{i_1 \dots i_n}^{p \dots p} = i^{n-1} \left[\prod_{l=1}^{n-1} \left(J_{i_l}^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_l}} + \frac{\hbar}{i} \frac{\delta}{\delta J_{i_l}^p} \right) \left(J_{i_n}^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}_{i_n}} \right) \Big|_{J^p=0} \right]_{q_a \rightarrow \bar{q}_a[J^q] + \frac{\hbar}{i} \Delta_{ab}^{qq}[J^q] \frac{\delta}{\delta q_b}} \quad (39)$$

In contrast to the corresponding equation for the one-point function, Eq. (29), (which did lead to the alternative set of DSEs) here the connected Generating Functional in the standard Lagrange formalism appears. Note that the latter does not depend on J_p anymore. From the above functional equation, any other m -point Green's function with n -external legs associated to the J^p 's is obtained by taking $m-n$ derivatives with regard to J_m^q and setting them to zero. In particular, the quantum average of the momentum fields becomes a functional of the averaged field \bar{q} ,

$$\bar{p}_i[\bar{q}] = \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \Big|_{q_m \rightarrow \bar{q}_i[J^q] + \frac{\hbar}{i} \Delta_{ij}^{qq}[J^q] \frac{\delta}{\delta q_j}}. \quad (40)$$

Already at this point we want to point out that the averaged momentum field is generally not given by the usual definition of a canonical momentum field $\mathbf{p}_i^{\text{can}}$ on the level of the Effective Action

$$\bar{p}_i[J=0] = \left\langle \frac{\delta S}{\delta \dot{q}} \right\rangle \neq \frac{\delta \Gamma}{\delta \dot{q}} \equiv \mathbf{p}_i^{\text{can}}. \quad (41)$$

This will be shown explicitly in subsection VI A.

In addition, the relation between the pp -correlation functions and those that appear in the standard formalism can be expressed as

$$\Delta_{ij}^{pp} = \delta_{ij} + i \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \right) \Big|_{q \rightarrow \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta q}}, \quad (42)$$

whereas the corresponding mixed qp -correlation function can be written as

$$\Delta_{ij}^{qp}[J^q] = \frac{\delta}{\delta J_i^q} \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \right) \Big|_{q_m \rightarrow \frac{\delta \mathcal{V}}{\delta J_m^q} + \frac{\hbar}{i} \frac{\delta}{\delta J^q}}. \quad (43)$$

By using the chain rule $\delta/\delta J_m^q = \delta q_n/\delta J_m^q \delta/\delta q_n$, Eq. (43) can also be given in the form

$$\Delta_{ij}^{qp}[J^q] = \Delta_{il}^{qq} \frac{\delta}{\delta \bar{q}_l} \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \Big|_{q \rightarrow \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta q}} \right) = \Delta_{il}^{qq} \frac{\delta \bar{p}_j}{\delta \bar{q}_l}. \quad (44)$$

The other mixed correlator Δ^{pq} can be immediately inferred from the bosonic nature of the fields, $\Delta_{ij}^{pq} = \Delta_{ji}^{qp}$.

Similarly, it is possible to determine the connected three point correlation functions

$$\Delta_{ijk}^{qpp}[J^q] = \Delta_{il}^{qq} \frac{\delta}{\delta \bar{q}_l} \Delta_{jk}^{pp}, \quad \Delta_{ijk}^{qpp}[J^q] = \Delta_{im}^{qq} \Delta_{jl}^{qq} \left\{ \frac{\delta^3 \Gamma}{\delta \bar{q}_l \delta \bar{q}_m \delta \bar{q}_n} \Delta_{nk}^{qp} + \frac{\delta^2 \bar{p}_k}{\delta \bar{q}_m \delta \bar{q}_l} \right\}, \quad (45)$$

$$\Delta_{ijk}^{ppp}[J^q] = i\delta_{ij}\bar{p}_k + i\delta_{ik}\bar{p}_j + i\delta_{jk}\bar{p}_i + i^2 \left(\frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} \frac{\delta \mathcal{S}_0}{\delta \dot{q}_j} \frac{\delta \mathcal{S}_0}{\delta \dot{q}_k} \right) \Big|_{q \rightarrow \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta q}} \quad (46)$$

where in the last expression we have used the previous results.

Employing Eqs. (44) and Eq. (45) the first and second derivatives of \bar{p} with respect to \bar{q} can be expressed as functionals of the remaining elements. In this way the vacuum expectation value of p can be expanded. In fact, up to second order in the classical field it reads

$$\bar{p}_m = - \frac{\delta^2 \Gamma}{\delta \bar{q}_j \delta \bar{q}_l} \Delta_{lm}^{qp} \bar{q}_j - \frac{1}{2!} \left[\frac{\delta^3 \Gamma}{\delta \bar{q}_i \delta \bar{q}_j \delta \bar{q}_l} \Delta_{lm}^{qp} - \frac{\delta^2 \Gamma}{\delta \bar{q}_i \delta \bar{q}_n} \frac{\delta^2 \Gamma}{\delta \bar{q}_j \delta \bar{q}_l} \Delta_{nm}^{qpp} \right] \bar{q}_i \bar{q}_j + \dots \quad (47)$$

Since Δ_{ij}^{qp} can be computed using the procedure detailed above, the expression (47) determines \bar{p} as a function of the second-order dressed correlation functions.

C. Inverting the matrix propagator \mathbb{D}

The determination of the proper functions within the canonical formalism as a functional of those appearing in the standard Lagrange framework is rather cumbersome. This task involves connected tensors of rank larger than two and depends on the possibility to invert the propagator \mathbb{D} . Once the individual elements of this propagator are computed, the proper two-point function is completely determined in terms of the elements of the Lagrange formalism. In case the external sources associated to the p -fields vanish, the inverse of $-\Delta^{qq}$ is the proper Green's function that arises in the standard path integral representation.

By direct inversion of the 2×2 block matrix one obtains

$$\mathbb{G} = \begin{pmatrix} \Gamma_{ij}^{pp} & \Gamma_{il}^{pp} \Delta_{lm}^{pq} \Gamma_{mj}^{qq} \\ \Gamma_{il}^{qq} \Delta_{lm}^{qp} \Gamma_{mj}^{pp} & \Gamma_{ij}^{qqH} \end{pmatrix} \quad (48)$$

where

$$\Gamma_{ij}^{pp} = -(\Delta^{pp} + \Delta^{pq} \Gamma^{qq} \Delta^{qp})_{ij}^{-1} \quad \text{and} \quad \Gamma_{ij}^{qqH} = \Gamma_{ij}^{qq} + \Gamma_{il}^{qq} \Delta_{lk}^{qp} \Gamma_{km}^{pp} \Delta_{mn}^{pq} \Gamma_{nj}^{qq}. \quad (49)$$

According to Eq. (48) there are several equivalent representations of the latter expression, *e.g.* $\Gamma_{ij}^{qqH} = (\delta_{im} + \Gamma_{il}^{qp} \Delta_{lm}^{pq}) \Gamma_{mj}^{qq}$. However, in what follows we will consider the most simple form given by

$$\Gamma_{ij}^{qqH} = \Gamma_{ij}^{qq} + \Gamma_{il}^{qp} (\Gamma_{lm}^{pp})^{-1} \Gamma_{mj}^{pq}, \quad (50)$$

where we have introduced a unity in the form $\mathbb{I} = (\Gamma^{pp})^{-1} \Gamma^{pp}$ in the last expression of Eq. (49), and furthermore the expressions for the off-diagonal elements of Eq. (48) have been used.

The method to obtain \mathbb{G} can be generalized to any proper m -point function. For instance, let us suppose that we want to compute the proper three-point function. We denote it as \mathbb{G}_3 and the correspondingly connected version as $\mathbb{D}^{(3)}$. By considering the action of the symbolic functional derivative $\delta/\delta J$ on Eq. (16) we get the equation $\mathbb{D}^{(3)}\mathbb{G} + \mathbb{D}\mathbb{D}\mathbb{G}_3 = 0$. By inversion of \mathbb{D} in the second term we obtain the desired form for the proper 3-point Green's function $\mathbb{G}_3 = -\mathbb{G}\mathbb{G}\mathbb{D}^{(3)}\mathbb{G}$. Proceeding in an analogous way it is possible to express the proper four-point Green's function \mathbb{G}_4 , and so on. Thereby, the number of variables within the first order formalism can be reduced. From that point of view, the initially cumbersome problem becomes simpler.

D. Connecting the Symmetry-Related identities

Let us now return to the symmetry identity. To this end we write Eq. (18) in the following form

$$0 = \int \mathcal{D}[q] \left\{ J_m^q \delta q_m \left[\frac{\hbar}{i} \frac{\delta}{\delta J^p}, q \right] + J_m^p \delta p_m \left[\frac{\hbar}{i} \frac{\delta}{\delta J^p}, q \right] \right\} \int \mathcal{D}[p] \exp \left[\frac{i}{\hbar} I[\phi, J] \right]. \quad (51)$$

In order to simplify the following analysis let us consider theories where the transformations are linear in the momentum fields. Under this condition and restricting to the class of Hamiltonians analyzed so far we obtain

$$0 = \int \mathcal{D}[q] \left\{ J_m^q \delta q_m \left[J^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}}, q \right] + J_m^p \delta p_m \left[J^p + \frac{\delta \mathcal{S}_0}{\delta \dot{q}}, q \right] \right\} \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q, J] \right]. \quad (52)$$

In particular, if $J^p = 0$ we find that the action \mathcal{S}_0 is invariant under a symmetry transformation δq_m which is a functional of q only, *i.e.* a symmetry transformation $\delta q_m \left[\frac{\delta \mathcal{S}_0}{\delta \dot{q}}, q \right] \rightarrow \delta q_m [q]$.

The structure of Eq. (52) allows to write the symmetry identities in a similar form to Eq. (20) and Eq. (21),

$$0 = \int d^4x \left\{ \frac{\delta \Gamma^H}{\delta \bar{p}_m} \delta p_m \left[\frac{\delta \mathcal{S}}{\delta \dot{q}} - \frac{\delta \Gamma^H}{\delta \bar{p}}, q \right] + \frac{\delta \Gamma^H}{\delta \bar{q}_m} \delta q_m \left[\frac{\delta \mathcal{S}_0}{\delta \dot{q}} - \frac{\delta \Gamma^H}{\delta \bar{p}}, q \right] \right\}_{\phi \rightarrow \bar{\phi}[J] + \frac{\hbar}{i} \Delta^{\phi\phi}[J] \frac{\delta}{\delta \phi}}.$$

This concludes the formal discussion of the functional symmetry identities. A complete derivation, especially for constrained systems as Coulomb gauge Yang-Mills theories, is presented below in Sect. VIII.

IV. DECOMPOSITION OF PROPER LAGRANGE CORRELATION FUNCTIONS

In this section we will show how a general proper correlation function in the Lagrange formalism can be decomposed into correlation functions in the Hamilton approach.

A. Relations between the bare elements

So far we have presented the formalism for a general field theory. In the present and forthcoming sections we will restrict ourselves to the most important class of renormalizable field theories. In four dimensions the most general renormalizable ‘‘canonical action’’ for a pure bosonic theory can be expressed as a functional Taylor expansion,

$$I_0[q, p] = I_{0ji}^{qp} p_i q_j + \frac{1}{2} I_{0ij}^{pp} p_i p_j + \frac{1}{2} I_{0ijk}^{pqq} p_i q_j q_k + \frac{1}{2} I_{0ij}^{qq} q_i q_j + \frac{1}{3!} I_{0ijk}^{qqq} q_i q_j q_k + \frac{1}{4!} I_{0ijkl}^{qqqq} q_i q_j q_k q_l. \quad (53)$$

Here $I_{0ji}^{qp} = \partial_{\tau_i} \delta_{ij}$ is such that

$$I_{0ji}^{qp} q_j = \dot{q}_i. \quad (54)$$

Clearly, the coefficients $I_{0i\dots}^{\phi\dots}$ are field independent. They are given by the functional derivatives of I_0 evaluated at $\phi = 0$, namely

$$I_{0ij\dots}^{\phi\phi} \equiv \left. \frac{\delta}{\delta \phi_i} \frac{\delta}{\delta \phi_j} \frac{\delta}{\delta \phi_k} \dots I_0 \right|_{\phi=0}.$$

We remark that I_{0ijk}^{ppq} as well I_{0ijkl}^{qqqq} are dimensionless tensor couplings whereas I_{0ij}^{qq} and I_{0ijk}^{qqq} have mass dimension 2 and 1, respectively, and do not involve time derivatives of the fields. Nevertheless, depending on the assumed theory the latter two might depend on ∇^2 and ∇ . We are considering bosonic theories, so that all coefficients are symmetric. In particular, the first term in Eq. (2) can be written as $I_{0ji}^{pq} p_j q_i$ where $I_{0ji}^{pq} = -\partial_{\tau_i} \delta_{ij}$ which leads to the functional relation $\partial_{\tau_j} \delta_{ji} = -\partial_{\tau_i} \delta_{ij}$.

We can identify the coefficients present in the Hamiltonian density Eq. (23) as

$$\mathcal{A}_{ij} = -I_{0ij}^{pp} = \delta_{ij}, \quad \mathcal{B}_i = -\frac{1}{2} I_{0ijk}^{ppq} q_j q_k \quad \text{and} \quad \mathcal{C} = -\frac{1}{2} I_{0ij}^{qq} q_i q_j - \frac{1}{3!} I_{0ijk}^{qqq} q_i q_j q_k - \frac{1}{4!} I_{0ijkl}^{qqqq} q_i q_j q_k q_l. \quad (55)$$

Substituting Eqs. (55) in Eq. (26) and collecting the terms of the same order in q we get that the action \mathcal{S}_0 can be written as a polynomial functional,

$$\mathcal{S}_0 = \frac{1}{2} \mathcal{S}_{0ij} q_i q_j + \frac{1}{3!} \mathcal{S}_{0ijk} q_i q_j q_k + \frac{1}{4!} \mathcal{S}_{0ijkl} q_i q_j q_k q_l. \quad (56)$$

In this context the following relations between the bare coefficients arise

$$\mathcal{S}_{0ij} \equiv \left. \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta q_j} \right|_{q=0} = I_{0il}^{qp} I_{0lj}^{pq} + I_{0ij}^{qq}, \quad (57)$$

$$\mathcal{S}_{0ijk} \equiv \left. \frac{\delta^3 \mathcal{S}_0}{\delta \bar{q}_i \delta \bar{q}_j \delta \bar{q}_k} \right|_{q=0} = I_{0ikn}^{qpp} I_{0nj}^{pq} + \bar{q}_k \leftrightarrow \bar{q}_j \text{permut.} + \bar{q}_i \leftrightarrow \bar{q}_j \text{permut.} + I_{0ijk}^{qqq}, \quad (58)$$

$$\mathcal{S}_{0ijkl} \equiv \left. \frac{\delta^4 \mathcal{S}_0}{\delta q_i \delta q_j \delta q_k \delta q_l} \right|_{q=0} = I_{0ikn}^{qpp} I_{0njl}^{ppq} + \bar{q}_i \leftrightarrow \bar{q}_l \text{perm.} + \bar{q}_k \leftrightarrow \bar{q}_l \text{permut.} + I_{0ijkl}^{qqqq} \quad (59)$$

Substituting the above equations in Eq. (56) and considering the relation given by Eq. (54) allows to find additional relations. In fact, taking the second, third and fourth functional derivatives of \mathcal{S}_0 and evaluating at $q = 0$, respectively, we obtain

$$\mathcal{S}_{0ij}^{q\dot{q}} \equiv \left. \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta \dot{q}_j} \right|_{q=0} = I_{0ij}^{qp} = \partial_{\tau_j} \delta_{ji}, \quad \mathcal{S}_{0ji}^{\dot{q}q} \equiv \left. \frac{\delta^2 \mathcal{S}_0}{\delta \dot{q}_j \delta q_i} \right|_{q=0} = I_{0ij}^{pq} = -\partial_{\tau_j} \delta_{ji}, \quad \mathcal{S}_{0ijk}^{qq\dot{q}} \equiv \left. \frac{\delta^3 \mathcal{S}_0}{\delta q_i \delta q_j \delta \dot{q}_k} \right|_{q=0} = I_{0ijk}^{qqp}, \quad (60)$$

$$\left. \frac{\delta^3 \mathcal{S}_0}{\delta q_i \delta q_j \delta \dot{q}_k} \right|_{q=0} = I_{0ijk}^{qqq} \quad \text{and} \quad \left. \frac{\delta^4 \mathcal{S}_0}{\delta q_i \delta q_j \delta q_k \delta q_l} \right|_{q=0} = I_{0ijkl}^{qqqq}. \quad (61)$$

In the last two relations we have introduced δ denoting a partial functional differentiation acting just on those terms in the action that do not involve the time derivative of the field.

To complete our analysis we point out that from Eq. (A7) the general form of the quantum canonical momentum fields in four-dimensional renormalizable theories is given by

$$p_i = \mathcal{S}_{0ji}^{q\dot{q}} q_j + \frac{1}{2} \mathcal{S}_{0kji}^{qq\dot{q}} q_j q_k. \quad (62)$$

As these polynomial representations allow to identify a priori the bare elements, the expansions given above prove to be very convenient to derive the DSEs for a general bosonic theory in both formulations.

B. Diagrammatic representation

The relations between the correlation functions in both formulations will be given in the following via explicit diagrammatic expressions. To enable this, we will first introduce a graphical representation in terms of the fundamental objects that characterize the theory in both formulations.

- I) As before we consider fields that involve all the irreducible representations in the theory. The fundamental fields are represented by solid lines whereas the corresponding momentum fields are denoted by zigzag lines.



- II) Dressed propagators are denoted by thick, whereas bare propagators and external lines by thin lines, respectively. Off-diagonal propagator components are represented by a thick, half solid and half zigzag line.



To keep the representation of the DSEs in the first order formalism concise we also include the matrix propagator Eq. (10) represented by a double line.



- III) All proper correlation functions (including proper 2-point functions) in the first and second order formalism are denoted by small and large filled blobs,



whereas bare vertex functions are represented by open blobs.



- IV) In our analysis it will become useful to introduce the inverse proper momentum 2-point function

$$\mathcal{D}_{ij}^{pp} \equiv - \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{p}_j} \right)^{-1} \quad (63)$$

which will be represented by a thick dotted line. Since it has the form of an alternative momentum propagator it can connect as an internal momentum line to proper vertices. Similarly all connected correlators which are one-particle-irreducible (1PI) in the fields but merely connected via the "propagator" \mathcal{D}^{pp} are called *p-connected* and are denoted by a blob labeled by a P.



C. Diagrammatic decomposition

To express proper functions in the Lagrange formalism in terms of those of the Hamilton formalism, we exploit the underlying equivalence between the first and second order formalism. Due to the equivalence of the Generating Functionals of connected Green's functions at vanishing sources J^p , the Effective Actions in the two formalisms are identical when \bar{p} is a functional of \bar{q} which in turn is implicitly given by the stationarity in \bar{p} ,

$$\Gamma[\bar{q}] \equiv \Gamma^H[\bar{p}(\bar{q}), \bar{q}] \quad \text{whenever} \quad \frac{\delta \Gamma^H}{\delta \bar{p}} = -J^p = 0. \quad (64)$$

Here Γ^H still depends explicitly on \bar{q} and \bar{p} , where the quantum average of the momentum fields is in general a complicated functional of the fundamental field. Any proper n -point function in the standard formalism can be determined by taking n derivatives with respect to the fields in Eq. (64) and evaluated at the vacuum expectation value. In particular applying the chain rule and Eq. (64) the first derivative reads

$$\frac{\delta \Gamma}{\delta \bar{q}_i} = \frac{\delta \Gamma^H}{\delta \bar{q}_i} + \frac{\delta \Gamma^H}{\delta \bar{p}_j} \frac{\delta \bar{p}_j}{\delta \bar{q}_i} = \frac{\delta \Gamma^H}{\delta \bar{q}_i}. \quad (65)$$

In the above defined graphical representation this equation reads

As usual in the standard formalism, a differentiation with respect to a field is equivalent to attaching an external leg in the graphical representation (*cf.*, *e.g.*, ref. [27, 35]). Next we consider the second derivative of the action given by

$$\frac{\delta^2 \Gamma}{\delta \bar{q}_i \delta \bar{q}_j} = \frac{\delta \bar{p}_n}{\delta \bar{q}_i} \frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{q}_j} + \frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{q}_j}. \quad (66)$$

The field derivative of the momentum field can alternatively to the previous result (47) also be obtained from a field derivative of the constraint equation in Eq. (64),

$$\frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{p}_j} = -\frac{\delta \bar{p}_n}{\delta \bar{q}_i} \frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{p}_j} \Rightarrow \frac{\delta \bar{p}_n}{\delta \bar{q}_i} = -\frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{p}_j} \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{p}_j} \right)^{-1}. \quad (67)$$

Inserting this in Eq. (66) it is expressed entirely in terms of proper first order Green's functions and takes the symmetric form

$$\frac{\delta^2 \Gamma}{\delta \bar{q}_i \delta \bar{q}_j} = \frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{q}_j} - \frac{\delta^2 \Gamma^H}{\delta \bar{q}_i \delta \bar{p}_j} \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{p}_j} \right)^{-1} \frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{q}_j} \quad (68)$$

in accordance with Eq. (50). The above equation yields the graphical representation of the decomposition of the proper two-point Green's function

Interestingly, in terms of the first order correlation functions there is in addition to the proper part also a connected contribution due to the fact that the fundamental field mixes with the corresponding momentum field. This is evident from the off-diagonal elements Eq. (43) in the propagator matrix in the first order formalism. Due to the mixing the arising propagator in Eq. (67) is not the elementary p -propagator but the propagator for a collective mode described by the inverse of the proper two-point momentum correlation function \mathcal{D}^{pp} which is represented by the dotted line and related to the actual momentum propagator via Eq. (49).

The result for the fundamental field derivative of a proper correlation function in the Hamilton formalism yields the replacement rule

In the next step a field derivative can also act on the propagator \mathcal{D}^{pp} . Its derivative is obtained from the derivative of the inverse of an operator as

$$\frac{\delta \mathcal{D}_{ij}^{pp}}{\delta \bar{q}_k} = -\frac{\delta}{\delta \bar{q}_k} \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{p}_j} \right)^{-1} = \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_i \delta \bar{p}_m} \right)^{-1} \frac{\delta^3 \Gamma^H}{\delta \bar{p}_m \delta \bar{p}_n \delta \bar{q}_k} \left(\frac{\delta^2 \Gamma^H}{\delta \bar{p}_n \delta \bar{p}_j} \right)^{-1} = \mathcal{D}_{im}^{pp} \Gamma_{mnk}^{ppq} \mathcal{D}_{nj}^{pp}$$

which yields the graphical replacement rule

Applying these two replacement rules in all possible ways on the right hand side of the above equation for the two-point vertex provides immediately the corresponding decomposition of the proper 3-point vertex

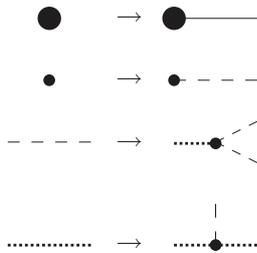


FIG. 1: The replacement rules that create the general decomposition of a proper correlation function in the Lagrange formalism in terms of Hamilton correlation functions.

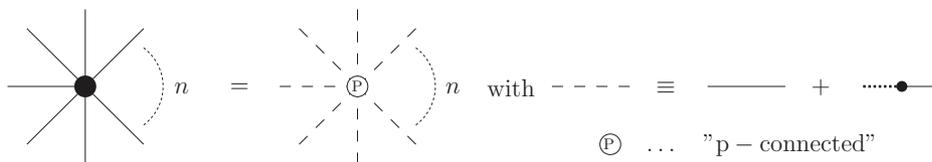


FIG. 2: The general result for the decomposition of a proper correlation function in the Lagrange framework in terms of correlators in the Hamilton formulation.

This yields directly a symmetric result, whereas the computation without the replacement Eq. (67) in each step would produce an asymmetric result involving higher derivatives of \bar{p} .

To derive the decomposition of higher order Lagrange correlation functions it is useful to note that both of the above replacement rules involve the external legs in the form of the composite expression

$$- - - - \equiv \text{---} + \text{.....}\bullet\text{---}$$

denoted by a dashed line. By introducing the composite external leg into the graphical representation, the decomposition of the 3-point function is given by a single graph with three of these new external legs. Via the previous rules it is easy to obtain a corresponding replacement rule for the composite external leg. Thereby the extension of a general n -point function by an additional leg can be obtained from the simplified set of rules in Fig. 1. Starting from the 3-point function these rules allow to derive the decomposition of arbitrary proper n -point functions in the Lagrange formalism. In the appendix we show that the graphs generated by these replacement rules have a very simple structure that can be summarized by the following general statement:

A proper n -point function in the Lagrange formalism can be decomposed into the sum of all p -connected n -point functions with composite external legs in the Hamilton framework.

This is shown in graphical form in Fig. 2.

The graphical representation manifestly shows that the propagation, decay and dispersion of particles are more complicated when analyzed in the Hamilton framework. At the tree level the arising propagators \mathcal{D}^{pp} are constant and correspondingly the dotted internal lines between the proper vertices represent merely contact terms. Similarly the external leg corrections involve no additional poles at tree level and the explicit energy dependence arising from the mixed correlator in Eq. (67) just cancels the one from the momentum field derivative of the initial vertex. In the fully dressed action, however, there could arise additional poles due to the actual propagation of the momentum fields via higher order kinetic terms, as well as an explicit energy dependence of the terms that cancels only after summing over all contributions.

Via graphical rules, one can therefore generate in a systematical way the connection between any proper Green's function in the standard second-order formalism to the appropriate correlation functions in the first-order formalism. We point out already here that the inverse relations for Green's functions in the latter context are more complicated and can involve loops related to connected correlation functions with both mixed and pure momentum fields. The method used in the last section allows us to represent them in terms of those appearing in the Lagrange formalism.

D. Inclusion of Grassmannian fields

As already mentioned we will consider for completeness also quantum field theories involving Grassmannian fields c_i and c_j^\dagger fulfilling the anti-commutation rules

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad (69)$$

$$\{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0. \quad (70)$$

The Grassmannian action for a renormalizable theory in four dimensions has the general form

$$I_0 = i\kappa_{ij}c_i^\dagger\dot{c}_j + i\lambda_{ijk}q_i c_j^\dagger c_k + i\alpha_{ij}c_i^\dagger c_j. \quad (71)$$

The quantum canonical momentum fields associated to the c_i are given by $p_i = \frac{\delta I_0}{\delta_R c_i} = i\kappa_{ij}c_j^\dagger$, where the suffix R denotes differentiation from the right. It is clear that the usual path integral representation for Grassmannian field is already of first-order form since the momentum fields are treated as independent variables. *Purely fermionic correlation functions are therefore trivially identical in the two formulations.*

Let us now study mixed correlation functions involving bosonic fields represented by the q_i in Eq. (71) when the bosonic path integral is written in the canonical form as well. Generally the tensors κ_{ij} , α_{ij} and λ_{ijk} are real and field independent. The kinetic tensor reads in particular

$$\kappa_{ij} = \begin{cases} \delta_{ij} & \text{with } c, c^\dagger \in \text{Fermions} \\ \delta_{ij}\sigma\partial_0 & \text{with } c, c^\dagger \in \text{Faddeev - Popov Ghosts} \end{cases}$$

where $\sigma \in \mathbb{R}$. Since it is a bosonic variable, \bar{p} is a functional of fermionic bilinears $\bar{c}_i^\dagger \Gamma_{ij} \bar{c}_j$ and the field \bar{q} . As a consequence the second term in Eq. (66) vanishes identically, which confirms the equality of the propagators in both formulations

$$\frac{\delta^2 \Gamma}{\delta \bar{c}_i \delta \bar{c}_j^\dagger} = \frac{\delta^2 \Gamma^H}{\delta \bar{c}_i \delta \bar{c}_j^\dagger}. \quad (72)$$

Analogous to the bosonic case the three-point vertex can be decomposed as

$$\frac{\delta^3 \Gamma}{\delta \bar{q}_i \delta \bar{c}_j \delta \bar{c}_k^\dagger} = \frac{\delta \bar{p}_n}{\delta \bar{q}_i} \frac{\delta^3 \Gamma^H}{\delta \bar{p}_n \delta \bar{c}_j \delta \bar{c}_k^\dagger} + \frac{\delta^3 \Gamma^H}{\delta \bar{q}_i \delta \bar{c}_j \delta \bar{c}_k^\dagger}. \quad (73)$$

Although the propagators are the same in both formulations the vertices can be different. The above chain rule again results in the appearance of composite legs for all bosons in the graphical representation. The decomposition of a general n -point correlation function is then again given by all p-connected graphs where only the bosonic propagators \mathcal{D}^{pp} are involved and only bosonic external legs are composite.

V. DECOMPOSITION OF CONNECTED HAMILTON CORRELATION FUNCTIONS

So far we gave in subsection III B general expressions for correlation functions in the Hamilton formalism. The goal in this section is to evaluate the general decomposition of the two-point functions in the Hamilton framework in case of a generic four-dimensional renormalizable quantum field theory in terms of Lagrange correlation functions. To do this, we start by computing the elements of Δ_{ij}^{qp} .

A. The mixed connected 2-point function

By considering Eq. (44) and Eq. (62) the mixed connected two-point function in the first order formalism can be written as (from now on we skip the explicit factor \hbar)

$$\Delta_{ij}^{qp} = \Delta_{il}^{qq} \left(\mathcal{S}_{0lj}^{q\dot{q}} - \frac{i}{2} \mathcal{S}_{0ukj}^{qq\dot{q}} \frac{\delta}{\delta \bar{q}_l} \Delta_{uk}^{qq} \right). \quad (74)$$

Using partial differentiation for the identity $\Delta_{il}^{qq}\Gamma_{lj}^{qq} = -\delta_{ij}$ we obtain

$$\frac{\delta}{\delta\bar{q}_l}\Delta_{uk}^{qq} = \Delta_{um}^{qq}\Gamma_{mln}^{qqq}\Delta_{nk}^{qq}, \quad (75)$$

and therefore in configuration space

$$\Delta_{ij}^{qp} = \Delta_{il}^{qq}\left(I_{0lj}^{qp} - \frac{i}{2}I_{0ukj}^{qqp}\Delta_{um}^{qq}\Gamma_{mln}^{qqq}\Delta_{nk}^{qq}\right), \quad (76)$$

where Eq. (60) has been used. According to the diagrammatic representation the above function has the decomposition given in Fig. 3.

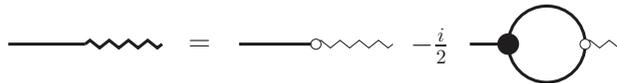


FIG. 3: The decomposition of the mixed Hamilton propagator in terms of Lagrange correlation functions.

Via the bosonic symmetry of the propagator it is easy to see that

$$\Delta_{ij}^{pq} = \left(I_{0li}^{qp} - \frac{i}{2}I_{0uki}^{qqp}\Delta_{um}^{qq}\Gamma_{mnl}^{qqq}\Delta_{nk}^{qq}\right)\Delta_{lj}^{qq}. \quad (77)$$

Clearly, Eq. (76) and Eq. (77) fulfill the condition $\Delta_{ij}^{qp} = \Delta_{ji}^{pq}$.

Our convention for the Fourier transform of a general two-point function (connected or proper) obeying translational invariance is

$$\Delta_{ij}^{\Phi\Phi} \equiv \Delta_{ij}^{\Phi\Phi}(x_i - x_j) = \int \bar{d}k \Delta_{ij}^{\Phi\Phi}(k) e^{-ik(x_i - x_j)}, \quad (78)$$

with $k(x_i - x_j) = k_0(x_{0i} - x_{0j}) - \mathbf{k} \cdot (\vec{x}_i - \vec{x}_j)$ and $\bar{d}k \equiv d^4k/(2\pi)^4$. As a consequence of this convention and the equivalence $\Delta_{ij}^{qp} = \Delta_{ji}^{pq}$ we obtain the relation

$$\Delta_{ij}^{qp}(k) = \Delta_{ji}^{pq}(-k) \quad (79)$$

where i and j represent the remaining internal indices. This yields the corresponding equation in momentum space

$$\Delta_{ij}^{qp}(k) = \Delta_{il}^{qq}(k) \left(ik_0\delta_{lj} - \frac{i}{2} \int \bar{d}\omega I_{0ukj}^{qqp}(k - \omega, \omega, -k) \Delta_{um}^{qq}(\omega - k) \Gamma_{mln}^{qqq}(\omega - k, k, -\omega) \Delta_{nk}^{qq}(\omega) \right). \quad (80)$$

In the derivation of the latter equation we have taken into account the Fourier transformation of the proper 3-point function

$$\Gamma_{\alpha\beta\gamma}^{\Phi\Phi\Phi} = \int \bar{d}k_\alpha \bar{d}k_\beta \bar{d}k_\gamma (2\pi)^4 \delta^{(4)}(k_\alpha + k_\beta + k_\gamma) \Gamma_{\alpha\beta\gamma}^{\Phi\Phi\Phi}(k_\alpha, k_\beta, k_\gamma) e^{-ik_\alpha x_\alpha - ik_\beta x_\beta - ik_\gamma x_\gamma} \quad (81)$$

where the δ -function expresses the momentum conservation.

B. The momentum propagator

The pure pp -correlator can be computed using Eqs. (42) and (62). Neglecting those terms that will eventually vanish when the sources are set to zero, it reads

$$\begin{aligned} \Delta_{ij}^{pp} &= \delta_{ij} + \mathcal{S}_{0li}^{qq}\mathcal{S}_{0mj}^{qq}\Delta_{lm}^{qq} - \frac{i}{2}\mathcal{S}_{0li}^{qq}\mathcal{S}_{0nmj}^{qqq}\Delta_{lu}^{qq}\frac{\delta}{\delta\bar{q}_u}\Delta_{nm}^{qq} - \frac{i}{2}\mathcal{S}_{0mj}^{qq}\mathcal{S}_{0kli}^{qqq}\Delta_{ku}^{qq}\frac{\delta}{\delta\bar{q}_u}\Delta_{lm}^{qq} - \frac{i}{2}\mathcal{S}_{0kli}^{qqq}\mathcal{S}_{0nmj}^{qqq}\Delta_{km}^{qq}\Delta_{ln}^{qq} \\ &\quad - \frac{1}{4}\mathcal{S}_{0kli}^{qqq}\mathcal{S}_{0nmj}^{qqq}\Delta_{ku}^{qq}\frac{\delta}{\delta\bar{q}_u}\left[\Delta_{lx}^{qq}\frac{\delta}{\delta\bar{q}_x}\Delta_{nm}^{qq}\right]. \end{aligned} \quad (82)$$

By iterated application of Eq. (75) and considering Eq. (60) this expression yields

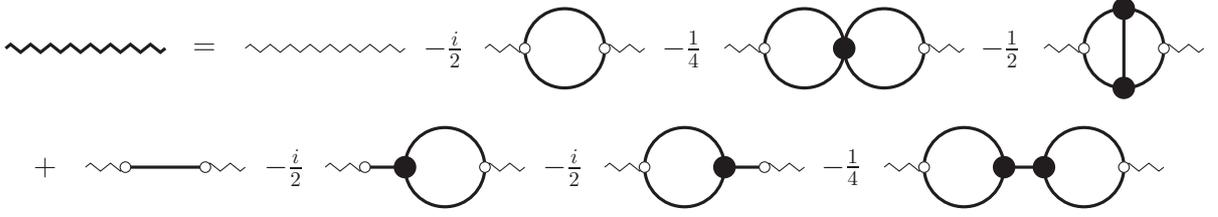


FIG. 4: The decomposition of the momentum propagator in terms of Lagrange correlation functions.

$$\begin{aligned}
\Delta_{ij}^{pp} = & \delta_{ij} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{km}^{qq} \Delta_{ln}^{qq} I_{0nmj}^{qqp} - \frac{1}{4} I_{0ikl}^{pqq} \Delta_{ku}^{qq} \Delta_{lx}^{qq} \Delta_{ny}^{qq} \Gamma_{yxuz}^{qqq} \Delta_{zm}^{qq} I_{0nmj}^{qqp} - \frac{1}{2} I_{0ikl}^{pqq} \Delta_{ku}^{qq} \Delta_{lx}^{qq} \Delta_{np}^{qq} \Gamma_{puq}^{qqq} \Delta_{qy}^{qq} \Gamma_{yxz}^{qqq} \Delta_{zm}^{qq} I_{0nmj}^{qqp} \\
& + I_{0il}^{pq} \Delta_{lm}^{qq} I_{0mj}^{qp} - \frac{i}{2} I_{0il}^{pq} \Delta_{lm}^{qq} \Gamma_{xmy}^{qqq} \Delta_{xn}^{qq} \Delta_{yu}^{qq} I_{0nuj}^{qqp} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{ku}^{qq} \Delta_{lx}^{qq} \Gamma_{xuy}^{qqq} \Delta_{ym}^{qq} I_{0mj}^{qp} \\
& - \frac{1}{4} I_{0ikl}^{pqq} \Delta_{ku}^{qq} \Delta_{lx}^{qq} \Gamma_{xuy}^{qqq} \Delta_{ym}^{qq} \Delta_{nf}^{qq} \Gamma_{fmz}^{qqq} \Delta_{zh}^{qq} I_{0nhk}^{qqp}. \tag{83}
\end{aligned}$$

Here we have already suppressed disconnected expressions involving tadpoles that cancel since Eq. (30) gives

$$\left. \frac{\delta \Gamma^H}{\delta \bar{p}_i} \right|_{q, \bar{p}=0} \sim I_{0mni}^{qqp} \Delta_{mn}^{qq} = 0. \tag{84}$$

The graphical representation of the decomposition of the momentum propagator in the Hamilton framework is shown in Fig. 4. Note that the loop graphs in the first line represent precisely the vacuum graphs of the n -particle irreducible (n PI) actions in the Lagrange formalism of order one and two with attached external p -legs. In addition to these proper contributions this result involves again connected graphs in the second line. These graphs correspond to the second and third line in Eq. (83) which can alternatively be expressed by the known mixed correlation functions derived in the last subsection as $-\Delta_{im}^{pq} \Gamma_{ml}^{qq} \Delta_{lj}^{pp}$ and correspondingly represented as a single graph.

After Fourier transformation of Eq. (83) we find the representation of Δ^{pp} in momentum space

$$\begin{aligned}
\Delta_{ij}^{pp}(k) = & \delta_{ij} - \Delta_{im}^{pq}(k) \Gamma_{ml}^{qq}(k) \Delta_{lj}^{pp}(k) - \frac{i}{2} \int \tilde{d}\omega I_{0ikl}^{pqq}(k, \omega - k, -\omega) I_{0nmj}^{qqp}(\omega, k - \omega, -k) \Delta_{km}^{qq}(\omega - k) \Delta_{ln}^{qq}(\omega) \\
& - \frac{1}{4} \int \tilde{d}\omega \tilde{d}\mu I_{0ikl}^{pqq}(k, \mu - k, -\mu) I_{0nmj}^{qqp}(-\omega, \omega + k, -k) \Delta_{ku}^{qq}(\mu - k) \Delta_{lx}^{qq}(-\mu) \Delta_{ny}^{qq}(-\omega) \Gamma_{yxuz}^{qqq}(\omega, \mu, k - \mu, -\omega - k) \\
& \times \Delta_{zm}^{qq}(-\omega - k) - \frac{1}{2} \int \tilde{d}\omega \tilde{d}\mu I_{0ikl}^{pqq}(k, \omega - k, -\omega) I_{0nmj}^{qqp}(\mu + k, -\mu, -k) \Delta_{ku}^{qq}(\omega - k) \Gamma_{puq}^{qqq}(-k - \mu, k - \omega, \omega + \mu) \\
& \times \Delta_{lx}^{qq}(-\omega) \Delta_{np}^{qq}(\mu + k) \Delta_{qy}^{qq}(\omega + \mu) \Gamma_{yxz}^{qqq}(-\omega - \mu, \omega, \mu) \Delta_{zm}^{qq}(\mu). \tag{85}
\end{aligned}$$

The summation over all field components indicated by the subindices in Eq. (80) (see also Eq. (83)) leads to multiple possibilities. Yet many of these might not be allowed due to the symmetries of the theory. The decomposition of higher order functions involving one or two external momentum fields is simply obtained by further derivatives of these generating equations. Via the chain rule a derivative w.r.t. to the source can be transformed into a derivative with respect to the averaged fields yielding in addition external propagators. Correspondingly the graphical decomposition of such correlation functions can be obtained recursively via the graphical replacement rules in the Lagrange framework given in [27, 35]. On the other hand, from Eq. (39) it is clear that higher order correlation functions involving n momentum fields involve graphs of loop order n . Therefore it is not possible to give a close result for all n -point functions.

To conclude this section we remark that the multiplication of Γ^{qq} on the left hand side amputates the Green's function Δ^{qq} . According to Eq. (47), this operation allows to derive

$$\frac{\delta \bar{p}_i}{\delta \bar{q}_j} = I_{0ji}^{qp} - \frac{i}{2} I_{0uki}^{qqp} \Delta_{um}^{qq} \Gamma_{mjn}^{qqq} \Delta_{nk}^{qq}. \tag{86}$$

Considering this expression and the diagrammatic rules in the standard path integral representation we get

$$\frac{\delta \bar{p}_i}{\delta \bar{q}_j \delta \bar{q}_k} = I_{0jki}^{qp} - \frac{i}{2} I_{0mui}^{qqp} \Delta_{mn}^{qq} \Gamma_{nkx}^{qqq} \Delta_{xy}^{qq} \Gamma_{yjt}^{qqq} \Delta_{tu}^{qq} + j \leftrightarrow k - \frac{i}{2} I_{0imu}^{pqq} \Delta_{mn}^{qq} \Gamma_{njkl}^{qqq} \Delta_{lu}^{qq}, \tag{87}$$

which can be represented graphically as



We note in passing, that similar to the composite collective propagator and composite leg in the last section this "vertex" allows to write the graphs in the first line by a single loop graph. Yet, here this is then surely no explicit decomposition in terms of second order correlation functions anymore.

VI. DECOMPOSITION OF PROPER HAMILTON CORRELATION FUNCTIONS

While in the context of the canonical formulation it is entirely possible to deduce the complete set of DSEs directly from Eqs. (35-37) we will follow a slightly less obvious path here. We proceed to give the diagrammatic and analytic expressions for the proper propagators, typically the ones of the first order formalism. Subsequently, we will show that such representations can be encoded into the usual ones, *i.e.* Eqs. (35-37), which then completes the proof of equivalence between both types of derivations.

A. The inverse propagators

The result given by Eq. (48) allows to analyze the structure of the inverse propagator. By considering the Fourier transformation of this equation we find that

$$\Gamma_{ij}^{pp}(k) = -(\Delta^{pp}(k) + \Delta^{pq}(k)\Gamma^{qq}(k)\Delta^{qp}(k))_{ij}^{-1}, \quad \Gamma_{ij}^{pq}(k) = \Gamma_{il}^{pp}(k)\Delta_{lm}^{pq}(k)\Gamma_{mj}^{qq}(k), \quad (88)$$

$$\Gamma_{ij}^{Hqq}(k) = \Gamma_{ij}^{qq}(k) + \Gamma_{il}^{qq}(k)\Delta_{lk}^{qp}(k)\Gamma_{km}^{pp}(k)\Delta_{mn}^{pq}(k)\Gamma_{nj}^{qq}(k). \quad (89)$$

For a field theory without a three-point interaction vertex involving the time derivative of the fields, one has $\Gamma^{pp} = -\mathbb{I}$, and Eq. (48) reduces to

$$\mathbb{G} = \begin{pmatrix} -\mathbb{I} & -ik_0\mathbb{I} \\ ik_0\mathbb{I} & \Gamma^{qq} - k_0^2\mathbb{I} \end{pmatrix}. \quad (90)$$

This simplified expression holds for theories like QED and/or self-interacting ϕ^4 -theory, where the vacuum expectation value of the momentum field is completely determined in terms of q , $\bar{p}_i = \dot{\bar{q}}_i$.

Actually, the expressions for Γ^{pp} , Γ^{qp} and Γ^{pq} in Eq. (90) encode a more general result because they could be obtained using Eq. (35) and Eq. (36) without setting variables to zero. As a consequence they show that dressed proper functions involving more than two external momentum legs are not present in such theories. This means that the mass and the coupling constant receive no contribution coming from the higher order corrections besides the usual ones given in the standard framework. On the other hand the absence of quantum corrections to Γ^{pp} , Γ^{qp} , and Γ^{pq} means that only wave-function renormalization contributes to these kinetic terms.

As in the present framework the quantum corrections to the propagators and vertices depend in general on temporal derivatives this shows that within the standard formalism there are several pieces in the Effective Action that depend on the time derivative of the averaged field. As a consequence, the canonical momentum fields defined on the level of the Effective Action differ from those given by the quantum average of p since the involved limiting processes do not commute.

B. Recovering the first order DSEs

Although the decomposition of proper Hamilton functions given above involves inversions that cannot be omitted in terms of second order correlation functions it is possible to transform these equations into a form where such matrix inversions are eliminated. This is done by explicitly introducing first order correlation functions on the right hand side again. As we will show in this subsection, this leads precisely to the first order DSEs. We start from Eq. (87) and use it to express Γ^{pp} as

$$\Gamma_{ij}^{pp} = - \left[\delta_{ij} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{km}^{qq} \Delta_{ln}^{qq} \frac{\delta \bar{p}_j}{\delta \bar{q}_n \delta \bar{q}_m} \right]^{-1}. \quad (91)$$

The multiplication of $-\Gamma^{pp-1}$ from the left hand side allows to write the above equation as

$$\left[\delta_{is} - \frac{i}{2} I_{0ikl}^{pqq} (\Delta_{kls}^{qqp} - \Delta_{km}^{qq} \Delta_{ln}^{qq} \Gamma_{umnn}^{qqq} \Delta_{us}^{qp}) \right] \Gamma_{sj}^{pp} = -\delta_{ij},$$

where Eq. (45) has been taken into account. The introduction of a \mathbb{I} as $-\Delta^{qq}\Gamma^{qq}$ in the last term in the bracket and the use of Eq. (45) makes it possible to express the above relation in the following form

$$\Gamma_{ij}^{pp} = -\delta_{ij} + \frac{i}{2} I_{0ikl}^{pqq} \{ \Delta_{kls}^{qqp} \Gamma_{sj}^{pp} + \Delta_{kls}^{qqq} \Gamma_{sj}^{qp} \}, \quad (92)$$

which can be translated to

$$\Gamma_{ij}^{pp} = -\delta_{ij} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{kn}^{q\phi} \Gamma_{njm}^{\phi p\phi} \Delta_{ml}^{\phi q}. \quad (93)$$

This presents the DSE for the proper momentum 2-point correlation function.

It is possible to obtain a similar equation for the pq -propagator:

$$\Gamma_{ij}^{pq} = \left[\delta_{il} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{km}^{qq} \Delta_{ln}^{qq} \frac{\delta^2 \bar{p}_l}{\delta \bar{q}_n \delta \bar{q}_m} \right]^{-1} \left(I_{0lj}^{pq} - \frac{i}{2} I_{0luk}^{pqq} \Delta_{um}^{qq} \Gamma_{mjn}^{qqq} \Delta_{nk}^{qq} \right). \quad (94)$$

The multiplication of $-\Gamma^{pp-1}$ from the right hand side allows to rewrite Eq. (94) as

$$\Gamma_{ij}^{pq} = I_{0ij}^{pq} - \frac{i}{2} I_{0iuk}^{pqq} \Delta_{um}^{qq} \Gamma_{mnj}^{qqq} \Delta_{nk}^{qq} + \frac{i}{2} I_{0iuk}^{pqq} \Delta_{um}^{qq} \Delta_{nk}^{qq} \frac{\delta \bar{p}_l}{\delta \bar{q}_m \delta \bar{q}_n} \Gamma_{lj}^{pq}. \quad (95)$$

This can be rewritten in the following form

$$\Gamma_{ij}^{pq} = I_{0ij}^{pq} - \frac{i}{2} I_{0iuk}^{pqq} \left\{ \Delta_{um}^{qq} \Gamma_{mnj}^{qqq} \Delta_{nk}^{qq} + \Delta_{um}^{qq} \Delta_{nk}^{qq} \Gamma_{mnl}^{qqq} \Delta_{lv}^{qp} \Gamma_{vj}^{pq} - \Delta_{ukl}^{qqp} \Gamma_{lj}^{pq} \right\}. \quad (96)$$

The introduction of a \mathbb{I} as $-\Delta^{qq}\Gamma^{qq}$ in the first and second term inside the brackets allows to write this as

$$\Gamma_{ij}^{pq} = I_{0ij}^{pq} + \frac{i}{2} I_{0iuk}^{pqq} \left\{ \Delta_{ukl}^{qqq} \left(\Gamma_{lj}^{qq} + \Gamma_{lm}^{qq} \Delta_{mn}^{qp} \Gamma_{nj}^{pq} \right) + \Delta_{ukl}^{qqp} \Gamma_{lj}^{pq} \right\}. \quad (97)$$

By considering Eq. (49) we get

$$\Gamma_{ij}^{pq} = I_{0ij}^{pq} + \frac{i}{2} I_{0iuk}^{pqq} \left\{ \Delta_{ukl}^{qqq} \Gamma_{lj}^{qqH} + \Delta_{ukl}^{qqp} \Gamma_{lj}^{pq} \right\} \quad (98)$$

which yields the result

$$\Gamma_{ij}^{pq} = I_{0ij}^{pq} - \frac{i}{2} I_{0ikl}^{pqq} \Delta_{kn}^{q\phi} \Gamma_{njm}^{\phi q\phi} \Delta_{ml}^{\phi q}. \quad (99)$$

We remark that the last terms of Eqs. (93) and (99) include several combinations of fundamental and momentum fields. The relations given by Eq. (93) and Eq. (99) are the DSEs of the proper pp - and pq -propagators expressed in terms of the usual elements of the canonical formulation.

It is straightforward to prove that Eq. (93) coincides with the corresponding equation derived using Eq. (35) and Eq. (62). This is different for Eq. (99). In fact, as discussed previously, we could in principle calculate these propagators from Eq. (37), nevertheless by considering the action Eq. (56) and the relations between the bare elements we find that the structure is more cumbersome than that given in Eq. (99). Instead, the latter one is in correspondence with Eq. (36) via the symmetry relation $\Gamma_{ij}^{pq} = \Gamma_{ji}^{qp}$ in case both \bar{q} and \bar{p} are bosonic fields.

The fact that we recover the standard first order propagator DSEs from the decomposition of proper Hamilton correlation functions has its origin in the equivalence between the canonical and Lagrange equations of motion at the quantum level. In appendix C it is shown that whenever they describe the same dynamical processes, the DSEs derived from them will be equivalent too. However, as we argued in the last subsection, the classical canonical momentum fields defined from the Effective Action and those given by the quantum average of p are not the same. This means that the equations expressed in term of \bar{p} and \bar{q} do not correspond to the classical canonical ones.

Based on this statement, the derivation of Γ_{ij}^{qqH} is considerably simpler using Eq. (38) than via the procedure performed in the last two cases. Indeed, by considering Eqs. (62) and (56) as well as the relations between the bare elements, we arrive at

$$\frac{\delta\Gamma^H}{\delta\bar{q}_i} = \left[I_{0il}^{qp} p_l + I_{0ijk}^{qpp} q_j p_k + I_{0ij}^{qq} q_j + \frac{1}{2} I_{ijk}^{qqq} q_j q_k + \frac{1}{3!} I_{0ijkl}^{qqqq} q_j q_k q_l \right]_{q \rightarrow \bar{q} + \frac{h}{4} \Delta^{q\phi} \frac{\delta}{\delta\phi}}. \quad (100)$$

Now, in order to give the explicit form of Γ_{ij}^{qqH} , we rewrite Eq. (100) as

$$\begin{aligned} \frac{\delta\Gamma^H}{\delta\bar{q}_i} &= I_{0ij}^{qp} \bar{p}_j + I_{0ij}^{qq} \bar{q}_j + I_{0ijk}^{qpp} \bar{q}_j \bar{p}_k + \frac{1}{2} I_{0ijk}^{qqq} \bar{q}_j \bar{q}_k + \frac{1}{3!} I_{0ijkl}^{qqqq} \bar{q}_j \bar{q}_k \bar{q}_l - i I_{0ijk}^{qpp} \Delta_{jk}^{qp}[J] - \frac{i}{2} I_{0ijk}^{qqq} \Delta_{jk}^{qq}[J] \\ &- \frac{1}{6} I_{0ijkl}^{qqqq} \Delta_{jm}^{q\phi}[J] \Delta_{ku}^{q\phi}[J] \Gamma_{mun}^{\phi\phi\phi}[\bar{q}, \bar{p}] \Delta_{nl}^{\phi q}[J_q] - \frac{i}{2} I_{0ijkl}^{qqqq} \bar{q}_j \Delta_{kl}^{qq}[J]. \end{aligned} \quad (101)$$

Taking the functional derivative with respect to \bar{q} and setting the vacuum expectation values of the fundamental fields to zero we get

$$\begin{aligned} \Gamma_{ij}^H &= I_{0ij}^{qq} - \frac{i}{2} I_{ijkl}^{qqqq} \Delta_{kl}^{qq} - i I_{0jlm}^{qpp} \Delta_{lu}^{q\phi} \Gamma_{uin}^{\phi\phi\phi} \Delta_{nm}^{\phi p} - \frac{i}{2} I_{0jlm}^{qqq} \Delta_{lu}^{q\phi} \Gamma_{uin}^{\phi\phi\phi} \Delta_{nm}^{\phi q} - \frac{1}{2} I_{jlmn}^{qqqq} \Delta_{lh}^{q\phi} \Gamma_{hfk}^{\phi\phi\phi} \Delta_{fy}^{\phi\phi} \Delta_{mu}^{q\phi} \\ &\times \Gamma_{uiy}^{\phi q\phi} \Delta_{kn}^{\phi q} - \frac{1}{6} I_{jlmn}^{qqqq} \Delta_{lh}^{q\phi} \Delta_{mf}^{q\phi} \Gamma_{hfik}^{\phi\phi\phi} \Delta_{kn}^{\phi q}. \end{aligned} \quad (102)$$

The Fourier transformation of Eqs. (93), (99) and (102) yields finally the corresponding DSEs in momentum space

$$\Gamma_{ij}^{pp}(k) = -\delta_{ij} - \frac{i}{2} \int \bar{d}\omega I_{0ikl}^{ppqq}(k, \omega - k, -\omega) \Delta_{kn}^{q\phi}(k - \omega) \Gamma_{njm}^{\phi\phi\phi}(k - \omega, -k, \omega) \Delta_{ml}^{\phi q}(-\omega) \quad (103)$$

$$\Gamma_{ij}^{pq}(k) = -ik_0 \delta_{ij} - \frac{i}{2} \int \bar{d}\omega I_{0ikl}^{ppqq}(k, \omega - k, -\omega) \Delta_{kn}^{q\phi}(k - \omega) \Gamma_{njm}^{\phi\phi\phi}(k - \omega, -k, -\omega) \Delta_{ml}^{\phi q}(-\omega) \quad (104)$$

$$\begin{aligned} \Gamma_{ij}^{qqH}(k) &= I_{0ij}^{qq}(k) - \frac{i}{2} \int \bar{d}\omega I_{0ijkl}^{qqqq}(k, -k, -\omega, \omega) \Delta_{kl}^{qq}(\omega) - i \int \bar{d}\omega I_{0ilm}^{qpp}(k, \omega - k, -\omega) \Delta_{lu}^{q\phi}(k - \omega) \Gamma_{ujn}^{\phi\phi\phi}(k - \omega, -k, \omega) \\ &\times \Delta_{nm}^{\phi p}(-\omega) - \frac{i}{2} \int \bar{d}\omega I_{0ilm}^{qqq}(k, \omega - k, -\omega) \Delta_{lu}^{q\phi}(k - \omega) \Gamma_{ujm}^{\phi\phi\phi}(k - \omega, -k, \omega) \Delta_{nm}^{\phi q}(-\omega) \\ &- \frac{1}{2} \int \bar{d}\omega \bar{d}\mu I_{0ilmn}^{qqqq}(k, -\mu - \omega, \omega - k, \mu) \Delta_{lh}^{q\phi}(\omega + \mu) \Gamma_{hfk}^{\phi\phi\phi}(\omega + \mu, -\omega, -\mu) \Delta_{fy}^{\phi\phi}(\omega) \Delta_{mu}^{q\phi}(k - \omega) \\ &\times \Gamma_{ujy}^{\phi q\phi}(k - \omega, -k, \omega) \Delta_{kn}^{\phi q}(\mu) - \frac{1}{6} \int \bar{d}\omega \bar{d}\mu I_{0ilmn}^{qqqq}(k, \omega - k - \mu, -\omega, \mu) \Delta_{lh}^{q\phi}(k + \mu - \omega) \Delta_{mf}^{q\phi}(\omega) \\ &\times \Gamma_{hfyk}^{\phi\phi\phi}(k + \mu - \omega, \omega, -k, -\mu) \Delta_{kn}^{\phi q}(\mu) \end{aligned} \quad (105)$$

We show the graphical representation of the complete set of DSEs in Fig. 5 where for conciseness we use the matrix propagator which yields all possible loop graphs involving physical vertices in accordance with the symmetries of the action as a consequence of the implied summation over repeated indices.

VII. THEORIES WITH AUXILIARY FIELDS

Before we come to the main application of the developed formalism in the context of Coulomb gauge QCD, we will show in this section that the above derived results also apply to theories involving auxiliary fields, *i.e.* the typical treatment of such theories represent a special case of the discussion presented here. However, it is not the kinetic but the interaction terms which are linearized. In the case of fermionic theories this is also referred to as bosonization [28]. The action is by construction only quadratic in the auxiliary fields and lacks kinetic terms for them. Therefore, as we detail below the above described analysis directly applies and gives general relations between correlation functions in the fundamental theory and the linearized form involving auxiliary fields.

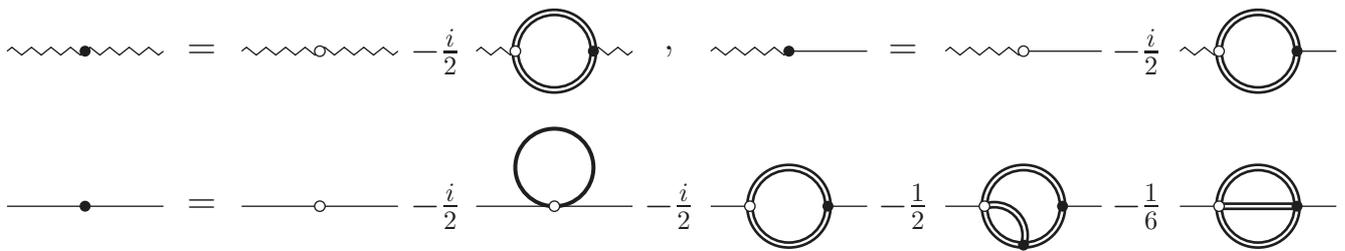


FIG. 5: Diagrammatic representation of the coupled system of DSEs within the first order formalism. The double lines represent the matrix propagator so that all possible graphs involving the individual propagators arise which are compatible with the symmetries and the restriction that the pqq -vertex is the only bare vertex involving momentum lines. In this form the formal equivalence to the second order equation is entirely manifest.

We will illustrate this in the case of fermionic theories with non-renormalizable, quartic interactions. Examples for this class of theories are the Nambu–Jona-Lasinio model [29] or the BCS theory of superconductivity [30]. The functional integral is given by

$$\int D\psi^\dagger D\psi \exp\left(\frac{i}{\hbar} \int d^4x (\mathcal{L}_\psi + J_{\bar{\psi}}(x)\bar{\psi}(x) + J_\psi(x)\psi(x))\right) \quad (106)$$

with the Lagrangian containing local quartic interactions

$$\mathcal{L}_\psi = \bar{\psi}(x)(i\partial - m_\psi)\psi(x) - \sum_i g_\psi^i (\bar{\psi}(x)\Gamma_i\psi(x))^2. \quad (107)$$

Here the Γ_i are Dirac matrices. The linearization of the fermionic interaction can be performed by formally introducing a one of the form

$$\mathbb{I} = \prod_i \int D\eta_i D\sigma_i \exp\left(i \int d^4x \sigma_i(x) (\eta_i(x) - \bar{\psi}(x)\Gamma_i\psi(x))\right) \quad (108)$$

into the fermionic path integral, where this path integral over σ_i enforces a functional δ -function that allows to rewrite the non-linear fermionic interaction in terms of η_i . Integrating then over the η_i yields the path integral of the corresponding linear sigma model

$$\int D\psi^\dagger D\psi D\sigma \exp\left(\frac{i}{\hbar} \int d^4x (\mathcal{L}_\sigma + J_{\bar{\psi}}(x)\bar{\psi}(x) + J_\psi(x)\psi(x) + J_\sigma^i(x)\sigma_i(x))\right)$$

where we have introduced additional sources for the auxiliary fields σ_i . After this bosonization procedure the Lagrangian of the corresponding "linear σ -model" reads

$$\mathcal{L}_\sigma = \bar{\psi}(x)\left(i\partial - m_\psi + g_\sigma^i \Gamma_i \sigma_i(x)\right)\psi(x) + \frac{m_\sigma^2}{2} \sigma_i(x)^2 \quad (109)$$

where $g_\psi = g_\sigma^2/(2m_\sigma^2)$. Here, there is, in contrast to the first order formalism, by construction no mixing between the fundamental and the auxiliary fields.

Analogous to Eq. (64) the Effective Actions of the two theories are again identical at vanishing sources $J_\sigma^i = 0$

$$\Gamma_\psi[\bar{\psi}^\dagger, \bar{\psi}] = \Gamma_\sigma[\bar{\psi}^\dagger, \bar{\psi}, \bar{\sigma}_i[\bar{\psi}^\dagger, \bar{\psi}]] \quad (110)$$

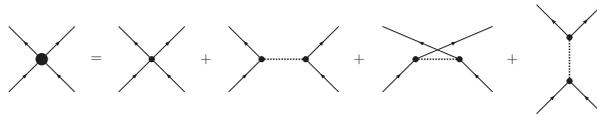
where the auxiliary fields are implicitly given by

$$\frac{\delta\Gamma_\sigma}{\delta\bar{\sigma}_i} = -J_\sigma^i = 0. \quad (111)$$

Via the chain rule of functional differentiation we obtain analogous expressions for Eqs. (65)-(69) with \bar{p}_i replaced by $\bar{\sigma}_i$, respectively. Correspondingly, the same graphical rules apply. With the simplification that there can be no mixing due to the different statistics of the fields, the external legs are not composite and the arising inverse 2-point function is the ordinary σ -propagator. Denoting as before connected correlation functions that are 1PI with respect to the fundamental fields and connected in the auxiliary fields as σ -connected, this leads to the analogous general result:

A proper n -point function in the fundamental theory can be decomposed into the sum of all σ -connected n -point functions in the linearized theory.

In particular the decomposition of the proper 4-fermion vertex in the fundamental theory Eq. (107) reads



where we represent in an analogous way the proper vertices in the fundamental and linearized theory by large and small blobs and the ordinary σ -propagator by the dotted line. This result confirms a well-known fact, namely, that no double-counting occurs in the linearized theory although the original fermion field and the auxiliary bosonic field are employed both. This redundancy in the description can also be prevented from the outset by partial re-bosonization [31] in the context of the functional renormalization group. In this approach the contribution of the fundamental degrees of freedom is entirely absorbed into the bosonized interactions even at the level of the effective action.

Since the Lagrangian of the linearized theory is at most quadratic and lacks kinetic terms for the σ -fields by construction, these fields can be trivially integrated out retaining their sources at this point. This leads to an analogous expression to Eq. (39) for the correlation function of n auxiliary fields $\langle \sigma_{i_1}(x_1) \cdots \sigma_{i_n}(x_n) \rangle$ after the sources are set to their vacuum expectation value. Due to the absence of 3-point interactions and mixing, the decomposition of the auxiliary σ -propagator is again simplified compared to the result given in Fig. 4,



where the different prefactors of the loop correction arise due to the fermionic nature of the fields. In contrast to the decomposition of the Hamilton propagators there is no difference between proper and connected 2-point functions in the case of the auxiliary field due to the absence of mixing, and the proper correlation functions is simply the inverse of the above equation. This concludes the demonstration of the developed formalism to the case of theories with auxiliary fields.

VIII. COULOMB GAUGE YANG-MILLS THEORY

A. The canonical action and the momentum correlation functions

As detailed in the Introduction the main motivation for developing the presented formalism is given by the fact that the first order formalism might be better suited for non-perturbative studies of Coulomb gauge Yang-Mills theory. As a first step into this direction we will apply the formalism developed so far to give the explicit relations between the two-point correlation functions of the transversal and longitudinal components of the conjugate momentum to the ones of the gauge field.

The starting point is the “canonical action” of Coulomb gauge Yang-Mills theory whose structure has been derived in [4, 8]

$$I_0 = \int d^4x \left(\mathbf{p}^a \cdot \dot{\mathbf{A}}^a - \frac{1}{2} (\mathbf{p}^a \cdot \mathbf{p}^a + \mathbf{B}^a \cdot \mathbf{B}^a) + \mathbf{p}^a \cdot \mathbf{D}^{ab} \sigma^b - \lambda^a \vec{\nabla} \cdot \mathbf{A}^a - \bar{c}^a \vec{\nabla} \cdot \mathbf{D}^{ab} c^b \right). \quad (112)$$

Here, \mathbf{p} is the conjugate momentum of the gauge field $A^{\mu,a} \equiv (\mathbf{A}^a, \sigma^a)$, \bar{c} and c are the Grassmann-valued Faddeev-Popov ghost fields introduced by fixing the gauge, and λ^a is a “colored” Lagrange multiplier field. Above $B_i^a = \epsilon_{ijk} \left(\nabla_j A_k^a - \frac{1}{2} g f^{abc} A_j^b A_k^c \right)$ represents the chromomagnetic field, σ^b the time-component of the gluon field, whereas $\mathbf{D}^{ac} = \vec{\nabla} \delta^{ac} - g f^{abc} \mathbf{A}^b$ is the covariant derivative in the adjoint representation, with the structure constants f^{abc} of the color-group $SU(3)$, and the gauge coupling g . The last two terms constitute the gauge fixing pieces introduced via the Faddeev-Popov procedure. However, as was originally pointed out by Gribov [32], the latter is plagued by the existence of equivalent gauge field configurations that are related by finite gauge transformations. A further condition must be imposed on the configuration space of gauge fields which restricts it to the so called “Gribov region”:

$$\Omega \equiv \left\{ \mathbf{A} : \vec{\nabla} \cdot \mathbf{A} = 0 \mid -\vec{\nabla} \cdot \mathbf{D} \geq 0 \right\}. \quad (113)$$

This simple domain is still not totally free of Gribov copies. Instead one should consider the fundamental modular region Λ [33]. It turns out though that functional integrals are dominated by configurations on the common boundary of Ω and Λ [4] so that, in practice, it is enough to consider the domain defined by Eq. (113).

Now, the term $\mathbf{p} \cdot \dot{\mathbf{A}}$ in Eq. (112) is equivalent to $p\dot{q}$ in Eq. (2), and the remaining part can be identified as a classical ‘‘Hamiltonian’’ density of Coulomb gauge Yang-Mills theory:

$$\mathcal{H}_{\text{Coul}} = \frac{1}{2} (\mathbf{p}^a \cdot \mathbf{p}^a + \mathbf{B}^a \cdot \mathbf{B}^a) - \mathbf{p}^a \cdot \mathbf{D}^{ab} \sigma^b + \lambda^a \vec{\nabla} \cdot \mathbf{A}^a + \bar{c}^a \vec{\nabla} \cdot \mathbf{D}^{ab} c^b. \quad (114)$$

Actually $\mathcal{H}_{\text{Coul}}$ is not the Coulomb gauge Hamiltonian density derived by Christ and Lee [2]. The derivation of the latter through Eq. (112) and the corresponding path integral representation requires further steps detailed e.g. in Ref. [34] and references therein. Although the Christ and Lee Hamiltonian has the desirable property that it involves only the physical degrees of freedom, the non-local character of the Color-Coulomb potential prevents an effective analysis of renormalizability in this framework. On the contrary, the canonical action given in Eq. (112) proved useful to study the renormalization properties of Coulomb gauge Yang-Mills theory [4, 5, 8, 18].

It is noteworthy that within the context of Eq. (112) the ghost sector is fully disconnected from \mathbf{p}^a which allows to write the ‘‘canonical action’’ as a functional Taylor expansion involving a pure bosonic piece like Eq. (53) and one containing the ghost field via Eq. (71). Therefore, we can identify $q = (\mathbf{A}^a, \sigma^a, \lambda^a)$ and employ the general expressions derived so far.

Except for the inverse, bare ghost two-point function

$$I_0^{ab(\bar{c}c)} = -\mathbf{k}^2 \delta^{ab}, \quad (115)$$

the remaining inverse tree level propagators of the theory are presented in Table I. Note that I_0 involves four-vertices that have the following form in momentum space

$$I_{0ij}^{abc(pA\sigma)} = -g f^{abc} \delta_{ij}, \quad I_{0i}^{abc(\bar{c}cA)} = ig f^{abc} k_{\bar{c}i}, \quad (116)$$

$$I_{0ijl}^{abc(AAA)} = ig f^{abc} [\delta_{ij}(k_a - k_b)_l + \delta_{jl}(k_b - k_c)_i + \delta_{li}(k_c - k_a)_j], \quad (117)$$

$$I_{0ijklm}^{abcd(AAAA)} = -g^2 \{ \delta_{ij} \delta_{lm} [f^{ace} f^{bde} - f^{ade} f^{bce}] + \delta_{il} \delta_{jm} [f^{abe} f^{cde} - f^{ade} f^{bce}] + \delta_{im} \delta_{jl} [f^{ace} f^{bde} - f^{abe} f^{cde}] \}, \quad (118)$$

with all momenta defined as incoming.

The decomposition of the conjugate momentum into transverse and longitudinal parts, $\mathbf{p}^a = \vec{\pi}^a - \vec{\nabla} \cdot \Omega^a$ (note that Ω^a must not be confused with the Gribov region), makes it convenient to study the required complete cancellation of the energy divergences [4, 5, 8] that emerge in a perturbative treatment of Coulomb gauge Yang-Mills theory. Certainly, such a decomposition increases the number of fields of the theory and leads to

$$\tilde{I}_0 = \int d^4x \left(\vec{\pi}^a \cdot \dot{\mathbf{A}}^a - \vec{\nabla} \Omega^a \cdot \dot{\mathbf{A}}^a + \frac{1}{2} \Omega^a \nabla^2 \Omega^a - \mathcal{H}_{\text{Coul}}(\mathbf{p} \rightarrow \vec{\pi}) - \tau^a \vec{\nabla} \cdot \vec{\pi}^a - \vec{\nabla} \Omega^a \cdot \mathbf{D}^{ab} \sigma^b \right), \quad (119)$$

where τ is a colored Lagrange multiplier field which appears via the transversality condition of $\vec{\pi}$. In this context, the general structure of the connected two-point functions is given in Table II. The latter may be found independent of any approximation but based solely on the principles of BRST invariance, spatial and time-reversal symmetries and transversality properties of the vector propagators. Each dressing function $\Lambda_{\phi\phi}$ is a dimensionless scalar function of k_0^2 and \mathbf{k}^2 except the ghost propagator

$$\Delta^{ab(\bar{c}c)} = \delta^{ab} \frac{\Lambda^c(\mathbf{k}^2)}{\mathbf{k}^2} \quad (120)$$

depending only on \mathbf{k}^2 . The tree-level propagators are obtained when

$$\Lambda_{\Omega\Omega} = \Lambda_{\Omega\lambda} = 0, \quad \Lambda_{AA} = \Lambda_{A\pi} = \Lambda_{\pi\pi} = \Lambda_{\sigma\sigma} = \Lambda_{\sigma\Omega} = \Lambda_{\sigma\lambda} = \Lambda^c = 1. \quad (121)$$

For a complete description, the reader is referred to ref. [4].

The general results of the preceding sections can in principle directly applied to Coulomb gauge QCD as well by decomposing the momentum field into its individual components. However, since the longitudinal momentum field features more complicated bare correlation functions that involve additional derivative operators we derive the corresponding expressions of subsection III B once more taking now the general action (A5) into account. To this

I_0	A_j	p_j	σ	λ
A_i	$-\mathbb{T}\mathbf{k}^2$	$ik_0\mathbb{I}$	$-k_0\mathbf{k}$	$-i\mathbf{k}$
p_i	$-ik_0\mathbb{I}$	$-\mathbb{I}$	$-i\mathbf{k}$	0
σ	0	$i\mathbf{k}$	0	0
λ	$i\mathbf{k}$	0	0	0

TABLE I: Tree level proper two point functions (without color factors) in momentum space.

end we first have to relate the bare momentum correlation functions with the corresponding ones for the individual momentum components. We formally express the decomposition via a the operator \mathcal{X}

$$p_i = \pi_i + \partial_i \Omega = \pi_i + \mathcal{X}_{ij} \Omega_j \quad \text{and} \quad \mathcal{X}_{ij} = \frac{\delta p_i}{\delta \Omega_j} = -\nabla_i \delta_{ij}. \quad (122)$$

Expanded in a local series in terms of the individual momentum components the general canonical action takes the form

$$I_0[q, p] = I_{0ji}^{qp} \pi_i q_j + \frac{1}{2} I_{0ij}^{pp} \pi_i \pi_j + \frac{1}{2} I_{0ijk}^{pqq} \pi_i q_j q_k + I_{0ji}^{qp} \mathcal{X}_{il} \Omega_l q_j + \frac{1}{2} \mathcal{X}_{il} I_{0ij}^{pp} \mathcal{X}_{jk} \Omega_l \Omega_k + \frac{1}{2} I_{0ijk}^{pqq} \mathcal{X}_{il} \Omega_l q_j q_k \\ + \frac{1}{2} I_{0ij}^{qq} q_i q_j + \frac{1}{3!} I_{0ijk}^{qqq} q_i q_j q_k + \frac{1}{4!} I_{0ijkl}^{qqqq} q_i q_j q_k q_l$$

with $I_{0ij}^{pp} = -\delta_{ij}$. The above expression yields for the transverse tree level correlators

$$I_{0ij}^{\pi q} = I_{0ij}^{pq}, \quad I_{0ij}^{\pi \pi} = I_{0ij}^{pp} \quad \text{and} \quad I_{0ijk}^{\pi qq} = \frac{\delta}{\delta \pi_i} \frac{\delta}{\delta q_j} \frac{\delta}{\delta q_k} I_0 \Big|_{\pi, \Omega, q=0} = I_{0ijk}^{pqq}, \quad (123)$$

as well as analogously for the longitudinal expressions

$$I_{0ij}^{\Omega q} = I_{0jl}^{qp} \mathcal{X}_{li}, \quad I_{0ij}^{\Omega \Omega} = \mathcal{X}_{li} I_{0lk}^{pp} \mathcal{X}_{kj} = -\mathcal{X}_{li} \delta_{lk} \mathcal{X}_{kj} = -\nabla_j^2 \delta_{ij} \quad (124)$$

and

$$I_{0ijk}^{\Omega qq} = \frac{\delta}{\delta \Omega_i} \frac{\delta}{\delta q_j} \frac{\delta}{\delta q_k} I_0 \Big|_{\pi, \Omega, q=0} = -I_{0ijk}^{pqq} \nabla_i \delta_{ii'}. \quad (125)$$

The Gaussian integration over π yields

$$S[q, \Omega, J^\pi, J^\Omega, J^q] = \mathcal{S}_0 + \frac{1}{2} J_i^\pi J_i^\pi + J_i^\pi \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} + I_{0ji}^{q\Omega} \Omega_i q_j + \frac{1}{2} \Omega_j I_{0ij}^{\Omega \Omega} \Omega_i + \frac{1}{2} I_{0ijk}^{\Omega qq} \Omega_i q_j q_k + J_i^\Omega \Omega_i + J_i^q q_i. \quad (126)$$

where \mathcal{S}_0 is given by Eq. (62) and J^i are the corresponding sources associated to the momentum fields. Similarly, the subsequent Gaussian integration over Ω gives

$$\tilde{S}[q, J^\pi, J^\Omega, J^q] = \mathcal{S}_0 + \frac{1}{2} J_i^\pi J_i^\pi + J_i^\pi \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} + J_i^q q_i - \frac{1}{2} J_i^\Omega s e (I_0^{\Omega \Omega})_{ij}^{-1} J_i^\Omega - J_i^\Omega (I_0^{\Omega \Omega})_{ij}^{-1} \left(I_{0uj}^{q\Omega} q_u + \frac{1}{2} I_{0juk}^{\Omega qq} q_u q_k \right) \\ - \frac{1}{2} I_{0ji}^{q\Omega} (I_0^{\Omega \Omega})_{il}^{-1} I_{0ml}^{q\Omega} q_j q_m - \frac{1}{2} I_{0ji}^{q\Omega} (I_0^{\Omega \Omega})_{il}^{-1} I_{0lmu}^{\Omega qq} q_j q_m q_u - \frac{1}{8} I_{0im\mu}^{\Omega qq} (I_0^{\Omega \Omega})_{ij}^{-1} I_{0jkl}^{\Omega qq} q_m q_u q_l q_k. \quad (127)$$

By collecting the terms of the same order in q we obtain the bare vertices in the second order formalism in terms of those that arise in the first order formalism. Thereby, the master equation for the momentum propagator Eq. (42) becomes in the case of the Ω -field

$$\Delta_{ij}^{\Omega \Omega} = (I_0^{\Omega \Omega})_{il}^{-1} \left[\delta_{lm} + i \left(I_{0ul}^{q\Omega} q_u + \frac{1}{2} I_{0lku}^{\Omega qq} q_k q_u \right) \left(I_{0sm}^{\Omega \Omega} q_s + \frac{1}{2} I_{0msd}^{\Omega qq} q_s q_d \right) \right] (I_0^{\Omega \Omega})_{mj}^{-1}$$

whereas the corresponding mixed version Eq. (44) is given by

$$\Delta_{ij}^{q\Omega} = -\Delta_{il}^{qq} \frac{\delta}{\delta q_l} \left(I_{0um}^{q\Omega} q_u + \frac{1}{2} I_{mku}^{\Omega qq} q_k q_u \right) (I_0^{\Omega \Omega})_{mj}^{-1}. \quad (128)$$

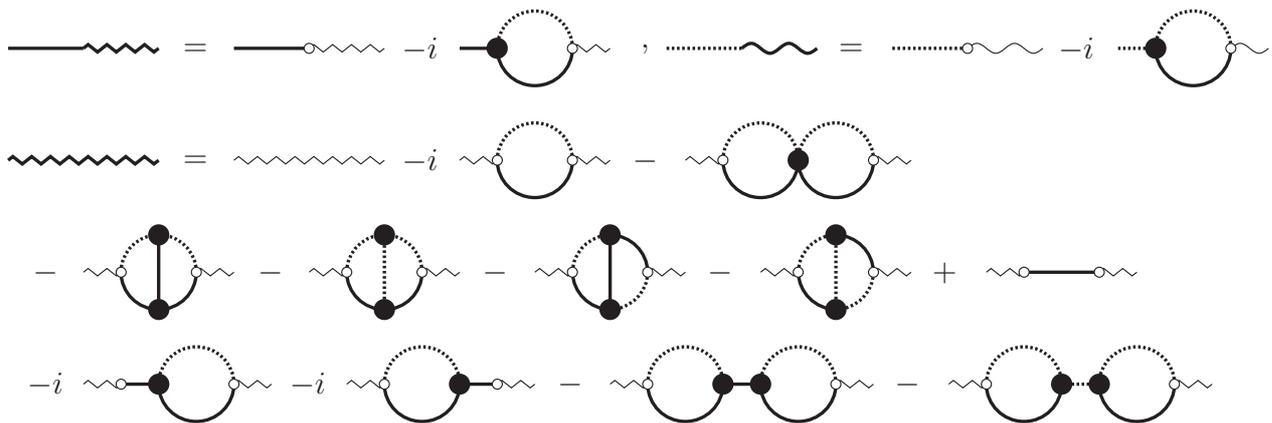


FIG. 6: Decomposition of the proper 2-point functions of Coulomb gauge QCD in the first order formalism in terms of the corresponding correlation functions of the second order representation. The spatial (\mathbf{A}) and temporal (σ) gauge fields are represented by solid respectively dotted lines whereas the corresponding transverse (π) and longitudinal (Ω) momenta by zigzag respectively wavy lines. The equations for the longitudinal momentum propagator are identical to the those for the transverse component and given by replacing zigzag by wavy lines.

\mathcal{W}^H	A_j	π_j	σ	Ω	λ	τ
A_i	$\mathbb{T} \frac{-\Lambda_{AA}}{(k_0^2 - \mathbf{k}^2)}$	$\mathbb{T} \frac{-ik_0^0 \Lambda_{A\pi}}{(k_0^2 - \mathbf{k}^2)}$	0	0	$\frac{-ik_i}{\mathbf{k}^2}$	0
π_i	$\mathbb{T} \frac{ik_0^0 \Lambda_{A\pi}}{(k_0^2 - \mathbf{k}^2)}$	$\mathbb{T} \frac{-\mathbf{k}^2 \Lambda_{\pi\pi}}{(k_0^2 - \mathbf{k}^2)}$	0	0	0	$\frac{-ik_i}{\mathbf{k}^2}$
σ	0	0	$\frac{-\Lambda_{\sigma\sigma}}{\mathbf{k}^2}$	$\frac{\Lambda_{\sigma\Omega}}{\mathbf{k}^2}$	$\frac{-ik_0^0 \Lambda_{\sigma\lambda}}{\mathbf{k}^2}$	0
Ω	0	0	$\frac{\Lambda_{\sigma\Omega}}{\mathbf{k}^2}$	$\frac{\Lambda_{\Omega\Omega}}{\mathbf{k}^2}$	$\frac{-ik_0^0 \Lambda_{\lambda}}{\mathbf{k}^2}$	$\frac{-1}{\mathbf{k}^2}$
λ	$\frac{ik_j}{\mathbf{k}^2}$	0	$\frac{ik_0^0 \Lambda_{\sigma\lambda}}{\mathbf{k}^2}$	$\frac{ik_0^0 \Lambda_{\Omega\lambda}}{\mathbf{k}^2}$	0	0
τ	0	$\frac{ik_j}{\mathbf{k}^2}$	0	$\frac{-1}{\mathbf{k}^2}$	0	0

TABLE II: General form of propagators in momentum space. The global color factor δ^{ab} has been extracted. All unknown functions $\Lambda_{\phi\phi}$ are dimensionless, scalar functions of k_0^2 and \mathbf{k}^2 . Here $\mathbb{T}_{ij} = \mathbb{I}_{ij} - k_i k_j / \mathbf{k}^2$ is the transverse projector in momentum space.

Here the q fields in Eqs. (128) and (128) again have to be replaced by

$$q \rightarrow \bar{q}[J^q] + \frac{\hbar}{i} \Delta^{qq}[J^q] \frac{\delta}{\delta \bar{q}}. \quad (129)$$

The corresponding equations involving the π field are identical to Eqs. (42) and (44). Performing the same algebraic steps as in sect. V yields the corresponding decomposition of the Hamilton propagators in Coulomb gauge QCD. The result is displayed in diagrammatic form in Fig. 6, it constitutes the main result of this section. We have also checked the structure of these expressions by an explicit projection of the general equations on the corresponding momentum components in appendix D.

Next we will consider the other direction of the connection discussed in Sect. IV, the one which expresses proper Lagrange correlators in terms of Hamiltonian ones. This is interesting in the context of Coulomb gauge QCD, since in the Hamilton framework a complete proof of the renormalizability of the theory seems possible due to explicit cancellations ensured by powerful Ward identities [8]. Here again it is the particular definition of the longitudinal momentum that complicates the issue and does not allow to give these equations explicitly. Nevertheless, one can easily convince oneself that these equations are given entirely by tree graphs that introduce no additional divergences. Therefore, once the renormalizability of the theory in the first order formalism is established, the corresponding connection immediately implies renormalizability of the theory in the Lagrange framework. Yet the cancellation mechanism of arising divergences might be far from obvious in the latter framework and could nevertheless prevent simple truncation schemes.

B. Connection between the renormalizability in Hamilton & Lagrange Coulomb gauge QCD

In the previous subsection we have derived general connections that give Coulomb gauge Greens functions in the first order formalism in terms of those in the second order formalism and vice versa. These general connections will allow to show the following statement:

Coulomb gauge is renormalizable in both formalisms if it is renormalizable in either one of them.

The renormalizability in the second order formalism is trivial if the first order formalism proves renormalizable since there are no loop graphs in the connection, as has already been noted.

In order to prove the other direction we assume that the Green's functions of the second order theory have been properly renormalized. As the following statements hold for both $\bar{\pi}^a$ and Ω^a we adopt the notation where p stands for both of these fields. We will show that the explicit expressions for the first order Greens functions given by the connections Fig. 6 and the respective ones for higher order Greens functions are likewise renormalizable. As a first step we show that there are no energy divergences in these connections. Energy divergences arise from loop graphs whose integrands are energy independent. In the case of the propagators given in Fig. 6, all loop graphs involve dressed spatial gluon propagators that according to table II are energy dependent. Without cancellations of the corresponding propagators these integrals therefore do not feature energy divergences. Actually this holds not only for the propagators but for the corresponding connection of arbitrary correlation functions in the first order formalism. To see this it is useful to consider the general structure of the loop graphs arising in these connection equations. The loop structure of these equations is determined by Eq. (39) which describes arbitrary pure momentum correlation functions. Mixed correlation functions are obtained from this equations by further derivatives with respect to the fields which correspond to attaching additional external legs to the graphs but do not alter the loop structure. The terms involving the momentum sources do not involve any summation and only yield the tree level correlators on the classical action. In Coulomb gauge Yang-Mills theory, the arising derivative of the bare action takes the form given in Eq. (62), i.e. it involves bare qp - and qpp -vertices where the external momentum leg is attached. The subsequent replacement of Eq. (129) in Eq. (39) produces a series of graphs. This latter replacement term in particular involves a summation (over the index b in Eq. (39)) that can lead to loop integrals.

As discussed in detail in (author?) [35] the field derivatives can either act on fields, propagators, or dressed vertices that were created before. Acting on a field simply attaches the associated propagator to the corresponding leg of the respective bare vertex. Acting on a dressed propagator splits it in two, inserts a dressed 3-point vertex and attaches the associated propagator to it. Finally acting on a dressed vertex attaches an additional leg and connects the associated propagator to it. Correspondingly, these individual rules effect merely that the dressed propagators in Eq. (39) are connected in all possible ways to the bare vertices or to each other via dressed vertices. All disconnected contributions vanish analogous to Eq. (84). With this graphical picture in mind it is easy to establish a few general properties of arbitrary connection graphs. Each external momentum leg is either attached indirectly via a tree level pq -mixing term and a field propagator to the residual graph or directly to a bare $p\sigma\mathbf{A}$ -vertex. In case the momentum leg is indirectly attached to a loop graph this is done via a dressed field-vertex. Loop graphs arise if a propagator is connected to the bare vertex it started from or if a chain of connected propagators returns to its starting point. Therefore, the number of loops in a graph is at most the number of its external momentum legs. Moreover, each loop graph includes at least one bare $p\sigma\mathbf{A}$ -vertex and correspondingly each loop graph also contains at least one spatial gluon propagator which according to table II is manifestly energy dependent.

Now let us exclude the possibility of cancellations. Since energy divergences arise from the unsuppressed integration over all modes they arise from the UV part of the loop integral. In the UV limit, however, the dressed proper correlators arising in the loop graphs take due to asymptotic freedom up to possible logarithmic corrections their bare form. Therefore, the perturbative pole of the transverse gluon propagator had to be cancelled by a corresponding factor in a vertex to remove the energy dependence of the propagator and render the loop energy divergent. Yet, contributions from the UV regime of graphs involving other than primitively divergent vertices vanish, whereas the bare form of the primitively divergent vertices, given in Eqs. (116) - (118), does not involve a corresponding factor that could cancel the bare propagator pole. Correspondingly it is clear that there cannot be cancellations that would remove the energy dependence of the integrand in the UV regime and so the loop integrals do not involve any explicit energy divergences. All loop integrals are thereby - aside from usual UV divergences that have to be renormalized - finite and well defined.

On dimensional grounds only propagator and 3-point functions involving momentum fields can feature ordinary UV divergences. These primitively divergent correlators are the qp - and pp -propagator in Fig. 6 and the qpp -vertices. Note first that they do not feature power-law divergences. This is trivial for graphs where the external momentum fields are all directly attached to the loop via a bare vertex, since the corresponding bare vertices are momentum independent. According to the above rules there is also the possibility that exactly one external momentum field is attached indirectly via a dressed 3-gluon vertex to a loop graph in the connection of one of the primitively divergent

correlation functions. If the corresponding external momentum is transverse the 3-gluon vertex is proportional to the energy and the linear divergent term vanishes in the symmetric integration. Similarly if the external momentum is longitudinal the corresponding 3-gluon vertex is proportional to the momentum and the transversality of the gluon propagator cancels this contribution. Correspondingly there is no linear divergence as should be the case for a gauge theory. The loop integrals appearing in the expressions for the primitively divergent momentum propagators can surely feature ordinary logarithmic UV divergences. Moreover, there could in principle be overlapping logarithmic divergences in these up to 3-loop graphs. Yet, the important point is that there are dedicated additional counterterms in the first order Lagrangian that allow to cancel any divergences explicitly by defining renormalized quantities. Finally there could in principle be overlapping divergences in subgraphs of higher order n -point functions, but due to the absence of energy divergences and the explicit renormalizability of all primitively divergent correlation functions we do not expect this to be a real issue. Clearly, a rigorous analysis of this problem requests to study the associated Slavnov-Taylor identities (derived in the last subsection). We finally note that since all connected Greens functions are finite after the renormalization process the corresponding proper Greens functions related via matrix inversion are likewise finite.

C. First order formalism and BRS-invariance

Next we consider the functional symmetry identity. The canonical action I_0 is BRST-invariant, *i.e.* invariant under

$$\begin{aligned} \delta \mathbf{A}^a &= \frac{1}{g} \mathbf{D}^{ac} c^c \delta \lambda, \quad \delta \sigma^a = -\frac{1}{g} \mathbf{D}^{0ac} c^c \delta \lambda, \quad \delta \bar{c}^a = \frac{1}{g} \lambda^a \delta \lambda, \quad \delta c^a = -\frac{1}{2} f^{abc} c^b c^c \delta \lambda, \\ \delta \mathbf{p}^a &= f^{abc} c^b \delta \lambda [(1-\alpha) \mathbf{p}^c - \alpha \mathbf{E}^c], \quad \delta \lambda^a = 0. \end{aligned} \quad (130)$$

Here $\delta \lambda$ is a Grassmannian infinitesimal parameter, whereas $\mathbf{D}^{0ac} = \delta^{ac} \partial_0 + g f^{abc} \sigma^b$. On the other hand $\mathbf{E}^a = -\vec{\nabla} \sigma^a - \mathbf{D}^{0ac} \mathbf{A}^c$ is the chromoelectric field. Note in addition that α is some color-singlet constant which in general could be some function of position.

By considering the above transformation the Slavnov-Taylor identity reads

$$\begin{aligned} 0 &= \int \mathcal{D}[\phi] \int d^4x \left\{ -\frac{1}{g} \rho^a \mathbf{D}^{0ab} c^b + \frac{1}{g} \mathbf{j}^a \cdot \mathbf{D}^{ab} c^b - \frac{1}{g} \lambda^a \eta^a - \frac{1}{2} f^{abc} \bar{\eta}^a c^b c^c + f^{abc} c^b [(1-\alpha) \mathbf{p}^c + \alpha \mathbf{E}^c] \cdot \mathbf{J}_p \right\} \\ &\times \exp \{ i I_0 + i I_s \}, \end{aligned} \quad (131)$$

where

$$I_s = \int d^4x \left\{ \rho^a \sigma^a + \mathbf{j}^a \cdot \mathbf{A}^a + \bar{c}^a \eta^a + \bar{\eta}^a c^a + \mathbf{p}^a \cdot \mathbf{J}_p \right\}. \quad (132)$$

Employing a procedure analogous to the one used in Sect. III D we obtain

$$\begin{aligned} 0 &= \int \mathcal{D}[\tilde{\phi}] \int d^4x \left\{ -\frac{1}{g} \rho^a \mathbf{D}^{0ab} c^b + \frac{1}{g} \mathbf{j}^a \cdot \mathbf{D}^{ab} c^b - \frac{1}{g} \lambda^a \eta^a - \frac{1}{2} f^{abc} \bar{\eta}^a c^b c^c + f^{abc} c^b [(1-\alpha) \mathbf{J}_p^c - \mathbf{E}^c] \cdot \mathbf{J}_p \right\} \\ &\times \exp \{ i S_0 + i S_s \}, \end{aligned} \quad (133)$$

where $\mathcal{D}[\tilde{\phi}]$ denotes the remaining integration measure of the fields c, \bar{c}, \mathbf{A} and σ . Expressed in the field strength tensor $\mathfrak{F}_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ one has

$$S_0 = \int d^4x \left\{ -\frac{1}{4} \mathfrak{F}^{a\mu\nu} \mathfrak{F}_{\mu\nu}^a - \lambda^a \vec{\nabla} \cdot \mathbf{A}^a - \bar{c}^a \vec{\nabla} \cdot \mathbf{D}^{ab} c^b \right\} \quad (134)$$

$$S_s = \int d^4x \left\{ \rho^a \sigma^a + \mathbf{j}^a \cdot \mathbf{A}^a + \bar{c}^a \eta^a + \bar{\eta}^a c^a - \mathbf{J}_p^a \cdot \mathbf{E}^a + \frac{1}{2} \mathbf{J}_p^a \cdot \mathbf{J}_p^a \right\}. \quad (135)$$

In addition we decompose the source $\mathbf{J}_a^p = \mathbf{J}_a^\pi - \frac{\vec{\nabla}}{-\nabla^2} J_a^\Omega$ into transversal and longitudinal components. Employing this decomposition the Slavnov-Taylor identity can be written as

$$\begin{aligned} 0 &= \int \mathcal{D}[\tilde{\phi}] \int d^4x \left\{ -\frac{1}{g} \rho^a \mathbf{D}^{0ab} c^b + \frac{1}{g} \mathbf{j}^a \cdot \mathbf{D}^{ab} c^b - \frac{1}{g} \lambda^a \eta^a \right. \\ &\left. - \frac{1}{2} f^{abc} \bar{\eta}^a c^b c^c + f^{abc} c^b (1-\alpha) \left[\mathbf{J}_c^\pi \cdot \mathbf{J}_a^\pi + \frac{\vec{\nabla}}{-\nabla^2} J_c^\Omega \frac{\vec{\nabla}}{-\nabla^2} J_a^\Omega \right] - f^{abc} c^b \left[\mathbf{E}_T^c \cdot \mathbf{J}_a^\pi - \mathbf{E}_L^c \frac{\vec{\nabla}}{-\nabla^2} J_a^\Omega \right] \right\} \exp \{ i S_0 + i S_s \}, \end{aligned} \quad (136)$$

where

$$E_T^{ci} = T^{ij} E^{ic} \quad \text{and} \quad E_L^{ci} = \frac{\partial^i \partial^j}{\nabla^2} E^{jc}. \quad (137)$$

This completes the application of the developed formalism to Coulomb gauge Yang-Mills theory.

IX. SUMMARY AND OUTLOOK

We conclude with our main result that given a quantum field theory in the context of the first order formalism it is possible to decompose all Green's functions in terms of those obtained from the second order formalism and vice versa. Whereas proper Lagrange correlation functions are given explicitly to all orders and involve only tree graphs involving dressed Hamilton correlation functions, the decomposition of Hamilton n -point functions involves loop graphs of loop order n . Although the structure of the latter equations seems to be somewhat cumbersome, they are still more compact and simple than the usual DSEs within the Hamilton formalism. We have discussed the connection between the Hamilton and the Lagrange formalism for a general quantum field theory and illustrated the detailed structure of the arising relations in the important case of a generic four-dimensional renormalizable field theory.

In accordance with the obtained equations we have argued that in theories where the quantum average of the momentum fields is completely determined as $\bar{p} = \dot{\bar{q}}$ the proper p -propagators receives quantum corrections only via wave-function renormalization. Additionally, it has been shown that the canonical momentum fields which can be defined from the Effective Action and those given by the quantum average of p are in general different.

As a demonstration of the general nature of the presented formalism we also showed that the results obtained in this paper can also be applied to the case of theories involving auxiliary fields from a linearization of the interaction part of the action. This yields general relations between correlation functions in the fundamental theory and the linearized form involving auxiliary fields. A major difference is that in this case the fundamental and the auxiliary fields do not mix which simplifies the connection considerably.

Clearly, the determination of Green's functions of Yang-Mills theories in the first order formalism is considerably more complicated compared to scalar or Abelian gauge theories. The presence of a coupling between the gauge fields and their time derivative as well as the subtleties of gauge fixing present major complications. In particular the renormalizability still poses a major challenge. The presented general connection of arbitrary Green functions in the two formulations allows to relate the structure of arising divergences in the different formulations. In particular, due to this relation the following important statement has been derived: *Coulomb gauge is renormalizable in both formalisms if it is renormalizable in either one of them.* This is useful since a proof of the renormalizability of the theory seems more feasible in the first order formalism where energy divergences explicitly cancel. The obtained connections should then help to understand the more intricate cancellation mechanism and thereby how to explicitly renormalize the theory in the second order formalism, where actual analytic or numerical calculations are considerably simpler. A detailed analysis of the renormalizability in Coulomb gauge Yang-Mills theory will be presented in a forthcoming publication.

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APPENDIX A: p -INTEGRATION OF $\mathcal{Z}[J]$.

In this appendix we perform the integration over the momentum fields in the vacuum-vacuum transition amplitude in presence of the classical sources. In order to do this, we assume a general quadratic Hamiltonian density given by Eq.(23). In order to evaluate the Gaussian functional integral over the momentum fields we first consider a finite dimensional generating functional

$$\mathfrak{Z}[J] = \mathcal{N}_J \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} d^4 q^j \prod_{j=0}^N d^4 p^j \exp \left[\frac{i}{\hbar} (\tilde{I}_0 - \mathcal{C} + J_q^i q^i) \right]. \quad (A1)$$

with the momentum field dependent part of the action

$$\tilde{I}_0 = -\frac{1}{2}p^i \mathcal{A}_{ij} p^j + (\dot{q}^i - \mathcal{B}^i + J_p^i) p^i. \quad (\text{A2})$$

By completing the squares in the above expression we arrive at

$$\tilde{I}_0 = \frac{1}{2}\pi^i (-\mathcal{A}_{ij}) \pi^j + \frac{1}{2} [\dot{q}^i - \mathcal{B}^i + J_p^i] \mathcal{A}_{ij}^{-1} [\dot{q}^j - \mathcal{B}^j + J_p^j], \quad (\text{A3})$$

where $\pi^i \equiv p^i - \mathcal{A}_{ij}^{-1} (\dot{q}^j - \mathcal{B}^j + J_p^j)$. By considering the change of the integration variables from p^i to π^i we obtain

$$\mathfrak{Z}[J] = \tilde{\mathcal{N}}_J \int \prod_{i=1}^{N-1} d^4 q^i \exp \left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q, J] \right] \quad \text{with} \quad \tilde{\mathcal{N}}_J = \mathcal{N}_J \int \prod_{i=0}^{N-1} d^4 \pi^i \exp \left[-\frac{i}{2} \pi^i \mathcal{A}^{ij} \pi^j \right] \quad (\text{A4})$$

and

$$\tilde{\mathcal{S}}[q, J] = \mathcal{S}_0[q] + \frac{1}{2} J_p^i \mathcal{A}_{ij}^{-1} [q] J_p^j + J_q^i q^i + J_p^i \mathcal{A}_{ij}^{-1} [q] (\dot{q}^j - \mathcal{B}^j [q]).$$

Here, the action \mathcal{S}_0 can be obtained by considering the limit of $\tilde{\mathcal{S}}[q, J]$ when $J \rightarrow 0$. Explicitly it reads

$$\mathcal{S}_0[q] = \frac{1}{2} \dot{q}^i \mathcal{A}_{ij}^{-1} [q] \dot{q}^j - \dot{q}^i \mathcal{A}_{ij}^{-1} [q] \mathcal{B}^j [q] - \int d^4 x \mathcal{V}[q], \quad (\text{A5})$$

where

$$\mathcal{V}[q] = -\frac{1}{2} \mathcal{B}^i [q] \mathcal{A}_{ij}^{-1} [q] \mathcal{B}^j [q] + \mathcal{C}[q]. \quad (\text{A6})$$

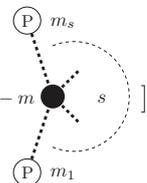
The canonical momentum fields defined as

$$p_n^{\text{can}} \equiv \frac{\delta \mathcal{S}_0}{\delta \dot{q}_i} = \mathcal{A}_{ij}^{-1} [q] (\dot{q}^j - \mathcal{B}^j [q]) \quad (\text{A7})$$

allow us to identify and substitute the last term in $\mathcal{S}[q, J]$ by its respective definition, and to eventually obtain the desired form of Eq. (25).

APPENDIX B: PROOF OF THE GENERAL FORM OF THE DECOMPOSITION OF PROPER LAGRANGE CORRELATION FUNCTIONS

In this appendix we prove the statement made in subsection IV C that the replacement rules precisely generate the p -connected correlators. As usual for a statement over the integers this is done by induction. The statement is trivially fulfilled in the case $n = 3$ given explicitly before. Now let us assume it is fulfilled for all integers $\leq n$ and show that this implies its validity for $n + 1$. In the p -connected diagrams with n legs we single out the proper vertex we started the iteration with. This vertex can be connected to other connected clusters so that all arising graphs have the general form

$$n \textcircled{\text{P}} = \sum_{m=0}^n \sum_{s=0}^{\lfloor m/2 \rfloor} \sum_{\mathcal{D}(\{m_i\})} [n-m \textcircled{\bullet} \textcircled{s}]$$


Here we suppress all external legs and only give their number next to the corresponding vertex. The sum over s counts the connected clusters attached to the considered proper vertex which in addition has $n - m$ external legs and the implicitly given sum labeled $\mathcal{D}(\{m_i\})$ runs over all ways to distribute the remaining m external legs to the s indistinguishable connected vertices (taking into account that a vertex has at least 3 legs).

Next we consider the attachment of an additional external leg via the rules given in Fig. 1, applied in all possible ways. The above representation then goes over to a $n + 1$ -point function of the following form

$$\begin{aligned}
n \textcircled{P} \rightarrow & \sum_{m=2}^n \sum_{s=1}^{\lfloor m/2 \rfloor} \sum_{\mathcal{D}(\{m_i\})} \sum_{i=1}^s ([\textcircled{P}^{R} \textcircled{P}^{m_s} \textcircled{P}^{m_1} \textcircled{P}^{m_i+1}] + [\textcircled{P}^{m_s} \textcircled{P}^{m_1} \textcircled{P}^{m_i}]) \\
& + \sum_{m=0}^n \sum_{s=0}^{\lfloor m/2 \rfloor} \sum_{\mathcal{D}(\{m_i\})} ([\textcircled{P}^{m_s} \textcircled{P}^{m_1} \textcircled{P}^{n-m+1}] + \sum_{i=1}^{n-m} [2 \textcircled{P}^{m_s} \textcircled{P}^{m_1} \textcircled{P}^{n-m}])
\end{aligned}$$

Here the connected cluster with $n + 1$ legs in the first class of graphs is labeled by the additional index R (for reduced) since one of the legs is connected to the p -propagator instead to a composite external leg. The derivative of this propagator is explicitly present via the second class of graphs. When the new proper vertex is absorbed in the connected cluster, according to the induction assumption ($m_i + 1 < n$) these two terms together precisely yield the full p -connected vertex with $m + 1$ external legs and without the index R . Since each connected cluster had before already at least two external legs, together with the new one there are now at least three. Terms where besides the new external leg there is only one other external leg at a connected cluster are explicitly given by the fourth class of graphs involving a new "connected cluster" that consists only of the proper 3-point vertex. Finally, the general expression at order $n + 1$ contains also graphs where the new external leg is attached to the proper vertex itself, given by the third class of graphs. Altogether, the sum of the different classes of graphs precisely yields all necessary graphs at order $n + 1$, which completes the proof.

APPENDIX C: FIRST ORDER DYSON-SCHWINGER EQUATIONS

In this appendix we derive the DSEs in the first order formalism directly. In order to do this we compute Eq. (14) with the canonical action given in Eq. (53)

$$\frac{\delta \Gamma^H}{\delta \bar{p}_i} = \frac{\delta I_0}{\delta p_i} \Big|_{\phi \rightarrow \bar{\phi} + \frac{\hbar}{i} \Delta \phi \frac{\delta}{\delta \phi}} = \left[-p_i + I_{0li}^{qp} q_l + \frac{1}{2} I_{0lki}^{qqp} q_l q_k \right]_{\phi \rightarrow \bar{\phi} + \frac{\hbar}{i} \Delta \phi \frac{\delta}{\delta \phi}}. \quad (C1)$$

We can identify the sum of the last two terms inside the bracket as the quantum canonical momentum fields, which allows to write the above relation as Eq. (30).

The substitution of Eq. (C1) in Eq. (34), allows to write the latter as

$$\frac{\delta \Gamma^H}{\delta \bar{q}_i} = \left[\frac{\delta \mathcal{S}_0}{\delta q_i} + \left(p_m - \frac{\delta \mathcal{S}_0}{\delta \dot{q}_m} \right) \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta \dot{q}_m} \right]_{\phi \rightarrow \bar{\phi} + \frac{\hbar}{i} \Delta \phi \frac{\delta}{\delta \phi}} \quad (C2)$$

where

$$\begin{aligned}
\frac{\delta \mathcal{S}_0}{\delta q_i} = & \mathcal{S}_{0ij}^{qq} q_j + \frac{1}{2} \mathcal{S}_{0ijk}^{qqq} q_j q_k + \frac{1}{3!} \mathcal{S}_{0ijkl}^{qqqq} q_j q_k q_l + I_{0jl}^{qp} I_{0li}^{pq} q_j + I_{0ij}^{qq} q_j + I_{0ijm}^{qqp} I_{0mk}^{pq} q_j q_k \\
& + \frac{1}{2} I_{0jkm}^{qqp} I_{0mi}^{pq} q_j q_k + \frac{1}{2} I_{0ijk}^{qqq} q_j q_k + \frac{1}{3!} I_{0ijkl}^{qqqq} q_j q_k q_l + \frac{1}{2} I_{0ijm}^{qqp} I_{0mkl}^{ppq} q_j q_k q_l
\end{aligned} \quad (C3)$$

and

$$\begin{aligned}
\left(p_m - \frac{\delta \mathcal{S}_0}{\delta \dot{q}_m} \right) \frac{\delta^2 \mathcal{S}_0}{\delta q_i \delta \dot{q}_m} = & I_{0im}^{qp} p_m + I_{0ijm}^{qqp} p_m q_j - I_{0jl}^{qp} I_{0li}^{pq} q_j - I_{0ijm}^{qqp} I_{0mk}^{pq} q_j q_k - \frac{1}{2} I_{0jkm}^{qqp} I_{0mi}^{pq} q_j q_k - \frac{1}{2} I_{0ijm}^{qqp} \\
& \times I_{0mkl}^{ppq} q_j q_k q_l.
\end{aligned} \quad (C4)$$

In the last two equations we have used the elementary decomposition of the bare elements and Eq. (62). Plugging Eqs. (C3) and (C4) into Eq. (C2) we arrive at the DSE (100).

APPENDIX D: EXPLICIT PROJECTION ON THE INDIVIDUAL MOMENTUM COMPONENTS

Defining the transverse projector $\mathbb{T}_{ij} = \delta_{ij} - \partial_i \partial_j / \nabla^2$ one has $\pi_i^a = \mathbb{T}_{ij} p_j^a$ and $\Omega^a = \frac{\vec{\nabla}}{-\nabla^2} \cdot \vec{p}^a$. As no time derivative appears one can obtain $\Delta_{ij}^{A\pi}$, $\Delta_{ij}^{\pi\pi}$, $\Delta_{ij}^{\sigma\Omega}$, and $\Delta_{ij}^{\Omega\Omega}$ by projecting p in the two-point functions Δ_{ij}^{Ap} , $\Delta_{ij}^{\sigma p}$ and Δ_{ij}^{pp} .

For instance to obtain $\Delta_{ij}^{A\pi}$ we decode the result given in Eq. (76). Let us first identify the fundamental field q with index i with A_i^a . We then expand the remaining sums over the fields involved in the q 's considering all possible combinations which generate the bare vertex $I_{0ijk}^{(p\sigma A)}$. The bosonic symmetry of the latter allows to write

$$\Delta_{ij}^{Ap} = \Delta_{il}^{AA} \left(I_{0lj}^{Ap} - i I_{0ukj}^{\sigma Ap} \Delta_{um}^{\sigma\sigma} \Gamma_{mln}^{\sigma AA} \Delta_{nk}^{AA} \right). \quad (D1)$$

Here no internal propagator like $\Delta_{ij}^{A\lambda}$ appears since no proper vertex functions with λ -derivative exist. The relation between this function and $\Delta_{ij}^{A\pi}$ is given by

$$\Delta_{ij}^{A\pi} = \mathbb{T}^{jr} \Delta_{ir}^{Ap} = \mathbb{T}^{rj} \Delta_{il}^{AA} \left(I_{0lr}^{Ap} - i I_{0ukr}^{\sigma Ap} \Delta_{um}^{\sigma\sigma} \Gamma_{mln}^{\sigma AA} \Delta_{nk}^{AA} \right). \quad (D2)$$

Obviously the case corresponding to $\Delta_{ij}^{\sigma\Omega}$ is obtained by replacing in Eq. (D1) $\vec{A} \rightarrow \sigma$ where appropriate and considering the relation

$$\Delta_{ij}^{\sigma\Omega} = \frac{\partial^j}{\nabla^2} \Delta_{ij}^{\sigma p} = \frac{\partial^j}{\nabla^2} \Delta_{il}^{\sigma\sigma} \left(I_{0lj}^{\sigma p} - i I_{0ukj}^{\sigma Ap} \Delta_{um}^{\sigma\sigma} \Gamma_{mln}^{\sigma\sigma A} \Delta_{nk}^{AA} \right). \quad (D3)$$

Note that the sum over r in Eq. (D2) and over j in Eq. (D3) must be understood over the discrete spatial indices only.

Clearly, by demanding a similar procedure we obtain that

$$\Delta_{ij}^{\pi\pi} = \mathbb{T}^{i'i'} \mathbb{T}^{j'j'} \Delta_{i'j'}^{pp} \quad \text{and} \quad \Delta_{ij}^{\Omega\Omega} = \frac{\partial^{i'}}{\nabla^2} \frac{\partial^{j'}}{\nabla^2} \Delta_{i'j'}^{pp} \quad (D4)$$

with

$$\begin{aligned} \Delta_{ij}^{pp} = & \delta_{ij} - \Delta_{im}^{pq} \Gamma_{ml}^{qq} \Delta_{lj}^{qp} - i I_{0ikl}^{pA\sigma} \Delta_{km}^{AA} \Delta_{ln}^{\sigma\sigma} I_{0nmj}^{\sigma Ap} - I_{0ikl}^{pA\sigma} \Delta_{ku}^{AA} \Delta_{lx}^{\sigma\sigma} \Delta_{ny}^{\sigma\sigma} \Gamma_{yxyz}^{\sigma\sigma AA} \Delta_{zm}^{AA} I_{0nmj}^{\sigma Ap} \\ & - I_{0ikl}^{pA\sigma} \Delta_{ku}^{AA} \Delta_{lx}^{\sigma\sigma} \Delta_{np}^{\sigma\sigma} \Gamma_{puq}^{Aq} \Delta_{qy}^{qq} \Gamma_{yz}^{\sigma A} \Delta_{zm}^{AA} I_{0nmj}^{\sigma Ap} - I_{0ikl}^{pA\sigma} \Delta_{ku}^{AA} \Delta_{lx}^{\sigma\sigma} \Delta_{np}^{AA} \Gamma_{puq}^{AAq} \Delta_{qy}^{qq} \Gamma_{yz}^{\sigma\sigma} \Delta_{zm}^{\sigma\sigma} I_{0nmj}^{\sigma Ap}. \end{aligned} \quad (D5)$$

Note that the sum over all field components q_i in the internal propagators leads to multiple possibilities.

The Fourier transformation of these expressions read therefore

$$\begin{aligned} \Delta_{ij}^{ab(A\pi)}(k) = & \Delta_{il}^{ar(AA)}(k) \left(ik_0 \mathbb{T}_{lj}(\mathbf{k}) \delta^{rb} - i \mathbb{T}_{jv}(\mathbf{k}) \int \tilde{d}\omega I_{0uv}^{cdb(\sigma Ap)}(k - \omega, \omega, -k) \Delta_{um}^{de(AA)}(\omega - k) \Gamma_{ml}^{erh(AA\sigma)}(\omega - k, k, -\omega) \right. \\ & \left. \times \Delta^{hc(\sigma\sigma)}(\omega) \right), \end{aligned} \quad (D6)$$

$$\begin{aligned} \Delta_{ij}^{ab(\sigma\Omega)}(k) = & \Delta_{il}^{al(\sigma\sigma)}(k) \left(-\delta^{lb} - \frac{k^j}{\mathbf{k}^2} \int \tilde{d}\omega I_{0uj}^{cdb(\sigma Ap)}(k - \omega, \omega, -k) \Delta_{um}^{de(AA)}(\omega - k) \Gamma_m^{elh(A\sigma\sigma)}(\omega - k, k, -\omega) \right. \\ & \left. \times \Delta^{hc(\sigma\sigma)}(\omega) \right), \end{aligned} \quad (D7)$$

$$\begin{aligned} \Delta_{ij}^{ab(\pi\pi)}(k) = & \mathbb{T}_{ij}(\mathbf{k}) \delta^{ab} - \Delta_{im}^{ae(\pi A)}(k) \Gamma_{ml}^{ec(AA)}(k) \Delta_{lj}^{cb(A\pi)}(k) - \mathbb{T}_{ii'}(\mathbf{k}) \mathbb{T}_{ij'}(\mathbf{k}) \left\{ i \int \tilde{d}\omega I_{0i'k}^{acd(pA\sigma)}(k, \omega - k, -\omega) \right. \\ & \times \Delta_{km}^{cf(AA)}(\omega - k) \Delta_{lm}^{de(\sigma\sigma)}(\omega) I_{0mj'}^{efb(\sigma Ap)}(\omega, k - \omega, -k) + \int \tilde{d}\omega \tilde{d}\mu I_{0i'k}^{acd(pA\sigma)}(k, \mu - k, -\mu) \Delta_{ku}^{cn(AA)}(\mu - k) \\ & \times \Delta_{lm}^{df(\sigma\sigma)}(-\mu) \Delta_{lm}^{ey(\sigma\sigma)}(-\omega) \Gamma_{uz}^{yfnl(\sigma\sigma AA)}(\omega, \mu, k - \mu, -\omega - k) \Delta_{zm}^{lf(AA)}(-\omega - k) I_{0mj'}^{efb(\sigma Ap)}(-\omega, \omega + k, -k) \\ & + \int \tilde{d}\omega \tilde{d}\mu I_{0i'k}^{acd(pA\sigma)}(k, \omega - k, -\omega) \Delta_{ku}^{cl(AA)}(\omega - k) \Gamma_{up}^{vlh(\sigma Aq)}(-k - \mu, k - \omega, \omega + \mu) \Delta_{lm}^{dx(\sigma\sigma)}(-\omega) \\ & \times \Delta_{lm}^{ev(\sigma\sigma)}(\mu + k) \Delta_{py}^{hs(qq)}(\omega + \mu) \Gamma_{yz}^{sxt(q\sigma A)}(-\omega - \mu, \omega, \mu) \Delta_{zm}^{tf(AA)}(\mu) I_{0mj'}^{efb(\sigma Ap)}(\mu + k, -\mu, -k) \\ & \left. + \int \tilde{d}\omega \tilde{d}\mu I_{0i'k}^{acd(pA\sigma)}(k, \omega - k, -\omega) \Delta_{ku}^{cl(AA)}(\omega - k) \Gamma_{zup}^{vlh(AAq)}(-k - \mu, k - \omega, \omega + \mu) \Delta_{zm}^{ev(AA)}(\mu + k) \right\} \end{aligned}$$

$$\times \Delta^{dx(\sigma\sigma)}(-\omega)\Delta^{hs(qq)}(\omega+\mu)\Gamma_y^{sxt(q\sigma\sigma)}(-\omega-\mu,\omega,\mu)\Delta^{tf(\sigma\sigma)}(\mu)I_{0mj'}^{efb(\sigma Ap)}(\mu+k,-\mu,-k), \quad (D8)$$

$$\begin{aligned} \Delta^{ab(\Omega\Omega)}(k) &= \frac{1}{\mathbf{k}^2}\delta^{ab} - \Delta_{im}^{ae(\Omega\sigma)}(k)\Gamma_{ml}^{ec(\sigma\sigma)}(k)\Delta_{lj}^{cb(\sigma\Omega)}(k) - \frac{k^i k^j}{\mathbf{k}^4} \left\{ i \int \tilde{d}\omega I_{0ik}^{acd(pA\sigma)}(k,\omega-k,-\omega) \right. \\ &\times \Delta_{km}^{cf(AA)}(\omega-k)\Delta^{de(\sigma\sigma)}(\omega)I_{0mj}^{efb(\sigma Ap)}(\omega,k-\omega,-k) + \int \tilde{d}\omega \tilde{d}\mu I_{0ik}^{acd(pA\sigma)}(k,\mu-k,-\mu)\Delta_{ku}^{cn(AA)}(\mu-k) \\ &\times \Delta^{df(\sigma\sigma)}(-\mu)\Delta^{ey(\sigma\sigma)}(-\omega)\Gamma_{uz}^{yfnl(\sigma\sigma AA)}(\omega,\mu,k-\mu,-\omega-k)\Delta_{zm}^{lf(AA)}(-\omega-k)I_{0mj}^{efb(\sigma Ap)}(-\omega,\omega+k,-k) \\ &+ \int \tilde{d}\omega \tilde{d}\mu I_{0ik}^{acd(pA\sigma)}(k,\omega-k,-\omega)\Delta_{ku}^{cl(AA)}(\omega-k)\Gamma_{up}^{vlh(\sigma Aq)}(-k-\mu,k-\omega,\omega+\mu)\Delta^{dx(\sigma\sigma)}(-\omega) \\ &\times \Delta^{ev(\sigma\sigma)}(\mu+k)\Delta_{py}^{hs(qq)}(\omega+\mu)\Gamma_{yz}^{sxt(q\sigma A)}(-\omega-\mu,\omega,\mu)\Delta_{zm}^{tf(AA)}(\mu)I_{0mj}^{efb(\sigma Ap)}(\mu+k,-\mu,-k) \\ &+ \int \tilde{d}\omega \tilde{d}\mu I_{0ik}^{acd(pA\sigma)}(k,\omega-k,-\omega)\Delta_{ku}^{cl(AA)}(\omega-k)\Gamma_{zup}^{vlh(AAq)}(-k-\mu,k-\omega,\omega+\mu)\Delta_{zm}^{ev(AA)}(\mu+k) \\ &\times \Delta^{dx(\sigma\sigma)}(-\omega)\Delta_{py}^{hs(qq)}(\omega+\mu)\Gamma_y^{sxt(q\sigma\sigma)}(-\omega-\mu,\omega,\mu)\Delta^{tf(\sigma\sigma)}(\mu)I_{0mj}^{efb(\sigma Ap)}(\mu+k,-\mu,-k). \end{aligned} \quad (D9)$$

The bare Greens functions involving the full momentum field can now be expressed by the corresponding expression for the individual components via the explicit expressions given in section VIII which results in the equations given in diagrammatic form in Fig. 6.

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