HAL open science

Central Limit Theorem by Higher Order Correlation Coefficients

René Blacher

▶ To cite this version:

René Blacher. Central Limit Theorem by Higher Order Correlation Coefficients. [Research Report] LJK. 2011. hal-00600639

HAL Id: hal-00600639
https://hal.science/hal-00600639

Submitted on 15 Jun 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Central Limit Theorem by Higher Order Correlation Coefficients

René BLACHER

René BLACHER
Laboratoire LMC, BP 53 38.041 Grenoble Cedex 9 FRANCE

Summary : The higher order correlation coefficients are able to detect any dependence. So, in a previous paper, we obtained conditions about these coefficients equivalent to the convergence of moments. We have deduced a central limit theorem with minimal assumptions. However, it was assumed that all random variables have the same distribution. In this report, we remove this condition. This allows us to reduce the assumptions necessary for the convergence of moments for martingales and even to replace this assumption by a weaker hypothesis. On the other hand, we shall prove that these assumptions can be simplified when the random variables are bounded.

On the other hand, we will compare the different assumptions of asymptotic independence between them, in particular, strong mixing condition, weak dependence and condition H_{mI} which we introduced in a previous paper We understand that it is this condition H_{mI} which is closest to the minimum conditions to ensure asymptotic normality. Finally, we see that, if one has a process whose moments converge, moments converge also for almost all processes which has only the same multilinear correlation coefficients that the first process.

Résumé : Les coefficients de corrélation d'ordre supérieur sont capables de détecter toute dépendance. Aussi, dans un article précédent, on a obtenu des conditions sur ces coefficients équivalentes à la convergence des moments. On en a déduit un théorème de la limite centrale avec des hypothèses minimales. On supposait cependant que tous les variables aléatoires aient la même loi. Dans ce rapport, nous supprimons cette condition. Cela nous permet de diminuer les hypothèses nécessaires à la convergence des moments pour les martingales et même de remplacer cette hypothèse par une hypothèse plus faible. D'autre part, nous montrons que l'on peut simplifier ces thèoremes lorsque on utilise des variables alèatoires bornèes.

D'autre part, nous allons comparer les différentes hypothèses d'indépendance asymptotique entre elles, en particulier, la condition fortement mélangeante, la faible dépendance et la condition H_{mI} que nous avons introduite dans un précédent article. On verra que c'est cette condition H_{mI} qui est la plus proche des conditions minimales permettant d'assurer la normalité asymptotique. Enfin, on verra aussi que si on a un processus dont les moments convergent, les moments convergent aussi pour presque tous les processus ayant seulement les mêmes coefficients de corrélation multilinéaire que ce premier processus.

KeyWords : Central Limit Theorem, moments, strongly mixing sequence, weak dependence, martingale, dependence density, higher order correlation coefficients.

Chapter 1

Higher Order Correlation Coefficients and MCLT

We first introduce the notations which we use throughout this report.

Notations 1.0.1 *Let X_n be a sequence of real random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. We suppose $\mathbb{E}\{X_s\} = 0$ for all $s \in \mathbb{N}^*$ and we set $\sigma(n)^2 = \mathbb{E}\{(X_1 + X_2 + ... + X_n)^2\}$ where $\mathbb{E}\{.\}$ is the expectation. We suppose $\mathbb{E}\{|X_s|^p\} < \infty$ for all $s \in \mathbb{N}^*$ and for all $p \in \mathbb{N}$.*

Hypothesis 1.0.1 *We assume that X_s has the law m_s for each $s \in \mathbb{N}^*$. Then, we denote by $\{P_j^s\}_{j\in\mathbb{N}}$ the family of orthonormal polynomials associated to m_s. We suppose that there exists.*

Notations 1.0.2 *Let Z_n be a sequence of real random variables. If Z_n converges in distribution to a random variable Z , one writes $Z_n \xrightarrow{d} Z$. If Z_n converges in probability to Z , one writes $Z_n \xrightarrow{P} Z$.*

If all the moments $\mathbb{E}\{Z_n^q\}$ converges to the real $\mathbb{E}\{Z^q\}$, one writes $Z_n \xrightarrow{M} Z$. Moreover, by misuse of our notations, one writes $Z_n \xrightarrow{M} N(0, M_2)$ if Z has the normal distribution $N(0, M_2)$.

1.1 Higher Order Correlation Coefficients

At first, we recall the definition of polynomial coefficients of correlation $\rho_{j_1,j_2,....,j_n}$, $(j_1, ..., j_n) \in N^n$.

Notations 1.1.1 *For all $n \in \mathbb{N}^*$, for all $(j_1, j_2, j_3, ..., j_n) \in \mathbb{N}^n$, we set $\rho_{j_1,j_2,j_3,.....,j_n} = \mathbb{E}\{P_{j_1}^1(X_1)P_{j_2}^2(X_2).....P_{j_n}^n(X_n)\}$ and $\alpha_{j_1,j_2,.....,j_n} = \mathbb{E}\{\tilde{P}_{j_1}^1(X_1)\tilde{P}_{j_2}^2(X_2).....\tilde{P}_{j_n}^n(X_n)\}$ where $\tilde{P}_j^s = \sigma_{s,j}P_j^s$ when $\sigma_{s,j} = \mathbb{E}\{(X_s)^j P_j^s(X_s)\ \}$. If $m_s = m$ for all $s \in \mathbb{N}^*$, we set $P_j^s = P_j$ and $\sigma_{s,j} = \sigma_j$.*

These dependence coefficients have been defined by Lancaster [21]. Each one measures a particular type of dependence between X_1, ...,X_n. For example, $\rho_{j_1,j_2,....,j_n} = 0$ if one of the X_j's is independent of the others. Moreover, if n=2, ρ_{j_1,j_2} is the polynomial correlation coefficient of order (j_1, j_2) between X_1 and X_2. In particular, $\alpha_{1,1}$ is the covariance and $\rho_{1,1}$ is the classical correlation coefficient : $\rho_{1,1}$ measures the linear dependence.

More generally the $\alpha_{j_1,j_2,...,j_n}$'s such that $j_s \leq 1$ measure multilinear dependence. Indeed, if $j_s = 0$ or 1, there existe $t_1, ..., t_p$ such that $\alpha_{j_1,j_2,...,j_n} = \mathbb{E}\{X_{t_1}X_{t_2}....X_{t_p}\}$.

Moreover, if $\{P_j^s\}$, $j \in \mathbb{N}$, is a basis of $L^2(\mathbb{R}, m_s)$ for each s, dependence is completely determined by these coefficients. For better understanding the part of the $\rho_{j_1,j_2,....,j_n}$ in dependence, we generalize the definitions of dependence density (cf [22]), i. e. the density with respect to $m^{\otimes} = m_1 \otimes m_2 \otimes ..... \otimes m_n$:

Definition 1.1.2 *Assume that, for all s, $\{P_j^s\}$, $j \in \mathbb{N}$, is a basis of $L^2(\mathbb{R}, m_s)$. Then, we call dependence density of $(X_1, ..., X_n)$ the formal series:*

$$f(x_1, ..., x_n) = 1 + \sum_{(j_1,...,j_n)\in N^n,\ at\ least\ 2\ j_s\neq 0} \rho_{j_1,j_2,....,j_n} P_{j_1}^1(x_1)....P_{j_n}^n(x_n) \ .$$

Indeed, one can generalize the results of [22] by the following way.

Proposition 1.1.1 *Let F_{X_s} and F_X be the distribution functions of X_s and $(X_1, ..., X_n)$. Then, for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$,*

$$F_X(x) = \int_{u\leq x}^{*} f(u) m^{\otimes}(du) \ ,$$

where

$$\int_{u\leq x}^{*} f(u) m^{\otimes}(du)$$
$$= F_{X_1}(x_1).....F_{X_n}(x_n)$$

$$+ \lim_{k_n\to\infty} \Big[\lim_{k_{n-1}\to\infty} \Big[..... \lim_{k_1\to\infty} \Big[\sum_{j_1\leq k_1,...,j_n\leq k_n} \rho_{j_1,j_2,....,j_n} \Big(\int_{-\infty}^{x_1} P_{j_1}^1 dm_1 \Big) .... \Big(\int_{-\infty}^{x_n} P_{j_n}^n dm_n \Big) \Big] ..... \Big] \Big] \ .$$

In particular, if $(X_1, ..., X_n)$ has a density f^* with respect to the product measure $m^{\otimes} = m_1 \otimes m_2 \otimes ..... \otimes m_n$, $f^* \in L^2(\mathbb{R}^n, m^{\otimes})$, $\sum_{j_1\leq k_1,...,j_n\leq k_n} \rho_{j_1,j_2,....,j_n} P_{j_1}^1(x_1)....P_{j_n}^n(x_n)$ converges in $L^2(\mathbb{R}^n, m^{\otimes})$ to f^*. Then, one can identify f and f^*.

On the other hand, $X_1, X_2, ..., X_n$ are independent if $f \equiv 1$, that is $\rho_{j_1,j_2,....,j_n} = 0$ for all $(j_1, j_2, ..., j_n) \neq (0, 0, ..., 0)$.

The use of dependence density allows a better understanding of the contribution of the $\rho_{j_1,...,j_n}$'s in dependence. Moreover, it simplifies the notations. Of course, f can be not a density because $\int^*$ is not inevitablty a Stieljes Riemann integral.

The interest of this definition is that the $\rho_{j_1,j_2,...,j_n}$'s are indeed dependence coefficients. As a matter of fact, the $\rho_{j_1,j_2,...,j_n}$'s measure polynomial dependence. For example $\rho_{1,2}$, $\rho_{2,1}$ and $\rho_{2,2}$ measure quadratic dependence, $\rho_{1,3}$, $\rho_{3,1}$, etc, measure cubic dependence. Moreover, $\sum_{j=1}^{\infty} \rho_{j,1}^2 \leq 1$ and $X_2 = g(X_1)$, $g \in L^2(\mathbb{R}, m_1)$, if and only if $\sum_{j=1}^{\infty} \rho_{j,1}^2 = 1$.

As a matter of fact, by using the $\rho_{j_1,j_2,...,j_n}$'s we can have a complete study of dependence. The most interesting property of these coefficients is that they can detect the most of the functional dependence.

The $\rho_{j_1,j_2,...,j_n}$'s have many applications and enable a better understanding of certain processes. For example, it is easy to express the fact that a process is a martingale because it is an orthogonal projection (cf appendix A.1.2).

Proposition 1.1.2 *let $\mathcal{F}_n$ be the σ-field generated by $X_1, ..., X_n$. Then, $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale if and only if $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$ for all $n \in \mathbb{N}$.*

So it is easy to make a connection between the fact that a martingale is an orthogonal projection and that the $\rho_{j_1,j_2,...,j_n}$'s are defined by using orthogonal polynomials P_j^s (cf appendix A.1.2).

Proposition 1.1.3 *Assume that $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale. Then, $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \geq 1$. Conversely, if $\{P_s^j\}$, $j \in \mathbb{N}$, is a basis of $L^2(\mathbb{R}, m_s)$ for all $s \in \mathbb{N}^*$, and if $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \in \mathbb{N}^*$ and for all $(j_1, ..., j_n)$, then $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale*

Now, the Fourier transform of orthogonal polynomials has a property very useful in the study of the MCLT (cf Theorem 1-2 of [23]).

Theorem 1 *For all $j \in \mathbb{N}$,*

$$\int e^{itx} P_j^s(x).m_s(dx) = \frac{\sigma_{s,j}}{j!}(it)^j + o(\ |t|^j)\ .$$

This property is very effective when we want to compute the law of sums of random variables. For example, suppose that $m_s = m$ for all s and $\mathbb{E}\{X_1^2\} = 1$. Let $\phi_m(t)$ be the characteristic function of X_1. Let f be the dependence density. Then, under certain simple assumptions, the characteristic function of $(X_1 + ... + X_n)/\sqrt{n}$ is

$$\int e^{it(x_1+....+x_n)/\sqrt{n}} f(x_1, ..., x_n) m(dx_1)....m(dx_n)$$

$$= \Big(\int e^{itx_1/\sqrt{n}} m(dx_1)\Big).....\Big(\int e^{itx_n/\sqrt{n}} m(dx_n)\Big)$$

$$+ \sum_{j_1,....,j_n} \rho_{j_1,j_2,....,j_n} \int e^{it(x_1+....+x_n)/\sqrt{n}} P_{j_1}^1(x_1)....P_{j_n}^n(x_n) m(dx_1)....m(dx_n)$$

$$= \Big(\int e^{itx_1/\sqrt{n}} m(dx_1)\Big).....\Big(\int e^{itx_n/\sqrt{n}} m(dx_n)\Big)$$

$$+ \sum_{q=0}^{\infty} \sum_{j_1+....+j_n=q} \rho_{j_1,j_2,....,j_n} \Big(\int e^{itx_1/\sqrt{n}} P_{j_1}(x_1) m(dx_1)\Big).....\Big(\int e^{itx_n/\sqrt{n}} P_{j_n}(x_n) m(dx_n)\Big)$$

$$= \phi_m(t/\sqrt{n})^n$$

$$+ \sum_{q=0}^{\infty} \sum_{j_1+....+j_n=q} \rho_{j_1,j_2,....,j_n} \Big(\frac{\sigma_{j_1}}{j_1!}\frac{(it)^{j_1}}{\sqrt{n^{j_1}}} + o(\ |t|^{j_1})\Big).....\Big(\frac{\sigma_{j_n}}{j_n!}\frac{(it)^{j_n}}{\sqrt{n^{j_n}}} + o(\ |t|^{j_n})\Big)\phi_m(t/\sqrt{n})^{n-q'}$$

(where $q' \leq q$)

$$\approx e^{-t^2/2} + \sum_{q=0}^{Q} \frac{(it)^q}{\sqrt{n^q}} \Big(\sum_{j_1+....+j_n=q} \frac{\sigma_{j_1}....\sigma_{j_n}\rho_{j_1,j_2,....,j_n}}{j_1!....j_n!} + o(\ |t|^q)\Big) e^{-t^2/2} + o(\ |t|^Q)\ .$$

Thanks to this result, a necessary and sufficient condition of convergence of moments was deduced in th 1-5 of [23] (cf also theorem 2).

This is not surprising: orthogonal polynomials have interesting applications in probability. Thus, we have obtained in [27] the exact distributions of quadratic forms by using the Hermite polynomials H_j and Laguerre polynomials L_j which have properties even stronger : $\frac{\int e^{itx} H_j(x).e^{-x^2/2}dx}{(2\pi)^{1/2}} = \frac{\sigma_j}{j!}(it)^j$. This has provided a simple formula to calculate the distributions of quadratic forms of Gaussian vectors without assuming the independence and whatever the dimension.

1.2 Central Limit Theorem

1.2.1 Case of random variables with the same distribution

One has proved in [23] the following theorem.

Theorem 2 *One assumes that, for all $s \in \mathbb{N}^*$, $m_s = m$. Then, all the moments $M_q^n = \mathbb{E}\Big\{\frac{(X_1+X_2+....+X_n)^q}{\sqrt{n^q}}\Big\}$ converges to $M_q \in \mathbb{R}$ if and only if, for all $q \in \mathbb{N}$, there existe $S_q \in \mathbb{R}$ such that*

$$\frac{q!}{\sqrt{n^q}} \sum_{j_1+j_2+...+j_n=q;\ j_s\leq 2} \alpha_{j_1,j_2,...,j_n} \to S_q \ .$$

Moreover, for all $q \in \mathbb{N}$, M_q is the moment of order q of $N(0, M_2)$ if and only if, for all $q \in \mathbb{N}$, S_q is the moment of order q of $N(0, S_2)$. In this case, $M_2 = S_2 + \sigma_0^2$ where $\sigma_0^2 = \mathbb{E}\{X_1^2\}$.

The interest of this theorem is that the $\rho_{j_1,j_2,...,j_n}$'s are indeed dependence coefficients. Now, theorem 2 gives only an equivalence to the convergence of the moments. In other words, we only turn this convergence into a condition on the dependence coefficients $\rho_{j_1,j_2,...,j_n}$. Then, in these theorems there is no asymptotical independence assumption. Besides, we can easily build up some sequences $\{X_n\}$ whose the moments converge without that the X_j are asymptotically independent. For example, let us take $X_n = e_n Y$ when Y has a distribution N(0,1) and $e_n = \pm 1$ is correctly chosen : $M_q^n \to M_q$ for all q. Though, in this case, the X_n's has the most strong dependence, the linear dependence with a linear correlation coefficient $\rho_{1,1} = \pm 1$.

Then, in order to have asymptotical independence condition it is enough to choose assumptions a little stronger on the $\rho_{j_1,j_2,...,j_n}$'s. By this method, we can obtain minimal conditions for the central limit theorem. For example the following theorem holds (cf [23]).

Theorem 3 *One assumes that, for all $s \in \mathbb{N}^*$, $m_s = m$. We suppose that*

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0 \ .$$

We suppose also that, for all $q \in \mathbb{N}^*$,

$$\frac{q!}{\sqrt{n^q}} \sum_{t_1=1}^{n} \sum_{t_2=t_1+1}^{n} ..... \sum_{t_q=t_{q-1}+1}^{n} \mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_q}\}$$

converges to the moment of order q of $N(0, S_2)$ and that

$$\frac{1}{\sqrt{n^q}} \sum_{j_1+j_2+...+j_n=q;\ j_s\leq 2,\ only\ one\ j_s=2} \rho_{j_1,j_2,...,j_n}$$

is bounded.

Then, $(X_1 + X_2 + ... + X_n)/\sqrt{n} \stackrel{M}{\to} N(0, M_2)$ with $M_2 = S_2 + \sigma_1^2$.

We recall that $(X_1 + X_2 + ... + X_n)/\sqrt{n} \stackrel{d}{\to} N(0, M_2)$ if $(X_1 + X_2 + ... + X_n)/\sqrt{n} \stackrel{M}{\to} N(0, M_2)$.

Remark that $S_2 < 0$ is possible because $M_2 = \sigma_0^2 + S_2$. In this case, the moment of order q of $N(0, S_2)$ is the moment of $i|S_2|Y_G$ where $Y_g \sim N(0, 1)$.

Note that the $\rho_{j_1,....,j_n}$'s or the $\alpha_{j_1,....,j_n}$'s appear well in each of these conditions. Indeed, there exists $(j_1, ..., j_n)$ where $j_s \leq 1$ such that $\mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_q}\} = \alpha_{j_1,....,j_n}$. Moreover, by proposition A.2.1, $n^{-2}\mathbb{E}\big\{\big[\sum_{s=1}^{n}\big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\big]^2\big\} \to 0$ is equivalent to

$$n^{-2} \sum_{j_1+j_2+...+j_n=4;\ j_s=2\ or\ 0} \rho_{j_1,j_2,...,j_n} \to 0$$

.

On the other hand, the conditions of Theorem 3 are actually stronger than those of Theorem 2 . Indeed, if $n^{-2}\mathbb{E}\{\big[\sum_{s=1}^{n}[X_s^2-\mathbb{E}\{X_s^2\}]\big]^2\}\to 0$ and if $(X_1+X_2+....+X_n)/\sqrt{n}\overset{M}{\to}N(0,M_2)$, by lemma 4-1 of [23],

$$\frac{1}{\sqrt{n^q}}\sum_{j_1+j_2+...+j_n=q;\ j_s\leq 2,\ at\ least\ 1\ j_s=2}\rho_{j_1,j_2,...,j_n}\to 0\ .$$

Remark that the condition $n^{-2}\mathbb{E}\{\big[\sum_{s=1}^{n}\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\big]^2\}\to 0$ is checked under weak assumptions. Indeed, by proposition A.2.2, it holds if $|\mathbb{E}\{X_s^2X_t^2\}-\mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}|\leq\alpha(|t-s|)$ where $\alpha(h)\to 0$ as $h\to\infty$.

Now, it seems natural to choose this condition in a CLT. Then the theorem 3 seems a theorem with minimum conditions of asymptotic independence for the MCLT. We can therefore assume that this is the case. In fact, we shall see in section 2.3.5 that this condition is maybe too weak because it does not require asymptotic normality.

1.2.2 Generalization

Theorems 2 and 3 are given under the assumption that the X_j's have the same law m. It is a too restrictive condition which prevents application of these theorems for martingales, for example. So we will study the case where the laws of the X_j's are different.

At first, we will need a sequence of normalization $\Psi(n)$ which can often be replaced by $\sigma(n)$.

Notations 1.2.1 *Let $\Psi(n)>0$. One supposes that $c_\Psi\sqrt{n}\leq\Psi(n)$ where $c_\Psi>0$. Let $M_p^n=\mathbb{E}\{\frac{(X_1+....+X_n)^p}{\Psi(n)^p}\}$. Let $h\in\mathbb{N}$. We set $B_h^n=max\{1,|M_h^n|\}$.*

Note that we could impose a weaker hypothesis than $c_\Psi\sqrt{n}\leq\Psi(n)$: in this case, we get conditions more complicated in the MCLT.

Now, because we study the case where the laws of the X_j's are different, we have to impose minimal assumptions in order to avoid, for example that $\mathbb{E}\{X_n^2\}\to\infty$. Also we will impose the following assumptions.

Hypothesis 1.2.1 *One supposes that, for all $p\in\mathbb{N}^*$, for all $j\geq 2$*

$$\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n}(X_t)^j}{\Psi(n)^j}\right|^p\right\}\leq C_n(j,p)\leq C(j,p)$$

where C(j,p) depends only on j and p and where $C_n(j,p)=\epsilon_n(j,p)\to 0$ as $n\to\infty$ if $j\geq 3$.

Let $\beta_s=\mathbb{E}\{X_s^2\}$. One assumes that $\sum_{s=1}^{n_m}\frac{\beta_s}{\Psi(n)^2}\to\sigma_0^2\in\mathbb{R}_+$.

Of course these conditions are checked if the m_r's have the same law. More generally, the first condition is checked if, for all $p\in\mathbb{N}^*$, there exists $C_1(p)>0$ such that $|\mathbb{E}\{X_n^p\}|\leq C_1(p)$.

Remark that condition "for all $p\in\mathbb{N}^*$, for all $j\geq 2$, $\mathbb{E}\{\big|\frac{\sum_{t=1}^{n}(X_t)^j}{\Psi(n)^j}\big|^p\}\leq C_n(j,p)$ " is equivalent to condition "for all $p\in\mathbb{N}^*$, for all $j\geq 2$, $\mathbb{E}\{\big[\frac{\sum_{t=1}^{n}(X_t)^j}{\Psi(n)^j}\big]^p\}\leq C_n(j,p)$ ". It suffices to consider p even and Holder's inequality.

Then, with these conditions, one can generalize theorem 3 by the following way.

Theorem 4 *We suppose that*

$$\Psi(n)^{-4}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}\to 0\ .$$

All the moments $M_q^n=\mathbb{E}\Big\{\frac{(X_1+X_2+....+X_n)^q}{\Psi(n)^q}\Big\}$ *converges to a real* M_q *if and only if, for all* $q\in\mathbb{N}$*, there existe* $S_q\in\mathbb{R}$ *and* $Sb_q^r\in\mathbb{R}$ *, r=2,3, such that*

$$\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^q}\to S_q$$

$$\Big|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2^{s_1}(X_{s_1})X_{s_2}.....X_{s_{q-1}}\}}{\Psi(n)^q}\Big|\leq Sb_q^2\ ,$$

$$\Big|\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\gamma_{s_1}\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^{q+1}}\Big|\leq Sb_q^3$$

where $\tilde{P}_2^{s_1}(x)=x^2-\gamma_{s_1}x-\beta_{s_1}$ *with* $\gamma_{s_1}=\mathbb{E}\{X_{s_1}^3\}/\mathbb{E}\{X_{s_1}^2\}$.

Moreover, M_q*, is the moment of order* q *of* $N(0,M_2)$ *if and only if, for all* $q\in\mathbb{N}^*$*,* $S_q=\nu_q$*, the moment of order* q *of* $N(0,S_2)$*. In this case* $M_2=\sigma_0^2+S_2$*.*

This theorem is proved in chapter 3.
Remark that if all the laws m_j's are the same, the third condition can be removed.

Now, when the X_j's are bounded, we shall prove a simpler theorem.

Theorem 5 *We suppose that there exists* $F>0$ *such that* $|X_s|\leq F$ *for all* $s\in\mathbb{N}^*$*. We suppose that*

$$\Psi(n)^{-4}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}\to 0\ .$$

All the moments $M_q^n=\mathbb{E}\Big\{\frac{(X_1+X_2+....+X_n)^q}{\Psi(n)^q}\Big\}$ *converges to a real* M_q *if and only if, for all* $q\in\mathbb{N}$*, there existe* $S_q\in\mathbb{R}$ *such that*

$$\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^q}\to S_q\ .$$

Moreover, M_q *is the moment of order* q *of* $N(0,M_2)$ *if and only if, for all* $q\in\mathbb{N}^*$*,* $S_q=\nu_q$*.*

Remark that if $|X_{m,t}|\leq F$, the hypothesis 1.2.1 holds.

Chapter 2

Applications

2.1 Process with the same first coefficients of correlation

Theorem 4 allows to better understand if an asymptotical independence condition is useful or not. For example, for fixed n, asymptotic normality depends only on a finite number of correlation coefficients : that is a countable number of those are useless.

We have a simple application of this result: if a sequence X_n satisfies the MCLT, an infinity of other sequences which have the same first correlation coefficients will also check MCLT.

Proposition 2.1.1 *Assume that, for all $s \in \mathbb{N}^*$, $m_s = m$. Let $\{Y_n\}$ be a process such that, for all $s \in \mathbb{N}^*$, Y_s has the same distribution m as X_s. Let $\rho_{j_1,j_2,....,j_n}$ and $\rho'_{j_1,j_2,....,j_n}$ be the higher order correlation coefficients associated to $\{X_n\}$ and $\{Y_n\}$, respectively.*

Assume that, for all $s \in \mathbb{N}^*$, $\{P_j^s\}_{j\in\mathbb{N}}$ is a basis of $L^2(\mathbb{R}, m_s)$. Assume that, for all $n \in \mathbb{N}^*$, the dependence density of the process $\{Y_n\}$ satisfies : for all n,

$$f_Y(x_1, ..., x_n)$$

$$= 1 + \sum_{(j_1,...,j_n),\ j_s\leq 2} \rho_{j_1,....,j_n} P^1_{j_1}(x_1)....P^n_{j_n}(x_n) + \sum_{(j_1,...,j_n),\ at\ least\ 1\ j_s>2} \rho'_{j_1,....,j_n} P^1_{j_1}(x_1)....P^n_{j_n}(x_n)$$

Then,

$$\frac{X_1 + ... + X_n}{\Psi(n)} \xrightarrow{M} N(0, M_2) \ \ if\ and\ only\ if\ \ \frac{Y_1 + ... + Y_n}{\Psi(n)} \xrightarrow{M} N(0, M_2)\ .$$

Thus we obtain a set of processes which satisfy the MCLT as soon as one of them satisfies it.

For example by using proposition A.2.1, we have the following propeties.

Example 2.1.1 *Let X_n be a bounded strictly stationary ϕ-mixing process such that $\sigma(n)^2 \geq c_\Psi^2 n$. Then, the MCLT holds (cf [14] and [15]).*

Then, $\frac{Y_1+.....+Y_n}{\sigma(n)} \xrightarrow{M} N(0,1)$ for all process $\{Y_n\}$ such that $\{Y_n\}$, for all n, the distribution of Y_n is m and has the dependence density

$$f_Y(x_1, ..., x_n)$$

$$= 1+ \sum_{(j_1,...,j_n),\ j_s\leq 1} \rho_{j_1,....,j_n} P_{j_1}(x_1)....P_{j_n}(x_n)+ \sum_{(j_1,...,j_n),\ at\ least\ one\ j_s\geq 2} \rho'_{j_1,....,j_n} P_{j_1}(x_1)....P_{j_n}(x_n)\ ,$$

when

$$\frac{1}{\sigma(n)^4} \sum_{j_1+j_2+...+j_n=4;\ j_s=2\ or\ 0} \rho'_{j_1,j_2,...,j_n} \to 0\ .$$

2.2 Martingale theory

We have understood in proposition 1.1.3 that, if $(X_1+...+X_n, \mathcal{F}_n)$ is a martingale, $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \in \mathbb{N}^*$. Then, $\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\} = 0$ for all $s_1 < ... < s_q$. The condition of theorem 4 $\sum_{s_1\neq s_2\neq.....\neq s_q} \frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^q} \to S_q$ is automatically checked.

It is therefore not surprising that we obtain quite simple CLT for martingale. This result clearly shows that the $\rho_{j_1,j_2,....,j_n}$'s which define all dependence, allow to better understand the importance of classical assumptions in the CLT and what they really mean.

Now we can also consider inovation processes : $X_{n+1} = Z_{n+1} - \mathbb{E}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$ where Z_n is any stochastic process : $\rho_{j_1,j_2,....,j_n,1} = 0$ for all n .

But in order that the MCLT holds, one can simplify this condition : in theorem 4 it is enough to assume $\rho_{j_1,....,j_n} = 0$ if $j_s \leq 1$ in order to obtain $\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\} = 0$. Then, one use the following notation.

Notations 2.2.1 *Let Z_n be a stochastic process. We denote by $\mathcal{P}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$ the orthogonal projection of Z_{n+1} onto the subspace generated by linear combination of random variables $Z_{t_1}Z_{t_2}....Z_{t_p}$, $t_1 < t_2 < .... < t_p \leq n$ where $p \in \mathbb{N}^*$.*

Indeed, one can use process much simpler than innovation process in order to apply theorem 4 with $\rho_{j_1,....,j_n} = 0$ if $j_s \leq 1$: one uses $X_{n+1} = Z_{n+1} - \mathcal{P}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$. This condition is less strong than the martingale assumption. Indeed, one can write $\mathbb{E}\{Z_{n+1}|Z_n, Z_{n-1}, ...\} = \mathcal{P}\{Z_{n+1}|Z_n, Z_{n-1}, ...\} + \mathcal{R}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$ where $\mathcal{R}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$ is orthogonal to $\mathcal{P}\{Z_{n+1}|Z_n, Z_{n-1}, ...\}$.

More generally, one can use process much simpler than martingales in order to apply theorem 4. Instead of assuming $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$, one can suppose $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ for all $n \in \mathbb{N}^*$, i.e. $\rho_{j_1,....,j_n} = 0$ if $j_s \leq 1$.

Example 2.2.1 *Let*

$$X_t = \sum_{i=0}^{\infty} C_i(\Theta_{t+i}) f_{i+1}(\Psi_{t+i})$$

where $C_n(x) = \sqrt{2}.cos(4^n x)$, where $\{\Theta_i\}$ is IID with uniform distribution on $[0, 2\pi]$, where $\{\Psi_i\}$ is a strictly stationary process independent of $\{\Theta_i\}$ and where $|f_{i+1}(y)| \leq \frac{1}{(i+1)^{1/2+a}}$ with $a > 0$.

Then, we shall prove in appendix B that $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ for all $n \in \mathbb{N}^*$.

Then, in order to apply theorem 4 with $\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\} = 0$, it is not necessary that X_n is a martingale. So we can state the following theorem.

Theorem 6 *Assume that the hypotheses 3.1.2 hold with $\psi(n) = n$. Assume that $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ for all $n \in \mathbb{N}^*$. We suppose that*

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0\ .$$

All the moments $M_q^n = \mathbb{E}\Big\{ \frac{(X_1+X_2+....+X_n)^q}{\sqrt{n^q}} \Big\}$ *converges to a moment of order* q *of* $N(0,\sigma_0^2)$ *if and only if, for all* $q \in \mathbb{N}$*, there existe* $Sb_q^2 \in \mathbb{R}$ *, such that*

$$\Bigg| \sum_{s_1 \neq s_2 \neq ..... \neq s_{q-1}} \frac{\mathbb{E}\{X_{s_1}^2 X_{s_2} ..... X_{s_{q-1}}\}}{\sqrt{n^q}} \Bigg| \leq Sb_q^2 \ ,$$

Theorem 7 *Assume that there exists* $F > 0$ *such that* $|X_s| \leq F$ *for all* s*. We suppose that* $\sum_{t=1}^n \frac{\mathbb{E}\{X_s^2\}}{n} \to \sigma_0^2$*. Assume that* $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ *for all* $n \in \mathbb{N}^*$*. We suppose that*

$$n^{-2}\mathbb{E}\Big\{ \Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big) \Big]^2 \Big\} \to 0 \ .$$

Then, all the moments $M_q^n = \mathbb{E}\Big\{ \frac{(X_1+X_2+....+X_n)^q}{\sqrt{n^q}} \Big\}$ *converges to a moment of order* q *of* $N(0,\sigma_0^2)$*.*

Compare these results to classical theorems about martingales (cf [20] pages 58 and 71).

Theorem 8 *Let* $\{\Sigma_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n, n \geq 1\})$ *be a zero mean square integrable martingale array with difference* X_{ni} *and let* η^2 *be an a.s. finite random variables. Assume that the* σ*-fields are nested :* $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ *for* $1 \leq i \leq k_n, n \geq 1$*.*

Assume that

$$A) \ \max_i \Big(|X_{ni}| \Big) \xrightarrow{P} 0 \ .$$

$$B) \ \mathbb{E}\Big\{ \max_i (X_{ni}^2) \Big\} \text{ is bounded in } n.$$

$$C) \ U_{n,k_n}^2 = \sum_i X_{ni}^2 \xrightarrow{P} \eta^2 \ .$$

Then,

$$\Sigma_{n,k_n} = \sum_i X_{ni} \xrightarrow{d} Z \ ,$$

where the random variable Z *has the characteristic function* $\mathbb{E}\{exp(-\eta^2 t^2/2)\}$*.*

For example, we can choose $k_n = n$, $\mathcal{F}_{n,i} = \mathcal{F}_i$, $\Sigma_{n,k_n} = (X_1 + ... + X_n)/\sqrt{n}$, $X_{ni} = X_i/\sqrt{n}$ and $\eta^2 = \sigma_0^2$. Then, in [20], we have also the following result about the convergence of moments.

Theorem 9 *Let* $p > 1$*. Let* μ_p *be the moment of order* p *of N(0,1). Assume that* $(X_1+...+X_n, \mathcal{F}_n)$ *is a martingale. Assume that the following conditions hold.*

$$A) \ \frac{1}{n} \max_{i \in \{2,3,....,n\}} \Big(\mathbb{E}\{X_i^2 | \mathcal{F}_{i-1}\} \Big) \xrightarrow{P} 0 \ .$$

$$B) \ \mathbb{E}\Big\{ \Big| \frac{1}{n} \sum_{n=1}^n \mathbb{E}\{X_i^2 | \mathcal{F}_{i-1}\} - \sigma_0^2 \Big|^p \Big\} \to 0 \ .$$

$$C) \ \mathbb{E}\Big\{ \Big| \frac{1}{n} \sum_{n=1}^n [X_i^2 - \sigma_0^2] \Big|^p \Big\} \to 0 \ .$$

Then,

$$\mathbb{E}\Big\{\Big|\frac{X_1+....+X_n}{\sqrt{n}}\Big|^{2p}\Big\} \to \mu_{2p}\sigma_0^{2p} \ .$$

Let us compare this theorem and theorem 6. At first, $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ holds if $(X_1+...+X_n, \mathcal{F}_n)$ is a martingale. It is a condition much weaker than the martingale assumption.

Moreover, by lemma A.2.1 , condition C) with p=2, involves that

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0 \ .$$

Now consider condition A) : $(1/n)\max_{i\in\{2,3,....,n\}}\big(\mathbb{E}\{X_i^2|\mathcal{F}_{i-1}\}\big) \stackrel{P}{\to} 0$. We know that $\mathbb{E}\{X_i^2|\mathcal{F}_{i-1}\}$ can be written with the $\rho_{j_1,....,j_n}$: e.g. $\mathbb{E}\{X_2^2|\mathcal{F}_1\} = \mathbb{E}\{X_2^2\} + \sigma_{1,2}\sum_j \rho_{j,2}P_j^1(X_1)$. That is the $\rho_{j_1,....,j_n}$ are implicitly in this theorem. But many are useless for the MCLT. The aim of theorem 4 is to suppress these useless parameters e.g. the $\rho_{j,2}$ such that $j > 2$.

Moreover, in theorem 7 we do not need of use the maximum as in theorem 9. On the other hand, we do not need condition B) for all p : $\mathbb{E}\big\{\big|\frac{1}{n}\sum_{n=1}^{n}\mathbb{E}\{X_i^2|\mathcal{F}_{i-1}\} - \sigma_0^2\big|^p\big\} \to 0$.

Then, clearly theorem 7 obtained by using the the $\rho_{j_1,....,j_n}$'s gives conditions much simpler than theorem 9.

Example 2.2.2 *Consider the sequences $X_t = \sum_{i=0}^{\infty} C_i(\Theta_{t+i})f_{i+1}(\Psi_{t+i})$ defined in example 2.2.1. One chooses $f_{i+1}(\Psi_t) = \frac{\Psi_t}{(1+i)^{1/2+a}}$. One supposes Ψ_t stricly stationary and bounded.*

We know that $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$ for all $n \in \mathbb{N}^*$.

Then, $(X_1 + ... + X_n)/\sqrt{n} \stackrel{M}{\to} N\big(0, \mathbb{E}\{(X_1)^2\}\big)$ if

$$\big|\mathbb{E}\{\Psi_t)^2(\Psi_{t'})^2\} - \mathbb{E}\{(\Psi_t)^2\}\mathbb{E}\{(\Psi_{t'})^2\}\big| \le \epsilon(t-t') \ ,$$

where $1 \ge \epsilon(t) > 0$ and where $\epsilon(t)$ is decreasing and converges to 0.

This condition of asymptotic independence about Ψ_t is therefore very weak, especially compared to the strong mixing condition or to the condition of weak dependence.

This shows clearly that the use of the $\rho_{j_1,....,j_n}$'s simplifies the CLT for martingales and allows also to better understand why the classical CLT conditions are relatively simple in the case of martingales.

2.3 Comparison of the conditions of asymptotic independence

2.3.1 Classical conditions

We first recall the definition of the strong mixing condition.

Definition 2.3.1 *: Assume that $\{X_n\}_{n\in\mathbb{N}^*}$ is a sequence of random variables. Then, $\{X_n\}$ is strongly mixing with coefficient α if*

$$\sup_{A\in\mathcal{M}_1^n,\ B\in\mathcal{M}_{n+h}^{\infty}}\Big|P(A\cap B) - P(A)P(B)\Big| = \alpha(h) \to 0$$

as $h \to \infty$, where for $a \le b$, $\mathcal{M}_a^b$ is the σ-field generated by $X_a, X_{a+1}, ..., X_b$.

For example, suppose now that the X_j's have the same law m and that X_n is strong mixing with coefficient α. Suppose that all the orthonormal polynomials P_j exist. We know that we can express the $\rho_{i_1,i_2,.....i_n}$'s in the form :

$$\rho_{j_1,j_2,.....j_n} = \mathbb{E}\big\{P_{j_1}(X_{t_1})P_{j_2}(X_{t_2}).....P_{j_e}(X_{t_e})P_{j_{e+1}}(X_{t_{e+1}})P_{j_{e+2}}(X_{t_{e+2}}).....P_{j_q}(X_{t_q})\big\}$$

where $t_s \in \mathbb{N}^*$, s=1,2,...,q, and $t_1 < t_2 < .... < t_q$.

On the other hand, $\big|\mathbb{E}\big\{[P_{j_1}(X_{v_1}).....P_{j_k}(X_{v_k})]^4\ \big\}\big|^k \leq \mathbb{E}\big\{P_{j_1}(X_{v_1})^{4k}\ \big\}.....\mathbb{E}\big\{P_{j_k}(X_{v_k})^{4k}\ \big\}$ which is equal to a constant $C_{j_1,...,.j_k}$. Then, by theorem 17-2-2 of [1], we know that the strong mixing condition involves

$$\big|\mathbb{E}\big\{P_{j_1}(X_{t_1})....P_{j_q}(X_{t_q})\big\}-\mathbb{E}\big\{P_{j_1}(X_{t_1})...P_{j_e}(X_{t_e})\big\}\mathbb{E}\big\{P_{j_{e+1}}(X_{t_{e+1}})...P_{j_q}(X_{t_q})\big\}\big| \leq K_a\alpha(t_{e+1}-t_e)^{1-a}$$

where K_a is a constant and $a > 0$ arbitrarily small.

Of course, this relationship is written with the $\rho_{j_1,j_2,.....j_n}$ such that $j_s = 0$ if $r = t_e < s < t_{e+1} = r + h$ as :

$$|\rho_{j_1,j_2,.....j_n} - \rho_{j_1,j_2,.....j_r,0,0,....0}\rho_{0,0,....,0,j_{r+h},j_{r+h+1},.....j_n}| \leq K_a\alpha(h)^{1-a}\ .$$

On the other hand, Doukhan and Louhichi [18] have introduced the $(\theta, \mathcal{L}, \Psi)$ weak-dependence.

Definition 2.3.2 *: Let $\mathcal{L} = \cup_{p=1}^{\infty}\mathcal{L}^p$ where $\mathcal{L}^p = \{f\ :\ \mathbb{R}^p \to \mathbb{R}\}$. Let $\Psi : \mathcal{L} \otimes \mathcal{L} \otimes (\mathbb{N}^*)^2 \to \mathbb{R}_+$ and $(\theta_r)_{r\in\mathbb{N}} \searrow 0$.*

The sequence $\{X_n\}_{n\in\mathbb{Z}}$ is $(\theta, \mathcal{L}, \Psi)$ weakly dependent if
$\forall r \in \mathbb{N}$, $\forall u, v \in \mathbb{N}^*$, $\forall (h, k) \in \mathcal{L}^u \otimes \mathcal{L}^v$,
$\forall\ i_1 < i_2 < ..... < i_u < i_u + r \leq j_1 < ..... < j_v$,

$$\big|Cov\big(h(X_{i_1}, ..., X_{i_u}), k(X_{j_1}, ..., X_{j_v})\ \big)\big| \leq \theta_r\Psi(h, k, u, v).$$

Clearly, under this assumption of weak dependence, we find the same kind of relationship as when the strong mixing condition holds :

$$\big|\mathbb{E}\big\{P_{j_1}(X_{t_1}).....P_{j_q}(X_{t_q})\big\} - \mathbb{E}\big\{P_{j_1}(X_{t_1}).....P_{j_e}(X_{t_e})\big\}\mathbb{E}\big\{P_{j_{e+1}}(X_{t_{e+1}}).....P_{j_q}(X_{t_q})\big\}\big| \leq C_1\theta_{t_{e+1}-t_e}\ ,$$

where C_1 depends on e, q-e, $j_1, j_2, ..., j_q$.

Now, remark that

$$\big|\mathbb{E}\big\{(X_t)^2(X_{t+h})^2\big\} - \mathbb{E}\big\{(X_t)^2\big\}\mathbb{E}\big\{(X_{t+h})^2\big\} \to 0\ .$$

In fact, it's as true for the strong mixing condition as for the weak dependence, That means by theorem 4, that, if all moments converge, then in addition to one or other of these conditions, it will be required inter alia that

$$\sum_{s_1\neq s_2\neq .....\neq s_q} \frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\sigma(n)^q} \to \nu_q\ .$$

2.3.2 Condition H_{mI}

Conditions of asymptotic independence H_{mI} and of asymptotic stationarity H_{mS} were introduced in [24] and [25].

Notations 2.3.3 *We denote by $\kappa(n) \in N$, an increasing sequence such that $\kappa(1) = 0$, $\kappa(n) \leq n$ and $\kappa(n)/n \to 0$ as $n \to \infty$. We define the sequences u(n) and $\tau(n)$ by : u(1)=1, $u(n) = max\{m \in N^* | 2m + \kappa(m) \leq n\}$ and $\tau(1) = 0$, $\tau(n) = n - 2u(n)$ if $n \geq 2$. Moreover, we simplify u(n) and $\tau(n)$ in $u_n = u$ and $\tau_n = \tau$.*

Let $\sigma(u)^2$ be the variance of $X_1 + X_2 + ... + X_u$. One sets $\Sigma_u = \frac{X_1+X_2+.....+X_u}{\sigma(u)}$, $\xi_u = \frac{X_{u+1}+X_{u+2}+.....+X_{u+\tau}}{\sigma(u)}$ and $\Sigma'_u = \frac{X_{u+\tau+1}+X_{u+\tau+2}+.....+X_{u+\tau+u}}{\sigma(u)}$.

In [24] , one has proved that $n/u \to 2$ and $\tau/u \to 0$ as $n \to \infty$. Moreover, one chooses $E\{(\xi_u)^2\} \to 0$.

Notations 2.3.4 *: We define conditions H_{mS} and H_{mI} by the following way :*

H_{mS} *: $\forall p \in \mathbf{N}$, $E\{(\Sigma_u)^p\} - E\{(\Sigma'_u)^p\} \to 0$ as $n \to \infty$.*

H_{mI} *: $\forall (p, q) \in (\mathbf{N}^*)^{\mathbf{2}}$, $E\{(\Sigma_u)^p(\Sigma'_u)^q\} - E\{(\Sigma_u)^p\}E\{(\Sigma'_u)^q\} \to 0$ as $n \to \infty$.*

In fact, in [25], we define conditions a little less strong because we consider the asymptotic independence of moments between $\Sigma_u + v_u$ and $\Sigma'_u + v'_u$ where $\{v_u\}$ and $\{v'_u\}$ are two sequences of random variables such that $E\{|v_u|^p\} + E\{|v'_u|^p\} \to 0$ for all $p \in \mathbb{N}$. Then, in [25] one has proved the following result.

Theorem 10 *: Assume that $E\{|\xi_u|^k\} \to 0$ as $n \to \infty$ for all $k \in N$. Assume that H_{mS} and H_{mI} hold. Then, $\Sigma_n \xrightarrow{M} N(0, 1)$.*

In fact, H_{mS} and H_{mI} implies also the convergence in dimension 2.

Corollary 2.3.1 *: Assume that $E\{|\xi_u|^k\} \to 0$ as $n \to \infty$ for all $k \in N$. Assume that H_{mS} and H_{mI} hold. Then, $(\Sigma_u, \Sigma'_u) \xrightarrow{M} N_2(0, I_2) = N(0, 1) \otimes N(0, 1)$.*

Proof By theorem 10, $\mathbb{E}\{(\Sigma_u)^k\} \to \mu_k$, the moments of order k of N(0,1). By H_{mS}, $\mathbb{E}\{(\Sigma'_u)^k\} \to \mu_k$. Then, $\mathbb{E}\{(\Sigma_u)^q\}\mathbb{E}\{(\Sigma'_u)^p)\} \to \mu_q\mu_p$. By H_{mI}, $\mathbb{E}\{(\Sigma_u)^q(\Sigma'_u)^p)\} - \mathbb{E}\{(\Sigma_u)^q\}\mathbb{E}\{(\Sigma'_u)^p)\} \to 0$. Then, $\mathbb{E}\{(\Sigma_u)^q(\Sigma'_u)^p)\} \to \mu_q\mu_p$. ■

Note that the convergence of moments involves the convergence in distribution

Corollary 2.3.2 *Assume that $E\{|\xi_u|^k\} \to 0$ as $n \to \infty$ for all $k \in N$. Assume that H_{mS} and H_{mI} hold. Then, $\Sigma_n \xrightarrow{d} N(0, 1)$.*

Proof By our assumptions, $\mathbb{E}\{(\Sigma_u)^k\} \to \mu_k$. By H_{mS}, $\mathbb{E}\{(\Sigma'_u)^k\} \to \mu_k$. By H_{mI}, for all $(a, b) \in \mathbb{R}^2$, $\mathbb{E}\{(a\Sigma_u + b\Sigma'_u)^k\}$ converges to the moments of order k of $N(0, a^2 + b^2)$. One deduces that $\Sigma_n \xrightarrow{d} N(0, 1)$.

Example 2.3.1 *Let $\zeta_t = \sum_{i=0}^{\infty} b^{i+1} h_i(\overline{\Theta_{t+i}})$, where $|h_i(\overline{\Theta_1})| \leq 1$, $|b| \leq 1/2$ and where $\overline{\Theta_t}$ is an IID sequence independent of another IID sequence Θ_t. Assume that ζ_t is not strong mixing.*

Assume that $k_i(\Theta_t, \zeta_t) = i^{-5/2} g_i(\Theta_t).sin(e(i)\zeta_t)$ where $E\{g_i(\Theta_1)\} = 0$ and $|e(i)| \leq 2\pi$ for all $i \in N$. Assume that $\sum_{s=0}^{\infty} \left| \sum_{i=s}^{\infty} k_{i+1}(\Theta_1, \zeta_1) \right| < C < \infty$.

Let $X_t = \sum_{i=0}^{\infty} k_{i+1}(\Theta_{t+i}, \zeta_{t+i})$. Then H_{mI} hold and $S_n \xrightarrow{M} N(0, 1)$ (cf [25])

2.3.3 Condition H_{mI} and correlation coefficients of higher order

We will compare these results about H_{mI} with the results about the correlation coefficients of higher order. We will see that we obtain almost minimal conditions more similar to classical conditions.

This is not surprising. We introduced the conditions H_{mI} by trying to find conditions slightly stronger than those about correlation coefficients of higher order (Partie B-I of [26]) and closer to the classical conditions : cf Partie B-II of [26].

So we obtain the following theorem.

Theorem 11 *Suppose that $|X_n| \leq F$ where $F > 0$. One assumes that there exists $c_\Psi > 0$ such that $\sigma(n) \geq c_\Psi \sqrt{n}$. One supposes that $\frac{1}{\sigma(u_n)^2} \sum_{s=1}^{u_n} \mathbb{E}\{X_s^2\} \to \sigma_0^2$ and $\frac{1}{\sigma(u_n)^2} \sum_{s=1+u_n+\tau_n}^{n} \mathbb{E}\{X_s^2\} \to \sigma_0^2$. One assumes that*

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{u_n} [(X_t)^2 - \mathbb{E}\{X_t^2\}]}{\sigma(u_n)^2} \right]^2 \right\} + \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{u_n} [(X_{u_n+\tau_n+t})^2 - \mathbb{E}\{X_{u_n+\tau_n+t}^2\}]}{\sigma(u_n)^2} \right]^2 \right\} \to 0 \ .$$

One assumes that, for all $k \in \mathbb{N}$,

$$\mathbb{E}\left\{ \left[\frac{X_{u+1} + X_{u+2} + ... + X_{u+\tau_n}}{\sigma(u_n)} \right]^k \right\} \to 0 \ .$$

Then, H_{mI} and H_{mS} hold if and only if for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$,

$$\sum_{s_1 \neq s_2 \neq .... \neq s_q, \ s_r \leq u_n} \ \sum_{t_1 \neq t_2 \neq .... \neq t_p, \ u_n+\tau_n < t_r \leq n} \frac{\mathbb{E}\{X_{s_1} X_{s_2} .... X_{s_q} X_{u_n+\tau_n+t_1} X_{u_n+\tau_n+t_2} .... X_{u_n+\tau_n+t_p}\}}{\sigma(u_n)^{p+q}}$$

converges to $\nu'_q \nu'_p$, where ν'_q is the moment of order p de la la loi $N(0, 1-\sigma_0^2)$.

Proof We apply corollaries 3.10.1 and 4.5.1 with $\Psi(n) = \sigma(n)$, $n_m = u_n$, $X_{m,t} = X_t$ for $t = 1, .., u_n$, $Y_{m,t} = X_{u_n+\tau_n+t}$ for $t = 1, .., u_n$.

Indeed, if H_{mI} and H_{mS} hold, by corollary 2.3.1, all the moments

$$M_{q,p}^n = \mathbb{E}\Big\{ \frac{(X_1 + ... + X_{u_n})^q (X_{u_n+\tau_n+1} + ... + X_{u_n+\tau_n+u})^p}{\sigma(u_n)^{p+q}} \Big\}$$

converges to $\mu_q \mu_p$. Then, all the conditions of corollary 4.5.1 are checked. That proves the necessary condition.

Conversely, let us prove the sufficient condition. Suppose that the conditions of this theorem are checked. By corollaries 3.10.1 and 4.5.1 , $M_{q,p}^n \to \mu_q \mu_p$, $M_q^n = \mathbb{E}\big\{ \frac{(X_1+....+X_u)^q}{\sigma(u)^q} \big\} \to \mu_q$ $M_p'^n = \mathbb{E}\big\{ \frac{(X_{u+\tau+1}+....+X_{u+\tau+u})^p}{\sigma(u)^{p+q}} \big\} \to \mu_p$. Therefore, $M_{q,p}^n - M_q^n M_p'^n \to 0$. Then, H_{mI} and H_{mS} hold. ■

If X_n is not bounded, one can use corollary 4.4.1 : conditions are more complicated. But this is hardly important: this theorem 11 suffices to show how H_{mI} results in terms of correlation coefficients of higher order. In particular, the main condition about the coefficients of multilinear correlation implies the following condition (when the sequences are bounded or not).

Corollary 2.3.3 *One assumes that there exists $c_\Psi > 0$ such that $\sigma(n) \geq c_\Psi \sqrt{n}$. One supposes that $(1/\sigma(u_n)^2) \sum_{s=1}^{u_n} \mathbb{E}\{X_s^2\} \to \sigma_0^2$ and $(1/\sigma(u_n)^2) \sum_{s=1+u_n+\tau_n}^{n} \mathbb{E}\{X_s^2\} \to \sigma_0^2$. One assumes that*

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{u_n} [(X_t)^2 - \mathbb{E}\{X_t^2\}]}{\sigma(u_n)^2} \right]^2 \right\} + \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{u_n} [(X_{u_n+\tau_n+t})^2 - \mathbb{E}\{X_{u_n+\tau_n+t}^2\}]}{\sigma(u_n)^2} \right]^2 \right\} \to 0 \ .$$

One assumes that, for all $k \in \mathbb{N}$,

$$\mathbb{E}\left\{ \left[\frac{X_{u+1} + X_{u+2} + ... + X_{u+\tau_n}}{\sigma(u_n)} \right]^k \right\} \to 0 \ .$$

Then, if H_{mI} and H_{mS} hold

$$\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \sum_{t_1 \neq t_2 \neq ..... \neq t_p} \frac{\mathbb{E}\{X_{s_1} X_{s_2} ..... X_{s_q} X_{u_n+\tau_n+t_1} X_{u_n+\tau_n+t_2} ..... X_{u_n+\tau_n+t_p}\}}{\sigma(u)^{p+q}}$$

$$-\Big[\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \frac{\mathbb{E}\{X_{s_1} X_{s_2} ..... X_{s_q}\}}{\sigma(u)^q} \Big] \Big[\sum_{t_1 \neq t_2 \neq ..... \neq t_p} \frac{\mathbb{E}\{X_{u_n+\tau_n+t_1} X_{u_n+\tau_n+t_2} ..... X_{u_n+\tau_n+t_p}\}}{\sigma(u)^p} \Big]$$

converges to 0

Finally we see that the condition H_{mI} leads to a condition about the $\rho_{j_1,....,j_n}$'s which is hardly stronger than that of Theorem 4.

2.3.4 Comparison of conditions

We will therefore compare the strength of the different conditions of asymptotic independence in spite of the fact that all are not directly comparable.

At first, it is not necessary that H_{mI} holds in the case of weak dependence a priori, at least. So we can not say that the condition H_{mI} is weaker than the condition of weak dependence (a priori, at least) and one can not directly compare these two conditions. It will be the same between the Martingale hypothesis and the assumptions of Theorem 4.

On the other hand, let us remark that if strong mixing condition holds and if the MCLT holds, it is necessary that H_{mI} holds.

Now, the $\rho_{j_1,....,j_n}$'s determine all dependence. So we must be able to formulate the various conditions of asymptotic independence as conditions about the $\rho_{j_1,....,j_n}$'s. However, it may be difficult to give an equivalence. So we shall just give some consequences that these conditions of asymptotic independence lead about the $\rho_{j_1,....,j_n}$'s and the MCLT [1].

Conditions of theorem 4 It is easy to understand that if the conditions of theorem 4 are checked, Q conditions about the $\rho_{j_1,....,j_n}$'s have to be checked in order that the first q moments converge where Q is approximately equal to 3q.

In this case the conditions which we obtain are conditions being about sums of $\rho_{j_1,....,j_n}$'s.

Conditions of theorem 4 and H_{mS} Clearly, if we impose moreover that H_{mS} holds Q' conditions about the $\rho_{j_1,....,j_n}$'s have to be checked in order that the first q moments converge where Q' is approximately equal to 6q (cf theorem 12).

In this case also the conditions which we obtain are conditions being about sums of $\rho_{j_1,....,j_n}$'s.

Conditions of theoreme 4 and H_{mI} If we impose moreover that H_{mI} holds Q" conditions about the $\rho_{j_1,....,j_n}$'s have to be checked in order that the first q moments $M^n_{q',p'}$, $q' + p' \leq q$, converge where Q" is approximately equal to $q^2/2$ (cf theorem 15).

In this case again the conditions which we obtain are conditions being about sums of $\rho_{j_1,....,j_n}$'s.

[1] Of course, in this case, one assumes $\Psi(n) = \sigma(n)$ and $\sigma(n)/\sqrt{n} \geq c_\psi$.

Strong mixing condition Now suppose that the X_j's have the same law m and that $\{X_j\}$ is strong mixing with coefficient α . Therefore, for all h, for all $t_1 < t_2 < .... < t_h$,

$$\left|\mathbb{E}\left\{P_{j_1}(X_{t_1})....P_{j_h}(X_{t_h})\right\}-\mathbb{E}\left\{P_{j_1}(X_{t_1})...P_{j_e}(X_{t_e})\right\}\mathbb{E}\left\{P_{j_{e+1}}(X_{t_{e+1}})...P_{j_h}(X_{t_h})\right\}\right| \leq K_a\alpha(t_{e+1}-t_e)^{1-a}.$$

So there is an countable number of relations about the $\rho_{j_1,....,j_n}$'s. Since only the first $\rho_{j_1,....,j_n}$'s are useful for the MCLT by theorem 2 ($j_s \leq 2$), so there is a countable number of unnecessary relationships which are also checked.

Remark also that $\left|\mathbb{E}\left\{(X_t)^2(X_{t+h})^2\right\} - \mathbb{E}\left\{(X_t)^2\right\}\mathbb{E}\left\{(X_{t+h})^2\right\} \to 0$. This means that the conditions of MCLT for strong mixing processes will be stronger than those of Theorem 4.

Moreover, it is easy to see that if all moments M_q^n of a strictly stationary strong mixing process are bounded, H_{mI} holds. Then the MCLT holds also.

Now, the conditions are relationship between groups of 3 $\rho_{j_1,....,j_n}$'s :

$$|\rho_{i_1,i_2,.....i_n} - \rho_{i_1,i_2,.....i_r,0,0,....0}\ \rho_{0,0,....,0,i_{r+h},i_{r+h+1},.....i_n}| \leq K_a\alpha(h)^{1-a}.$$

Furthermore it is the supremum which converges to 0. Then, these relations are stronger than those of sums of $\rho_{j_1,....,j_n}$'s.

Weak dependence Now, we suppose that the sequence $\{X_n\}_{n\in\mathbb{Z}}$ is $(\theta, \mathcal{L}, \Psi)$ weakly dependent. Then, for all $t_1 < t_2 < .... < t_h$,

$$\left|\mathbb{E}\left\{P_{j_1}(X_{t_1}).....P_{j_h}(X_{t_h})\right\} - \mathbb{E}\left\{P_{j_1}(X_{t_1}).....P_{j_e}(X_{t_e})\right\}\mathbb{E}\left\{P_{j_{e+1}}(X_{t_{e+1}}).....P_{j_h}(X_{t_h})\right\}\right|$$

$$\leq C_1\theta_{t_{e+1}-t_e}\ .$$

By the same way, $\left|\mathbb{E}\left\{X_t)^2(X_{t+h})^2\right\} - \mathbb{E}\left\{X_t)^2\right\}\mathbb{E}\left\{X_{t+h})^2\right\} \to 0$.

We obtain the same conclusions as for strong mixing processes. However, it is not sure a priori that, if all moments M_q^n converges, H_{mI} holds.

In the case of weak dependence, conditions that we have about the $\rho_{j_1,....,j_n}$'s are always relations between 3 groups of $\rho_{j_1,....,j_n}$'s.

Martingale if $\{X_j\}$ is a martingale, $\rho_{j_1,j_2,....,j_n,1} = 0$, i.e. $\mathbb{E}\left\{P_{j_1}^{t_1}(X_{t_1}).....P_{j_{h-1}}^{t_{h-1}}(X_{t_{h-1}})X_{t_h}\right\} = 0$ for all h, for all $t_1 < t_2 < .... < t_h$. Clearly, in this case also, there is a countable number of relations useless for the MCLT.

Moreover, these conditions are equalities on some $\rho_{j_1,....,j_n}$: $\rho_{j_1,j_2,....,j_n,1} = 0$. These relations are much stronger than the convergence of sums of $\rho_{j_1,....,j_n}$.

Processus such that $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$. In this case, $\mathbb{E}\left\{X_{t_1}X_{t_2}.....X_{t_h}\right\} = 0$ for all h, for all $t_1 < t_2 < .... < t_h$. It is a weaker condition than the martingale condition, but stronger than those of theorem 4 and furthermore, there are a countable number of relations necessary for the convergence of the moment of order q when we consider that the relationship must be true for all n.

In this case the conditions that we have are still equalities on some $\rho_{j_1,....,j_n}$. But there are less than for martingales.

Conclusion

All these conditions are not always directly comparable. For example, a martingale does not necessarily satisfy the MCLT.

But there is a way to get an idea of the strength of each of the hypotheses: by using the correlation coefficients of higher order.

For example, one could say that a condition is stronger than another if it requires more relations about the $\rho_{j_1,....,j_n}$'s.

Now we can also consider what type of relationship it is. In this case we understand that the conditions of theorem 4 are conditions which we can consider as a minimum. But we also understand that the condition H_{mI} is almost minimal.

In fact we shall even wonder if the true minimum condition is not H_{mI}.

2.3.5 Condition H_{mI} and minimal condition

Minimal condition It arises indeed a question: Is that the conditions of theorem 4 are actually conditions which can be considered as conditions of asymptotic independence? These conditions which are directly related to the correlation coefficients of higher order are actually stronger than the conditions of theorem 2 (which give an equivalence to the convergence of moments and which involve no dependence a priori).

But they have a default: the conditions of theorem 4 does not necessarily mean that asymptotic distributions are normal unless we impose the S_q are normal moments.

But to impose that the S_q's are normal moments is more like a chance because it does not change much about the condition of asymptotic independence itself. Indeed, to say that $\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^q}\to S_q$ where S_q is arbitrary and $\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n)^q}\to \nu_q$ are two conditions which require the same type of asymptotic independence.

If one wants this condition of asymptotic normality, we wonder if we should not therefore impose conditions slightly stronger. One wonders if, as a matter of fact, condition H_{mI} is the minimum condition for the asymptotic independence in order that MCLT holds with asymptotic normality.

Study of conditions H_{mI} and H_{mS} At first, note that if the conditions of Theorem 4 are checked, it is necessary that other aditional condition about the correlation coefficients of higher order holds in order that H_{mS} holds. However, if the conditions of Theorem 4 are checked and if H_{mS} does not hold, we will have a very strange case where we have $\Sigma_n \overset{M}{\to} N(0,1)$, $\Sigma_u \overset{M}{\to} N(0,1)$ and $\Sigma'_u \overset{M}{\not\to} N(0,1)$.

Intuitively, we feel that it is logical to assume moreover that H_{mS} holds if we want a minimal regularity in the asymptotic convergence.

Now, if the conditions of Theorem 4 are checked and if H_{mI} does not hold, there will be also conditions on the correlation coefficients of higher order which seems rather strange.

For example suppose that $\sigma(n)^2 = n$, that X_n is strictly stationary, and that it is the moment of order 3 which does not check H_{mI}, but that conditions of theorem 4 holds. Then,

$$\sum_{r<s<t\leq n}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}\sim\sum_{r<s<t\leq u}\frac{\mathbb{E}\{X_rX_sX_t\}}{u^{3/2}}\sim\sum_{u+\tau<r<s<t}\frac{\mathbb{E}\{X_rX_sX_t\}}{u^{3/2}}\to 0$$

where we set $x_n\sim y_n$ if $x_n-y_n\to 0$ as $n\to\infty$ for all real sequences x_n et y_n. Therefore,

$$\sum_{r<s<t\leq n}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}\sim\sum_{r<s\leq u<u+\tau<t\leq n}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}+\sum_{r\leq u<u+\tau<s<t}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}\to 0\ .$$

Then, under H_{mI}, by corollary 2.3.3,

$$\sum_{r<s\leq u<u+\tau<t\leq n}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}\to 0\ ,\ \textit{and}\ \sum_{r\leq u<u+\tau<s<t}\frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}}\to 0\ .$$

On the other hand, under the assumption of theorem 4 ,

$$\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} + \sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \to 0 \ .$$

Therefore, if H_{mI} does not hold for the moment of order 3 and if the hypotheses of theorem 4 are checked

$$\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} + \sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \to 0$$

$$\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \not\to 0 \ \ and \ \sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \not\to 0 \ .$$

This is a case which seems strange when we admit that there is some asymptotic independence. Indeed, is what one can speak of asymptotic independence if $\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n} \not\to 0$ and $\sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n} \not\to 0$?

Thus, if X_n is strictly stationary, it will be difficult to find examples where H_{mI} does not hold for the moment of order 3 if the assumptions of theorem 4 are checked.

In order to find more easily a such example, we must give up some of our assumptions.

Example 2.3.2 *We suppose that $X_{n+1} = \sqrt{n+1}Z_{n+1} - \sqrt{n}Z_n$, $X_1 = Z_1$ where Z_n is IID and Z_1 has the distribution N(0,1).*

Study Clearly X_n is 2-dependent, and then there exists a simple condition about the $\rho_{j_1,....,j_n}$'s : $\mathbb{E}\{P_{j_1}^{t_1}(X_{t_1}).....P_{j_p}^{t_p}(X_{t_p})\} = \mathbb{E}\{P_{j_1}^{t_1}(X_{t_1}).....P_{j_i}^{t_i}(X_{t_i})\}\mathbb{E}\{P_{j_{i+1}}^{t_{i+1}}(X_{t_{i+1}}).....P_{j_p}^{t_p}(X_{t_p})\}$ if there exists i such that $t_{i+1} - t_i > 2$.

Moreover, $X_1+X_2 = Z_1+\sqrt{2}Z_2-\sqrt{1}Z_1 = \sqrt{2}Z_2$, $X_1+X_2+X_3 = \sqrt{2}Z_2+\sqrt{3}Z_3-\sqrt{2}Z_2 = \sqrt{3}Z_3$.

Therefore, $X_1 + X_2 + X_3 + ... + X_n = \sqrt{n}Z_n$ and one can choose $\Psi(n) = \sigma(n) = \sqrt{n}$.

Moreover, $(\Sigma_u, \Sigma'_u) = \left[(\sqrt{u_n}Z_{u_n})/\sqrt{u_n} \ , \ \sqrt{n}Z_n - \sqrt{u_n+\tau_n}Z_{u_n+\tau_n})/\sqrt{u_n}\right]$ converge to the same distribution as $\left[Z_{u_n} \ , \ (\sqrt{2}Z_n - Z_{u_n+\tau_n})\right]$ which does not converge to $N(0,1) \otimes N(0,1)$.

Remark that, in this case, $(1/n)\sum \beta_s$ does not converge because

$$\sum_{s=0}^{n} \mathbb{E}\{X_{s+1}^2\} = \sum_{s=1}^{n} \mathbb{E}\{(\sqrt{s+1}Z_{s+1} - \sqrt{s}Z_s)^2\} + \mathbb{E}\{Z_1^2\} = \sum_{s=1}^{n}[(s+1)+s] + 1 \ . \blacksquare$$

Conclusion If $\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \not\to 0$ and $\sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \not\to 0$, is that the condition $\sum_{r<s\leq u<u+\tau<t\leq n} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} + \sum_{r\leq u<u+\tau<s<t} \frac{\mathbb{E}\{X_rX_sX_t\}}{n^{3/2}} \to 0$ is sufficient to say that there is asymptotic independence? If it is not the case, we must choose conditions a little stronger. Precisely, a condition a little stronger is the condition H_{mI}.

On the other hand, intuitively, the condition $(\Sigma_u, \Sigma'_u) \stackrel{M}{\to} N_2(0, I_2)$ is a condition which appears minimum as a condition of asymptotic independence.

Yet in terms of correlation coefficients of higher order for the MCLT, the condition H_{mI} is not minimal : it is only almost minimal. But we understood that, if we impose only assumptions of theorem 4, there will be quite strange conditions. Moreover, there is no asymptotic normality. That shows indeed that the condition H_{mI} is very close to conditions of asymptotic independence which can be required for convergence to the normal law.

Conclude by saying that the convergence of (Σ_u, Σ'_u) to $N(0, I_2)$ is a requirement nearly asymptotic independence which can perhaps be considered as minimal for the asymptotic normality.

Chapter 3

MCLT in dimension 1

In this chapter, we prove theorems 4 and 5 .

Then, we will study the case where the laws of X_j's are possibly different. Unfortunately, in this case, we can not apply the same technique as in [23]. It is then easier to prove theorems without using the correlation coefficients of higher order $\rho_{j_1,....,j_n}$ and orthogonal polynomials. Unfortunately the proof is much longer.

Then, we shall prove several MCLT before deducting the MCLT with the correlation coefficients of higher order.

On the other hand, in order to prove Theorem 11, the easiest way is to prove a MCLT in dimensions 2 for sequences with double array $X_{m,s}, Y_{m,s}$. Then, it is easier to prove these results under larger assumptions which we will introduce now.

3.1 Notations and assumptions

Notations 3.1.1 *Let x_n and y_n be two real sequences. We set $x_n \sim y_n$ if $x_n - y_n \to 0$ as $n \to \infty$. In particular, $x_n \sim x$ if $x_n \to x$ as $n \to \infty$.*

Let Z_n and T_n be two sequences of random variables defined on $(\Omega, \mathcal{A}, P)$. We set $Z_n \sim T_n$ if Z_n and T_n have asymptotically the same distribution.

By misuse of our notations, we set also $S_n \sim N(0, \sigma^2)$ if S_n has asymptotically the distribution $N(0, \sigma^2)$.

In chapter 4, we shall generalize by natural way these notations to double triangular array of random variables $(X_{m,s}, Y_{m,s})$.

Notations 3.1.2 *Let $X_{m,s}$, $s = 1, 2, ..., n_m$, m=1,2,.. be a triangular array of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$.*

We suppose $\mathbb{E}\{X_{m,s}\} = 0$ and $|\mathbb{E}\{(X_{m,s})^p\}| < \infty$ for all $p \in \mathbb{N}$.

Hypothesis 3.1.1 *Let $\Psi(n) > 0$. We suppose that $\sqrt{c_\Psi n} \leq \Psi(n)$ where $c_\Psi > 0$. We set $B_h^{n_m} = max\{1, |M_h^{n_m}|\}$.*

Hypothesis 3.1.2 *We suppose that, for all $p \in \mathbb{N}^*$, for all $j \geq 2$*

$$\mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n_m} (X_{m,t})^j}{\Psi(n_m)^j} \right|^p \right\} \leq C_{n_m}(j,p) \leq C(j,p) \ ,$$

where C(j,p) depends only on j and p and where $C_{n_m}(j,p) = \epsilon_m(j,p) \to 0$ as $m \to \infty$ if $j \geq 3$.

Hypothesis 3.1.3 *Let $\beta_{m,s} = \mathbb{E}\{X_{m,t}^2\}$. One assumes that $\sum_{t=1}^{n_m} \frac{\beta_{m,t}}{\Psi(n_m)^2} \to \sigma_0^2 \in \mathbb{R}_+$.*

Indeed, if $\mathbb{E}\Big\{\frac{(X_{m,1}+X_{m,2}+X_{m,3}+....+X_{m,n})^2}{\Psi(n_m)^2}\Big\}$ converges and $\frac{\sum_{s\neq t}X_{m,s}X_{m,t}}{\Psi(n_m)^2}$ converges also, then, $(1/\Psi(n_m)^2)\sum_s \mathbb{E}(X_{m,s}^2\}$ converges also. Now, in all the MCLT of this report, we impose that $\frac{\sum_{s\neq t}X_{m,s}X_{m,t}}{\Psi(n_m)^2}$ converges.

3.2 General lemma

3.2.1 Lemma about sets

At first, we need the following notations.

Notations 3.2.1 *Let k and r be two integers such that $1 \leq r \leq k$. We set*

$$t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_k$$
$$= \big\{(t_1, t_2, ..., t_r, t_{r+1}, ..., t_k) \in \{1, 2, ..., n_m\}^k \;\big|\; t_s \neq t_{s'} \;\; if \;\; s < s' \leq r\big\} \;,$$

$$t_1 \neq t_2 \neq .... \neq t_{r-1}, t_{r+1}, ..., t_k$$
$$= \big\{(t_1, t_2, ..., t_{r-1}, t_{r+1}, ..., t_k) \in \{1, 2, ..., n_m\}^{k-1} \;\big|\; t_s \neq t_{s'} \;\; if \;\; s < s' \leq r-1\big\} \;.$$

In particular, $t_1, t_2, ..., t_k = \big\{1, 2, ..., n_m\big\}^k$. Moreover, if r=2, $t_1 \neq t_2 \neq .... \neq t_{r-1}, t_{r+1}, ..., t_k = t_1, t_3, t_4, ..., t_k$ and if r=1, $t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_k = t_1, t_2, ..., t_k$.

Lemma 3.2.1 *Let $r \geq 3$. Then,*

$$\Big\{t_1 \neq t_2 \neq .... \neq t_{r-1}, t_r, ..., t_k\Big\}$$

$$= \Big\{t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_k\Big\}$$
$$\cup\Big\{t_1 = t_r \neq t_2 \neq ... \neq t_{r-1}, t_{r+1}, .., t_k\Big\}$$
$$\cup\Big\{t_1 \neq t_2 = t_r \neq ... \neq t_{r-1}, t_{r+1}, .., t_k\Big\}$$
$$.....$$
$$\cup\Big\{t_1 \neq t_2 \neq ... \neq t_{r-1} = t_r, t_{r+1}, .., t_k\Big\} \;.$$

For example, the following lemma holds.

Lemma 3.2.2 *We simplify $X_{m,t}$ in X_t. Let $k \in \mathbb{N}$, $k \geq 3$ and $h \geq k$. For all $s \in \{1, 2, ..., k\}$, we denote by R_s^t, $t = 1, 2, ..., n_m$, s=1,...,k, a sequence of polynomials of degree j_s. Let $r \geq 3$. Then, for all $k \leq h$,*

$$\mathbb{E}\Big\{\frac{\sum_{t_1\neq t_2\neq ....\neq t_{r-1}, t_r, ..., t_k} R_1^{t_1}(X_{t_1})R_2^{t_2}(X_{t_2}).....R_k^{t_k}(X_{t_k})}{\Psi(n)^h}\Big\}$$

$$= \mathbb{E}\Big\{\frac{\sum_{t_1\neq t_2\neq ....\neq t_r, t_{r+1}, ..., t_k} R_1^{t_1}(X_{t_1})R_2^{t_2}(X_{t_2}).....R_k^{t_k}(X_{t_k})}{\Psi(n)^h}\Big\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq\ldots\neq t_{r-1},t_{r+1},..,t_k}[\mathbf{R_1^{t_1}(X_{t_1})R_r^{t_1}(X_{t_1})}]\mathrm{R_2^{t_2}(X_{t_2})...R_{r-1}^{t_{r-1}}(X_{t_{r-1}})R_{r+1}^{t_{r+1}}(X_{t_{r+1}})...R_k^{t_k}(X_{t_k})}}{\Psi(n)^h}\right\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq\ldots\neq t_{r-1},t_{r+1},..,t_k} R_1^{t_1}(X_{t_1})[\mathbf{R_2^{t_2}(X_{t_2})R_r^{t_2}(X_{t_2})}]\mathrm{...R_{r-1}^{t_{r-1}}(X_{t_{r-1}})R_{r+1}^{t_{r+1}}(X_{t_{r+1}})...R_k^{t_k}(X_{t_k})}}{\Psi(n)^h}\right\}$$

+...

$$+\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq\ldots\neq t_{r-1},t_{r+1},..,t_k} R_1^{t_1}(X_{t_1})R_2^{t_2}(X_{t_2})...[\mathbf{R_{r-1}^{t_{r-1}}(X_{t_{r-1}})R_r^{t_{r-1}}(X_{t_{r-1}})}]\mathrm{R_{r+1}^{t_{r+1}}(X_{t_{r+1}})...R_k^{t_k}(X_{t_k})}}{\Psi(n)^h}\right\}.$$

Example Suppose r=3. Then,

$$\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2,t_3,\ldots,t_k} X_{t_1}X_{t_2}.........X_{t_k}}{\sqrt{n^h}}\right\}$$

$$=\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq t_3,t_4,\ldots,t_k} X_{t_1}X_{t_2}X_{t_3}X_{t_4}.........X_{t_k}}{\sqrt{n^h}}\right\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2,t_4,\ldots,t_k} X_{t_1}^2X_{t_2}X_{t_4}.........X_{t_k}}{\sqrt{n^h}}\right\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2,t_4,\ldots,t_k} X_{t_1}X_{t_2}^2X_{t_4}.........X_{t_k}}{\sqrt{n^h}}\right\}.$$

If r=2, the following lemma hold.

Lemma 3.2.3 *We suppose r=2. Then,*

$$\{t_1,t_2,...,t_k\}=\{t_1\neq t_2,t_3,...,t_k\}\cup\{\ t_1=t_2,t_3,..,t_k\}\ .$$

Lemma 3.2.4 *The following equalities holds.*

$$\mathbb{E}\left\{\frac{\sum_{t_1,t_2,\ldots,t_k} R_1^{t_1}(X_{t_1})R_2^{t_2}(X_{t_2}).....R_k^{t_k}(X_{t_k})}{\Psi(n)^h}\right\}$$

$$=\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2,t_3,\ldots,t_k} R_1^{m_1}(X_{t_1})R_2^{t_2}(X_{t_2}).....R_k^{t_k}(X_{t_k})}{\Psi(n)^h}\right\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1,t_3,..,t_k}[\mathbf{R_1^{t_1}(X_{t_1})R_2^{t_1}(X_{t_1})}]\mathrm{R_3^{t_3}(X_{t_3})...R_k^{t_k}(X_{t_k})}}{\Psi(n)^h}\right\}.$$

3.2.2 Number of coefficients of moments

We simplify n_m in n. Because

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}=\sum_{j_1+.....+j_{n_m}=q}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}\ ,$$

one wants to study the sets $\{j_1+.....+j_{n_m}=q, j_s\leq 1\}$ in order to know the sums

$$\sum_{j_1+.....+j_{n_m}=q, j_s\leq 1}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^{2q}}\Big\}\ .$$

First study We study the set

$$\{j_1+.....+j_n=q, j_s\leq 1\}=\{(j_1,.....,j_n)\in\mathbb{N}^n\ |j_1+.....+j_n=q, j_s\leq 1\}$$

when q=2, n=6. We have

$$\begin{gathered}\{j_1+.....+j_n=q, j_s\leq 1\}\\ =\{(1,1,0,0,0,0),(1,0,1,0,0,0),.....,(1,0,0,0,0,1)\}\\ \cup\{(0,1,1,0,0,0),(0,1,0,1,0,0),.....,(0,1,0,0,0,1)\}\\ \cup\{(0,0,1,1,0,0),(0,0,1,0,1,0),(0,0,1,0,0,1)\}\\ \cup\{(0,0,0,1,1,0),(0,0,0,1,0,1)\}\\ \cup\{(0,0,0,0,1,1)\}\ .\end{gathered}$$

Its cardinal is $(n-1)+(n-2)+(n-3)+.....+2+1=\frac{(n-1)n}{2}=C_n^2$. Indeed, there is C_n^2 ways to select two terms among n.

Now, we choose q=3, n=8. We have

$$\{j_1+.....+j_n=q, j_s\leq 1\}$$

$$\begin{gathered}=\{(1,1,1,0,0,0,0,0),(1,1,0,1,0,0,0,0),.....,(1,1,0,0,0,0,0,1)\}\ :\ n-2\ events\\ \cup\{(1,0,1,1,0,0,0,0),(1,0,1,0,1,0,0,0),.....,(1,0,1,0,0,0,0,1)\}\ :\ n-3\ events\\ \cup\{(1,0,0,1,1,0,0,0),(1,0,0,1,0,1,0,0),.....(1,0,0,1,0,0,0,1)\}\ :\ n-4\ events\\ .....\\ \cup\{(1,0,0,0,0,0,1,1)\}\ :\ n-2-(n-2-1)\ events\end{gathered}$$

$$\begin{gathered}\cup\{(0,1,1,1,0,0,0,0),(0,1,1,0,1,0,0,0),.....,(0,1,1,0,0,0,0,1)\}\ :\ n-3\ events\\ \cup\{(0,1,0,1,1,0,0,0),(0,1,0,1,0,1,0,,0),.....,(0,1,0,1,0,0,0,1)\}\ :\ n-4\ events\\ \cup\{(0,1,0,0,1,1,0,0),(0,1,0,0,1,0,1,0),.....(0,1,0,0,1,0,0,1)\}\ :\ n-5\end{gathered}$$

.....

$$\cup\{(0,1,0,0,0,0,1,1)\} \ : \ n-3-(n-3-1) \ events.$$

.....

.....

The number of possible combinations is

$$Card$$
$$= \frac{(n-2)(n-1)}{2} + \frac{(n-3)(n-2)}{2} + ... + \frac{(n-(n-2))(n-(n-2)+1)}{2} + \frac{(n-(n-1))(n-(n-1)+1)}{2}$$
$$= \frac{(n-2)(n-2)}{2} + \frac{(n-3)(n-3)}{2} + ... + \frac{(n-(n-2))(n-(n-2))}{2} + \frac{(n-(n-1))(n-(n-1))}{2}$$
$$+ \frac{(n-2)}{2} + \frac{(n-3)}{2} + ... + \frac{(n-(n-2))}{2} + \frac{(n-(n-1))}{2} \ .$$

Now, $\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)n}{6}$. Donc $\sum_{i=1}^{n-2} i^2 = \frac{(2(n-2)+1)((n-2)+1)(n-2)}{6} = \frac{(2n-3)(n-1)(n-2)}{6}$.

Therefore,

$$Card = (1/2)\Big[\frac{(2n-3)(n-1)(n-2)}{6} + \frac{(n-2)(n-1)}{2}\Big]$$
$$= (1/2)\Big[\frac{(2n-3)(n-1)(n-2)}{6} + \frac{3(n-2)(n-1)}{6}\Big]$$
$$= (1/2)\Big[\frac{(2n-3+3)(n-1)(n-2)}{6}\Big]$$
$$= \frac{n(n-1)(n-2)}{6} = \frac{n!}{3!(n-3)!} = C_n^3 \ .$$

This is normal : it is C_n^3 ways to select three terms among n.

Finally $card(\{j_1 + ... + j_n = q, j_s \leq 1\}) = C_n^q$. Moreover, for all random variables $X_1, ..., X_n$,

$$\big\{X_1^{j_1} ....X_n^{j_n} \ \big| \ j_1 + ... + j_n = q, j_s \leq 1\big\}$$
$$= \big\{X_{s_1} ....X_{s_q} \ \big| \ (s_1, s_2, ..., s_q) \in \{1, 2, ..., n\}^q \ , \ 1 \leq s_1 < s_2 < ..... < s_q \leq n\big\} \ .$$

Lemma 3.2.5 *Let $p \in \mathbb{N}^*$. We simplify $X_{m,s}$ in X_s, n_m in n. Then,*

$$\sum_{j_1+.....+j_n=pq, \ j_s=0 \ ou \ p} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}}{\Psi(n)^{pq}}\Big\} = \frac{1}{q!} \sum_{t_1 \neq t_2 \neq .... \neq t_q} \mathbb{E}\Big\{\frac{X_{t_1}^p.....X_{t_q}^p}{\Psi(n)^{pq}}\Big\}$$

Proof One can suppose p=1. When $p > 1$, it is enough to set $X_t = Y_t^p$.
Then,

$$\big\{X_1^{j_1}....X_n^{j_n} \;\big|\; j_1 + ... + j_n = q, j_s \leq 1\big\} = \big\{X_{s_1}....X_{s_q} \;\big|\; 1 \leq s_1 < s_2 < ..... < s_q \leq n\big\}.$$

Therefore,

$$\big\{\mathbb{E}\{(X_1^{j_1}....X_n^{j_n})\} \;\big|\; j_1 + ... + j_n = q, j_s \leq 1\big\} = \big\{\mathbb{E}\{(X_{s_1}....X_{s_q})\} \;\big|\; 1 \leq s_1 < s_2 < ..... < s_q \leq n\big\}.$$

Now, let $\mathcal{P}_q$ the set of permutations of q terms. Then,

$$\big\{(s_{p(1)}, ...s_{p(q)}) \in \{1, ..., n\}^q \;\big|\; s_1 < s_2 < ..... < s_q,\; p \in \mathcal{P}_q\big\}$$

$$= \big\{(s_1, ..., s_q) \in \{1, ..., n\}^q \;\big|\; s_1 \neq s_2 \neq .... \neq s_q\big\}.$$

Moreover, if $p \in \mathcal{P}_q$, then, $\mathbb{E}\{(X_{s_{p(1)}}....X_{s_{p(q)}})\} = \mathbb{E}\{(X_{s_1}....X_{s_q})\}$.
Therefore, because there is q! permutations which belongs to $\mathcal{P}_q$,

$$\sum_{j_1+.....+j_n=q,\; j_s=0 \;\; ou \;\; 1} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}}{\Psi(n)^q}\Big\} = \sum_{t_1<t_2<....<t_q} \mathbb{E}\Big\{\frac{X_{t_1}.....X_{t_q}}{\Psi(n)^q}\Big\}$$

$$= \frac{1}{q!} \sum_{t_1 \neq t_2 \neq .... \neq t_q} \mathbb{E}\Big\{\frac{X_{t_1}.....X_{t_q}}{\Psi(n)^q}\Big\}\;. \blacksquare$$

Second study We study the sets $\{j_1 + ... + j_n = q, j_s \leq 2,\; one\; j_s = 2\}$. At first, choose q=4. We have

$$\{j_1 + ... + j_n = q, j_s \leq 2,\; un\; j_s = 2\}$$

$$= \{(2,1,1,0,0,0,....,0), (2,1,0,1,0,0,....,0), ..., (2,1,0,0,....,0,0,1)\}$$

$$\cup\{(2,0,1,1,0,0,....,0), (2,0,1,0,1,0,....,0), ..., (2,0,1,0,....,0,0,1)\}$$

$$.....$$

$$\cup\{(2,0,0,....,0,1,1,0), (2,0,0,....,0,1,0,1)\}$$

$$\cup\{(2,0,0,0,....,0,1,1)\}$$

$$\cup\{(1,2,1,0,0,0,....,0), (1,2,0,1,0,0,....,0), ..., (1,2,0,0,....,0,0,1)\}$$

$$\cup\{(0,2,1,1,0,0,....,0), (0,2,1,0,1,0,....,0), ..., (0,2,1,0,....,0,0,1)\}$$

$$.....$$

$$\cup\{(0,2,0,....,0,1,1,0), (0,2,0,....,0,1,0,1)\}$$

$$\cup\{(0,2,0,0,....,0,1,1)\}$$

$$\cup\{(1,1,2,0,0,0,....,0), (1,0,2,1,0,0,....,0), ..., (1,0,2,0,....,0,0,1)\}$$

$$\cup\{(0,1,2,1,0,0,....,0), (0,1,2,0,1,0,0,....,0), ..., (0,1,2,0,0,....,0,0,1)\}$$

$$\cup\{(0,0,2,1,1,0,0,....,0), (0,0,2,1,0,1,0,....,0), ..., (0,0,2,1,0,....,0,0,1)$$

$$\cup\{(0,0,2,0,1,1,0,0,....,0),(0,0,2,0,1,0,1,0,....,0),.....,(0,0,2,0,1,0,....,0,0,1)$$
$$.....$$
$$\cup\{(0,0,2,0,....,0,1,1,0),(0,0,2,0,....,0,1,0,1)\}$$
$$\cup\{(0,0,2,0,0,....,0,1,1)\}$$

$$\cup\{(1,1,0,0,....0,2,0),(1,0,1,0,....0,2,0),....,(1,0,0,....,1,2,0),(1,0,0,....,0,2,1)\}$$
$$\cup\{(0,1,1,0,....0,2,0),(0,1,0,1,0,....0,2,0),....,(0,1,0,....,1,2,0),(0,1,0,....,0,2,1)\}$$
$$.....$$
$$.....$$
$$\cup\{(0,0,0,....,0,1,1,2,0),(0,0,0,....,1,0,2,1)\}$$
$$\cup\{(0,0,0,0,....,0,1,2,1)\}$$

$$\cup\{(1,1,0,0,....0,2),(1,0,1,0,....0,2),....,(1,0,0,....,1,2)\}$$
$$\cup\{(0,1,1,0,....0,2),(0,1,0,1,0,....0,2),....,(0,1,0,....,1,2)\}$$
$$.....$$
$$.....$$
$$\cup\{(0,0,0,....,0,1,1,0,2),(0,0,0,....,1,0,1,2)\}$$
$$\cup\{(0,0,0,0,....,0,1,1,2)\}\ .$$

Therefore,

$$\{X_1^{j_1}.....X_n^{j_n}\ |\ j_1+.....+j_n=q, j_s\leq 2,\ one\ j_s=2\}$$
$$=\{X_{s_1}X_{s_2}X_{u_1}^2\ |\ (s_1,s_2,u_1)\in\{1,2,....,n\}^q\ |\ s_1<s_2,\quad s_i\neq u_1\ for\ i=1,2\}\ .$$

Then, it is clear that to know $\{j_1+.....+j_n=q, j_s\leq 2,\ "k"\ j_s=1,\quad "h"\ j_s=2\}$, it is the same thing as to know all the k-tuple $S=(s_1,....,s_k)\in\{1,2,....,n\}^k$ and all the h-tuple $U_S=(u_1,....,u_h)\in\{\{1,2,....,n\}\backslash S\}^h$ in the n-k remaining elements. It is clear that the order within the h-tuples and k-tuples has no interest. Therefore, we have to consider the $s_1<....<s_k$ and the $u_1<...<u_h$.

Now, the following lemma holds.

Lemma 3.2.6 *Let* $\mathbb{S}^*=\{S=(s_1,....,s_k)\in\{1,2,....,n\}^k|s_1<s_2<....<s_k\}$ *and* $\mathbb{U}_S^*=\{U_S=(u_1,....,u_h)\in\{\{1,2,....,n\}\backslash S\}^h\ |\ u_1<u_2<....<u_h\}$. *Then,*

$$\{j_1+.....+j_n=q, j_s\leq 2,\ "k"\ j_s=1,\quad "h"\ j_s=2\}$$
$$=\cup_{S\in\mathbb{S}^*}\cup_{U_S\in\mathbb{U}_S^*}\{(j_1,.....,j_n)\ |\ j_{s_i}=1\ if\ s_i\in S,\quad j_{u_i}=2\ if\ u_i\in U_S,\quad j_i=0\ if\ not\}\ .$$

Lemma 3.2.7 *Let*

$$\mathbb{S}=\{S=(s_1,....,s_k)\in\{1,2,....,n\}^k|s_1\neq s_2\neq....\neq s_k\}$$
$$and\ \mathbb{U}_S=\{U_S=(u_1,....,u_h)\in\{\{1,2,....,n\}\backslash S\}^h\ |\ u_1\neq u_2\neq....\neq u_h\}.$$

Then, we have

$$\{j_1 + ... + j_n = q, j_s \leq 2,\ "k"\ j_s = 1,\ \ "h"\ j_s = 2\}$$
$$= \cup_{S\in\mathbb{S}^*} \cup_{U_S\in\mathbb{U}_S^*} \{(j_1, ..., j_n) \mid j_{s_i} = 1\ if\ s_i \in S,\ \ j_{u_i} = 2\ if\ u_i \in U_S,\ \ j_i = 0\ if\ not\}$$
$$= \cup_{S\in\mathbb{S}} \cup_{U_S\in\mathbb{U}_S} \{(j_1, ..., j_n) \mid j_{s_i} = 1\ if\ s_i \in S,\ \ j_{u_i} = 2\ if\ u_i \in U_S,\ \ j_i = 0\ if\ not\}\ .$$

Lemma 3.2.8 *We simplify $X_{m,s}$ in X_s. Then,*

$$\sum_{j_1+.....+j_n=q,\ j_s=0,1,2,\ "k"\ j_s=1,\ "h"\ j_s=2} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}}{\Psi(n)^q}\Big\}$$
$$= \frac{1}{h!k!} \sum_{s_1\neq s_2 \not\cdot .. \neq s_k \neq u_1 \neq u_2 \not\cdot .. \neq u_h} \mathbb{E}\Big\{\frac{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2}{\Psi(n)^q}\Big\}\ .$$

Proof We have the following equalities

$$\{X_1^{j_1}.....X_n^{j_n} \mid j_1 + ... + j_n = q,\ j_s = 0, 1, 2,\ "k"\ j_s = 1,\ "h"\ j_s = 2\}$$
$$= \{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2 \mid (s_1, s_2, ..., s_k, u_1, ..., u_h) \in \{1, 2, ..., n\}^{k+h} \cap \mathcal{B}\}$$

where

$$\mathcal{B} = \{(s_1, s_2, ..., s_k, u_1, ..., u_h) \mid s_1 < s_2 < ..... < s_k,\ u_1 < u_2 < ..... < u_h,\ s_i \neq u_j\}\ .$$

Let $\mathcal{P}_k$ be the set of the permutations of k elements. Then,

$$\Big\{(s_1, s_2, ..., s_k, u_1, ..., u_h) \in \{1, ..., n\}^{k+h} \ \Big|\ \ s_1 \neq s_2\ \not\cdot ... \neq s_k \neq u_1 \neq u_2\ \not\cdot ... \neq u_h\Big\}.$$
$$= \Big\{(s_{p(1)}, ...s_{p(k)}, u_{p'(1)}, ...u_{p'(h)}) \ \Big|\ \ s_1 < ..... < s_k, u_1 < ..... < u_h,\ s_i \neq u_j,\ p \in \mathcal{P}_k,\ p' \in \mathcal{P}_h\Big\}.$$

Then, if $p \in \mathcal{P}_k$, $p' \in \mathcal{P}_h$, we have
$\mathbb{E}\{X_{s_{p(1)}}....X_{s_{p(k)}}X_{u_{p'(1)}}^2.....X_{u_{p'(h)}}^2\} = \mathbb{E}\{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2\}$.
Then, because there are q! permutations belonging to $\mathcal{P}_q$,

$$\sum_{j_1+.....+j_n=q,\ j_s=0,1,2,\ r\ j_s=1,\ t\ j_s=2} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}}{\Psi(n)^q}\Big\}$$
$$= \sum_{s_1<s_2<.....<s_k,\ u_1<u_2<.....<u_h,\ s_i\neq u_j} \mathbb{E}\Big\{\frac{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2}{\Psi(n)^q}\Big\}$$
$$= \frac{1}{h!k!} \sum_{s_1\neq s_2 \not\cdot .. \neq s_k \neq u_1 \neq u_2 \not\cdot .. \neq u_h} \mathbb{E}\Big\{\frac{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2}{\Psi(n)^q}\Big\}\ . \blacksquare$$

One can generalize easily this lemma

Lemma 3.2.9 *Let $p \in \mathbb{N}^*$. Then,*

$$\sum_{j_1+.....+j_n=q,\ j_i\leq p,\ "h_t"\ j_i=t,\ t=1,2,...,p} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}}{\Psi(n)^q}\Big\}$$
$$= \frac{1}{h_1!h_2!....h_p!} \sum_{s_1^t\neq s_2^t \not\cdot .. \neq s_{h_t}^t,\ s_i^t\neq s_j^{t'}\ \ if\ \ t\neq t'} \mathbb{E}\Big\{\frac{\prod_{t=1}^p (X_{s_1^t}^t.....X_{s_{h_t}^t}^t)}{\Psi(n)^q}\Big\}\ .$$

3.3 First equivalence to bounded moments

3.3.1 Lemma of recurence

In all these lemma, we shall use the following notations.

Notations 3.3.1 *For all $s \in \{1,2,...,k\}$, for all $m \in \mathbb{N}^*$, we denote by $R_s^{m,t}$, $t = 1,2,...,n_m$, a sequence of polynomials of degree j_s. We set $k_0 = card\{j_s = 0\}$, $k_1 = card\{j_s = 1\}$, $k_2 = card\{j_s = 2\}$, and $k_3 = card\{j_s > 2\}$. Let $k'_t = card\{j_s = t\}$ and $h = 2k'_0 + \sum_t tk'_t$.*

Then, we have the following lemma.

Lemma 3.3.1 *We assume $k_0 = 0$. Let $H_1 \geq k$. Then,*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_k}\left(\prod_{s=1}^k R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right|$$
$$\leq \left[\prod_{j_s=1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})}{\Psi(n_m)}\right|^{H_1}\right\}^{\frac{1}{H_1}}\right]\left[\prod_{j_s\neq 1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right|^{H_1}\right\}^{\frac{1}{H_1}}\right].$$

Proof Of course, $h \geq k$. Then, we can write

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_k}\left(\prod_{s=1}^k R_s^{m,t_s}(X_{t_s})\right)}{\Psi(n_m)^h}\right\}\right| = \left|\mathbb{E}\left\{\prod_{s=1}^k\left[\frac{\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right]\right\}\right|$$
$$= \left|\mathbb{E}\left\{\left[\prod_{j_s=1}\left(\frac{\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right)\right]\left[\prod_{j_s\neq 1}\left(\frac{\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right)\right]\right\}\right|$$
$$\leq \left[\prod_{j_s=1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n} R_s^{m,t}(X_{m,t})}{\Psi(n_m)}\right|^{H_1}\right\}^{\frac{1}{H_1}}\right]\left[\prod_{j_s\neq 1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n} R_s^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right|^{H_1}\right\}^{\frac{1}{H_1}}\right],$$

by Holder's inequality. ■

Then, by using hypothesis 3.1.2, we have the following corollary.

Lemma 3.3.2 *For all $s \in \{1,2,...,k\}$, we suppose that $R_s^{m,t}(x) = x^{j_s}$, $t = 1,2,...,n_m$. We assume $k_0 = 0$. Then,*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_k}\left(\prod_{s=1}^k R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right| \leq \left[\prod_{j_s>1} C_{n_m}(j_s,H_1)^{\frac{1}{H_1}}\right]\left[\prod_{j_s=1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^{H_1}\right\}\right]^{\frac{1}{H_1}}.$$

Then, by using lemma 3.2.2 and 3.2.4 , we deduce :

Lemma 3.3.3 *For all $s \in \{1,2,...,k\}$, we suppose that $R_s^{m,t}(x) = x^{j_s}$, $t = 1,2,...,n_m$. We assume $k_0 = 0$. Then, for all r,*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq ....\neq t_r,t_{r+1},...,t_k}\left(\prod_{s=1}^k R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right| \leq e(n)K_1\ Max\left[1\ ,\ \prod_{j_s=1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^{H_1}\right\}^{\frac{1}{H_1}}\right],$$

where $K_1 > 0$ and where $e(n) \to 0$ if there exists s such that $j_s \geq 3$.

Proof By lemma 3.3.2 , the lemma holds for r=1.

In order to prove the lemma if $r \geq 1$, one uses lemma 3.2.2 and 3.2.4. Then, by lemma 3.3.2,

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq ....\neq t_r, t_{r+1},...,t_k}\left(\prod_{s=1}^{k} R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right|$$

is increased by sums of products of terms themselves bounded by some $C_{n_m}(j,p)$'s and by terms of the form $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^c\right\}^{1/c}$ where $c \leq H_1$.

Now, let $c \leq d = H_1$. Suppose $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^d\right\} \geq 1$. Then, $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^c\right\}^{1/c} \leq \mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^d\right\}^{1/d}$ by Holder Inequality.

Suppose now $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^d\right\} \leq 1$. Then, $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^c\right\}^{1/c} \leq \mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}(X_{m,t})}{\Psi(n_m)}\right|^d\right\}^{1/d} \leq 1$.

It is enough to prove the lemma. ■

Lemma 3.3.4 *We suppose that $R_s^{m,t}(x) = x^{j_s}$ for $t = 1, 2, ..., n_m$ where $j_s > 0$. We assume $k_3 > 0$ or $k_2 \geq 2$ if $k_3 = 0$.*

We define H^* by $H^* = h - 2$. Then, we define H by $H = H^*$ if H^* is even and $H = H^* + 1$ if not. Then, there exists $K_3 > 0$ and $K_4 > 0$ which do not depend on m, such that

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_k}\left(\prod_{s=1}^{k} R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right| \leq e(n_m)K_3B_H^{n_m} \ ,$$

where $e(n) \leq K_4$ and $e(n_m) \to 0$ as $m \to \infty$ if $k_3 > 0$.

Proof Indeed, H is even. Moreover, by our assumptions, $k \leq h-2$. Then, $H \geq k \geq k_1$. Then, one can choose $H_1 = H$ in lemma 3.3.2. Then, $k_1/H \leq 1$. Moreover,

$$\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n_m)}\right|^H\right\} = \mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n_m)}\right)^H\right\} .$$

If $\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n_m)}\right)^H\right\} \leq 1$, $\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n_m)}\right)^H\right\}^{\frac{k_1}{H}} \leq 1$.

If not, $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n)}\right|^H\right\}^{\frac{k_1}{H}} \leq \mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n)}\right)^H\right\}$.

Then, $\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m} X_{m,t}}{\Psi(n_m)}\right|^H\right\}^{\frac{k_1}{H}} \leq B_H^{n_m}$. Then, it is enough to apply lemma 3.3.2. ■

Lemma 3.3.5 *Under the assumptions of lemma 3.3.4 , there exists $K_3' > 0$ and $K_4' > 0$ which do not depend on m, such that, for all $r \in \mathbb{N}$, $1 \leq r \leq k$,*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq ....\neq t_r, t_{r+1},...,t_k}\left(\prod_{s=1}^{k} R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right| \leq e'(n_m)K_3'.B_H^{n_m} \ ,$$

where $e'(n_m) \leq K_4'$ and $e'(n_m) \to 0$ as $m \to \infty$ if $k_3 > 0$.

Proof Indeed by lemma 3.3.4 , this result holds for r=1. Now suppose that lemma 3.3.5 holds for all $r' \leq r-1$. Then, it is enough to apply lemma 3.2.2, 3.2.4 and 3.3.4 and to use $|B_h^{n_m}| \leq |B_{h+1}^{n_m}|$. ■

3.3.2 First proposition about bounded moments

By using the previous lemma, we can prove the following proposition.

Proposition 3.3.1 *All the moments* $M_q^{n_m} = \mathbb{E}\Big\{\frac{(X_{m,1}+X_{m,2}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}$ *are bounded by a real* $B_q > 0$ *if and only if, for all* $q \in \mathbb{N}$*, there exists* $Sb_q^1 \in \mathbb{R}_+$ *and* $Sb_q^2 \in \mathbb{R}_+$ *such that*

$$\Bigg|\sum_{s_1\neq s_2\neq .....\neq s_q} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\Bigg| \leq Sb_q^1 \ ,$$

$$\Bigg|\sum_{s_1\neq s_2\neq .....\neq s_{q-1}} \frac{\mathbb{E}\{X_{m,s_1}^2X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\Bigg| \leq Sb_q^2 \ .$$

Proof of the sufficiency condition of prop 3.3.1 We prove the theorem by recurrence on q. If q=0,1 and 2, it is obvious.

So, we suppose that it holds for all $q' \leq q-1$.

Let $\mathcal{S}_q^*$, be the substitution of $\big\{(j_1, j_2, ..., j_n) \in \mathbb{N}^n \big| j_1+....+j_n = q\big\}$ defined by $\mathcal{S}_q^*(j_1, j_2..., j_n) = \{u_1, u_2, ..., u_n\}$ where $u_1 \geq u_2 \geq .... \geq u_n$.

We suppose $n > q$ because $n_m \to \infty$. Then, $u_{q+1} = u_{q+2} = ... = u_n = 0$ and we define $\mathcal{S}_q$ by $\mathcal{S}_q(j_1, j_2..., j_n) = \{u_1, u_2, ..., u_q\}$. Let $\mathcal{P}_q = \big\{\mathcal{S}_q(j_1, j_2, ..., j_n) \ \big| j_1 + j_2 + ... + j_n = q\big\}$.

Then, by lemma 3.2.9 ,

$$M_q^{n_m} = \mathbb{E}\Big\{\frac{(X_{m,1} + X_{m,2} + ... + X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{j_1+....+j_{n_m}=q} \frac{q!}{j_1!...j_{n_m}!} \ \mathbb{E}\Big\{\frac{X_{m,1}^{j_1}....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{\mathcal{O}_q\in\mathcal{P}_q} \ \sum_{(j_1,....,j_{n_m}):\ \mathcal{S}_q^*(j_1,....,j_{n_m})=\mathcal{O}_q} \frac{q!}{j_1!...j_{n_m}!} \ \mathbb{E}\Big\{\frac{X_{m,1}^{j_1}....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{\mathcal{O}_q=(u_1,...,u_q)\in\mathcal{P}_q} N'_{\mathcal{O}_q} \sum_{s_1\neq....\neq s_q} \frac{q!}{u_1!...u_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,s_1}^{u_1}....X_{m,s_q}^{u_q}}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{\mathcal{O}_q=(u_1,...,u_q)\in\mathcal{P}_q} N_{\mathcal{O}_q} \sum_{s_1\neq....\neq s_q} \mathbb{E}\Big\{\frac{X_{m,s_1}^{u_1}....X_{m,s_q}^{u_q}}{\Psi(n_m)^q}\Big\} \ ,$$

where $N'_{\mathcal{O}_q} \in \mathbb{R}$ and $N_{\mathcal{O}_q} \in \mathbb{R}$.

Let $\{u_1, u_2, ..., u_k\}$ where $u_1 \geq 3$ or $u_2 \geq 2$. By Lemma 3.3.5

$$\mathbb{E}\left[\frac{\sum_{t_1\neq t_2\neq....\neq t_k} X_{m,t_1}^{u_1}....X_{m,t_k}^{u_k}}{\Psi(n_m)^q}\right]$$

is bounded.

By our assumption,

$$\left|\sum_{s_1\neq s_2\neq .....\neq s_q}\frac{\mathbb{E}\{X_{s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\right|\leq Sb_q^1\ ,$$

$$\left|\sum_{s_1\neq s_2\neq .....\neq s_{q-1}}\frac{\mathbb{E}\{X_{m,s_1}^2X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2\ .$$

Therefore

$$\sum_{\mathcal{O}_q=(u_1,...,u_q):\ u_1+....+u_q=q}N_{\mathcal{O}_q}\sum_{s_1\neq ....\neq s_q}\mathbb{E}\Big\{\frac{X_{m,s_1}^{u_1}....X_{m,s_q}^{u_q}}{\Psi(n_m)^q}\Big\}$$

is bounded. ■

Proof of the necessity condition of propostion 3.3.1 Now we suppose that all the moments are bounded. Then, by lemma 3.3.3, for all $\mathcal{O}_h=(u_1,....,u_h)$,

$$|\mathbb{E}\Big\{\sum_{s_1\neq ....\neq s_h}\frac{X_{m,s_1}^{u_1}....X_{m,s_h}^{u_h}}{\Psi(n_m)^q}\Big\}|$$

is bounded

In particular

$$\left|\sum_{s_1\neq s_2\neq .....\neq s_h}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_h}\}}{\Psi(n_m)^h}\right|$$

$$\left|\sum_{s_1\neq s_2\neq .....\neq s_{h-1}}\frac{\mathbb{E}\{X_{m,s_1}^2X_{m,s_2}.....X_{m,s_{h-1}}\}}{\Psi(n_m)^h}\right|$$

are bounded. ■

3.4 Second equivalence to bounded moments

3.4.1 Lemma

Lemma 3.4.1 *Let* $h=2k_0'+\sum_{t=1}^{k-1}tk_t'$. *We assume* $R_1^{m,t}=\beta_{m,t}=\mathbb{E}\{X_{m,t}^2\}$ *for all* $s\in\{1,2,...,k\}$, *and, for* $s\geq 2$, $R_s^{m,t}(x)=x^{j_s}$ *where* $j_s\geq 1$. *Then, there exists* $K_0>0$ *and* $H_2\in\mathbb{N}$, $H_2\leq h-2$ *such that*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_{k-1}}\left(\prod_{s=1}^{k-1}R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right|\leq\left|\left(\frac{\sum_{t=1}^n\beta_{m,t}}{\Psi(n_m)^2}\right)B_{H_2}^{n_m}K_0\right|\ .$$

Proof Let $H=card\{j_s|j_s\geq 1\}$. Then, $H\leq h-2$. If $H=h-2$, the result is obvious. Indeed,

$$\mathbb{E}\left\{\frac{\sum_{t_1,t_2,...,t_{k-1}}\left(\prod_{s=1}^{k-1}R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}=\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m}\beta_{m,t}}{\Psi(n_m)^2}\right)\left[\prod_{s=2}^{k-1}\left(\frac{\sum_{t=1}^{n_m}X_{m,t}}{\Psi(n_m)}\right)\right]\right\}\ .$$

If $H<h-2$, we set $H_2=H$ if H is even and $H_2=H+1$ if H is odd. Then we can write

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_2,\ldots,t_{k-1}}\left(\prod_{s=1}^{k-1}R_s^{m,t_s}(X_{m,t_s})\right)}{\Psi(n_m)^h}\right\}\right|$$

$$=\left(\frac{\sum_{t=1}^{n_m}\beta_{m,s}}{\Psi(n_m)^2}\right)\left|\mathbb{E}\left\{\left[\prod_{j_s=1}\left(\frac{\sum_{t=1}^{n_m}R_{m,s}^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right)\right]\left[\prod_{j_s>1}\left(\frac{\sum_{t=1}^{n_m}R_{m,s}^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right)\right]\right\}\right|$$

$$\leq\left(\frac{\sum_{t=1}^{n_m}\beta_{m,s}}{\Psi(n_m)^2}\right)\left[\prod_{j_s=1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}X_{m,t}}{\Psi(n_m)}\right|^{H_2}\right\}^{1/H_2}\right]\left[\prod_{j_s>1}\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}R_{m,s}^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right|^{H_2}\right\}\right]^{1/H_2}.$$

By hypothesis 3.1.2, if $j_s\geq 1$, $\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m}R_{m,s}^{m,t}(X_{m,t})}{\Psi(n_m)^{j_s}}\right)^{H_2}\right\}$ is bounded.
Moreover, $\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m}X_{m,t}}{\Psi(n_m)}\right)^{H_2}\right\}^{k_1/H_2}\leq B_{H_2}^{n_m}$. ■

Lemma 3.4.2 *Pour tout $j\geq 2$, $\frac{\sum_{t=1}^{n_m}\beta_{m,s}^j}{\Psi(n_m)^{2j}}\to 0$.*

Proof By hypothesis 3.1.2 and by Holder Inequality,

$$\frac{\sum_{t=1}^{n_m}\beta_{m,s}^j}{\Psi(n_m)^{2j}}=\frac{\sum_{t=1}^{n_m}\mathbb{E}\{X_{m,t}^2\}^j}{\Psi(n_m)^{2j}}\leq\frac{\sum_{t=1}^{n_m}\mathbb{E}\{(X_{m,t})^{2j}\}}{\Psi(n_m)^{2j}}$$

$$=\mathbb{E}\left\{\frac{\sum_{t=1}^{n_m}(X_{m,t})^{2j}}{\Psi(n_m)^{2j}}\right\}=\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}(X_{m,t})^{2j}}{\Psi(n_m)^{2j}}\right]^1\right\}\leq\epsilon_m(2j,1)\to 0\ as\ n\to\infty\ .\ ■$$

Lemma 3.4.3 *Let Q, u_1,$v_1\in\mathbb{N}$ where $Q>0$. If $u_1>0$ and $v_1>0$,*

$$\mathbb{E}\left\{\left|\frac{\sum_{s_1}\beta_{m,s_1}^{u_1}X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\right|^Q\right\}\leq C_{n_m}(2v_1,2Q)^{1/4}\Big(\sum_{s_1}\frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\Big)^{Q/2}\to 0\ .$$

If $u_1=0$ and $v_1>1$ or if $u_1>0$ and $v_1=0$, $\mathbb{E}\left\{\left|\frac{\sum_{s_1}\beta_{m,s_1}^{u_1}X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\right|^Q\right\}$ is bounded. If $u_1=0$ and $v_1=1$, $\mathbb{E}\left\{\left|\frac{\sum_{s_1}\beta_{m,s_1}^{u_1}X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\right|^Q\right\}$ is bounded if all the moments $M_q^{n_m}$ are bounded.

Proof We suppose $u_1>0$ and $v_1>0$. We have the following relations

$$\mathbb{E}\left\{\left|\frac{\sum_{s_1}\beta_{m,s_1}^{u_1}X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\right|^Q\right\}\leq\mathbb{E}\left\{\left|\Big(\sum_{s_1}\frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\Big)^{1/2}\Big(\sum_{s_1}\frac{X_{m,s_1}^{2v_1}}{\Psi(n_m)^{2v_1}}\Big)^{1/2}\right|^Q\right\}$$

$$\leq\Big(\sum_{s_1}\frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\Big)^{Q/2}\mathbb{E}\left\{\left|\Big(\sum_{s_1}\frac{X_{m,s_1}^{2v_1}}{\Psi(n_m)^{2v_1}}\Big)\right|^{Q/2}\right\}$$

$$\leq\Big(\sum_{s_1}\frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\Big)^{Q/2}\mathbb{E}\left\{\Big(\sum_{s_1}\frac{X_{m,s_1}^{2v_1}}{\Psi(n_m)^{2v_1}}\Big)^{2Q}\right\}^{1/4}$$

$$\leq C_{n_m}(2v_1,2Q)^{1/4}\Big(\sum_{s_1}\frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\Big)^{Q/2}\ ,$$

by hypothesis 3.1.2. Now, $C_{n_m}(2v_1, 2Q)$ is bounded because $v_1 > 0$ and, moreover, converges to 0 if $v_1 > 1$. Moreover, by lemma 3.4.2, $\sum_{s_1} \frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}} \to 0$ if $u_1 \geq 1$.

Therefore, $C_{n_m}(2v_1, 2Q)^{1/4}\big(\sum_{s_1} \frac{\beta_{m,s_1}^{2u_1}}{\Psi(n_m)^{4u_1}}\big)^{Q/2}$ converges always to 0 if $u_1 > 0$ and $v_1 > 0$.
Moreover, if $u_1 = 0$ and $v_1 > 0$ or if $u_1 > 0$ and $v_1 = 0$, the lemma is obvious. ■

Lemma 3.4.4 *Let $0 \leq q" \leq q'$ and $h = \sum_{s=1}^{q"}(2u_s + v_s) + (q' - q")$. One assumes that all the moments M_q^n are bounded. Then, for all $P \in \mathbb{N}^*$,*

$$\mathbb{E}\Big\{\Big(\sum_{s_1,....s_{q'}} \frac{\beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1} \beta_{m,s_2}^{u_2} X_{m,s_2}^{v_2} .... \beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}} X_{m,s_{q"+1}} ..... X_{m,s_{q'}}}{\Psi(n_m)^h}\Big)^P\Big\}$$

is bounded and converges to 0 if $u_1 \geq 1$ and $v_1 \geq 1$.

Proof Suppose P=1. We have

$$\Big|\sum_{s_1,....s_{q'}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1} \beta_{m,s_2}^{u_2} X_{m,s_2}^{v_2} .... \beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}} X_{m,s_{q"+1}} ..... X_{m,s_{q'}}}{\Psi(n_m)^h}\Big\}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\frac{\sum_{s_1} \beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\Big) ..... \Big(\frac{\sum_{s_{q"}} \beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}}}{\Psi(n_m)^{2u_{q"}+v_{q"}}}\Big)\Big(\frac{\sum_{s_{q"+1}} X_{m,s_{q"+1}}}{\Psi(n_m)}\Big) ..... \Big(\frac{\sum_{s_{q'}} X_{m,s_{q'}}}{\Psi(n_m)}\Big)\Big\}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\frac{\sum_{s_1} \beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\Big) ..... \Big(\frac{\sum_{s_{q"}} \beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}}}{\Psi(n_m)^{2u_{q"}+v_{q"}}}\Big)\Big(\frac{\sum_{s_{q"+1}} X_{m,s_{q"+1}}}{\Psi(n_m)}\Big)^{q'-q"}\Big\}\Big|$$

$$\leq \Big[\prod_{r=1}^{q"} \mathbb{E}\Big\{\Big|\frac{\sum_{s_r} \beta_{m,s_r}^{u_r} X_{m,s_r}^{v_r}}{\Psi(n_m)^{2u_r+v_r}}\Big|^{q"+1}\Big\}^{\frac{1}{q"+1}}\Big]\mathbb{E}\Big\{\Big|\frac{\sum_{s_{q"+1}} X_{m,s_{q"+1}}}{\Psi(n_m)}\Big|^{(q'-q")(q"+1)}\Big\}^{\frac{1}{q"+1}}$$

$$\leq |M_{2(q'-q")(q"+1)}^{n_m}|^{\frac{1}{2(q"+1)}} \mathbb{E}\Big\{\Big|\frac{\sum_{s_1} \beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\Big|^{q"+1}\Big\}^{\frac{1}{q"+1}} ..... \mathbb{E}\Big\{\Big|\frac{\sum_{s_{q"}} \beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}}}{\Psi(n_m)^{2u_{q"}+v_{q"}}}\Big|^{q"+1}\Big\}^{\frac{1}{q"+1}} .$$

It is enough to apply lemma 3.4.3 in order to conclude.
In order to prove the lemma when $P > 1$, it is enough to use the same technique. ■

Lemma 3.4.5 *For all $s \in \{1, 2, ..., k-1\}$, we assume $R_s^{m,t}(x) = x$ for $s \geq 2$ and $R_1^{m,t} = \beta_{m,t}$ for $t = 1, 2, 3, ..., n_m$. We set H=k-2 if k is even and H=k-1, if not.*
Then, for all r,

$$\Big|\mathbb{E}\Big\{\frac{\sum_{t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_{k-1}} \Big(\prod_{s=1}^{k-1} R_s^{m,t_s}(X_{m,t_s})\Big)}{\Psi(n_m)^k}\Big\}\Big| \leq K_5 B_H^{n_m} .$$

where K_5 is a constants $K_5 > 0$.

Proof By hypothesis 3.1.3, $\Big(\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\Big)$ is bounded. Then, by lemma 3.4.1 this lemma is proved if r=1.

Now, we prove this lemma by recurence on r. For example, for r=2, and for

$$\mathbb{E}\left\{\frac{\sum_{t_1,t_2,t_3,\ldots,t_{k-1}}\beta_{m,t_1}X_{m,t_2}.....X_{m,t_{k-1}}}{\Psi(n_m)^k}\right\}$$

$$=\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2,t_3,t_4,\ldots,t_{k-1}}\beta_{m,t_1}X_{m,t_2}X_{m,t_3}X_{m,t_4}.....X_{m,t_{k-1}}}{\Psi(n_m)^k}\right\}$$

$$+\mathbb{E}\left\{\frac{\sum_{t_1,t_3,t_4,\ldots,t_{k-1}}\beta_{m,t_1}X_{m,t_1}X_{m,t_3}X_{m,t_4}.....X_{m,t_{k-1}}}{\Psi(n_m)^k}\right\},$$

where

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1,t_3,t_4,\ldots,t_{k-1}}\beta_{m,t_1}X_{m,t_1}X_{m,t_3}X_{m,t_4}.....X_{m,t_{k-1}}}{\Psi(n_m)^k}\right\}\right|$$

$$=\left|\mathbb{E}\left\{\left(\frac{\sum_{t_1}\beta_{m,t_1}X_{m,t_1}}{\Psi(n_m)^{3/2}}\right)\left(\frac{\sum_{t_3}X_{m,t_3}}{\Psi(n_m)}\right).....\left(\frac{\sum_{t_{k-1}}X_{m,t_{k-1}}}{\Psi(n_m)}\right)\right\}\right|$$

$$=\left|\mathbb{E}\left\{\left(\frac{\sum_{t_1}\beta_{m,t_1}X_{m,t_1}}{\Psi(n_m)^{3/2}}\right)\left(\frac{\sum_{t}X_{m,t}}{\Psi(n_m)}\right)^{k-3}\right\}\right|$$

$$\leq\mathbb{E}\left\{\left|\frac{\sum_{t_1}\beta_{m,t_1}X_{m,t_1}}{\Psi(n_m)^{3/2}}\right|^{k-2}\right\}^{1/(k-2)}\mathbb{E}\left\{\left|\frac{\sum_{t}X_{m,t}}{\Psi(n_m)}\right|^{k-2}\right\}^{(k-3)/(k-2)}$$

$$\leq\mathbb{E}\left\{\left|\frac{\sum_{t_1}\beta_{m,t_1}X_{m,t_1}}{\Psi(n_m)^{3/2}}\right|^{k-2}\right\}^{1/(k-2)}B_H^{n_m},$$

where $\mathbb{E}\left\{\left|\frac{\sum_{t_1}\beta_{m,t_1}X_{m,t_1}}{\Psi(n_m)^{3/2}}\right|^{k-2}\right\}^{1/(k-2)}\to 0$ by lemma 3.4.3.

Or, in order to prove the recurrence for r=2, by lemma 3.4.1 , one proves the result for sums of the type

$$\mathbb{E}\left\{\frac{\sum_{t_1,t_2,t_3,\ldots,t_{k-2}}\beta_{m,t_1}X_{m,t_2}^2.....X_{m,t_{k-2}}}{\Psi(n_m)^k}\right\}. \blacksquare$$

3.4.2 Second proposition about bounded moments

Now, one can prove the following proposition.

Proposition 3.4.1 *All the moments* $M_q^{n_m}=\mathbb{E}\left\{\frac{(X_{m,1}+X_{m,2}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\right\}$ *are bounded by a real* $B_q>0$ *if and only if, for all* $q\in\mathbb{N}$*, there existe* $Sb_q^r\in\mathbb{R}$ *, r=1,2, such that*

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\right|\leq Sb_q^1$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{[(X_{m,s_1})^2-\beta_{m,s_1}]X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2 .$$

Proof of the sufficiency condition of proposition 3.4.1 We prove the result by recurrence on q. When q=0, 1 and 2, it is obvious.

One supposes that all the moment of order q' are bounded if $q' \leq q-1$. Then, by lemma 3.4.5, and by recurence assumption,

$$\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\mathbb{E}\{\ \beta_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q}$$

is bounded. Therefore

$$\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{\ X^2_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q}$$

is also bounded. Therefore, it is enough to refer tot the proof of the recurrence in proposition 3.3.1 which proves that $M_q^{n_m}$ is bounded. That proves the recurence.

Proof of the necessity condition of proposition 3.4.1 Now we suppose that all the moments are bounded. Then, by proposition 3.3.1 ,

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_q}\}}{\Psi(n_m)^q} \right|$$

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{X^2_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right|$$

are bounded

Moreover, by lemma 3.4.5,

$$\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\mathbb{E}\{\ \beta_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q}$$

is bounded.

Therefore,

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{[(X_{m,s_1})^2 - \beta_{m,s_1}] X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right|$$

is bounded. ■

3.5 Third equivalence to bounded moments

We now give the condition expressed in terms of $\rho_{j_1,\ldots,j_n}$'s.

Lemma 3.5.1 *Let* $\tilde{P}_2^{m,s_1}(x) = x^2 - \gamma_{m,s_1} x - \beta_{m,s_1}$ *where* $\gamma_{m,s_1} = \mathbb{E}\{X^3_{m,s_1}\}/\mathbb{E}\{X^2_{m,s_1}\}$. *We suppose that* $\sum_{s=1}^{n_m} \frac{\gamma^2_{m,s}}{\Psi(n)^2}$ *is bounded. Then, for all* $h \in \mathbb{N}^*$, $\mathbb{E}\big\{\big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s}(X_{m,s})^h}{\Psi(n)^{h+1}}\big)^2\big\}$ *is bounded.*

Proof For all $h \in \mathbb{N}^*$,

$$\Big|\mathbb{E}\Big\{\Big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s}(X_{m,s})^h}{\Psi(n)^{h+1}}\Big)^2\Big\}\Big| \leq \Big|\mathbb{E}\Big\{\Big(\sum_{s=1}^{n_m} \frac{\gamma^2_{m,s}}{\Psi(n)^2}\Big)\Big(\sum_{s=1}^{n_m} \frac{(X_{m,s})^{2h}}{\Psi(n)^{2h}}\Big)\Big\}\Big|$$

$$\leq \Big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s}^2}{\Psi(n)^2}\Big) C(2h,1)$$

which is bounded by our assumptions. ■

Then, we have the following proposition.

Proposition 3.5.1 *We suppose that* $\sum_{s=1}^{n_m} \frac{\gamma_{m,s}^2}{\Psi(n)^2}$ *is bounded. Then, all the moments* $M_q^{n_m} = \mathbb{E}\Big\{\frac{(X_{m,1}+X_{m,2}+....+X_{m,n_m})^q}{\Psi(n)^q}\Big\}$ *are bounded by a real* $B_q > 0$ *if and only if, for all* $q \in \mathbb{N}$*, there existe* $Sb_q^r \in \mathbb{R}$ *, r=1,2,3, such that*

$$\Big|\sum_{s_1 \neq s_2 \neq .....\neq s_q} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n)^q}\Big| \leq Sb_q^1$$

$$\Big|\sum_{s_1 \neq s_2 \neq .....\neq s_{q-1}} \frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n)^q}\Big| \leq Sb_q^2 \ ,$$

$$\Big|\sum_{s_1 \neq s_2 \neq .....\neq s_q} \frac{\gamma_{m,s_1}\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n)^{q+1}}\Big| \leq Sb_q^3 \ .$$

These conditions are indeed conditions about the $\rho_{j_1,...,j_n}$'s. Indeed, we know that $\tilde{P}_2^{m,s}$ is an orthogonal polynomial with degree 2 associated to $X_{m,s}$. Therefore, there exists $(j_1,...,j_n)$ with $j_s \leq 2$ and only one $j_t = 2$ such that $\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}.....X_{m,s_{q-1}}\} = \alpha_{j_1,...,j_n}$.

Proof of the sufficiency of proposition 3.5.1 By our assumption,

$$\sum_{s_1 \neq s_2 \neq .....\neq s_q} \frac{\mathbb{E}\{\ \gamma_{m,s_1}X_{m,s_1}X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n)^q}$$

is bounded. Therefore,

$$\sum_{s_1 \neq s_2 \neq .....\neq s_q} \frac{\mathbb{E}\{\ [X_{m,s_1}^2 - \beta_{m,s_1}]X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n)^q}$$

is bounded. Then, all the conditions of proposition 3.4.1 are checked. Then the sufficient condition is proved.

Proof of the necessity of proposition 3.5.1 Now we suppose that all the moments are bounded. Then, by proposition 3.4.1 ,

$$\Big|\sum_{s_1 \neq s_2 \neq .....\neq s_h} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_h}\}}{\Psi(n)^h}\Big|$$

$$\Big|\sum_{s_1 \neq s_2 \neq .....\neq s_{h-1}} \frac{\mathbb{E}\{[X_{m,s_1}^2 - \beta_{m,s_1}]X_{m,s_2}.....X_{m,s_{h-1}}\}}{\Psi(n)^h}\Big|$$

are bounded

Moreover,

$$\Big|\sum_{s_1,s_2,....,s_{q-1}} \frac{\mathbb{E}\{\ \gamma_{m,s_1}X_{m,s_1}X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n)^q}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\sum_{s_1}\frac{\gamma_{m,s_1}X_{m,s_1}}{\Psi(n)^2}\Big)\Big(\sum_{s_2}\frac{X_{m,s_2}}{\Psi(n)}\Big).....\Big(\sum_{s_{q-1}}\frac{X_{m,s_{q-1}}}{\Psi(n)}\Big)\Big\}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\sum_{s_1}\frac{\gamma_{m,s_1}X_{m,s_1}}{\Psi(n)^2}\Big)\Big(\sum_{s_2}\frac{X_{m,s_2}}{\Psi(n)}\Big)^{q-2}\Big\}\Big|$$

$$\leq \mathbb{E}\Big\{\Big(\sum_{s_1}\frac{\gamma_{m,s_1}X_{m,s_1}}{\Psi(n)^2}\Big)^2\Big\}^{1/2}\mathbb{E}\Big\{\Big(\sum_{s_2}\frac{X_{m,s_2}}{\Psi(n)}\Big)^{2(q-2)}\Big\}^{1/2}$$

which is bounded by lemma 3.5.1.

Then, by lemma 3.2.2 and 3.2.4 , we shall understand, by using the same technique with the lemma 3.5.1 and by using lemma 3.3.2 that it is true also for

$$\Bigg|\sum_{s_1\neq s_2\neq.....\neq s_q} \frac{\mathbb{E}\{\ \gamma_{m,s_1}X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n)^q}\Bigg| .$$

That proves necessary condition. ■

3.6 Fourth equivalence to bounded moments

In this section, we suppose that there exists $F > 0$ such that $|X_{m,t}| \leq F$.

3.6.1 Lemma

Lemma 3.6.1 *Let $j_1 \geq 2$ and $j_s \geq 1$ for s=2,3,....,k. We define H^* by $H^* = k-1$. Then, we define H by $H = H^*$ if H^* is even and $H = H^*+1$ if not. Then, there exists $K_6 > 0$ such that*

$$\Bigg|\mathbb{E}\Bigg\{\frac{\sum_{t_1,t_2,...,t_k}\Big(\prod_{s=1}^k (X_{m,t_s})^{j_s}\Big)}{\Psi(n_m)^h}\Bigg\}\Bigg| \leq K_6 B_H^{n_m} \ . \ ■$$

Proof We have

$$\Bigg|\mathbb{E}\Bigg\{\frac{\sum_{t_1,t_2,...,t_k}\Big(\prod_{s=1}^k (X_{m,t_s})^{j_s}\Big)}{\Psi(n_m)^h}\Bigg\}\Bigg|$$

$$\leq \mathbb{E}\Bigg\{\Bigg|\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_1}}{\Psi(n_m)^{j_1}}\Bigg|\Bigg|\prod_{s\geq 2}\Bigg(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\Bigg)\Bigg|\Bigg\}$$

$$\leq \Bigg|\frac{n_m F^{j_1}}{c_\Psi^{j_1}(n_m)^{j_1/2}}\Bigg|\mathbb{E}\Bigg\{\Bigg|\prod_{s\geq 2}\Bigg(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\Bigg)\Bigg|\Bigg\}$$

$$\leq \frac{F^{j_1}}{c_\Psi^{j_1}}\mathbb{E}\Bigg\{\Bigg|\prod_{s\geq 2}\Bigg(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\Bigg)\Bigg|\Bigg\}$$

$$\leq \frac{F^{j_1}}{c_\Psi^{j_1}}\mathbb{E}\left\{\left|\prod_{s\geq 2,\ j_s\geq 2}\left(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\right)\prod_{s\geq 2,\ j_s=1}\left(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\right)\right|\right\}$$

(therefore for the same reasons as previously, there exists $K_5 > 0$ such that)

$$\leq K_6\mathbb{E}\left\{\left|\prod_{s\geq 2,\ j_s=1}\left(\frac{\sum_{t=1}^{n_m}(X_{m,t})^{j_s}}{\Psi(n_m)^{j_s}}\right)\right|\right\}$$

$$\leq K_6\mathbb{E}\left\{\left|\frac{\sum_{t=1}^{n_m}X_{m,t}}{\Psi(n_m)}\right|^{k_1}\right\}$$

(then, because $k_1 \leq H^* \leq H$ where H is even)

$$\leq K_6\mathbb{E}\left\{\left(\frac{\sum_{t=1}^{n_m}X_{m,t}}{\Psi(n_m)}\right)^H\right\}^{\frac{k_1}{H}} \leq K_6 B_H^{n_m}\ . \blacksquare$$

Lemma 3.6.2 *Let $k \in \mathbb{N}^*$, $k \geq 2$. We define H^* by $H^* = k-1$. Then, we define H by $H = H^*$ if H^* is even and $H = H^*+1$ if not. There exists $K_7 > 0$ such that*

$$\left|\mathbb{E}\left\{\frac{\sum_{t_1\neq t_2\neq \ldots \neq t_r, t_{r+1},\ldots,t_k}\left(X_{m,t_1}^2 X_{m,t_2}X_{m,t_3}\ldots X_{m,t_k}\right)}{\Psi(n_m)^{k+1}}\right\}\right| \leq K_7.B_H^{n_m}\ .$$

Proof By lemma 3.6.1, this result holds for r=1. Now suppose that it holds for all $r' \leq r-1$. Then, it is enough to apply lemma 3.2.2, 3.3.4, 3.2.4 and 3.6.1. $\blacksquare$

3.6.2 Fourth proposition about bounded moments

Remark that if $|X_{m,t}| \leq F$, the hypothesis 3.1.2 holds.

Proposition 3.6.1 *One supposes that there exists $F > 0$ such that $|X_{m,t}| \leq F$.*

All the moments $M_q^{n_m} = \mathbb{E}\left\{\frac{(X_{m,1}+X_{m,2}+\ldots+X_{m,n_m})^q}{\Psi(n)^q}\right\}$ are bounded by a real $B_q > 0$ if and only if, for all $q \in \mathbb{N}$, there existe $Sb_q^1 \in \mathbb{R}$ such that

$$\left|\sum_{s_1\neq s_2\neq \ldots \neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}\ldots\ldots X_{m,s_q}\}}{\Psi(n)^q}\right| \leq Sb_q^1$$

Proof of the sufficiency condition of proposition 3.6.1 We prove this proposition by recurence. It holds for q=0,1,2. Suppose that it holds for all $q' \leq q-1$. Then, by lemma 3.6.2,

$$\sum_{s_1\neq s_2\neq \ldots \neq s_q}\frac{\mathbb{E}\{\ X_{m,s_1}^2 X_{m,s_2}\ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n)^q}$$

is bounded.

By proof of the sufficient condition of proposition 3.3.1,

$$M_q^{n_m} = \sum_{\mathcal{O}_q=(u_1,...,u_q)\in\mathcal{P}_q} N_{\mathcal{O}_q} \sum_{s_1\neq....\neq s_q} \mathbb{E}\Big\{\frac{X_{m,s_1}^{u_1}....X_{m,s_q}^{u_q}}{\Psi(n_m)^q}\Big\} ,$$

where $N_{\mathcal{O}_q} \in \mathbb{R}$.

Let $\{u_1, u_2, ..., u_k\}$ where $u_1 \geq 3$ or $u_2 \geq 2$. By lemma 3.3.3

$$\mathbb{E}\left[\frac{\sum_{t_1\neq t_2\neq....\neq t_k} X_{m,t_1}^{u_1}....X_{m,t_k}^{u_k}}{\Psi(n_m)^q}\right]$$

is bounded.

By our assumption,

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_q} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\right| \leq Sb_q^1 .$$

Then, $M_q^{n_m}$ is bounded. The sufficient condition is proved.

Proof of the necessity condition of proposition 3.6.1 Now we suppose that all the moments are bounded. Then, it is enough to use proposition 3.3.1. ■

3.7 First Theorem of Convergence

3.7.1 Lemma

We shall need the following lemma.

Lemma 3.7.1 *We suppose that all the moments $M_q^{n_m}$ are bounded. We suppose that $R_1^{m,t}(x) = Q_2^{m,t}(x) = x^2 - \beta_{m,t}$ and that $R_s^{m,t}$ is a polynomial of degree j_s. If $j_s = 0$, one assume that $(1/\Psi(n_m)^2)\sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})$ is bounded. If $j_s \geq 1$, one assume that $R_s^{m,t}(x) = x^{j_s}$. Moreover let $h = \sum_{s=1}^k j'_s$ where $j'_s = j_s$ if $j_s \geq 1$ and $j'_0 = 2$ if $j_s = 0$.*

We assume $\frac{1}{\Psi(n_m)^4}\mathbb{E}\{\big[\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})\big]^2\} \to 0$. Then, for all r, $1 \leq r \leq k$,

$$\frac{1}{\Psi(n_m)^h}.\mathbb{E}\Big\{\sum_{t_1\neq t_2\neq....\neq t_r,t_{r+1},...,t_k} \Big[\prod_{s=1}^k R_s^{m,t_s}(X_{m,t_s})\Big]\Big\}$$

converges to 0.

Proof By the Schwartz inequality and by the Holder Inequality

$$\frac{1}{\Psi(n_m)^{2h}}\mathbb{E}\Big\{\sum_{t_1\ t_2,...,t_k} \Big[\prod_{s=1}^k R_s^{m,t_s}(X_{m,t_s})\Big]\Big\}^2$$

$$= \frac{1}{\Psi(n_m)^{2h}}\mathbb{E}\Big\{\prod_{s=1}^k \Big[\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s})\Big]\Big\}^2$$

$$\leq \mathbb{E}\Big\{\frac{1}{\Psi(n_m)^4}\Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t})\Big)^2\Big\}\mathbb{E}\Big\{\frac{1}{\Psi(n_m)^{2h-4}}\Big[\prod_{s=2}^k\Big(\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s})\Big)\Big]^2\Big\}$$

$$\leq \mathbb{E}\Big\{\frac{1}{\Psi(n_m)^4}\Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t})\Big)^2\Big\}\prod_{s=2}^{k}\mathbb{E}\Big\{\Big[\frac{\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s})}{\Psi(n_m)^{j_s}}\Big]^{2(k-1)}\Big\}^{\frac{1}{k-1}} .$$

By assumption and by hypothesis 3.1.2 , the term on right-hand side is bounded. Moreover,

$$\frac{1}{\Psi(n_m)^4}\mathbb{E}\Big\{\Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t})\Big)^2\Big\} \to 0 .$$

We deduce the lemma if r=1.

When $r > 1$, we prove the result by recurrence by using the lemma 3.2.2, 3.2.4 and 3.4.3. ■

Lemma 3.7.2 *We assume* $\frac{1}{\Psi(n_m)^4}\mathbb{E}\big\{\big[\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})\big]^2\big\} \to 0$. *We suppose that all moments* M_q^n *are bounded. Then,*

$$\sum_{s_1\neq\ldots\neq s_{q-e}} \mathbb{E}\Big\{\frac{X_{m,s_1}^2\ldots\ldots X_{m,s_e}^2 X_{m,s_{e+1}}\ldots X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\} - \sum_{s_1\neq\ldots\neq s_{q-e}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\beta_{m,s_e} X_{m,s_{e+1}}\ldots X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\} \to 0 .$$

Proof For example,

$$\sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{X_{m,s_1}^2 X_{m,s_2}^2 X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{(X_{m,s_1}^2-\beta_{m,s_1}) X_{m,s_2}^2 X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\}$$
$$+ \sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1} X_{m,s_2}^2 X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\}$$

$$= \sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{(X_{m,s_1}^2-\beta_{m,s_1}) X_{m,s_2}^2 X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\}$$
$$+ \sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}(X_{m,s_2}^2-\beta_{m,s_2}) X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\}$$
$$+ \sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,s_2} X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\} .$$

Then for example by lemma 3.7.1

$$\sum_{s_1\neq\ldots\neq s_{q-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}(X_{m,s_2}^2-\beta_{m,s_2}) X_{m,s_3}\ldots X_{m,s_{q-2}}}{\Psi(n_m)^q}\Big\} \to 0 . ■$$

Lemma 3.7.3 *Let $q \in \mathbb{N}$ and $e \in \mathbb{N}$ such that $1 \leq e \leq q-e$. We suppose that all the moments $M_h^{n_m}$ are bounded. We suppose that, for all $q' \leq q-1$,*

$$\sum_{s_1 \neq s_2 \neq \ldots \neq s_{q'}} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_{q'}}\}}{\Psi(n_m)^{q'}} \to S_{q'} \ .$$

We set $S_0 = 1$. Then,

$$\sum_{s_1 \neq \ldots \neq s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_1} \beta_{m,s_2} \ldots\ldots \beta_{m,s_e} X_{m,s_{e+1}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\} \to \sigma_0^{2e} S_{q-2e} \ .$$

Proof We set

$$\{s_1 \neq \ldots \neq s_e, s_{e+1} \neq \ldots \neq \ldots s_{q-e}\}$$
$$= \big\{(s_1, s_2, \ldots, s_e, s_{e+1}, .., s_{q-e}) \in \{1, 2, \ldots, n_m\}^{q-e} \ \big| \ s_r \neq s_{r'} \ if \ r < r' \leq e \ or \ e+1 \leq r < r' \leq q-e\big\}.$$

Then, we study at first

$$\sum_{s_1 \neq \ldots \neq s_e, s_{e+1} \neq \ldots \neq s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_1} \beta_{m,s_2} \ldots \beta_{m,s_e} X_{m,s_{e+1}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\}$$
$$= \sum_{s_1 \neq \ldots \neq s_e} \sum_{s_{e+1} \neq \ldots \neq s_{q-e}} \Big(\frac{\beta_{m,s_1} \beta_{m,s_2} \ldots \beta_{m,s_e}}{\Psi(n_m)^{2e}} \Big) \mathbb{E}\Big\{ \frac{X_{m,s_{e+1}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^{q-2e}} \Big\} \ .$$

It is easy to understand that

$$\sum_{s_1 \neq \ldots \neq s_e} ` \Big(\frac{\beta_{m,s_1} \beta_{m,s_2} \ldots \beta_{m,s_e}}{\Psi(n_m)^{2e}} \Big) \to \sigma_0^{2e} \ .$$

Indeed,

$$\sum_{s_1, \ldots, s_e} ` \Big(\frac{\beta_{m,s_1} \beta_{m,s_2} \ldots \beta_{m,s_e}}{\Psi(n_m)^{2e}} \Big) = \Big(\frac{\sum_{s_1} \beta_{m,s_1}}{\Psi(n_m)^2} \Big) \ldots\ldots \Big(\frac{\sum_{s_e} \beta_{m,s_e}}{\Psi(n_m)^2} \Big) \to \sigma_0^{2e} \ ,$$

and, for example, by lemma 3.4.2, $\frac{\sum_{s_1} \beta_{m,s_1}^2}{\Psi(n_m)^4} \to 0$. Then it is enough to use lemma 3.2.2 and 3.2.4.

Therefore,

$$\sum_{s_1 \neq \ldots \neq s_e, s_{e+1} \neq \ldots \neq \ldots s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_1} \beta_{m,s_2} \ldots \beta_{m,s_e} X_{m,s_{e+1}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\} \to \sigma_0^{2e} S_{q-2e} \ .$$

Now, in order to obtain $\sum_{s_1 \neq \ldots \neq s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_1} \ldots \beta_{m,s_e} X_{m,s_{e+1}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\}$, we shall have to add or to subtract sums such that there exists i, j satisfying $s_i = s_{e+j}$. For example, we shall have to subtract terms of the form

$$\sum_{s_2 \neq \ldots \neq s_e, s_{e+1} \neq \ldots \neq \ldots s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_2} \ldots \beta_{m,s_e} [\beta_{m,s_{e+1}} X_{m,s_{e+1}}] X_{m,s_{e+2}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\} \ .$$

One will prove that this sum converges to 0 by recurence. With this aim, we study first

$$\sum_{s_2 \ldots, s_e, s_{e+1}, \ldots, s_{q-e}} \mathbb{E}\Big\{ \frac{\beta_{m,s_2} \ldots \beta_{m,s_e} \beta_{m,s_{e+1}} X_{m,s_{e+1}} X_{m,s_{e+2}} \ldots X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\}$$

which can be written as

$$\Big(\frac{\sum_{s_2}\beta_{m,s_2}}{\Psi(n_m)^2}\Big).....\Big(\frac{\sum_{s_e}\beta_{m,s_e}}{\Psi(n_m)^2}\Big)\sum_{s_{e+1},....,s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}$$

Now, $\frac{\sum_{s_2}\beta_{m,s_2}}{\Psi(n_m)^2}$ is bounded and $\sum_{s_{e+1},....,s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}\to 0$ is bounded by lemma 3.4.4.

It is general for all the steps of recurrence: thanks to lemma 3.2.2 and 3.2.4, one can always write the terms

$$\sum_{s_2\neq....\neq s_r,s_{e+1}\neq...\neq s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_2}....\beta_{m,s_e}[\beta_{m,s_{e+1}}X_{m,s_{e+1}}]X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}$$

as a sum of products. Or by using lemma 3.4.4, we always find in these products, terms which are bounded by products of bounded expectations (cf lemma 3.3.1 and hypothesis 3.1.2) in the form $\mathbb{E}\Big\{\Big(\frac{\sum_{s_1}\beta^u_{m,s}X^v_{m,s}}{\Psi(n_m)^{2u+v}}\Big)^{2w}\Big\}$, $u,v,w\in\mathbb{N}$: among these terms there is at least one term of the form $\sum_{s_{e+1}}\frac{\beta^u_{m,s_{e+1}}X^v_{m,s_{e+1}}}{\Psi(n_m)^{2u+v}}$ which converges to 0. That proves the recurence, and then, that

$$\sum_{s_2\neq....\neq s_e,s_{e+1}\neq....\neq...s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_2}....\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}\to 0\ .$$

We deduce that

$$\sum_{s_1\neq.....\neq s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,s_2}....\beta_{m,s_e}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}$$

converges to the same limit as

$$\sum_{s_1\neq....\neq s_e,s_{e+1}\neq....\neq...s_{q-e}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,s_2}....\beta_{m,s_e}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}}{\Psi(n_m)^q}\Big\}\ ,$$

i.e. as

$$\Big(\frac{\sum_{s_1}\beta_{m,s_1}}{\Psi(n_m)^2}\Big).....\Big(\frac{\sum_{s_e}\beta_{m,s_e}}{\Psi(n_m)^2}\Big)\mathbb{E}\Big\{\Big(\sum_{s_{e+1}\neq....\neq s_{q-e}}\frac{X_{m,s_{e+1}}....X_{m,s_{q-e}}}{\Psi(n_m)^{q-2e}}\Big)\Big\}\to\sigma_0^{2e}S_{q-2e}\ .\ \blacksquare$$

3.7.2 First theorem of convergence.

Theorem 12 *One assumes that*

$$\mathbb{E}\Bigg\{\Bigg[\frac{\sum_{t=1}^{n_m}\big[(X_{m,t})^2-\mathbb{E}\{(X_{m,t})^2\}\big]}{\Psi(n_m)^2}\Bigg]^2\Bigg\}\to 0\ .$$

All the moments $M_q^{n_m}=\mathbb{E}\Big\{\frac{(X_{m,1}+X_{m,2}+....+X_{m,n})^q}{\Psi(n_m)^q}\Big\}$ *converges to a real* M_q *if and only if, for all* $q\in\mathbb{N}$, *there exists* $S_q\in\mathbb{R}$ *and* $Sb_q^2\in\mathbb{R}_+$ *:*

$$\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\to S_q$$

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2 X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq S b_q^2 \ .$$

Proof of sufficient Condition of theorem 12 By proposition 3.3.1, all the moments are bounded.

Thanks to lemma 3.7.3 and 3.7.2, we deduce that, for all q and all e,

$$\sum_{s_1 \neq \ldots. \neq s_{q-e}} \mathbb{E}\Big\{ \frac{X_{m,s_1}^2 X_{m,s_2}^2 \ldots. X_{m,s_e}^2 X_{m,s_{e+1}} \ldots. X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\} \to \sigma_0^{2e} S_{q-2e} \ .$$

By lemma 3.3.5 , $\sum_{s_1 \neq \ldots. \neq s_k} \mathbb{E}\Big\{ \frac{X_{m,s_1}^{u_1} \ldots. X_{m,s_k}^{u_k}}{\Psi(n_m)^h} \Big\} \to 0$ if there exist t such that $u_t \geq 3$. Therefore $\sum_{s_1 \neq \ldots. \neq s_k} \mathbb{E}\Big\{ \frac{X_{m,s_1}^{u_1} \ldots. X_{m,s_k}^{u_k}}{\Psi(n_m)^h} \Big\}$ converges for all $(u_1, ..., u_k)$.

Therefore, because

$$\mathbb{E}\Big\{ \frac{(X_{m,1} + \ldots. + X_{m,n})^q}{\psi(n_m)^q} \Big\} = \sum_{j_1 + \ldots\ldots + j_{n_m} = q} \frac{q!}{j_1! \ldots\ldots. j_{n_m}!} \mathbb{E}\Big\{ \frac{X_{m,1}^{j_1} \ldots\ldots X_{m,n_m}^{j_{n_m}}}{\psi(n_m)^q} \Big\} \ ,$$

all the moments converge ■

Proof of Necessary Condition of theorem 12 We suppose that all the moments converge. Then, the second relation is a consequence of proposition 3.3.1 .

Then, we prove the first relation by recurrence. If q= 1 or 2, it is obvious. Then, we suppose that it holds for all $q' \leq q-1$.

We deduce from lemma 3.7.3 and 3.7.2 that, for all $e \geq 1$,

$$\sum_{s_1 \neq \ldots. \neq s_{q-e}} \mathbb{E}\Big\{ \frac{X_{m,s_1}^2 X_{m,s_2}^2 \ldots. X_{m,s_e}^2 X_{m,s_{e+1}} \ldots. X_{m,s_{q-e}}}{\Psi(n_m)^q} \Big\} \to \sigma_0^{2e} S_{q-2e} \ .$$

By lemma 3.3.5 , $\sum_{s_1 \neq \ldots. \neq s_k} \mathbb{E}\Big\{ \frac{X_{m,s_1}^{u_1} \ldots. X_{m,s_k}^{u_k}}{\Psi(n_m)^h} \Big\} \to 0$ if there exist t such that $u_t \geq 3$. Therefore $\sum_{s_1 \neq \ldots. \neq s_k} \mathbb{E}\Big\{ \frac{X_{m,s_1}^{u_1} \ldots. X_{m,s_k}^{u_k}}{\Psi(n_m)^h} \Big\}$ converges for all $(u_1, ..., u_k)$ such that there exists $u_t \geq 2$.

Therefore, because

$$\mathbb{E}\Big\{ \frac{(X_{m,1} + \ldots. + X_{m,n})^q}{\psi(n_m)^q} \Big\} = \sum_{j_1 + \ldots\ldots + j_{n_m} = q} \frac{q!}{j_1! \ldots\ldots. j_{n_m}!} \mathbb{E}\Big\{ \frac{X_{m,1}^{j_1} \ldots\ldots X_{m,n_m}^{j_{n_m}}}{\psi(n_m)^q} \Big\} \ ,$$

then, $\sum_{s_1 \neq \ldots. \neq s_q} \mathbb{E}\Big\{ \frac{X_{m,s_1} \ldots. X_{m,s_q}}{\Psi(n_m)^q} \Big\}$ converges. ■

3.8 Convergence to the normal distribution

One can specify theorem 12 in the case of convergence to the normal distribution.

Proposition 3.8.1 *One assumes that*

$$\mathbb{E}\Big\{\Big[\frac{\sum_{t=1}^{n_m}(\big[(X_{m,t})^2-\mathbb{E}\{(X_{m,t})^2\}\big]}{\Psi(n_m)^2}\Big]^2\Big\}\to 0\ .$$

The moments of order q, $M_q^{n_m}$ converges to the moment of order q of $N(0,\sigma_0^2+S_2)$ where $S_2\in\mathbb{R}$, if and only if, for all $q\in\mathbb{N}^*$,

$$\sum_{s_1\neq s_2\neq .....\neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}\}}{\Psi(n_m)^q}\to S_q\ ,$$

where $S_q=\nu_q$, the moment of order q of $N(0,S_2)$ for $q\geq 1$ and if there exists $Sb_q^2>0$ such that, for all $q\in\mathbb{N}^*$,

$$\Big|\sum_{s_1\neq s_2\neq .....\neq s_{q-1}}\frac{\mathbb{E}\{(X_{m,s_1})^2X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\Big|\leq Sb_q^2\ .$$

3.8.1 Proof

We know that

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}=\sum_{j_1+.....+j_{n_m}=q}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}\ .$$

Now, because, by theorem 12 , all the moments converges, then, by lemma 3.3.5, if there exists s such that $j_s\geq 3$,

$$\sum_{j_1+.....+j_n=q,\ at\ least\ one\ j_s\geq 3}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}\to 0\ .$$

Then, it is enough to study the sums where $j_s\leq 2$.

Then, we have the following lemma

Lemma 3.8.1 *We set $S_0=1$. Let q" $=\lfloor q/2\rfloor$, the integer part of q/2. Then,*

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}$$

$$\sim\frac{q!}{1}\frac{\sigma_0^0}{0!}\frac{S_q}{(q)!}+\frac{q!}{2}\frac{\sigma_0^2}{1!}\frac{S_{q-2}}{(q-2)!}+\frac{q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_{q-4}}{(q-4)!}+.....+\frac{q!}{2^{q"}}\frac{\sigma_0^4}{q"!}\frac{S_{q-2q"}}{(q-2q")!}\ .\ \blacksquare$$

Proof We have

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q}{\Psi(n_m)^q}\Big\}=\sum_{j_1+.....+j_{n_m}=q}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$\sim\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 1}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$+\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ one\ j_s=2}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$+\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ two\ j_s=2}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

.....

.....

$$+\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ q"\ j_s=2}\frac{q!}{j_1!.....j_{n_m}!}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$\sim\frac{q!}{1}\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 1}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$+\frac{q!}{2}\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ one\ j_s=2}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$+\frac{q!}{2^2}\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ two\ j_s=2}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

.....

.....

$$+\frac{q!}{2^{q"}}\sum_{j_1+.....+j_{n_m}=q,\ j_s\leq 2,\ q"\ j_s=2}\mathbb{E}\Big\{\frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}}}{\Psi(n_m)^q}\Big\}$$

$$\sim\frac{q!}{1}\frac{\sigma_0^0}{0!}\frac{S_q}{(q)!}+\frac{q!}{2}\frac{\sigma_0^2}{1!}\frac{S_{q-2}}{(q-2)!}+\frac{q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_{q-4}}{(q-4)!}+.....+\frac{q!}{2^{q"}}\frac{\sigma_0^{2q"}}{q"!}\frac{S_{q-2q"}}{(q-2q")!}\ ,$$

by lemma 3.7.2, 3.7.3 and 3.2.9. ■

Proof of sufficient condition If q is odd, q-2e is odd and $S_{q-2e}=0$. Then, by lemma 3.8.1 $M_q^{n_m}\to 0$.

Now, we study the case where q is even. Then, by lemma 3.8.1 and because $\nu_{2q}=\frac{\nu_2^q(2q)!}{2^q q!}$,

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^{2q}}{\Psi(n_m)^{2q}}\Big\}$$

$$\sim\frac{2q!}{1}\frac{\sigma_0^0}{0!}\frac{S_{2q}}{(2q)!}+\frac{2q!}{2}\frac{\sigma_0^2}{1!}\frac{S_{2q-2}}{(2q-2)!}+\frac{2q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_{2q-4}}{(2q-4)!}+.....+\frac{2q!}{2^q}\frac{\sigma_0^{2q}}{q!}\frac{S_0}{0!}$$

$$\sim\frac{2q!}{1}\frac{\sigma_0^0}{0!}\frac{S_2^q(2q)!}{(2q)!2^q q!}+\frac{2q!}{2}\frac{\sigma_0^2}{1!}\frac{S_2^{q-1}(2q-2)!}{(2q-2)!2^{q-1}(q-1)!}+\frac{2q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_2^{q-2}(2q-4)!}{(2q-4)!2^{q-2}(q-2)!}+.....+\frac{2q!}{2^q}\frac{\sigma_0^{2q}}{q!}\frac{S_2^{q-q}(2q-2q)!}{0!2^{q-q}(q-q)!}$$

$$\sim \frac{2q!}{1}\frac{\sigma_0^0}{0!}\frac{S_2^q}{2^q q!} + \frac{2q!}{2}\frac{\sigma_0^2}{1!}\frac{S_2^{q-1}}{2^{q-1}(q-1)!} + \frac{2q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_2^{q-2}}{2^{q-2}(q-2)!} + ... + \frac{2q!}{2^q}\frac{\sigma_0^{2q}}{q!}\frac{S_2^{q-q}}{2^{q-q}(q-q)!}$$

$$\sim \frac{2q!}{2^q}\Big[\frac{S_2^q}{0!q!} + \frac{\sigma_0^2 S_2^{q-1}}{1!(q-1)!} + \frac{\sigma_0^4 S_2^{q-2}}{2!(q-2)!} + ... + \frac{\sigma_0^{2q} S_2^0}{q!0!}\Big]$$

$$\sim \frac{2q!}{q!2^q}\ (\sigma_0^2 + S_2)^q$$

which is the moment of order 2q of $N(0, \sigma_0^2 + S_2)$. ■

Proof of necessary condition We suppose that the moments $M_q^{n_m}$ converge to M_q which is the moment of order q of $N(0, M_2)$. Then, we prove by recurence that S_q is the moment of order q of $N(0, S_2)$. Indeed, it is true if q=1. If q=2,

$$\mathbb{E}\Big\{\frac{(X_{m,1} + ... + X_{m,n_m})^2}{\Psi(n_m)^2}\Big\} = \sum_s \mathbb{E}\Big\{\frac{X_{m,s}^2}{\Psi(n_m)^2}\Big\} + \sum_{s\neq t}\mathbb{E}\Big\{\frac{X_{m,s}X_{m,t}}{\Psi(n_m)^2}\Big\} \to \sigma_0^2 + S_2 = M_2\ .$$

Then, we suppose that $S_{q'}$ is the moment of order q' of $N(0, S_2)$ if $q' \leq 2q-2$. By lemma 3.8.1, if q is even,

$$M_{2q} = \frac{M_2^q(2q!)}{2^q q!} = \frac{(S_2+\sigma_0^2)^q(2q!)}{2^q q!} \sim \mathbb{E}\Big\{\frac{(X_{m,1} + ... + X_{m,n_m})^{2q}}{\Psi(n_m)^{2q}}\Big\}$$

$$\sim \frac{2q!}{1}\frac{\sigma_0^0}{0!}\frac{S_{2q}}{(2q)!} + \frac{2q!}{2}\frac{\sigma_0^2}{1!}\frac{S_{2q-2}}{(2q-2)!} + \frac{2q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_{2q-4}}{(2q-4)!} + ... + \frac{2q!}{2^q}\frac{\sigma_0^{2q}}{q!}\frac{S_0}{0!}$$

$$\sim \frac{2q!}{1}\frac{\sigma_0^0}{0!}\frac{S_{2q}}{(2q)!} + \frac{2q!}{2}\frac{\sigma_0^2}{1!}\frac{S_2^{q-1}}{2^{q-1}(q-1)!} + \frac{2q!}{2^2}\frac{\sigma_0^4}{2!}\frac{S_2^{q-2}}{2^{q-2}(q-2)!} + ... + \frac{2q!}{2^q}\frac{\sigma_0^{2q}}{q!}\frac{S_2^{q-q}}{2^{q-q}(q-q)!}$$

$$\sim S_{2q} + \frac{2q!}{2^q}\Big[\frac{\sigma_0^2 S_2^{q-1}}{1!(q-1)!} + \frac{\sigma_0^4 S_2^{q-2}}{2!(q-2)!} + ... + \frac{\sigma_0^{2q} S_2^0}{q!0!}\Big]\ .$$

Therefore

$$(S_2+\sigma_0^2)^q = \frac{S_{2q}2^q q!}{(2q!)} + q!\Big[\frac{\sigma_0^2 S_2^{q-1}}{1!(q-1)!} + \frac{\sigma_0^4 S_2^{q-2}}{2!(q-2)!} + ... + \frac{\sigma_0^{2q} S_2^0}{q!0!}\Big]\ .$$

Therefore,

$$(S_2+\sigma_0^2)^q = \frac{S_{2q}2^q q!}{(2q!)} + q!\Big[\frac{\sigma_0^0 S_2^q}{q!} + \frac{\sigma_0^2 S_2^{q-1}}{1!(q-1)!} + \frac{\sigma_0^4 S_2^{q-2}}{2!(q-2)!} + ... + \frac{\sigma_0^{2q} S_2^0}{q!0!}\Big] - S_2^q\ .$$

Therefore, $\frac{S_{2q}2^q q!}{(2q!)} = S_2^q$ and $S_{2q} = S_2^q\frac{S_2^q(2q!)}{2^q q!}$.

If q is odd, this result is easier to prove because $S_{2q+1} = 0$ and $M_{2q+1} = 0$. ■

3.9 Second theorem of convergence

In this section, we assume that that $\sum_{s=1}^{n_m} \frac{\gamma_{m,s}^2}{\Psi(n)^2}$ is bounded.

Moreover, we suppose that $\mathbb{E}\big\{\big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s}X_{m,s}}{\Psi(n)^2}\big)^2\big\} \to 0$. For example, it is an hypothesis which holds as soon as $\gamma_{m,s}$ is bounded, as $\mathbb{E}\big\{X_{m,s}^2\big\}$ is bounded and as $|\mathbb{E}\big\{X_{m,s}X_{m,t}\big\}| \leq \alpha(s-t)$ where $\alpha(h) \to 0$ when $h \to \infty$

3.9.1 Lemma

Lemma 3.9.1 *We suppose that* $\mathbb{E}\big\{\big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s}X_{m,s}}{\Psi(n)^2}\big)^2\big\}$ *is bounded. Then*

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\} \to 0 \ .$$

if and only if

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\} \to 0 \ .$$

Proof We know that $\tilde{P}_2^{m,s_1}(x) = x^2 - \gamma_{m,s_1}x - \beta_{m,s_1}$ where $\gamma_{m,s_1} = \mathbb{E}\{X_{m,s_1}^3\}/\mathbb{E}\{X_{m,s_1}^2\}$. Then,

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\}$$

$$= \mathbb{E}\left\{\left[\frac{\sum_{s=1}^{n_m}(X_{m,s}^2 - \gamma_{m,s}X_{m,s} - \beta_{m,s})}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \gamma_{m,t}X_{m,t} - \beta_{m,t})}{\Psi(n_m)^2}\right]\right\}$$

$$= \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2 - 2\left[\frac{\sum_{s=1}^{n_m}\gamma_{m,s}X_{m,s}}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right] + \left[\frac{\sum_{s=1}^{n_m}\gamma_{m,s}X_{m,s}}{\Psi(n_m)^2}\right]^2\right\} .$$

By our assumption

$$\mathbb{E}\left\{\left[\frac{\sum_{s=1}^{n_m}\gamma_{m,s}X_{m,s}}{\Psi(n_m)^2}\right]^2\right\} \to 0 \ .$$

Therefore, by Schwartz inequality,

$$\mathbb{E}\left\{\left[\frac{\sum_{s=1}^{n_m}\gamma_{m,s}X_{m,s}}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]\right\}^2$$

$$\leq \mathbb{E}\left\{\left[\frac{\sum_{s=1}^{n_m}\gamma_{m,s}X_{m,s}}{\Psi(n_m)^2}\right]^2\right\}\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]^2\right\}$$

where

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]^2\right\} = \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m}(X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]\right\}$$

$$= \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} X_{m,t}^2}{\Psi(n_m)^2}\right]^2\right\} - 2\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m} X_{m,t}^2}{\Psi(n_m)^2}\right]\right\} + \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right]^2\right\}$$

$$= \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} X_{m,t}^2}{\Psi(n_m)^2}\right]^2\right\} - 2\left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right]\mathbb{E}\left\{\frac{\sum_{t=1}^{n_m} X_{m,t}^2}{\Psi(n_m)^2}\right\} + \left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right]^2$$

$$\leq C(2,2) + 2C(2,1)\left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right] + \left[\frac{\sum_{t=1}^{n_m} \beta_{m,t}}{\Psi(n_m)^2}\right]^2$$

which is bounded by hypothesis 3.1.3.

Therefore

$$\mathbb{E}\left\{\left[\frac{\sum_{s=1}^{n_m} \gamma_{m,s} X_{m,s}}{\Psi(n_m)^2}\right]\left[\frac{\sum_{t=1}^{n_m} (X_{m,t}^2 - \beta_{m,t})}{\Psi(n_m)^2}\right]\right\}^2 \to 0.$$

We deduce that

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\} \to 0$$

is equivalent to

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\} \to 0 \ . \ \blacksquare$$

3.9.2 Statement and proof of the theorem

This second theorem is expressed in terms of $\rho_{j_1,\ldots,j_n}$.

Theorem 13 *We assume that that* $\sum_{s=1}^{n_m} \frac{\gamma_{m,s}^2}{\Psi(n)^2}$ *is bounded and that* $\mathbb{E}\{\big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s} X_{m,s}}{\Psi(n)^2}\big)^2\} \to 0$. *One assumes that*

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\} \to 0 \ .$$

All the moments $M_q^{n_m} = \mathbb{E}\left\{\frac{(X_{m,1}+X_{m,2}+\ldots+X_{m,n_m})^q}{\Psi(n_m)^q}\right\}$ *converge to a real* M_q *if and only if, for all* $q \in \mathbb{N}$, *there exists* $S_q \in \mathbb{R}$ *and* $Sb_q^r \in \mathbb{R}$ *, r=2,3, such that*

$$\sum_{s_1 \neq s_2 \neq \ldots \neq s_q} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} \ldots\ldots X_{m,s_q}\}}{\Psi(n)^q} \to S_q$$

$$\left|\sum_{s_1 \neq s_2 \neq \ldots \neq s_{q-1}} \frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1}) X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n)^q}\right| \leq Sb_q^2 \ ,$$

$$\left|\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\gamma_{m,s_1} \mathbb{E}\{X_{s_1} X_{s_2} \ldots\ldots X_{s_q}\}}{\Psi(n)^{q+1}}\right| \leq Sb_q^3 \ .$$

Proof of the sufficiency condition of theorem 13 By lemma 3.9.1,

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}\tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\}\to 0\ .$$

is equivalent to

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\}\to 0\ .$$

By proposition 3.5.1 , all the moments are bounded. Then, by proposition 3.3.1

$$\left|\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_{q-1}}\frac{\mathbb{E}\{(X_{m,s_1})^2X_{m,s_2}\ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2\ .$$

Then, by theorem 12, all the moments converge.

Proof of the necessity condition of theorem 13 Now we suppose that all the moments converge. Then, by proposition 3.5.1 ,

$$\left|\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}\ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n)^q}\right|\leq Sb_q^2\ ,$$

$$\left|\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_q}\frac{\gamma_{m,s_1}\mathbb{E}\{X_{s_1}X_{s_2}\ldots\ldots X_{s_q}\}}{\Psi(n)^{q+1}}\right|\leq Sb_q^3\ .$$

Moreover, by theorem 12,

$$\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}\ldots\ldots X_{m,s_q}\}}{\Psi(n)^q}\to S_q\ .\ \blacksquare$$

By using proposition 3.8.1, it is easy to deduce the following theorem.

Corollary 3.9.1 *One assumes that*

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}(\tilde{P}_2^{m,t}(X_{m,t}))}{\Psi(n_m)^2}\right]^2\right\}\to 0\ .$$

All the moments of order q $M_q^{n_m}$ converge to M_q, the moment of order q of $N(0,\sigma_0^2+S_2)$, $S_2\in\mathbb{R}$, if and only if, for all $q\in\mathbb{N}$,

$$\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_q}\frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}\ldots\ldots X_{m,s_q}\}}{\Psi(n_m)^q}\to S_q$$

where $S_q=\nu_q$ the moment of order q of $N(0,S_2)$,

$$\left|\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}\ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n)^q}\right|\leq Sb_q^2\ ,$$

$$\left|\sum_{s_1\neq s_2\neq\ldots\ldots\neq s_q}\frac{\gamma_{m,s_1}\mathbb{E}\{X_{s_1}X_{s_2}\ldots\ldots X_{s_q}\}}{\Psi(n)^{q+1}}\right|\leq Sb_q^3\ .$$

3.10 Third theorem of Convergence

Theorem 14 *One supposes that there exists $F > 0$ such that $|X_{m,t}| \leq F$. One assumes that*

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M_q^{n_m} = \mathbb{E}\left\{ \frac{(X_{m,1}+X_{m,2}+....+X_{m,n_m})^q}{\Psi(n_m)^q} \right\}$ converges to a real M_q if and only if, for all $q \in \mathbb{N}$, there existe $S_q \in \mathbb{R}$ such that

$$\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} ..... X_{m,s_q}\}}{\Psi(n)^q} \to S_q \ .$$

Proof of the sufficient condition of theorem 14 By proposition 3.6.1, all the moments $M_q^{n_m}$ are bounded. By proposition 3.3.1,

$$\left| \sum_{s_1 \neq s_2 \neq ..... \neq s_{q-1}} \frac{\mathbb{E}\{X_{m,s_1}^2 X_{m,s_2} ..... X_{m,s_{q-1}}\}}{\Psi(n)^q} \right|$$

is bounded. By theorem 12, all the moments $M_q^{n_m}$ converges.

Proof of the Necessary condition of theorem 14 By theorem 12,

$$\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} ..... X_{m,s_q}\}}{\Psi(n)^q} \to S_q \ .$$

By using proposition 3.8.1, it is easy to deduce the following theorem.

Corollary 3.10.1 *We suppose that there exists $F > 0$ such that $|X_{m,t}| \leq F$.*
One assumes that

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M_q^{n_m} = \mathbb{E}\left\{ \frac{(X_{m,1}+X_{m,2}+....+X_{m,n_m})^q}{\Psi(n_m)^q} \right\}$ converge to the moment of order q of $N(0, \sigma_0^2 + S_2)$, $S_2 \in \mathbb{R}$, if and only if, for all $q \in \mathbb{N}^*$,

$$\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} ..... X_{m,s_q}\}}{\Psi(n_m)^q} \to S_q \ ,$$

where $S_q = \nu_q$, the moment of order q of $N(0, S_2)$.

Chapter 4

MCLT in dimension 2

4.1 Notations and assumptions

We use again notation introduced in the previous chapter. Moreover, they are completed by the following way.

Notations 4.1.1 *Let $(X_{m,s}, Y_{m,s}) \in \mathbb{R}^2$, $s = 1, 2, ..., n_m$, m=1,2,.. be a triangular array of random vectors defined on a probability space $(\Omega, \mathcal{A}, P)$ such that $n_m \to \infty$.*

One supposes that $\mathbb{E}\{X_{m,s}\} = \mathbb{E}\{Y_{m,s}\} = 0$ and $|\mathbb{E}\{(X_{m,s})^p\}| < \infty$ and $|\mathbb{E}\{(Y_{m,s})^p\}| < \infty$ for all $p \in \mathbb{N}$.

Hypothesis 4.1.1 *One keeps the notations of hypothesis 3.1.2. Then, we suppose that, for all $p \in \mathbb{N}^*$, for all $j \geq 2$*

$$\mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n_m} (X_{m,t})^j}{\Psi(n_m)^j} \right|^p \right\} \leq C_{n_m}(j,p) \leq C(j,p) \ .$$

$$\mathbb{E}\left\{ \left| \frac{\sum_{t=1}^{n_m} (Y_{m,t})^j}{\Psi(n_m)^j} \right|^p \right\} \leq C_{n_m}(j,p) \leq C(j,p) \ .$$

Hypothesis 4.1.2 *One supposes that there exists σ_0^2 and σ_1^2 such that $(1/\Psi(n)^2) \sum_s \mathbb{E}\{X_{m,s}^2\} \to \sigma_0^2$ and $(1/\Psi(n)^2) \sum_s \mathbb{E}\{Y_{m,s}^2\} \to \sigma_1^2$. We set $\beta'_{m,s} = \mathbb{E}\{Y_{m,s}^2\}$.*

In the case of random vectors, higher order correllation coefficients are defined by the same way.

Notations 4.1.2 *For all $n, n' \in \mathbb{N}^*$, for all $(j_1, j_2, ..., j_n) \in \mathbb{N}^n$, for all $(i_1, i_2, ..., i_{n'}) \in \mathbb{N}^{n'}$, we set*

$$\rho_{j_1,j_2,...,j_n,i_1,i_2,...,i_{n'}}$$
$$= \mathbb{E}\{P_{j_1}^{m,1}(X_{m,1}) P_{j_2}^{m,2}(X_{m,2}) .... P_{j_n}^{m,n}(X_{m,n}) P_{i_1}'^{m,1}(Y_{m,1}) P_{i_2}'^{m,2}(Y_{m,2}) .... P_{i_{n'}}'^{m,n'}(Y_{m,n'})\} \ ,$$

where $\{P_j'^{m,t}\}$, $j \in \mathbb{N}$, is the family of orthonormal polynomials associated to $Y_{m,t}$.

4.2 First theorem of Convergence

4.2.1 lemmas about sets

One generalizes easily the lemmas to the case with two dimensions by using natural notations. Then, the following lemma holds.

Lemma 4.2.1 *Let $r \geq 3$. Then, for all $k \leq h$ and all $r' \leq r-1$,*

$$\Big\{t_1 \neq t_2 \neq .... \neq t_{r-1}, t_r, ..., t_k,\ s_1 \neq s_2 \neq .... \neq s_{r'}, s_{r'+1}, ..., s_{k'}\Big\}$$

$$= \Big\{t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_k,\ s_1 \neq s_2 \neq .... \neq s_{r'}, s_{r'+1}, ..., s_{k'}\Big\}$$

$$\cup\Big\{t_1 = t_r \neq t_2 \neq ... \neq t_{r-1}, t_{r+1}, .., t_k,\ s_1 \neq s_2 \neq .... \neq s_{r'}, s_{r'+1}, ..., s_{k'}\Big\}$$

$$\cup\Big\{t_1 \neq t_2 = t_r \neq ... \neq t_{r-1}, t_{r+1}, .., t_k,\ s_1 \neq s_2 \neq .... \neq s_{r'}, s_{r'+1}, ..., s_{k'}\Big\}$$

$$.....$$

$$\cup\Big\{t_1 \neq t_2 \neq ... \neq t_{r-1} = t_r, t_{r+1}, .., t_k,\ s_1 \neq s_2 \neq .... \neq s_{r'}, s_{r'+1}, ..., s_{k'}\Big\}\ .$$

It is enough to sum under the sign $\mathbb{E}$ and to use the proof of lemma 3.2.9 in order to obtain the following propositions.

Lemma 4.2.2 *We simplify $X_{m,t}$ in X_t and $Y_{m,t}$ in Y_t. Let $p, q \in \mathbb{N}^*$. Then,*

$$\sum_{j_1+...+j_n=q,\ j_s\leq 2,\ "k"\ j_s=1,\ "h"\ j_s=2} \ \sum_{i_1+...+i_n=p,\ i_s\leq 2,\ "a"\ i_s=1,\ "b"\ i_s=2} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{k+2h+a+2b}}\Big\}$$

$$= \frac{1}{h!k!a!b!} \sum_{s_1\neq....\neq s_k\neq u_1\neq....\neq u_h} \ \sum_{t_1\neq....\neq t_a\neq w_1\neq....\neq w_b} \mathbb{E}\Big\{\frac{X_{s_1}.....X_{s_k}X_{u_1}^2.....X_{u_h}^2Y_{t_1}.....Y_{t_a}Y_{w_1}^2.....Y_{w_b}^2}{\Psi(n)^{k+2h+a+2b}}\Big\}\ .$$

Lemma 4.2.3 *Let $p, q \in \mathbb{N}^*$. Then,*

$$\sum_{j_1+...+j_n=q,\ "h_t"\ j_s=t,\ t=0,1,..,r} \ \sum_{i_1+...+i_n=p,\ "a_t"\ i_s=t,\ t=0,1,..,o} \mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{p+q}}\Big\}$$

$$= \frac{1}{h_1!...h_r!a_1!...a_o!} \sum_{s_1^t\neq....\neq s_{h_t}^t,\ s_i^t\neq s_j^{t'}} \ \sum_{e_1^t\neq....\neq e_{a_t}^t,\ e_i^t\neq e_j^{t'}} \mathbb{E}\Big\{\frac{\big[\prod_{t=1}^r X_{s_1^t}^t.....X_{s_{h_t}^t}^t\big]\big[\prod_{t=1}^o Y_{e_1^t}^t.....Y_{e_{a_t}^t}^t\big]}{\Psi(n)^{p+q}}\Big\}\ .$$

Lemma 4.2.4 *Let $h = r_1 + ... + r_k$ and $h' = r'_1 + ... + r'_{k'}$ We suppose that all the moments $M_{q,p}^{n_m}$ are bounded. We suppose that there exists $d_s \geq 3$ or $d'_s \geq 3$. Then,*

$$\sum_{u_1\neq u_2\neq....\neq u_r, u_{r+1},...,u_k} \ \sum_{v_1\neq v_2\neq....\neq v_{r'}, v_{r'+1},...,v_{k'}} \mathbb{E}\Big\{\frac{X_{m,u_1}^{d_1}....X_{m,u_k}^{d_k}Y_{m,v_1}^{d'_1}....Y_{m,v_{k'}}^{d'_{k'}}}{\Psi(n)^{h+h'}}\Big\} \to 0\ .$$

Proof This lemma holds if r=r'=1. For example, if $d_1 \geq 3$, by Holder Inequality

$$\Big|\sum_{u_1\neq u_2\neq....\neq u_r, u_{r+1},...,u_k} \ \sum_{v_1\neq v_2\neq....\neq v_{r'}, v_{r'+1},...,v_{k'}} \mathbb{E}\Big\{\frac{X_{m,u_1}^{d_1}....X_{m,u_k}^{d_k}Y_{m,v_1}^{d'_1}....Y_{m,v_{k'}}^{d'_{k'}}}{\Psi(n)^{h+h'}}\Big\}\Big|$$

$$= \left| \mathbb{E}\Big\{ \Big(\frac{\sum_s X_{m,s}^{d_1}}{\Psi(n)^{d_1}} \Big) ..... \Big(\frac{\sum_s X_{m,s}^{d_k}}{\Psi(n)^{d_k}} \Big) \Big(\frac{\sum_s Y_{m,s}^{d'_1}}{\Psi(n)^{d'_1}} \Big) ..... \Big(\frac{\sum_s Y_{m,s}^{d'_{k'}}}{\Psi(n)^{d'_{k'}}} \Big) \Big\} \right|$$

$$\leq \mathbb{E}\Big\{ \Big| \frac{\sum_s X_{m,s}^{d_1}}{\Psi(n)^{d_1}} \Big|^{k+k'} \Big\}^{\frac{1}{k+k'}} .....\mathbb{E}\Big\{ \Big| \frac{\sum_s X_{m,s}^{d_k}}{\Psi(n)^{d_k}} \Big|^{k+k'} \Big\}^{\frac{1}{k+k'}} \mathbb{E}\Big\{ \Big| \frac{\sum_s Y_{m,s}^{d'_1}}{\Psi(n)^{d'_1}} \Big|^{k+k'} \Big\}^{\frac{1}{k+k'}} .....\mathbb{E}\Big\{ \Big| \frac{\sum_s Y_{m,s}^{d'_{k'}}}{\Psi(n)^{d'_{k'}}} \Big|^{k+k'} \Big\}^{\frac{1}{k+k'}}$$

$$\leq \Bigg[\mathbb{E}\Big\{ \Big(\frac{\sum_s X_{m,s}^{d_1}}{\Psi(n)^{d_1}} \Big)^{2k+2k'} \Big\} .... \mathbb{E}\Big\{ \Big(\frac{\sum_s X_{m,s}^{d_k}}{\Psi(n)^{d_k}} \Big)^{2k+2k'} \Big\} \mathbb{E}\Big\{ \Big(\frac{\sum_s Y_{m,s}^{d'_1}}{\Psi(n)^{d'_1}} \Big)^{2k+2k'} \Big\} ..... \mathbb{E}\Big\{ \Big(\frac{\sum_s Y_{m,s}^{d'_{k'}}}{\Psi(n)^{d'_{k'}}} \Big)^{2k+2k'} \Big\} \Bigg]^{\frac{1}{2k+2k'}}$$

$$\leq \epsilon(n) \Bigg[\mathbb{E}\Big\{ \Big(\frac{\sum_s X_{m,s}^{d_2}}{\Psi(n)^{d_2}} \Big)^{2k+2k'} \Big\} .... \mathbb{E}\Big\{ \Big(\frac{\sum_s X_{m,s}^{d_k}}{\Psi(n)^{d_k}} \Big)^{2k+2k'} \Big\} \mathbb{E}\Big\{ \Big(\frac{\sum_s Y_{m,s}^{d'_1}}{\Psi(n)^{d'_1}} \Big)^{2k+2k'} \Big\} .... \mathbb{E}\Big\{ \Big(\frac{\sum_s Y_{m,s}^{d'_{k'}}}{\Psi(n)^{d'_{k'}}} \Big)^{2k+2k'} \Big\} \Bigg]^{e}$$

where $\epsilon(n) \to 0$ and $e = \frac{1}{2k+2k'}$. Moreover, by our assumption and by hypothesis 3.1.2, all the other terms are bounded. Then, the lemma is proved when r=r'=1. For $r \geq 1$ or $r' \geq 1$, it is enough to apply lemma 3.2.3 or 4.2.1 for example.

Lemma 4.2.5 *We suppose that all the moments $M_{q,p}^{n_m}$ are bounded. We suppose $R_1^{m,t}(x) = Q_2^{m,t}(x) = x^2 - \beta_{m,t}$ and $R_s^{m,t}$ (resp $S_s^{m,t}$) is a polynomial of degree $j_s \geq 0$ (resp, $i_s \geq 0$). If $j_s = 0$ (resp $i_s = 0$), one assume that $(1/\Psi(n_m)^2) \sum_{t=1}^{n_m} R_s^{m,t}(X_{m,t})$ (resp $(1/\Psi(n_m)^2) \sum_{t=1}^{n_m} S_s^{m,t}(X_{m,t})$) is bounded. If $j_s \geq 1$ (resp $i_s \geq 1$) , one assume that $R_s^{m,t}(x) = x^{j_s}$ (resp $S_s^{m,t}(x) = x^{i_s}$). Moreover let $h = \sum_{s=1}^{k} j'_s$ where $j'_s = j_s$ if $j_s \geq 1$ and $j'_s = 2$ if $j_s = 0$ and $h' = \sum_{s=1}^{k'} i'_s$ where $i'_s = i_s$ if $i_s \geq 1$ and $i'_s = 2$ if $i_s = 0$.*

We assume $\frac{1}{\Psi(n_m)^4} \mathbb{E}\big\{ \big(\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t}) \big)^2 \big\} \to 0$. Then, for all r, r' , $1 \leq r \leq k$, $1 \leq r' \leq k'$,

$$\frac{\mathbb{E}\Big\{ \sum_{t_1 \neq t_2 \neq .... \neq t_r, t_{r+1}, ..., t_k} \sum_{o_1 \neq o_2 \neq .... \neq o_{r'}, o_{r'+1}, ..., o_{k'}} \Big[\prod_{s=1}^{k} R_s^{m,t_s}(X_{m,t_s}) \Big] \Big[\prod_{s'=1}^{k'} S_{s'}^{m,o_{s'}}(Y_{m,o_{s'}}) \Big] \Big\}}{\Psi(n_m)^{h+h'}}$$

converges to 0.

Proof By the Schwartz inequality and the Holder Inequality,

$$\frac{1}{\Psi(n_m)^{2h+2h'}} . \mathbb{E}\Big\{ \sum_{t_1 \, t_2, ..., t_k} \sum_{o_1, o_2 ..., o_{k'}} \Big[\prod_{s=1}^{k} R_s^{m,t_s}(X_{m,t_s}) \Big] \Big[\prod_{s'=1}^{k'} S_{s'}^{m,o_{s'}}(Y_{m,o_{s'}}) \Big] \Big\}^2$$

$$\leq \mathbb{E}\Big\{ \frac{1}{\Psi(n_m)^{h+h'}} \Big[\prod_{s=1}^{k} \Big(\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s}) \Big) \Big] \Big[\prod_{s=1}^{k'} \Big(\sum_{o_s=1}^{n_m} S_s^{m,o_s}(Y_{m,o_s}) \Big) \Big] \Big\}^2$$

$$\leq \mathbb{E}\Big\{ \frac{1}{\Psi(n_m)^4} \Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t}) \Big)^2 \Big\}$$

$$\mathbb{E}\Big\{ \frac{1}{\Psi(n_m)^{2h+2h'-4}} \Big[\prod_{s=2}^{k} \Big(\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s}) \Big)^2 \Big] \Big[\prod_{s=1}^{k'} \Big(\sum_{o_s=1}^{n_m} S_s^{m,o_s}(Y_{m,o_s}) \Big)^2 \Big] \Big\}$$

$$\leq \mathbb{E}\Big\{\frac{1}{\Psi(n_m)^4}\Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t})\Big)^2\Big\}\Big[\prod_{s=2}^{k}\mathbb{E}\Big\{\Big(\frac{\sum_{t_s=1}^{n_m} R_s^{m,t_s}(X_{m,t_s})}{\Psi(n_m)^{j_s}}\Big)^{2(k+k'-1)}\Big\}^{\frac{1}{k+k'-1}}\Big]$$

$$\cdot \qquad \Big[\prod_{s'=1}^{k'}\mathbb{E}\Big\{\Big(\frac{\sum_{o_{s'}=1}^{n_m} S_{s'}^{m,o_{s'}}(Y_{m,o_{s'}})}{\Psi(n_m)^{i_{s'}}}\Big)^{2(k+k'-1)}\Big\}^{\frac{1}{k+k'-1}}\Big] .$$

By our assumption and by hypothesis 3.1.2 , the term on right-hand side is bounded. Moreover,

$$\frac{1}{\Psi(n_m)^4}\mathbb{E}\Big\{\Big(\sum_{t=1}^{n_m} R_1^{m,t}(X_{m,t})\Big)^2\Big\} \to 0 .$$

We deduce the lemma if r=1 and r'=1.

When $r > 1$ or $r' > 1$, we prove the result by recurrence by using the lemma 3.2.2, 3.2.4 , 4.2.1 and 3.4.3. ■

Lemma 4.2.6 *Let $0 \leq q" \leq q'$, $0 \leq p" \leq p'$. Let $h = \sum_{s=1}^{q"}(2u_s + v_s) + (q' - q")$ and let $h' = \sum_{s=1}^{p"}(2c_s + d_s) + (p' - p")$. One assumes that $M_{q,p}^{n_m}$ is bounded pour tout q,p. Let*

$$S_{\mathcal{H}}^{n_m} = \sum_{t_1,\ldots.t_{p'}\in\mathcal{H}} \beta_{m,t_1}^{c_1} Y_{m,t_1}^{d_1}\beta_{m,t_2}^{c_2} Y_{m,t_2}^{d_2}\ldots.\beta_{m,t_{p"}}^{c_{p"}} Y_{m,t_{p"}}^{d_{p"}} Y_{m,t_{p"}+1}\ldots\ldots X_{m,t_{p'}} .$$

Then,

$$\sum_{s_1,\ldots.s_{q'}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}\beta_{m,s_2}^{u_2} X_{m,s_2}^{v_2}\ldots.\beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}} X_{m,_{q"}+1}\ldots\ldots X_{m,s_{q'}} S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h+h'}}\Big\} \to 0$$

if $u_1 \geq 1$ and $v_1 \geq 1$.

Proof By lemma 3.4.4 $\mathbb{E}\{(S_{\mathcal{H}}^{n_m}/\Psi(n_m)^{h'})^P\}$ is bounded for all P. Then, we have

$$\Big|\sum_{s_1,\ldots.s_{q'}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}\beta_{m,s_2}^{u_2} X_{m,s_2}^{v_2}\ldots.\beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}} X_{m,s_{q"}+1}\ldots\ldots X_{m,s_{q'}} S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h+h'}}\Big\}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\frac{\sum_{s_1}\beta_{s_1}^{u_1} X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\Big)\ldots\ldots\Big(\frac{\sum_{s_{q"}}\beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}}}{\Psi(n_m)^{2u_{q"}+v_{q"}}}\Big)\Big(\frac{\sum_{s_{q"}+1} X_{m,s_{q"}+1}}{\Psi(n_m)}\Big)\ldots\ldots\Big(\frac{\sum_{s_{q'}} X_{m,s_{q'}}}{\Psi(n_m)}\Big)\Big(\frac{S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h'}}\Big)\Big\}\Big|$$

$$= \Big|\mathbb{E}\Big\{\Big(\frac{\sum_{s_1}\beta_{m,s_1}^{u_1} X_{m,s_1}^{v_1}}{\Psi(n_m)^{2u_1+v_1}}\Big)\ldots\ldots\Big(\frac{\sum_{s_{q"}}\beta_{m,s_{q"}}^{u_{q"}} X_{m,s_{q"}}^{v_{q"}}}{\Psi(n_m)^{2u_{q"}+v_{q"}}}\Big)\Big(\frac{\sum_{s_{q"}+1} X_{m,s_{q"}+1}}{\Psi(n_m)}\Big)^{q'-q"}\Big(\frac{S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h'}}\Big)\Big\}\Big|$$

$$\leq \Big[\prod_{r=1}^{q"}\mathbb{E}\Big\{\Big|\frac{\sum_{s_r}\beta_{m,s_r}^{u_r} X_{m,s_r}^{v_r}}{\Psi(n_m)^{2u_r+v_r}}\Big|^{q"+2}\Big\}^{\frac{1}{q"+2}}\Big]\mathbb{E}\Big\{\Big|\frac{\sum_{s_{q"}+1} X_{m,s_{q"}+1}}{\Psi(n_m)}\Big|^{(q'-q")(q"+2)}\Big\}^{\frac{1}{q"+2}}\mathbb{E}\Big\{\Big|\frac{S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h'}}\Big|^{q"+2}\Big\}^{\frac{1}{q"+2}}$$

$$\leq |M_{2(q'-q")(q"+2)}^{n_m}|^{\frac{1}{2(q"+2)}}\Big[\prod_{r=1}^{q"}\mathbb{E}\Big\{\Big|\frac{\sum_{s_r}\beta_{m,s_r}^{u_r} X_{m,s_r}^{v_r}}{\Psi(n_m)^{2u_r+v_r}}\Big|^{q"+2}\Big\}^{\frac{1}{q"+2}}\Big]\mathbb{E}\Big\{\Big(\frac{S_{\mathcal{H}}^{n_m}}{\Psi(n_m)^{h'}}\Big)^{2q"+4}\Big\}^{\frac{1}{2q"+4}} .$$

It is enough to apply lemma 3.4.3 in order to conclude. ■

Lemma 4.2.7 *Let $\beta'_{m,s} = \mathbb{E}\{Y^2_{m,s}\}$. We suppose that all the moments $M^{n_m}_{q,p}$ are bounded. We assume that $\frac{1}{\Psi(n_m)^4}\mathbb{E}\{\big(\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})\big)^2\} \to 0$. Then,*

$$\sum_{s_1\neq....\neq s_{q-e}}\ \sum_{v_1\neq....\neq v_{q'-f}} \mathbb{E}\Big\{\frac{X^2_{m,s_1}.....X^2_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}Y^2_{m,v_1}.....Y^2_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$-\sum_{s_1\neq....\neq s_{q-e}}\ \sum_{v_1\neq....\neq v_{q'-f}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}.....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

converges to 0

Proof For example,

$$\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{X^2_{m,s_1}X^2_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}Y^2_{m,v_1}Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$=\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{(X^2_{m,s_1}-\beta_{m,s_1})X^2_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}Y^2_{m,v_1}Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$+\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}X^2_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}Y^2_{m,v_1}Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$=\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{(X^2_{m,s_1}-\beta_{m,s_1})X^2_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}Y^2_{m,v_1}Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$+\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}(X^2_{m,s_2}-\beta_{m,s_2})X_{m,s_3}....X_{m,s_{q-2}}Y^2_{m,v_1}Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$+\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}(Y^2_{m,v_1}-\beta'_{m,v_1})Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$+\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,2}X_{m,s_3}....X_{m,s_{q-2}}\beta'_{m,v_1}(Y^2_{m,v_2}-\beta'_{m,v_2})Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$+\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,s_2}X_{m,s_3}....X_{m,s_{q-2}}\beta'_{v_1}\beta'_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}\ .$$

Then for example by lemma 4.2.5

$$\sum_{s_1\neq....\neq s_{q-2}}\ \sum_{v_1\neq v_2\neq....\neq v_{q'-2}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}\beta_{m,2}X_{m,s_3}....X_{m,s_{q-2}}\beta'_{m,v_1}(Y^2_{m,v_2}-\beta'_{m,v_2})Y^2_{m,v_2}Y_{m,v_3}.....Y_{m,v_{q'-2}}}{\Psi(n_m)^{q+q'}}\Big\}$$

converges to 0. ■

Lemma 4.2.8 *We suppose that all the moments $M^{n_m}_{h,k}$ are bounded. One assumes that, for all, p, q,*

$$\sum_{s_1\neq.....\neq s_q}\ \sum_{t_1\neq.....\neq t_p} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}Y_{m,t_1}Y_{m,t_2}.....Y_{m,t_p}\}}{\Psi(n_m)^{p+q}} \to S_{q,p}\ .$$

We set $S_{0,0}=1$. Then,

$$\sum_{s_1\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_{p-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}.....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{p-f}}}{\Psi(n_m)^{q+p}}\Big\}$$

converges to $\sigma_0^{2e}\sigma_1^{2e}S_{q-2e,p-2f}$

Proof At first, we study

$$\sum_{s_1\neq....\neq s_e,s_{e+1}\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_f,v_{f+1}\neq....\neq v_{q'-f}}$$
$$\mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$=\sum_{s_1\neq....\neq s_e}\sum_{s_{e+1}\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_f}\sum_{v_{f+1}\neq....\neq v_{q'-f}}$$
$$\Big(\frac{\beta_{m,s_1}....\beta_{m,s_e}}{\Psi(n_m)^{2e}}\Big)\Big(\frac{\beta'_{m,v_1}....\beta'_{m,v_f}}{\Psi(n_m)^{2f}}\Big)\mathbb{E}\Big\{\frac{X_{m,s_{e+1}}....X_{m,s_{q-e}}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q-2e}\Psi(n_m)^{q'-2f}}\Big\}\ .$$

It is easy to understand that

$$\sum_{s_1\neq....\neq s_e}\Big(\frac{\beta_{m,s_1}\beta_{m,s_2}....\beta_{m,s_e}}{\Psi(n_m)^{2e}}\Big)\to\sigma_0^{2e}\ .$$

$$\sum_{v_1\neq....\neq v_f}\Big(\frac{\beta'_{m,v_1}....\beta'_{m,v_f}}{\Psi(n_m)^{2f}}\Big)\to\sigma_1^{2f}\ .$$

Indeed,

$$\sum_{s_1,...,s_e}\Big(\frac{\beta_{m,s_1}\beta_{m,s_2}....\beta_{m,s_e}}{\Psi(n_m)^{2e}}\Big)=\Big(\frac{\sum_{m,s_1}\beta_{m,s_1}}{\Psi(n_m)^2}\Big).....\Big(\frac{\sum_{s_e}\beta_{m,s_e}}{\Psi(n_m)^2}\Big)\to\sigma_0^{2e}\ ,$$

and, for example, by lemma 3.4.2,

$$\frac{\sum_{s_1}\beta^2_{m,s_1}}{\Psi(n_m)^4}\to 0\ .$$

Therefore,

$$\sum_{s_1\neq....\neq s_e,s_{e+1}\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_f,v_{f+1}\neq....\neq v_{q'-f}}$$
$$\mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

converges to $\sigma_0^{2e}\sigma_1^{2f}S_{q-2e,q'-2f}$.

Now, in order to obtain
$\sum_{s_1\neq\ldots\neq s_{q-e}}\sum_{v_1\neq\ldots\neq v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\beta_{m,s_e}X_{m,s_{e+1}}\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$,
we shall have to add or to subtract sums such that there exists i, j satisfying $s_i = s_{e+j}$ or i',j' such that $v_{i'} = v_{f+j'}$. For example, , we shall have to subtract the sum

$$\sum_{s_2\neq\ldots\neq s_e,s_{e+1}\neq\ldots\neq s_{q-e}}\ \sum_{v_1\neq\ldots\neq v_f,v_{f+1}\neq\ldots\neq v_{q'-f}}$$
$$\mathbb{E}\Big\{\frac{\beta_{m,s_2}\ldots\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}.$$

One will prove that this sum converges to 0 by recurence. With this aim, we study first

$$\sum_{s_2\ldots\ldots,s_{q-e}}\ \sum_{v_1,\ldots\ldots,v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_2}\ldots\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

which can be written as

$$\prod_{i=2}^{e}\Big(\frac{\sum_{s_i}\beta_{m,s_i}}{\Psi(n_m)^2}\Big)\prod_{j=1}^{f}\Big(\frac{\sum_{s_j}\beta'_{m,s_j}}{\Psi(n_m)^2}\Big)\mathbb{E}\Big\{\Big(\frac{\sum_{s_{e+1}}\beta_{m,s_{e+1}}X_{m,s_{e+1}}}{\Psi(n_m)^3}\Big)$$
$$\Big(\sum_{s_{e+2},\ldots,s_{q-e}}\frac{X_{m,s_{e+2}}\ldots X_{m,s_{q-e}}}{\Psi(n_m)^{q-e-1}}\Big)\Big(\sum_{v_1,\ldots,v_{q'-f}}\frac{Y_{m,v_{f+1}}\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q'-f}}\Big)\Big\}.$$

Now, $\frac{\sum_{s_2}\beta_{m,s_2}}{\Psi(n_m)^2}$ is bounded and

$$\mathbb{E}\Big\{\Big(\sum_{s_{e+2},\ldots,s_{q-e}}\frac{X_{m,s_{e+2}}\ldots X_{m,s_{q-e}}}{\Psi(n_m)^{q-2e-1}}\Big)\Big\}^2=\mathbb{E}\Big\{\Big(\sum_s\frac{X_{m,s}}{\Psi(n_m)}\Big)^{q-2e-1}\Big\}^2$$
$$\leq\mathbb{E}\Big\{\Big(\sum_s\frac{X_{m,s}}{\Psi(n_m)}\Big)^{2(q-2e-1)}\Big\}$$

is bounded by our assumption. At last, $\mathbb{E}\Big\{\frac{\sum_{s_1}\beta_{m,s_{e+1}}X_{m,s_{e+1}}}{\Psi(n_m)^3}\Big\}\to 0$ by lemma 3.4.3. It is enough to apply Holder Inequality to conclude.

It is general for all the steps of recurrence : thanks to lemma 3.2.2 and 3.2.4, one can always write the terms

$$\sum_{s_1\neq\ldots\neq s_r,s_{r+1}\ldots,s_{q-e}}\ \sum_{v_1\neq\ldots\neq v_{r'},v_{r'+1},\ldots,v_{q'-f}}$$
$$\mathbb{E}\Big\{\frac{\beta_{m,s_2}\ldots\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

as a sum of products. Or by using proof of lemma 4.2.6, we always find in these products, terms which are bounded by products of bounded expectations (cf lemma 3.3.1 and hypothesis 3.1.2) for example, in the form $\mathbb{E}\Big\{\Big(\frac{\sum_s\beta^u_{m,s}X^v_{m,s}}{\Psi(n_m)^{2u+v}}\Big)^{2w}\Big\}$, $u,v,w\in\mathbb{N}$: among these terms there is at least one term of the form $\sum_{s_1}\frac{\beta^u_{m,s_1}X^v_{m,s_1}}{\Psi(n_m)^{2u+v}}$ which converges to 0. That proves the recurence, and then, that

$$\sum_{s_2\neq....\neq s_e, s_{e+1}\neq....\neq s_{q-e}} \sum_{v_1\neq....\neq v_f, v_{f+1}\neq....\neq v_{q'-f}}$$

$$\mathbb{E}\Big\{\frac{\beta_{m,s_2}.....\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

converges to 0.

One can reason by the same way about other terms, for example terms such

$$\sum_{s_3\neq....\neq s_e, s_{e+1}\neq....\neq s_{q-e}} \sum_{v_1\neq....\neq v_f, v_{f+1}\neq....\neq v_{q'-f}}$$

$$\mathbb{E}\Big\{\frac{\beta_{m,s_3}.....\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}\beta_{m,s_{e+2}}X_{m,s_{e+2}}X_{m,s_{e+3}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

or

$$\sum_{s_2\neq..\neq s_e, s_{e+1}\neq....\neq s_{q-e}} \sum_{v_2\neq....\neq v_f, v_{f+1}\neq....\neq v_{q'-f}}$$

$$\mathbb{E}\Big\{\frac{\beta_{m,s_2}.....\beta_{m,s_e}\beta_{m,s_{e+1}}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}\beta'_{m,2}....\beta'_{m,v_f}\beta'_{m,v_{f+1}}Y_{m,v_{f+1}}Y_{m,v_{f+2}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}.$$

We deduce that

$$\sum_{s_1\neq....\neq s_{q-e}} \sum_{v_1\neq...\neq v_{q'-f}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

converges to the same limit as

$$\sum_{s_2\neq....\neq s_e, s_{e+1}\neq....\neq s_{q-e}} \sum_{v_1\neq....\neq v_f, v_{f+1}\neq....\neq v_{q'-f}}$$

$$\mathbb{E}\Big\{\frac{\beta_{m,s_2}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}\ ,$$

i.e. converges to $\sigma_0^{2e}\sigma_1^{2f}S_{q-2e,q'-2f}$. ■

Lemma 4.2.9 *We suppose that all the moments $M_{h,k}^{n_m}$ converge. Then, for all e, for all f, for all p, for all q,*

$$\sum_{s_1\neq....\neq s_{q-e}} \sum_{v_1\neq....\neq v_{p-f}} \mathbb{E}\Big\{\frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}\beta'_{m,v_1}.....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{p-f}}}{\Psi(n_m)^{q+p}}\Big\}$$

converges.

Proof We study at first

$$\sum_{s_1,\ldots\ldots,s_{q-e}}\sum_{v_1,\ldots\ldots,v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\ldots\beta_{m,s_e}X_{m,s_{e+1}}\ldots\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

$$=\Big(\sum_s\frac{\beta_{m,s}}{\Psi(n_m)^2}\Big)^e\Big(\sum_s\frac{\beta'_{m,s}}{\Psi(n_m)^2}\Big)^f\mathbb{E}\Big\{\Big(\sum_s\frac{X_{m,s}}{\Psi(n_m)}\Big)^{q-2e}\Big(\sum_s\frac{Y_{m,s}}{\Psi(n_m)}\Big)^{p-2f}\Big\}$$

which converges to $\sigma_0^{2e}\sigma_1^{2f}M_{q-2e,q'-2f}$.

Now, in order to obtain

$$\sum_{s_1\neq\ldots\ldots\neq s_{q-e}}\sum_{v_1\neq\ldots\ldots\neq v_{p-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\ldots\beta_{m,s_e}X_{m,s_{e+1}}\ldots\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{p-f}}}{\Psi(n_m)^{q+p}}\Big\}\ ,$$

we shall have to add or to subtract sums such that there exists i, j satisfying $s_i=s_{e+j}$ or i',j' such that $v_{i'}=v_{j'}$.

For example, we shall have to subtract terms of the form

$$\sum_{s_1,\ldots\ldots,s_e,s_{e+2},\ldots,s_{q-e}}\sum_{v_1,\ldots,v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\ldots\beta_{m,s_e}X^2_{m,s_{e+2}}X_{m,s_{e+3}}\ldots\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

which, by using the same technique as in lemma 4.2.7, converges to the same limit as

$$\sum_{s_1,\ldots\ldots,s_e,s_{e+2},\ldots,s_{q-e}}\sum_{v_1,\ldots,v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_1}\ldots\ldots\beta_{m,s_e}\beta_{m,s_{e+2}}X_{m,s_{e+3}}\ldots\ldots X_{m,s_{q-e}}\beta'_{m,v_1}\ldots\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}.$$

Then, we prove the result by using a recurence on e.

Now, we shall have also to substract the sum

$$\sum_{s_2,\ldots\ldots,s_{q-e}}\sum_{v_1,\ldots,v_{q'-f}}\mathbb{E}\Big\{\frac{\beta_{m,s_2}\ldots\ldots\beta_{m,e}[\beta_{m,s_{e+1}}X_{m,s_{e+1}}]X_{m,s_{e+2}}\ldots\ldots X_{m,s_{q-e}}\beta'_{m,1}\ldots\ldots\beta'_{m,v_f}Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}}\Big\}$$

which is written as

$$\prod_{i=2}^{e}\Big(\frac{\sum_{s_i}\beta_{m,s_i}}{\Psi(n_m)^2}\Big)\prod_{j=1}^{f}\Big(\frac{\sum_{s_j}\beta'_{m,s_j}}{\Psi(n_m)^2}\Big)$$
$$\mathbb{E}\Big\{\Big(\frac{\sum_{s_1}\beta_{m,s_{e+1}}X_{m,s_{e+1}}}{\Psi(n_m)^3}\Big)\Big(\sum_{s_{e+2},\ldots\ldots,s_{q-e}}\frac{X_{m,s_{e+2}}\ldots\ldots X_{m,s_{q-e}}}{\Psi(n_m)^{q-e-1}}\Big)\Big(\sum_{v_1,\ldots,v_{q'-f}}\frac{Y_{m,v_{f+1}}\ldots\ldots Y_{m,v_{q'-f}}}{\Psi(n_m)^{q'-f}}\Big)\Big\}.$$

Now, $\frac{\sum_{s_2}\beta_{m,s_2}}{\Psi(n_m)^2}$ is bounded and

$$\mathbb{E}\Big\{\Big(\sum_{s_{e+2},\ldots\ldots,s_{q-e}}\frac{X_{m,s_{e+2}}\ldots\ldots X_{m,s_{q-e}}}{\Psi(n_m)^{q-2e-1}}\Big)\Big\}^2=\mathbb{E}\Big\{\Big(\sum_s\frac{X_{m,s}}{\Psi(n_m)}\Big)^{q-2e-1}\Big\}^2\leq\mathbb{E}\Big\{\Big(\sum_s\frac{X_{m,s}}{\Psi(n_m)}\Big)^{2(q-2e-1)}\Big\}$$

is bounded by our assumption. At last, $\mathbb{E}\Big\{\frac{\sum_{s_1}\beta_{m,s_{e+1}}X_{m,s_{e+1}}}{\Psi(n_m)^3}\Big\}\to 0$ by lemma 3.4.3.

More generally, we shall have to substract tems of the form

$$\sum_{s_{1+c},....,s_e,s_{e+d+1},...,s_{q-e}} \sum_{v_1,...,v_{q'-f}}$$

$$\mathbb{E}\Big\{ \frac{\beta_{s_{1+c}}....\beta_{m,s_e}[\beta^c_{m,s_{e+1}} X^d_{m,s_{e+1}}]X_{m,s_{e+d+1}}...X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}} \Big\}$$

which converges to 0 by the same reasoning, but by applying lemma 3.4.4 .

We shall have also to substract tems of the form

$$\sum_{s_2,....,s_{q-e}} \sum_{v_1,...,v_{q'-f}}$$

$$\mathbb{E}\Big\{ \frac{\beta^2_{m,s_2}\beta_{m,s_3}.....\beta_{m,s_e}X_{m,s_{e+1}}X_{m,s_{e+2}}....X_{m,s_{q-e}}\beta'_{m,v_1}....\beta'_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}} \Big\}$$

which converges to 0 because by lemma 3.4.2, $\frac{\sum_{s_2}\beta^2_{m,s_2}}{\Psi(n_m)^4} \to 0$.

We shall deduce that

$$\sum_{s_1\neq.....\neq s_{q-e}} \sum_{v_1\neq....\neq v_{q'-f}} \mathbb{E}\Big\{ \frac{\beta_{m,s_1}.....\beta_{m,s_e}X_{m,s_{e+1}}....X_{m,q-e}\beta'_{m,1}....\beta'_{m,f}Y_{m,v_{f+1}}.....Y_{m,v_{q'-f}}}{\Psi(n_m)^{q+q'}} \Big\}$$

converges. ■

4.2.2 Statement and proof of first theorem

Theorem 15 *Let $Q_2^{m,t}(x) = x^2 - \beta_{m,t}$, $Q_2'^{m,t}(y) = y^2 - \beta'_{m,t}$ where $\beta'_{m,t} = \mathbb{E}\{Y^2_{m,t}\}$. One assumes that*

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ , \ \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2'^{m,t}(Y_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M^{n_m}_{q,p} = \mathbb{E}\Big\{ \frac{(X_{m,1}+....+X_{m,n_m})^q(Y_{m,1}+....+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}} \Big\}$ converges to a real $M_{q,p}$ if, for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$, there exists $S_{q,p} \in \mathbb{R}$ and Sb^2_q such that

$$\sum_{s_1\neq s_2\neq.....\neq s_q} \sum_{t_1\neq t_2\neq.....\neq t_p} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}Y_{m,t_1}Y_{m,t_2}.....Y_{m,t_p}\}}{\Psi(n_m)^{p+q}} \to S_{q,p}$$

$$\left| \sum_{s_1\neq s_2\neq.....\neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq Sb^2_q \ ,$$

$$\left| \sum_{t_1\neq t_2\neq.....\neq t_{p-1}} \frac{\mathbb{E}\{(Y_{m,t_1})^2Y_{m,t_2}.....Y_{m,t_{p-1}}\}}{\Psi(n_m)^p} \right| \leq Sb^2_p \ .$$

Proof of sufficient condition of theorem 15 By proposition 3.3.1, all the moments of marginal distributions are bounded. By using Holder inequality, we deduce that all the moments are bounded.

By lemma 4.2.8 and 4.2.7 we deduce that

$$\sum_{s_1\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_{p-f}}\mathbb{E}\Big\{\frac{X^2_{m,s_1}....X^2_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}Y^2_{m,v_1}.....Y^2_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{p-f}}}{\Psi(n_m)^{q+p}}\Big\}$$

converges to $\sigma_0^{2e}\sigma_1^{2e}S_{q-2e,p-2f}$.

By lemma 4.2.4, if there exists $r_s\geq 3$ or $r'_s\geq 3$,

$$\sum_{u_1\neq u_2\neq.....\neq u_k}\sum_{v_1\neq v_2\neq....\neq v_{k'}}\mathbb{E}\left\{\frac{X^{r_1}_{m,u_1}....X^{r_k}_{m,u_k}Y^{r'_1}_{m,v_1}....Y^{r'_{k'}}_{m,v_{k'}}}{\Psi(n_m)^{h+h'}}\right\}\to 0\ .$$

Therefore

$$\sum_{u_1\neq u_2\neq.....\neq u_k}\sum_{v_1\neq v_2\neq....\neq v_{k'}}\mathbb{E}\left\{\frac{X^{r_1}_{m,u_1}....X^{r_k}_{m,u_k}Y^{r'_1}_{m,v_1}....Y^{r'_{k'}}_{m,v_{k'}}}{\Psi(n_m)^{h+h'}}\right\}$$

converges for all $(u_1,...,u_k,v_1,...,v_{k'})$.

Then, because

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q(Y_{m,1}+....+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}}\Big\}$$

$$=\sum_{j_1+.....+j_{n_m}=q}\sum_{i_1+.....+i_{n_m}=p}\frac{q!}{j_1!.....j_{n_m}!}\ \frac{p!}{i_1!.....i_{n_m}!}\mathbb{E}\Big\{\frac{X^{j_1}_{m,1}.....X^{j_{n_m}}_{m,n_m}Y^{i_1}_{m,1}.....Y^{i_{n_m}}_{m,n_m}}{\Psi(n_m)^{q+p}}\Big\}\ ,$$

all the moments converge. ■

Proof of necessary condition of theorem 15 We suppose that all the moments converge. Then, the second and thirth relations are a consequence of proposition 3.3.1 .

By lemma 4.2.9 and 4.2.7 , we deduce that, for all e, for all f,

$$\sum_{s_1\neq....\neq s_{q-e}}\sum_{v_1\neq....\neq v_{p-f}}\mathbb{E}\Big\{\frac{X^2_{m,s_1}....X^2_{m,s_e}X_{m,s_{e+1}}....X_{m,s_{q-e}}Y^2_{m,v_1}.....Y^2_{m,v_f}Y_{m,v_{f+1}}.....Y_{m,v_{p-f}}}{\Psi(n_m)^{q+p}}\Big\}$$

converges.

By lemma 4.2.4, if there exists $r_s\geq 3$ or $r'_s\geq 3$,

$$\sum_{u_1\neq u_2\neq.....\neq u_k}\sum_{v_1\neq v_2\neq....\neq v_{k'}}\mathbb{E}\left\{\frac{X^{r_1}_{m,u_1}....X^{r_k}_{m,u_k}Y^{r'_1}_{m,v_1}....Y^{r'_{k'}}_{m,v_{k'}}}{\Psi(n_m)^{p+q}}\right\}\to 0\ .$$

Therefore,

$$\sum_{u_1\neq u_2\neq.....\neq u_k}\sum_{v_1\neq v_2\neq....\neq v_{k'}}\mathbb{E}\left\{\frac{X^{r_1}_{m,u_1}....X^{r_k}_{m,u_k}Y^{r'_1}_{m,v_1}....Y^{r'_{k'}}_{m,v_{k'}}}{\Psi(n_m)^{p+q}}\right\}$$

converges for all $(u_1,...,u_k,v_1,...,v_{k'})$ if there exists $u_t\geq 2$ or $v_t\geq 2$.

Because

$$\mathbb{E}\Big\{\frac{(X_{m,1}+....+X_{m,n_m})^q(Y_{m,1}+....+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}}\Big\}$$

$$= \sum_{j_1+.....+j_{n_m}=q} \sum_{i_1+.....+i_{n_m}=q'} \frac{q!}{j_1!.....j_{n_m}!} \frac{q'!}{i_1!.....i_{n_m}!} \mathbb{E}\Big\{ \frac{X_{m,1}^{j_1}.....X_{m,n_m}^{j_{n_m}} Y_{m,1}^{i_1}.....Y_{m,n_m}^{i_{n_m}}}{\Psi(n_m)^{q+q'}} \Big\} ,$$

then,

$$\sum_{s_1\neq s_2\neq.....\neq s_q} \sum_{t_1\neq t_2\neq.....\neq t_p} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}Y_{m,t_1}Y_{m,t_2}.....Y_{m,t_p}\}}{\Psi(n_m)^{p+q}}$$

converges. ■

4.3 Convergence to the normal distribution

In case of convergence to the normal distribution, one specifies theorem 15 by the following way.

Proposition 4.3.1 *One assume $\sigma_0^2 = \sigma_1^2$. One assumes that*

$$\mathbb{E}\Big\{ \Big[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \Big]^2 \Big\} \to 0 \ , \ \mathbb{E}\Big\{ \Big[\frac{\sum_{t=1}^{n_m} Q_2^{\prime m,t}(Y_{m,t})}{\Psi(n_m)^2} \Big]^2 \Big\} \to 0 \ .$$

All the moments $M_{q,p}^n$ converges to a real $M_{q,p} = \mu_q\mu_p$, where μ_q is the moment of order q of $N(0,1)$, if and only if , for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$,

$$\sum_{s_1\neq s_2\neq.....\neq s_q} \sum_{t_1\neq t_2\neq.....\neq t_p} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}.....X_{m,s_q}Y_{m,t_1}Y_{m,t_2}.....Y_{m,t_p}\}}{\Psi(n_m)^{p+q}} \to \nu_q\nu_p \ ,$$

where ν_q is the moment of order q of $N(0,\nu_2)$ with $1 = \sigma_0^2 + \nu_2$, and, if there exists $Sb_q^2 \in \mathbb{R}$

$$\Big| \sum_{s_1\neq s_2\neq.....\neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2 X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \Big| \leq Sb_q^2 \ ,$$

$$\Big| \sum_{t_1\neq t_2\neq.....\neq t_{q-1}} \frac{\mathbb{E}\{(Y_{m,t_1})^2 Y_{m,t_2}.....Y_{m,t_{p-1}}\}}{\Psi(n_m)^p} \Big| \leq Sb_p^2 \ .$$

4.3.1 Proof of proposition 4.3.1

In this proof we simplify $X_{m,t}$ in X_t and $Y_{m,t}$ in Y_t and n_m in n. Of course, we use theorem 15. At first, under necessary or sufficient assumption, we have the following lemma.

Lemma 4.3.1 *Let q" $= \lfloor q/2 \rfloor$, p" $= \lfloor p/2 \rfloor$. Then,*

$$\mathbb{E}\Big\{ \frac{(X_1 + ... + X_{n_m})^q (Y_1 + ... + Y_{n_m})^p}{\Psi(n_m)^{q+p}} \Big\}$$

$$\sim \frac{q!p!}{2^0 2^0} \frac{\sigma_0^0\sigma_1^0}{0!0!} \frac{S_{q,p}}{q!p!} + \frac{q!p!}{2^1 2^0} \frac{\sigma_0^2\sigma_1^0}{1!0!} \frac{S_{q-2,p}}{(q-2)!p!} + ... + \frac{q!p!}{2^{q"} 2^0} \frac{\sigma_0^{2q"}\sigma_1^0}{q"!0!} \frac{S_{q-2q",p}}{(q-2q")!p!}$$

$$+ \frac{q!p!}{2^0 2} \frac{\sigma_0^0\sigma_1^2}{0!1!} \frac{S_{q,p-2}}{q!(p-2)!} + \frac{q!p!}{2^1 2} \frac{\sigma_0^2\sigma_1^2}{1!1!} \frac{S_{q-2,p-2}}{(q-2)!(p-2)!} + ... + \frac{q!p!}{2^{q"} 2} \frac{\sigma_0^{2q"}\sigma_1^2}{q"!1!} \frac{S_{q-2q",p-2}}{(q-2q")(p-2)!}$$

$$+\frac{q!p!}{2^0 2^2}\frac{\sigma_0^0\sigma_1^4}{0!2!}\frac{S_{q,p-4}}{q!(p-4)!}+\frac{q!p!}{2^1 2^2}\frac{\sigma_0^2\sigma_1^4}{1!2!}\frac{S_{q-2,p-4}}{(q-2)!(p-4)!}+\dots\dots\dots+\frac{q!p!}{2^{q"}2^2}\frac{\sigma_0^{2q"}\sigma_1^4}{q"!2!}\frac{S_{q-2q",p-4}}{(q-2q")!(p-4)!}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+\frac{q!p!}{2^0 2^{p"}}\frac{\sigma_0^0\sigma_1^{2p"}}{0!p"!}\frac{S_{q,p-2p"}}{q!(p-2p")!}+\frac{q!p!}{2^1 2^{p"}}\frac{\sigma_0^2\sigma_1^{2p"}}{1!p"!}\frac{S_{q-2,p-2p"}}{(q-2)!(p-2p")!}+\dots\dots\dots$$

$$\dots\dots\dots+\frac{q!p!}{2^{q"}2^{p"}}\frac{\sigma_0^{2q"}\sigma_1^{2p"}}{q"!p"!}\frac{S_{q-2q",p-2p"}}{(q-2q")!(p-2p")!}\ .$$

Proof In order to prove necessary or sufficient condition, by theorem 15

$$\sum_{s_1\neq s_2\neq\dots\neq s_q}\ \sum_{t_1\neq t_2\neq\dots\neq t_p}\frac{\mathbb{E}\{X_{s_1}X_{s_2}\dots\dots X_{s_q}Y_{t_1}Y_{t_2}\dots\dots Y_{t_p}\}}{\Psi(n)^{p+q}}\to S_{q,p}\ .$$

By lemma 4.2.4, if there exists s such that $j_s\geq 3$ or $i_s\geq 3$, for example,

$$\sum_{j_1+\dots+j_{n_m}=q,\ at\ least\ one\ j_s\geq 3}\ \sum_{i_1+\dots+i_{n_m}=p}\frac{q!}{j_1!\dots\dots j_{n_m}!}\ \frac{p!}{i_1!\dots\dots i_{n_m}!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}\to 0\ .$$

Then, it is enough to study the sums where $j_s\leq 2$ and $i_t\leq 2$.

Then,

$$\mathbb{E}\Big\{\frac{(X_1+\dots+X_n)^q(Y_1+\dots+Y_n)^p}{\Psi(n)^{q+p}}\Big\}$$

$$\sim\sum_{j_1+\dots+j_n=q,\ j_s\leq 1}\ \sum_{i_1+\dots+i_n=p,\ i_t\leq 1}\frac{q!}{j_1!\dots\dots j_n!}\ \frac{p!}{i_1!\dots\dots i_{n_m}!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+\dots+j_n=q,\ 1\ j_s=2}\ \sum_{i_1+\dots+i_n=p,\ i_t\leq 1}\frac{q!}{j_1!\dots\dots j_n!}\ \frac{p!}{i_1!\dots\dots i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$+\sum_{j_1+\dots+j_n=q,\ q"\ j_s=2}\ \sum_{i_1+\dots+i_n=p,\ i_t\leq 1}\frac{q!}{j_1!\dots\dots j_n!}\ \frac{p!}{i_1!\dots\dots i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+\dots+j_n=q,\ j_s\leq 1}\ \sum_{i_1+\dots+i_n=p,\ 1\ i_t=2}\frac{q!}{j_1!\dots\dots j_n!}\ \frac{p!}{i_1!\dots\dots i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+\dots+j_n=q,\ 1\ j_s=2}\ \sum_{i_1+\dots+i_n=p,\ 1\ i_t=2}\frac{q!}{j_1!\dots\dots j_n!}\ \frac{p!}{i_1!\dots\dots i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}\dots X_n^{j_n}Y_1^{i_1}\dots Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$+\sum_{j_1+.....+j_n=q,\ q"\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 1\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ 2\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+.....+j_n=q,\ 1\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 2\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\sum_{j_1+.....+j_n=q,\ q"\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 2\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

.....

.....

$$+\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ p"\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\sum_{j_1+.....+j_n=q,\ 1\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ p"\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\sum_{j_1+.....+j_n=q,\ q"\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ p"\ i_t=2}\frac{q!}{j_1!.....j_n!}\ \frac{p!}{i_1!.....i_n!}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$=\frac{q!p!}{2^0 2^0}\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ i_t\leq 1}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^1 2^0}\sum_{j_1+.....+j_n=q,\ 1\ j_s=2}\ \sum_{i_1+.....+i_n=p,\ i_t\leq 1}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\frac{q!p!}{2^{q"}2^{0}}\sum_{j_1+.....+j_n=q,\ q"\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ i_t\leq 1}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^{0}2}\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ 1\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^{1}2}\sum_{j_1+.....+j_n=q,\ 1\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 1\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\frac{q!p!}{2^{q"}2}\sum_{j_1+.....+j_n=q,\ q"\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 1\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^{0}2^{2}}\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ 2\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^{1}2^{2}}\sum_{j_1+.....+j_n=q,\ 1\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 2\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\frac{q!p!}{2^{q"}2^{2}}\sum_{j_1+.....+j_n=q,\ q"\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ 2\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

.....

.....

$$+\frac{q!p!}{2^{0}2^{p"}}\sum_{j_1+.....+j_n=q,\ j_s\leq 1}\ \sum_{i_1+.....+i_n=p,\ p"\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$+\frac{q!p!}{2^{1}2^{p"}}\sum_{j_1+.....+j_n=q,\ 1\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ p"\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

.....

.....

$$+\frac{q!p!}{2^{q"}!2^{p"}}\sum_{j_1+.....+j_n=q,\ q"\ \ j_s=2}\ \sum_{i_1+.....+i_n=p,\ p"\ \ i_t=2}\mathbb{E}\Big\{\frac{X_1^{j_1}.....X_n^{j_n}Y_1^{i_1}.....Y_n^{i_n}}{\Psi(n)^{q+p}}\Big\}$$

$$\sim \frac{q!p!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{S_{q,p}}{q!p!}+\frac{q!p!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{S_{q-2,p}}{(q-2)!p!}+\text{............}+\frac{q!p!}{2^{q"}2^0}\frac{\sigma_0^{2q"}\sigma_1^0}{q"!0!}\frac{S_{q-2q",p}}{(q-2q")!p!}$$

$$+\frac{q!p!}{2^0 2}\frac{\sigma_0^0\sigma_1^2}{0!1!}\frac{S_{q,p-2}}{q!(p-2)!}+\frac{q!p!}{2^1 2}\frac{\sigma_0^2\sigma_1^2}{1!1!}\frac{S_{q-2,p-2}}{(q-2)!(p-2)!}+\text{................}+\frac{q!p!}{2^{q"}2}\frac{\sigma_0^{2q"}\sigma_1^2}{q"!1!}\frac{S_{q-2q",p-2}}{(q-2q")(p-2)!}$$

$$+\frac{q!p!}{2^0 2^2}\frac{\sigma_0^0\sigma_1^4}{0!2!}\frac{S_{q,p-4}}{q!(p-4)!}+\frac{q!p!}{2^1 2^2}\frac{\sigma_0^2\sigma_1^4}{1!2!}\frac{S_{q-2,p-4}}{(q-2)!(p-4)!}+\text{................}+\frac{q!p!}{2^{q"}2^2}\frac{\sigma_0^{2q"}\sigma_1^4}{q"!2!}\frac{S_{q-2q",p-4}}{(q-2q")!(p-4)!}$$

$$\text{..}$$

$$+\frac{q!p!}{2^0 2^{p"}}\frac{\sigma_0^0\sigma_1^{2p"}}{0!p"!}\frac{S_{q,p-2p"}}{q!(p-2p")!}+\frac{q!p!}{2^1 2^{p"}}\frac{\sigma_0^2\sigma_1^{2p"}}{1!p"!}\frac{S_{q-2,p-2p"}}{(q-2)!(p-2p")!}+\text{................}$$

$$\text{....................................}+\frac{q!p!}{2^{q"}2^{p"}}\frac{\sigma_0^{2q"}\sigma_1^{2p"}}{q"!p"!}\frac{S_{q-2q",p-2p"}}{(q-2q")!(p-2p")!},$$

by lemma 4.2.3, 4.2.8, 4.2.9 and 4.2.7. ■

Proof of sufficient condition If q or p is odd, $S_{p-2e,q-2f}=\nu_{q-2e}\nu'_{q-2f}=0$. Therefore, by lemma 4.3.1, if p or q is odd, $M_{q,p}=0$.

Now, suppose that q and p are even : in order to simplify, we study the moment of order (2q,2p). By lemma 4.3.1,

$$\mathbb{E}\Big\{\frac{(X_1+....+X_n)^{2q}(Y_1+....+Y_n)^{2p}}{\Psi(n)^{2q+2p}}\Big\}\Big\}$$

$$\sim \frac{2q!2p!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{\nu_{2q}}{(2q)!}\frac{\nu_{2p}}{(2p)!}$$
$$+\frac{2q!2p!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{\nu_{2q-2}}{(2q-2)!}\frac{\nu_{2p}}{(2p)!}$$
$$+\frac{2q!2p!}{2^2 2^0}\frac{\sigma_0^4\sigma_1^0}{2!0!}\frac{\nu_{2q-4}}{(2q-4)!}\frac{\nu_{2p}}{(2p)!}$$
$$+\text{..}$$
$$+\text{..}$$
$$\frac{2q!2p!}{2^q 2^0}\frac{\sigma_0^{2q}\sigma_1^0}{q!0!}\frac{\nu_{2q-2q}}{(2q-2q)!}\frac{\nu_{2p}}{(2p)!}$$

$$+\frac{2q!2p!}{2^0 2}\frac{\sigma_0^0\sigma_1^2}{0!1!}\frac{\nu_{2q}}{(2q)!}\frac{\nu_{2p-2}}{(2p-2)!}$$
$$+\frac{2q!2p!}{2^1 2}\frac{\sigma_0^2\sigma_1^2}{1!1!}\frac{\nu_{2q-2}}{(2q-2)!}\frac{\nu_{2p-2}}{(2p-2)!}$$

$$+\frac{2q!2p!}{2^2 2}\ \frac{\sigma_0^4\sigma_1^2}{2!1!}\ \frac{\nu_{2q-4}}{(2q-4)!}\ \frac{\nu_{2p-2}}{(2p-2)!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2}\ \frac{\sigma_0^{2q}\sigma_1^2}{q!1!}\ \frac{\nu_{2q-2q}}{(2q-2q)!}\ \frac{\nu_{2p-2}}{(2p-2)!}$$

$$+\frac{2q!2p!}{2^0 2^2}\frac{\sigma_0^0\sigma_1^4}{0!2!}\frac{\nu_{2q}}{(2q)!}\frac{\nu_{2p-4}}{(2p-4)!}$$

$$+\frac{2q!2p!}{2^1 2^2}\frac{\sigma_0^2\sigma_1^4}{1!2!}\frac{\nu_{2q-2}}{(2q-2)!}\frac{\nu_{2p-4}}{(2p-4)!}$$

$$+\frac{2q!2p!}{2^2 2^2}\ \frac{\sigma_0^4\sigma_1^4}{2!2!}\ \frac{\nu_{2q-4}}{(2q-4)!}\ \frac{\nu_{2p-4}}{(2p-4)!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2^2}\ \frac{\sigma_0^{2q}\sigma_1^4}{q!2!}\ \frac{\nu_{2q-2q}}{(2q-2q)!}\ \frac{\nu_{2p-4}}{(2p-4)!}$$

.....

.....

.....

.....

$$+\frac{2q!2p!}{2^0 2^p}\frac{\sigma_0^0\sigma_1^{2p}}{0!p!}\frac{\nu_{2q}}{(2q)!}\frac{\nu_{2p-2p}}{(2p-2p)!}$$

$$+\frac{2q!2p!}{2^1 2^p}\frac{\sigma_0^2\sigma_1^{2p}}{1!p!}\frac{\nu_{2q-2}}{(2q-2)!}\frac{\nu_{2p-2p}}{(2p-2p)!}$$

$$+\frac{2q!2p!}{2^2 2^p}\ \frac{\sigma_0^4\sigma_1^{2p}}{2!p!}\ \frac{\nu_{2q-4}}{(2q-4)!}\ \frac{\nu_{2p-2p}}{(2p-2p)!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2^p}\ \frac{\sigma_0^{2q}\sigma_1^{2p}}{q!p!}\ \frac{\nu_{2q-2q}}{(2q-2q)!}\ \frac{\nu_{2p-2p}}{(2p-2p)!}$$

$$\sim \frac{2q!2p!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{\nu_2^q}{2^q q!}\frac{\nu_2^p}{2^p p!}$$

$$+\frac{2q!2p!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{\nu_2^{q-1}}{2^{q-1}(q-1)!}\frac{\nu_2^p}{2^p p!}$$

$$+\frac{2q!2p!}{2^2 2^0}\,\frac{\sigma_0^4\sigma_1^0}{2!0!}\,\frac{\nu_2^{q-2}}{2^{q-2}(q-2)!}\,\frac{\nu_2^p}{2^p p!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2^0}\,\frac{\sigma_0^{2q}\sigma_1^0}{q!0!}\,\frac{\nu_2^{q-q}}{2^{q-q}(q-q)!}\,\frac{\nu_2^p}{2^p p!}$$

$$+\frac{2q!2p!}{2^0 2}\,\frac{\sigma_0^0\sigma_1^2}{0!1!}\,\frac{\nu_2^q}{2^q(q)!}\,\frac{\nu_2^{p-1}}{2^{p-1}(p-1)!}$$

$$+\frac{2q!2p!}{2^1 2}\,\frac{\sigma_0^2\sigma_1^2}{1!1!}\,\frac{\nu_2^{q-1}}{2^{q-1}(q-1)!}\,\frac{\nu_2^{p-1}}{2^{p-1}(p-1)!}$$

$$+\frac{2q!2p!}{2^2 2}\,\frac{\sigma_0^4\sigma_1^2}{2!1!}\,\frac{\nu_2^{q-2}}{2^{q-2}(q-2)!}\,\frac{\nu_2^{p-1}}{2^{p-1}(p-1)!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2}\,\frac{\sigma_0^{2q}\sigma_1^2}{q!1!}\,\frac{\nu_2^{q-q}}{2^{q-q}(q-q)!}\,\frac{\nu_2^{p-1}}{2^{p-1}(p-1)!}$$

$$+\frac{2q!2p!}{2^0 2^2}\,\frac{\sigma_0^0\sigma_1^4}{0!2!}\,\frac{\nu_2^q}{2^q(q)!}\,\frac{\nu_2^{p-2}}{2^{p-2}(p-2)!}$$

$$+\frac{2q!2p!}{2^1 2^2}\,\frac{\sigma_0^2\sigma_1^4}{1!2!}\,\frac{\nu_2^{q-1}}{2^{q-1}(q-1)!}\,\frac{\nu_2^{p-2}}{2^{p-2}(p-2)!}$$

$$+\frac{2q!2p!}{2^2 2^2}\,\frac{\sigma_0^4\sigma_1^4}{2!2!}\,\frac{\nu_2^{q-2}}{2^{q-2}(q-2)!}\,\frac{\nu_2^{p-2}}{2^{p-2}(p-2)!}$$

.....

.....

$$+\frac{2q!2p!}{2^q 2^2}\,\frac{\sigma_0^{2q}\sigma_1^4}{q!2!}\,\frac{\nu_2^{q-q}}{2^{q-q}(q-q)!}\,\frac{\nu_2^{p-2}}{2^{p-2}(p-2)!}$$

.....

.....

.....

.....

$$+\frac{2q!2p!}{2^0 2^p}\,\frac{\sigma_0^0\sigma_1^{2p}}{0!p!}\,\frac{\nu_2^q}{2^q(q)!}\,\frac{\nu_2^{p-p}}{2^{p-p}(p-p)!}$$

$$+\frac{2q!2p!}{2^1 2^p}\,\frac{\sigma_0^2\sigma_1^{2p}}{1!p!}\,\frac{\nu_2^{q-1}}{2^{q-1}(q-1)!}\,\frac{\nu_2^{p-p}}{2^{p-p}(p-p)!}$$

$$+\frac{2q!2p!}{2^2 2^p}\ \frac{\sigma_0^4\sigma_1^{2p}}{2!p!}\ \frac{\nu_2^{q-2}}{2^{q-2}(q-2)!}\ \frac{\nu_2^{p-p}}{2^{p-p}(p-p)!}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{2q!2p!}{2^q 2^p}\ \frac{\sigma_0^{2q}\sigma_1^{2p}}{q!p!}\ \frac{\nu_2^{q-q}}{2^{q-q}(q-q)!}\ \frac{\nu_2^{p-p}}{2^{p-p}(p-p)!}$$

$$\sim \frac{2q!2p!}{2^p 2^q}\Big[\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{\nu_2^q}{q!}\frac{\nu_2^p}{p!}+\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{\nu_2^{q-1}}{(q-1)!}\frac{\nu_2^p}{p!}+\frac{\sigma_0^4\sigma_1^0}{2!0!}\ \frac{\nu_2^{q-2}}{(q-2)!}\ \frac{\nu_2^p}{p!}+\cdots\cdots+\ \frac{\sigma_0^{2q}\sigma_1^0}{q!0!}\ \frac{\nu_2^{q-q}}{(q-q)!}\ \frac{\nu_2^p}{p!}$$

$$+\frac{\sigma_0^0\sigma_1^2}{0!1!}\frac{\nu_2^q}{(q)!}\frac{\nu_2^{p-1}}{(p-1)!}+\frac{\sigma_0^2\sigma_1^2}{1!1!}\frac{\nu_2^{q-1}}{(q-1)!}\frac{\nu_2^{p-1}}{(p-1)!}+\frac{\sigma_0^4\sigma_1^2}{2!1!}\ \frac{\nu_2^{q-2}}{(q-2)!}\ \frac{\nu_2^{p-1}}{(p-1)!}+\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+\frac{\sigma_0^{2q}\sigma_1^2}{q!1!}\frac{\nu_2^{q-q}}{(q-q)!}\ \frac{\nu_2^{p-1}}{(p-1)!}$$

$$+\frac{\sigma_0^0\sigma_1^4}{0!2!}\frac{\nu_2^q}{(q)!}\frac{\nu_2^{p-2}}{(p-2)!}+\frac{\sigma_0^2\sigma_1^4}{1!2!}\frac{\nu_2^{q-1}}{(q-1)!}\frac{\nu_2^{p-2}}{(p-2)!}+\frac{\sigma_0^4\sigma_1^4}{2!2!}\ \frac{\nu_2^{q-2}}{(q-2)!}\ \frac{\nu_2^{p-2}}{(p-2)!}+\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+\ \frac{\sigma_0^{2q}\sigma_1^4}{q!2!}\ \frac{\nu_2^{q-q}}{(q-q)!}\ \frac{\nu_2^{p-2}}{(p-2)!}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{\sigma_0^0\sigma_1^{2p}}{0!p!}\frac{\nu_2^q}{(q)!}\ \frac{\nu_2^{p-p}}{(p-p)!}+\frac{\sigma_0^2\sigma_1^{2p}}{1!p!}\frac{\nu_2^{q-1}}{(q-1)!}\frac{\nu_2^{p-p}}{(p-p)!}+\frac{\sigma_0^4\sigma_1^{2p}}{2!p!}\ \frac{\nu_2^{q-2}}{(q-2)!}\ \frac{\nu_2^{p-p}}{(p-p)!}+\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+\ \frac{\sigma_0^{2q}\sigma_1^{2p}}{q!p!}\ \frac{\nu_2^{q-q}}{(q-q)!}\ \frac{\nu_2^{p-p}}{(p-p)!}\Big]$$

$$\sim \frac{2q!2p!}{2^p 2^q}\Big[\frac{\nu_2^p}{p!}\frac{\sigma_1^0}{0!}\Big(\frac{\sigma_0^0}{0!}\frac{\nu_2^q}{q!}+\frac{\sigma_0^2}{1!}\frac{\nu_2^{q-1}}{(q-1)!}+\frac{\sigma_0^4}{2!}\ \frac{\nu_2^{q-2}}{(q-2)!}+\cdots\cdots+\ \frac{\sigma_0^{2q}}{q!}\ \frac{\nu_2^{q-q}}{(q-q)!}\Big)$$

$$+\frac{\nu_2^{p-1}}{(p-1)!}\frac{\sigma_1^2}{1!}\Big(\frac{\sigma_0^0}{0!}\frac{\nu_2^q}{q!}+\frac{\sigma_0^2}{1!}\frac{\nu_2^{q-1}}{(q-1)!}+\frac{\sigma_0^4}{2!}\ \frac{\nu_2^{q-2}}{(q-2)!}+\cdots\cdots+\ \frac{\sigma_0^{2q}}{q!}\ \frac{\nu_2^{q-q}}{(q-q)!}\Big)$$

$$+\frac{\nu_2^{p-2}}{(p-2)!}\frac{\sigma_1^4}{2!}\Big(\frac{\sigma_0^0}{0!}\frac{\nu_2^q}{q!}+\frac{\sigma_0^2}{1!}\frac{\nu_2^{q-1}}{(q-1)!}+\frac{\sigma_0^4}{2!}\ \frac{\nu_2^{q-2}}{(q-2)!}+\cdots\cdots+\ \frac{\sigma_0^{2q}}{q!}\ \frac{\nu_2^{q-q}}{(q-q)!}\Big)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{\nu_2^{p-p}}{(p-p)!}\frac{\sigma_1^{2p}}{p!}\Big(\frac{\sigma_0^0}{0!}\frac{\nu_2^q}{q!}+\frac{\sigma_0^2}{1!}\frac{\nu_2^{q-1}}{(q-1)!}+\frac{\sigma_0^4}{2!}\ \frac{\nu_2^{q-2}}{(q-2)!}+.....+\ \frac{\sigma_0^{2q}}{q!}\ \frac{\nu_2^{q-q}}{(q-q)!}\Big)\Big]$$

$$\sim\frac{2q!2p!}{2^p2^q}\Big[\frac{\nu_2^p}{p!}\frac{\sigma_1^0}{0!}+\frac{\nu_2^{p-1}}{(p-1)!}\frac{\sigma_1^2}{1!}+\frac{\nu_2^{p-2}}{(p-2)!}\frac{\sigma_1^4}{2!}+.....+\frac{\nu_2^{p-p}}{(p-p)!}\frac{\sigma_1^{2p}}{p!}\Big]\frac{\left(\sigma_0^2+\nu_2\right)^q}{q!}$$

$$\sim\frac{2q!2p!}{2^p2^q}\frac{\left(\sigma_0^2+\nu_2\right)^q}{q!}\ \frac{\left(\sigma_1^2+\nu_2\right)^p}{p!}$$

which is the moment of order (2q,2p) of $N(0,\sigma_0^2+\nu_2)\otimes N(0,\sigma_1^2+\nu_2)$. ■

Now we want to prove necessary condition. Then, we need the following lemma.

Lemma 4.3.2 *We suppose that*

$$\mu_q\mu_p$$

$$=\frac{q!p!}{2^02^0}\frac{\sigma_0^0\sigma_0^0}{0!0!}\frac{S_{q,p}}{q!p!}+\frac{q!p!}{2^12^0}\frac{\sigma_0^2\sigma_0^0}{1!0!}\frac{\nu_{q-2}\nu_p}{(q-2)!p!}+.....+\frac{q!p!}{2^{q"}2^0}\ \frac{\sigma_0^{2q"}\sigma_0^0}{q"!0!}\ \frac{\nu_{q-2q"}\nu_p}{(q-2q")!p!}$$

$$+\frac{q!p!}{2^02}\frac{\sigma_0^0\sigma_0^2}{0!1!}\frac{\nu_q\nu_{p-2}}{q!(p-2)!}+\frac{q!p!}{2^12}\frac{\sigma_0^2\sigma_0^2}{1!1!}\frac{\nu_{q-2}\nu_{p-2}}{(q-2)!(p-2)!}+.....+\frac{q!p!}{2^{q"}2}\ \frac{\sigma_0^{2q"}\sigma_0^2}{q"!1!}\ \frac{\nu_{q-2q"}\nu_{p-2}}{(q-2q")(p-2)!}$$

$$+\frac{q!p!}{2^02^2}\frac{\sigma_0^0\sigma_0^4}{0!2!}\frac{\nu_q\nu_{p-4}}{q!(p-4)!}+\frac{q!p!}{2^12^2}\frac{\sigma_0^2\sigma_0^4}{1!2!}\frac{\nu_{q-2}\nu_{p-4}}{(q-2)!(p-4)!}+.....+\frac{q!p!}{2^{q"}2^2}\ \frac{\sigma_0^{2q"}\sigma_0^4}{q"!2!}\ \frac{S_{q-2q",p-4}}{(q-2q")!(p-4)!}$$

$$.....$$

$$+\frac{q!p!}{2^02^{p"}}\frac{\sigma_0^0\sigma_0^{2p"}}{0!p"!}\frac{\nu_q\nu_{p-2p"}}{q!(p-2p")!}+\frac{q!p!}{2^12^{p"}}\frac{\sigma_0^2\sigma_0^{2p"}}{1!p"!}\frac{\nu_{q-2}\nu_{p-2p"}}{(q-2)!(p-2p")!}+.....$$

$$.....+\frac{q!p!}{2^{q"}2^{p"}}\ \frac{\sigma_0^{2q"}\sigma_0^{2p"}}{q"!p"!}\ \frac{\nu_{q-2q"}\nu_{p-2p"}}{(q-2q")!(p-2p")!}\ .$$

Then, $S_{q,p}=\nu_q\nu_p$.

Proof We know that $1=M_2=\sigma_0^2+\nu_2$. If q=2h and p=2k, one can write

$$\frac{q!p!}{2^02^0}\frac{\sigma_0^0\sigma_0^0}{0!0!}\frac{S_{q,p}}{q!p!}+\frac{q!p!}{2^12^0}\frac{\sigma_0^2\sigma_0^0}{1!0!}\frac{\nu_{q-2}\nu_p}{(q-2)!p!}+.....+\frac{q!p!}{2^{q"}2^0}\ \frac{\sigma_0^{2q"}\sigma_0^0}{q"!0!}\ \frac{\nu_{q-2q"}\nu_p}{(q-2q")!p!}$$

$$+\frac{q!p!}{2^02}\frac{\sigma_0^0\sigma_0^2}{0!1!}\frac{\nu_q\nu_{p-2}}{q!(p-2)!}+\frac{q!p!}{2^12}\frac{\sigma_0^2\sigma_0^2}{1!1!}\frac{\nu_{q-2}\nu_{p-2}}{(q-2)!(p-2)!}+.....+\frac{q!p!}{2^{q"}2}\ \frac{\sigma_0^{2q"}\sigma_0^2}{q"!1!}\ \frac{\nu_{q-2q"}\nu_{p-2}}{(q-2q")(p-2)!}$$

$$.....$$

$$+\frac{q!p!}{2^02^{p"}}\frac{\sigma_0^0\sigma_0^{2p"}}{0!p"!}\frac{\nu_q\nu_{p-2p"}}{q!(p-2p")!}+\frac{q!p!}{2^12^{p"}}\frac{\sigma_0^2\sigma_0^{2p"}}{1!p"!}\frac{\nu_{q-2}\nu_{p-2p"}}{(q-2)!(p-2p")!}+.....$$

$$.....+\frac{q!p!}{2^{q"}2^{p"}}\ \frac{\sigma_0^{2q"}\sigma_0^{2p"}}{q"!p"!}\ \frac{\nu_{q-2q"}\nu_{p-2p"}}{(q-2q")!(p-2p")!}$$

$$= \frac{q!p!}{2^0 2^0}\frac{\sigma_0^0\sigma_0^0}{0!0!}\frac{S_{q,p}}{q!p!} + \frac{q!p!}{2^1 2^0}\frac{\sigma_0^2\sigma_0^0}{1!0!}\frac{\nu_2^{h-1}\nu_2^k}{2^{h-1}(h-1)!2^k k!} + \ldots\ldots\ldots\ldots + \frac{q!p!}{2^{q"}2^0}\frac{\sigma_0^{2q"}\sigma_0^0}{q"!0!}\frac{\nu_2^{h-q"}\nu_2^k}{2^{h-q"}(h-q")!2^k k!}$$

$$+\frac{q!p!}{2^0 2}\frac{\sigma_0^0\sigma_0^2}{0!1!}\frac{\nu_2^h\nu_2^{k-1}}{2^h h!2^{k-1}(k-1)!}+\frac{q!p!}{2^1 2}\frac{\sigma_0^2\sigma_0^2}{1!1!}\frac{\nu_2^{h-1}\nu_2^{k-1}}{2^{h-1}(h-1)!2^{k-1}(k-1)!}+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + \frac{q!p!}{2^{q"}2}\frac{\sigma_0^{2q"}\sigma_0^2}{q"!1!}\frac{\nu_2^{h-q"}\nu_2^{k-1}}{2^{h-q"}(h-q")!2^{k-1}(k-1)!}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{q!p!}{2^0 2^{p"}}\frac{\sigma_0^0\sigma_0^{2p"}}{0!p"!}\frac{\nu_2^h\nu_2^{k-p"}}{2^h h!2^{k-p"}(k-p")!}+\frac{q!p!}{2^1 2^{p"}}\frac{\sigma_0^2\sigma_0^{2p"}}{1!p"!}\frac{\nu_2^{h-1}\nu_2^{k-p"}}{2^{h-1}(h-1)!2^{k-p"}(k-p")!}+\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots + \frac{q!p!}{2^{q"}2^{p"}}\frac{\sigma_0^{2q"}\sigma_0^{2p"}}{q"!p"!}\frac{\nu_2^{h-q"}\nu_2^{k-p"}}{2^{h-q"}(h-q")!2^{k-p"}(k-p")!}$$

$$= \frac{q!p!}{2^0 2^0}\frac{\sigma_0^0\sigma_0^0}{0!0!}\frac{S_{q,p}}{q!p!} + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^0\nu_2^{h-1}\nu_2^k}{1!0!(h-1)!k!} + \ldots\ldots\ldots\ldots + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^0\nu_2^{h-q"}\nu_2^k}{q"!0!(h-q")!k!}$$

$$+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^0\sigma_0^2\nu_2^h\nu_2^{k-1}}{0!1!h!(k-1)!} + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^2\nu_2^{h-1}\nu_2^{k-1}}{1!1!(h-1)!(k-1)!} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^2\nu_2^{h-q"}\nu_2^{k-1}}{q"!1!(h-q")!(k-1)!}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^0\sigma_0^{2p"}\nu_2^h\nu_2^{k-p"}}{0!p"!h!(k-p")!} + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^{2p"}\nu_2^{h-1}\nu_2^{k-p"}}{1!(h-1)!p"(k-p")!} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^{2p"}\nu_2^{h-q"}\nu_2^{k-p"}}{q"!p"!(h-q")!(k-p")!}$$

$$= \mu_q\mu_p = \frac{q!p!}{2^h h!2^k k!} = \frac{q!p!(\sigma_0^2+\nu_2)^h(\sigma_0^2+\nu_2)^k}{2^h h!2^k k!}$$

$$= \frac{q!p!}{2^h 2^k}\frac{\sigma_0^0\sigma_0^0\nu_2^h\nu_2^k}{0!0!h!k!} + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^0\nu_2^{h-1}\nu_2^k}{1!0!(h-1)!k!} + \ldots\ldots\ldots\ldots + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^0\nu_2^{h-q"}\nu_2^k}{q"!0!(h-q")!k!}$$

$$+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^0\sigma_0^2\nu_2^h\nu_2^{k-1}}{0!1!h!(k-1)!} + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^2\nu_2^{h-1}\nu_2^{k-1}}{1!1!(h-1)!(k-1)!} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^2\nu_2^{h-q"}\nu_2^{k-1}}{q"!1!(h-q")!(k-1)!}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^0\sigma_0^{2p"}\nu_2^h\nu_2^{k-p"}}{0!p"!h!(k-p")!}+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^2\sigma_0^{2p"}\nu_2^{h-1}\nu_2^{k-p"}}{1!(h-1)!p"(k-p")!}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{q!p!}{2^h 2^k}\frac{\sigma_0^{2q"}\sigma_0^{2p"}\nu_2^{h-q"}\nu_2^{k-p"}}{q"!p"!(h-q")!(k-p")!}\ .$$

We deduce that $S_{q,p} = \frac{q!p!\nu_2^h\nu_2^k}{2^h 2^k h!k!}$.

If q (or p) is odd, it is easier to prove because $\mu_q = 0 = \nu_{q-2e}$. ■.

Proof of necessary condition We want to prove by recurence that $S_{q,p} = \nu_q \nu_p$ for all (q,p). It is easy to understand that the result is true if q = 0 or p = 0 : it is the result in dimension 1 . By definition, it is true also if p = q = 1.

At first, we will see by recurrence that it is true if p = 1 or q = 1. For example, if p=1, it is thus true for q=1. Then, we suppose that it holds for all $q' \leq q$. Let Q=q+2. Let $Q" = \lfloor Q/2 \rfloor$. Then, by lemma 4.3.1,

$$0 = \mathbb{E}\Big\{\frac{(X_1 + ... + X_n)^Q (Y_1 + ... + Y_n)}{\Psi(n_m)^{Q+1}}\Big\}$$

$$\sim \frac{Q!1!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{S_{Q,1}}{Q!1!} + \frac{Q!p!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{\nu_{Q-2}\nu_1}{(Q-2)!1!} + ... + \frac{Q!1!}{2^{Q"}2^0}\frac{\sigma_0^{2Q"}\sigma_1^0}{Q"!0!}\frac{\nu_{Q-2Q"}\nu_1}{(Q-2Q")!1!}$$

$$= \frac{Q!1!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{S_{Q,1}}{Q!1!} = 0 \ .$$

Then, the results holds if p=1 or q=1.

Then, we suppose that the result holds for all (q',p') such that $q' \leq q$ and $p' \leq p = q$. Let Q=q+2 and Q"=q"+1. Then, p-2p" = 0 or 1. Therefore, $S_{Q,p-2p"} = \nu_Q \nu_{p-2p"}$. Let P= p-2p"+2. Then, by lemma 4.3.1,

$$\mu_Q \mu_{p-2(p"-1)} = \mathbb{E}\Big\{\frac{(X_1 + ... + X_n)^Q (Y_1 + ... + Y_n)^{p-2(p"-1)}}{\Psi(n_m)^{Q+p-2p"+2}}\Big\}$$

$$\sim \frac{Q!P!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{S_{Q,p-2p"+2}}{Q!(p-2p"+2)!} + \frac{Q!P!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{S_{Q-2,p-2p"+2}}{(Q-2)!(p-2p"+2)!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"}2^0}\frac{\sigma_0^{2Q"}\sigma_1^0}{Q"!0!}\frac{S_{Q-2Q",p-2p"+2}}{(Q-2Q")!(p-2p"+2)!}$$

$$+ \frac{Q!P!}{2^0 2^1}\frac{\sigma_0^0\sigma_1^2}{0!1!}\frac{S_{Q,p-2p"}}{Q!(p-2p")!} + \frac{Q!P!}{2^1 2^1}\frac{\sigma_0^2\sigma_1^2}{1!1!}\frac{S_{Q-2,p-2p"}}{(Q-2)!(p-2p")!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"}2^1}\frac{\sigma_0^{2Q"}\sigma_1^2}{Q"!1!}\frac{S_{Q-2Q",p-2p"}}{(Q-2Q")!(p-2p")!}$$

$$\sim \frac{Q!P!}{2^0 2^0}\frac{\sigma_0^0\sigma_1^0}{0!0!}\frac{S_{Q,p-2p"+2}}{Q!(p-2p"+2)!} + \frac{Q!P!}{2^1 2^0}\frac{\sigma_0^2\sigma_1^0}{1!0!}\frac{\nu_{Q-2}\nu_{p-2p"+2}}{(Q-2)!(p-2p"+2)!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"}2^0}\frac{\sigma_0^{2Q"}\sigma_1^0}{Q"!0!}\frac{\nu_{Q-2Q"}\nu_{p-2p"+2}}{(Q-2Q")!(p-2p"+2)!}$$

$$+ \frac{Q!P!}{2^0 2^1}\frac{\sigma_0^0\sigma_1^2}{0!1!}\frac{\nu_Q\nu_{p-2p"}}{Q!(p-2p")!} + \frac{Q!P!}{2^1 2^1}\frac{\sigma_0^2\sigma_1^2}{1!1!}\frac{\nu_{Q-2}\nu_{p-2p"}}{(Q-2)!(p-2p")!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"}2^1}\frac{\sigma_0^{2Q"}\sigma_1^2}{Q"!1!}\frac{\nu_{Q-2Q"}\nu_{p-2p"}}{(Q-2Q")!(p-2p")!}$$

By lemma 4.3.2, we deduce that $S_{Q,p-2p"+2} = \nu_Q \nu_{p-2p"+2}$.

We proceed by the same way for $S_{Q,p-2p"+4}$: $S_{Q,p-2p"+4} = \nu_Q \nu_{p-2p"+4}$, and so on by recurrence untill $S_{Q,p} = \nu_Q \nu_p = S_{q+2,q} = \nu_{q+2}\nu_q$. Therefore, $S_{q+2,p'} = \nu_{q+2}\nu_{p'}$ for all $p' \leq p = q$.

We proceed by the same way for $S_{q,p+2}$ by reversing p and q and we prove that $S_{q',p+2} = \nu_{q'}\nu_{p+2}$ for all $q' \leq q$.

It remains to apply a last time lemma 4.3.1, and 4.3.2 in order to prove that $S_{q+2,q+2} = \nu_{q+2}\nu_{p+2}$. Indeed, let P=p+2. Then,

$$\mathbb{E}\Big\{ \frac{(X_1 + ... + X_n)^Q (Y_1 + ... + Y_n)^P}{\Psi(n_m)^{Q+P}} \Big\}$$

$$\sim \frac{Q!P!}{2^0 2^0} \frac{\sigma_0^0 \sigma_1^0}{0!0!} \frac{S_{Q,P}}{Q!P!} + \frac{Q!P!}{2^1 2^0} \frac{\sigma_0^2 \sigma_1^0}{1!0!} \frac{\nu_{Q-2}\nu_P}{(Q-2)!P!} + ... + \frac{Q!P!}{2^{Q"} 2^0} \frac{\sigma_0^{2Q"} \sigma_1^0}{Q"!0!} \frac{\nu_{Q-2Q"}\nu_P}{(Q-2Q")!P!}$$

$$+ \frac{Q!P!}{2^0 2} \frac{\sigma_0^0 \sigma_1^2}{0!1!} \frac{\nu_Q \nu_{P-2}}{Q!(P-2)!} + \frac{Q!P!}{2^1 2} \frac{\sigma_0^2 \sigma_1^2}{1!1!} \frac{\nu_{Q-2}\nu_{P-2}}{(Q-2)!(P-2)!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"} 2} \frac{\sigma_0^{2Q"} \sigma_1^2}{Q"!1!} \frac{\nu_{Q-2Q"}\nu_{P-2}}{(Q-2Q")(P-2)!}$$

$$+ \frac{Q!P!}{2^0 2^2} \frac{\sigma_0^0 \sigma_1^4}{0!2!} \frac{\nu_{Q-4}\nu_{P-4}}{Q!(P-4)!} + \frac{Q!P!}{2^1 2^2} \frac{\sigma_0^2 \sigma_1^4}{1!2!} \frac{\nu_{Q-2}\nu_{P-4}}{(Q-2)!(P-4)!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"} 2^2} \frac{\sigma_0^{2Q"} \sigma_1^4}{Q"!2!} \frac{S_{Q-2Q",P-4}}{(Q-2Q")!(P-4)!}$$

$$.....$$

$$+ \frac{Q!P!}{2^0 2^{P"}} \frac{\sigma_0^0 \sigma_1^{2P"}}{0!P"!} \frac{\nu_Q \nu_{P-2P"}}{Q!(P-2P")!} + \frac{Q!P!}{2^1 2^{P"}} \frac{\sigma_0^2 \sigma_1^{2P"}}{1!P"!} \frac{\nu_{Q-2}\nu_{P-2P"}}{(Q-2)!(P-2P")!} + ...$$

$$..... + \frac{Q!P!}{2^{Q"} 2^{P"}} \frac{\sigma_0^{2Q"} \sigma_1^{2P"}}{Q"!P"!} \frac{\nu_{Q-2Q"}\nu_{P-2P"}}{(Q-2Q")!(P-2P")!} .$$

This result proves the recurence, and then, the necessary condition. ■

4.4 Second theorem of convergence

Now we have a MCLT with conditions about the $\rho_{j_1,....,j_n}$'s.

Theorem 16 *Let $\tilde{P}_2'^{m,s}(y) = y^2 - \gamma'_{m,s} y - \beta_{m,s}$ where $\gamma'_{m,s} = \mathbb{E}\{Y_{m,s}^3\}/\mathbb{E}\{Y_{m,s}^2\}$. We assume that $\sum_{s=1}^{n_m} \frac{\gamma_{m,s}^2}{\Psi(n)^2}$ and $\sum_{s=1}^{n_m} \frac{\gamma'^2_{m,s}}{\Psi(n)^2}$ are bounded. We assume that $\mathbb{E}\big\{\big(\sum_{s=1}^{n_m} \frac{\gamma_{m,s} X_{m,s}}{\Psi(n)^2}\big)^2\big\} \to 0$ and $\mathbb{E}\big\{\big(\sum_{s=1}^{n_m} \frac{\gamma'_{m,s} Y_{m,s}}{\Psi(n)^2}\big)^2\big\} \to 0$. One assumes that*

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ , \ \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} \tilde{P}_2'^{m,t}(Y_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M_{q,p}^{n_m} = \mathbb{E}\Big\{ \frac{(X_{m,1}+....+X_{m,n_m})^q (Y_{m,1}+....+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}} \Big\}$ converges to a real $M_{q,p}$ if, for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$, there exists $S_{q,p} \in \mathbb{R}$ and $Sb_q^r \in \mathbb{R}$, r=2,3,4,5, such that

$$\sum_{s_1 \neq s_2 \neq ..... \neq s_q} \sum_{t_1 \neq t_2 \neq ..... \neq t_p} \frac{\mathbb{E}\{X_{m,s_1} X_{m,s_2} ..... X_{m,s_q} Y_{m,t_1} Y_{m,t_2} ..... Y_{m,t_p}\}}{\Psi(n_m)^{p+q}} \to S_{q,p} \ ,$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2\ ,$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2'^{m,s_1}(Y_{m,s_1})Y_{m,s_2}.....Y_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^4\ ,$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\gamma_{m,s_1}\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}\}}{\Psi(n_m)^{q+1}}\right|\leq Sb_q^3\ .$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_q}\frac{\gamma'_{m,s_1}\mathbb{E}\{Y_{s_1}Y_{s_2}.....Y_{s_q}\}}{\Psi(n_m)^{q+1}}\right|\leq Sb_q^5\ .$$

Proof of sufficient Condition of theorem 16 The conditions of theorem are checked if p=0 or q=0. Therefore, all the conditions of theorem 13 are checked for the sequences $X_{m,t}$ and $Y_{m,t}$. Therefore

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{(X_{m,s_1})^2X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2\ ,$$

$$\left|\sum_{t_1\neq t_2\neq.....\neq t_{p-1}}\frac{\mathbb{E}\{(Y_{m,t_1})^2Y_{m,t_2}.....Y_{m,t_{p-1}}\}}{\Psi(n_m)^p}\right|\leq Sb_p^2\ .$$

Then, it is enough to apply theorem 15 in order to prove the sufficient condition. ■

Proof of necessary condition of theorem 16 We suppose that all the moments converge. Therefore, the first condition is checked by theorem 15. Then, the moments of marginal distributions converge. Then, for the four other conditions, it is enough to use theorem 13 for sequences $X_{m,t}$ and $Y_{m,t}$. ■

Corollary 4.4.1 *One assumes that $\sigma_0^2=\sigma_1^2$. We assume that $\sum_{s=1}^{n_m}\frac{\gamma_{m,s}^2}{\Psi(n)^2}$ and $\sum_{s=1}^{n_m}\frac{\gamma'^2_{m,s}}{\Psi(n)^2}$ are bounded. We assume that $\mathbb{E}\{(\sum_{s=1}^{n_m}\frac{\gamma_{m,s}X_{m,s}}{\Psi(n)^2})^2\}\to 0$ and $\mathbb{E}\{(\sum_{s=1}^{n_m}\frac{\gamma'_{m,s}Y_{m,s}}{\Psi(n)^2})^2\}\to 0$. One assumes that*

$$\mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}\tilde{P}_2^{m,t}(X_{m,t})}{\Psi(n_m)^2}\right]^2\right\}\to 0\ ,\ \mathbb{E}\left\{\left[\frac{\sum_{t=1}^{n_m}\tilde{P}_2'^{m,t}(Y_{m,t})}{\Psi(n_m)^2}\right]^2\right\}\to 0\ .$$

All the moments $M_{q,p}^{n_m}=\mathbb{E}\left\{\frac{(X_{m,1}+....+X_{m,n_m})^q(Y_{m,1}+....+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}}\right\}$ converges to $\mu_q\mu_p$ if and only if , for all $q\in\mathbb{N}$, for all $p\in\mathbb{N}$, there exists $Sb_q^r\in\mathbb{R}$, r=2,3,4,5, such that

$$\sum_{s_1\neq s_2\neq.....\neq s_q}\sum_{t_1\neq t_2\neq.....\neq t_p}\frac{\mathbb{E}\{X_{s_1}X_{s_2}.....X_{s_q}Y_{t_1}Y_{t_2}.....Y_{t_p}\}}{\Psi(n_m)^{p+q}}\to\nu_q\nu_p\ ,$$

$$\left|\sum_{s_1\neq s_2\neq.....\neq s_{q-1}}\frac{\mathbb{E}\{\tilde{P}_2^{m,s_1}(X_{m,s_1})X_{m,s_2}.....X_{m,s_{q-1}}\}}{\Psi(n_m)^q}\right|\leq Sb_q^2\ ,$$

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{\tilde{P}_2'^{m,s_1}(Y_{m,s_1})Y_{m,s_2}\ldots\ldots Y_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq Sb_q^4 \ ,$$

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\gamma_{m,s_1}\mathbb{E}\{X_{s_1}X_{s_2}\ldots\ldots X_{s_q}\}}{\Psi(n_m)^{q+1}} \right| \leq Sb_q^3 \ .$$

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \frac{\gamma'_{m,s_1}\mathbb{E}\{Y_{s_1}Y_{s_2}\ldots\ldots Y_{s_q}\}}{\Psi(n_m)^{q+1}} \right| \leq Sb_q^5 \ .$$

Proof of sufficient Condition of Corollary 4.4.1 If the conditions of corollary are checked, all the moments converge by theorem 16 .

Therefore, all the conditions of theorem 15 are checked for the sequences $X_{m,t}$ and $Y_{m,t}$. Therefore

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2 X_{m,s_2}\ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq Sb_q^2 \ ,$$

$$\left| \sum_{t_1 \neq t_2 \neq \ldots\ldots \neq t_{p-1}} \frac{\mathbb{E}\{(Y_{m,t_1})^2 Y_{m,t_2}\ldots\ldots Y_{m,t_{p-1}}\}}{\Psi(n_m)^p} \right| \leq Sb_p^2 \ .$$

Then, it is enough to use proposition 4.3.1 in order to prove the sufficient condition. ■

Proof of necessary condition of Corollary 4.4.1 We suppose that all the moments converge to the moments of $N(0, I_2)$. Then, by proposition 4.3.1, the first condition is checked. By theorem 16, the other conditions are also checked. ■

4.5 Third theorem of convergence

Now we suppose that $X_{m,t}$ and $Y_{m,t}$ are bounded. Remak that, in this case, hypothesis 4.1.1 holds.

Theorem 17 *We suppose that there exists $F > 0$ such that $|X_{m,t}| \leq F$ and $|Y_{m,t}| \leq F$.*

One assumes that

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ , \ \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2'^{m,t}(Y_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M_{q,p}^{n_m} = \mathbb{E}\left\{ \frac{(X_{m,1}+\ldots+X_{m,n_m})^q (Y_{m,1}+\ldots+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}} \right\}$ converges to a real $M_{q,p}$ if, for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$, there exists $S_{q,p} \in \mathbb{R}$:

$$\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \sum_{t_1 \neq t_2 \neq \ldots\ldots \neq t_p} \frac{\mathbb{E}\{X_{m,s_1}X_{m,s_2}\ldots\ldots X_{m,s_q}Y_{m,t_1}Y_{m,t_2}\ldots\ldots Y_{m,t_p}\}}{\Psi(n_m)^{p+q}} \to S_{q,p} \ .$$

Proof of sufficient Condition of theorem 17 The conditions of theorem are checked when p=0 or q=0. Therefore, all the conditions of proposition 3.6.1 are checked by sequences $X_{m,t}$ and $Y_{m,t}$. Then, all the moments are bounded. Then, by proposition, 3.3.1

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2 X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq Sb_q^2 \ ,$$

$$\left| \sum_{t_1 \neq t_2 \neq \ldots\ldots \neq t_{p-1}} \frac{\mathbb{E}\{(Y_{m,t_1})^2 Y_{m,t_2} \ldots\ldots Y_{m,t_{p-1}}\}}{\Psi(n_m)^p} \right| \leq Sb_p^2 \ .$$

Then, it is enough to apply theorem 15 in order to prove the sufficient condition. ■

Proof of necessary condition of theorem 17 We suppose that all the moments converge. Then, the condition is proved by theorem 15. ■

Corollary 4.5.1 *One assumes $\sigma_0^2 = \sigma_1^2$. We suppose that there exists $F > 0$ such that $|X_{m,t}| \leq F$ and $|Y_{m,t}| \leq F$.*

One assumes that

$$\mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{m,t}(X_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ , \ \mathbb{E}\left\{ \left[\frac{\sum_{t=1}^{n_m} Q_2^{\prime m,t}(Y_{m,t})}{\Psi(n_m)^2} \right]^2 \right\} \to 0 \ .$$

All the moments $M_{q,p}^{n_m} = \mathbb{E}\left\{ \frac{(X_{m,1}+\ldots.+X_{m,n_m})^q (Y_{m,1}+\ldots.+Y_{m,n_m})^p}{\Psi(n_m)^{p+q}} \right\}$ converges to $\mu_q \mu_p$ if and only if , for all $q \in \mathbb{N}$, for all $p \in \mathbb{N}$,

$$\sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_q} \sum_{t_1 \neq t_2 \neq \ldots\ldots \neq t_p} \frac{\mathbb{E}\{X_{s_1} X_{s_2} \ldots\ldots X_{s_q} Y_{t_1} Y_{t_2} \ldots\ldots Y_{t_p}\}}{\Psi(n_m)^{p+q}} \to \nu_q \nu_p \ .$$

Proof of sufficient Condition of Corollary 4.5.1 If the conditions of the corollary are checked, all the moments converge by theorem 17 .

Therefore, all the conditions of theorem 15 are checked by the sequences $X_{m,t}$ and $Y_{m,t}$. Therefore,

$$\left| \sum_{s_1 \neq s_2 \neq \ldots\ldots \neq s_{q-1}} \frac{\mathbb{E}\{(X_{m,s_1})^2 X_{m,s_2} \ldots\ldots X_{m,s_{q-1}}\}}{\Psi(n_m)^q} \right| \leq Sb_q^2 \ ,$$

$$\left| \sum_{t_1 \neq t_2 \neq \ldots\ldots \neq t_{p-1}} \frac{\mathbb{E}\{(Y_{m,t_1})^2 Y_{m,t_2} \ldots\ldots Y_{m,t_{p-1}}\}}{\Psi(n_m)^p} \right| \leq Sb_p^2 \ .$$

Then, it is enough to apply proposition 4.3.1 in order to prove sufficient condition. ■

Proof of necessary condition of Corollary 4.5.1 We suppose that all the moments converge. Therefore, by proposition 4.3.1 , the condition is proved. ■

Appendix A

Some demonstrations

A.1 Correlation coefficients of higher ordrer

A.1.1 First proposition

Now we prove proposition 1.1.1, that is that

$$F_X(x) = \int_{u \geq x}^{*} f(u) m^{\otimes}(du)$$

$$= F_{X_1}(x_1).....F_{X_n}(x_n)$$

$$+ \lim_{k_n \to \infty} \Big[\lim_{k_{n-1} \to \infty} \Big[..... \lim_{k_1 \to \infty} \Big[\sum_{j_1 \leq k_1, ..., j_n \leq k_n} \rho_{j_1, j_2, ..., j_n} \Big(\int_{-\infty}^{x_1} P_{j_1}^1 dm_1 \Big) .... \Big(\int_{-\infty}^{x_n} P_{j_n}^n dm_n \Big) \Big] ..... \Big] \Big] .$$

We need the following lemma.

Lemma A.1.1 *Let M_X be the probability of $(X_1, X_2, ..., X_n)$. Let $f \in L^4(\mathbb{R}^n, M_X)$. Let $g^s \in L^{4(n-1)}(\mathbb{R}, m_s)$ for s=1,2,...,n. For s=1,2,....,n, let g_k^s be a sequence of $L^{4(n-1)}(\mathbb{R}, m_s)$ such that $g_k^s \to g^s$ in $L^2(\mathbb{R}, m_s)$. Then,*

$$\lim_{k_n \to \infty} \Big[\lim_{k_{n-1} \to \infty} \Big[..... \lim_{k_1 \to \infty} \int \Big(f(x_1, x_2, ..., x_n) g_{k_1}^1(x_1) g_{k_2}^2(x_2) ..... g_{k_n}^n(x_n) M_X(dx_1, ..., dx_n) \Big) \Big] .... \Big] \Big]$$

$$= \int \Big(f(x_1, x_2, ..., x_n) g^1(x_1) g^2(x_2) ..... g^n(x_n) M_X(dx_1, ..., dx_n) \ .$$

Proof By Holder's Inequality,

$$\Big[\int f(x_1, x_2, ..., x_n) g_{k_2}^2(x_2) g_{k_3}^3(x_3) ..... g_{k_n}^n(x_n) [g_{k_1}^1(x_1) - g^1(x_1)] M_X(dx_1 dx_2 ..... dx_n) \Big]^4$$

$$\leq \Big[\int f(x_1, ..., x_n)^4 M_X(dx_1 ..... dx_n) \Big] \Big[\int \big(g_{k_2}^2(x_2) ..... g_{k_n}^n(x_n) \big)^4 M_X(dx_2 ..... dx_n) \Big]$$

$$\Big[\int [g_{k_1}^1(x_1) - g^1(x_1)]^2 M_X(dx_1 ..... dx_n) \Big]^2$$

$$\leq \Big[\int f(x_1,...,x_n)^4 M_X(dx_1.....dx_n)\Big]^4 \Big[\int g_{k_2}^2(x_2)^{4n-4} m_2(dx_2)\Big]^{\frac{1}{n-1}} .....$$

$$.....\Big[\int g_{k_n}^n(x_n)^{4n-4} m_n(dx_n)\Big]^{\frac{1}{n-1}} \Big[\int [g_{k_1}^1(x_1) - g^1(x_1)]^2 M_X(dx_1.....dx_n)\Big]^2$$

converges to 0 as $k_1 \to \infty$.

By the same method applied for k_s, s=2,3,....,n, one proves the result. For example,

$$\int f(x_1,x_2,...,x_n) g_{k_3}^3(x_3).....g_{k_n}^n(x_n)[g_{k_2}^2(x_2) - g^2(x_2)]g^1(x_1)M_X(dx_1dx_2.....dx_n)\Big] \to 0 \ .$$

Then, for example, for n=3,

$$\int f(x_1,x_2,x_3) g_{k_1}^1(x_1) g_{k_2}^2(x_2) g_{k_3}^3(x_3) M_X(dx_1dx_2.....dx_n)$$

$$= \int f(x_1,x_2,x_3) g_{k_2}^2(x_2) g_{k_3}^3(x_3)[g_{k_1}^1(x_1) - g^1(x_1)]M_X(dx_1dx_2.....dx_n)$$

$$+ \int f(x_1,x_2,x_3) g_{k_3}^3(x_3)[g_{k_2}^2(x_2) - g^2(x_2)]g^1(x_1)M_X(dx_1dx_2.....dx_n)$$

$$+ \int f(x_1,x_2,x_3)[g_{k_3}^3(x_3) - g_3(x_3)]g^2(x_2)g^1(x_1)M_X(dx_1dx_2.....dx_n)$$

$$+ \int f(x_1,x_2,x_3) g^1(x_1)g^2(x_2)g^3(x_3)M_X(dx_1dx_2.....dx_n) \ .$$

Then, for $\epsilon > 0$, one chooses k_3 such that
$|\int f(x_1,x_2,x_3)[g_3^{k_3}(x_3) - g_3(x_3)]g^2(x_2)g^1(x_1)M_X(dx_1dx_2.....dx_n)| \leq \epsilon/3$.

After, for this k_3, one chooses k_2 such that
$|\int f(x_1,x_2,x_3) g_{k_3}^3(x_3)[g_{k_2}^2(x_2) - g^2(x_2)]g^1(x_1)M_X(dx_1dx_2.....dx_n)| \leq \epsilon/3$.

After, for these k_3 and k_2 one chooses k_1 such that
$|\int f(x_1,x_2,x_3) g_2^{k_2}(x_2) g_3^{k_3}(x_3)[g_1^{k_1}(x_1) - g^1(x_1)]M_X(dx_1dx_2.....dx_n)| \leq \epsilon/3$. ■

Proof of proposition 1.1.1 Let B_s, s=1,2,...,n, be n Borel sets of $\mathbb{R}$. Assume that $B_s = [-\infty, x_s]$. Then, $\mathbb{1}_{B_s} = \sum_{j=0}^{\infty} \gamma_j^s P_j^s$ with $\gamma_j^s = \int_{B_s} P_j^s dm_s$.

Then, one proves proposition 1.1.1 by using lemma A.1.1 with f=1 and $g_s = \mathbb{1}_{B_s}(x_s)$. ■

A.1.2 Second and third propositions

Now, we prove proposition 1.1.2 : *$(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale if and only if $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\}$ = 0 for all $n \geq 1$.*

Proof of proposition 1.1.2 By definition, $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale if, for all n, $\mathbb{E}\{X_1 + ... + X_n + X_{n+1}|\mathcal{F}_n\} = X_1 + ... + X_n$. Then, $X_1 + ... + X_n = \mathbb{E}\{X_1 + ... + X_n + X_{n+1}|\mathcal{F}_n\} = \mathbb{E}\{X_1 + ... + X_n|\mathcal{F}_n\} + \mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = X_1 + ... + X_n + \mathbb{E}\{X_{n+1}|\mathcal{F}_n\}$. Then, $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$.

Reciprocally, if $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$, $\mathbb{E}\{X_1 + ... + X_n + X_{n+1}|\mathcal{F}_n\} = \mathbb{E}\{X_1 + ... + X_n|\mathcal{F}_n\} + \mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = \mathbb{E}\{X_1 + ... + X_n|\mathcal{F}_n\} = X_1 + ... + X_n$. ■

Now, we prove proposition 1.1.3, i.e. :

If $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale, $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \geq 1$. Conversely, if $\{P_s^j\}$ is a basis of $L^2(\mathbb{R}, m_s)$ and if $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \in \mathbb{N}^*$ and for all $(j_1, ..., j_n)$, then $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale.

Proof of proposition 1.1.3 If $(X_1 + ... + X_n, \mathcal{F}_n)$ is a martingale, for all n, $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$. Then, X_{n+1} is orthogonal to the space L^2 generated by X_1,.....,X_n. In particular X_{n+1} is orthogonal to the function $P_{j_1}^1(X_1).....P_{j_n}^n(X_n)$. Then, $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $n \in \mathbb{N}^*$.

Reciprocally, assume that, for all s, $\{P_j^s\}$ is a basis of $L^2(\mathbb{R}, m_s)$, and that $\rho_{j_1,j_2,....,j_n,1} = 0$ for all $(j_1, ..., j_n)$ and for all $n \in \mathbb{N}^*$.

Then one uses lemma A.1.1 with $f(x_1, ..., x_n, x_{n+1}) = x_{n+1}$. One uses the same notations as in the proof of proposition 1.1.1 : Let B_s, s=1,2,...,n, be n Borel sets of $\mathbb{R}$. Then, $\mathbb{1}_{B_t} = \sum_{j=0}^{\infty} \gamma_j^s P_j^s$ with $\gamma_j^s = \int_{B_s} P_j^s dm_s$. One sets $g_s = \mathbb{1}_{B_s}$ and $g_{k_s}^s = \sum_{j=0}^{k_s} \gamma_j^s P_j^s$.

By by lemma A.1.1 , for all Borel sets B_1,....,B_n,

$$\int x_{n+1} \mathbb{1}_{B_1}(x_1).....\mathbb{1}_{B_n}(x_n) M_X(dx_1, ..., dx_n) = 0 \ .$$

Therefore $\mathbb{E}\{X_{n+1}|\mathcal{F}_n\} = 0$. ■

A.2 Equivalences

In this section, we suppose that $m_s = m$ and $X_{m,t} = X_t$. Then, we have the following proposition.

Proposition A.2.1 *We suppose that M_2^n converge with $\Psi(n)^2 = n$. Then,*

$$n^{-2} \sum_{j_1+j_2+...+j_n=4;\ j_s=2\ or\ 0} \rho_{j_1,j_2,...,j_n} \to 0$$

if and only if

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n} \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0 \ .$$

Proof We know that

$$\mathbb{E}\Big\{\Big(\sum_{s=1}^{n} \frac{X_1 + ... + X_n}{\sqrt{n}}\Big)^2\Big\} = \mathbb{E}\Big\{\sum_{s\neq t} \frac{X_s X_t}{n}\Big\} + \mathbb{E}\Big\{\frac{X_1^2 + ... + X_n^2}{n}\Big\}$$

$$= \mathbb{E}\Big\{\sum_{s\neq t} \frac{X_s X_t}{n}\Big\} + \mathbb{E}\{X_1^2\} \ .$$

Therefore, if M_2^n converges, $\sum_{s\neq t} \mathbb{E}\{\frac{X_s X_t}{n}\}$ converges. Therefore, $\sum_{s\neq t} \mathbb{E}\{\frac{X_s X_t}{n^2}\} \to 0$.

Moreover

$$\mathbb{E}\Big\{\sum_{s\neq t} \frac{X_s\big(X_t^2 - \mathbb{E}\{X_t^2\}\big)}{n^2}\Big\}$$

$$= \mathbb{E}\Big\{\sum_{s,t} \frac{X_s\big(X_t^2 - \mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} - \mathbb{E}\Big\{\sum_{s=1}^{n} \frac{X_s\big(X_s^2 - \mathbb{E}\{X_s^2\}\big)}{n^2}\Big\}$$

$$= \frac{1}{\sqrt{n}}\mathbb{E}\Big\{\Big(\sum_{s=1}^{n}\frac{X_s}{\sqrt{n}}\Big)\Big(\sum_{t=1}^{n}\frac{X_t^2-\mathbb{E}\{X_t^2\}}{n}\Big)\Big\} - \sum_{s=1}^{n}\frac{\mathbb{E}\{X_s^3\}}{n^2} + \sum_{s=1}^{n}\frac{\mathbb{E}\{X_s\}\mathbb{E}\{X_s^2\}}{n^2}$$

$$= \frac{1}{\sqrt{n}}\mathbb{E}\Big\{\Big(\sum_{s=1}^{n}\frac{X_s}{\sqrt{n}}\Big)\Big(\sum_{t=1}^{n}\frac{X_t^2-\mathbb{E}\{X_t^2\}}{n}\Big)\Big\} - \frac{\mathbb{E}\{X_1^3\}}{n} + \frac{\mathbb{E}\{X_1\}\mathbb{E}\{X_1^2\}}{n} \ .$$

Now, by Schwartz Inequality,

$$\frac{1}{\sqrt{n}}\mathbb{E}\Big\{\Big(\sum_{s=1}^{n}\frac{X_s}{\sqrt{n}}\Big)\Big(\sum_{t=1}^{n}\frac{X_t^2-\mathbb{E}\{X_t^2\}}{n}\Big)\Big\}^2$$

$$\leq \frac{1}{\sqrt{n}}\mathbb{E}\Big\{\Big(\sum_{s=1}^{n}\frac{X_s}{\sqrt{n}}\Big)^2\Big\}\mathbb{E}\Big\{\Big(\sum_{t=1}^{n}\frac{X_t^2-\mathbb{E}\{X_t^2\}}{n}\Big)^2\Big\}$$

$$\leq \frac{M_2^n}{\sqrt{n}}\mathbb{E}\Big\{\Big(\sum_{s,t}\frac{\big[X_s^2-\mathbb{E}\{X_s^2\}\big]\big[X_t^2-\mathbb{E}\{X_t^2\}\big]}{n^2}\Big\}$$

$$\leq \frac{M_2^n}{\sqrt{n}}\sum_{s,t}\frac{\mathbb{E}\{X_s^2X_t^2\}-\mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}}{n^2}$$

$$\leq \frac{M_2^n}{\sqrt{n}}\Big[\sum_{s,t}\frac{\mathbb{E}\{X_s^4\}^{1/2}\mathbb{E}\{X_t^4\}^{1/2}}{n^2} - \mathbb{E}\{X_1^2\}^2\Big]$$

$$\leq \frac{M_2^n}{\sqrt{n}}\Big[\ \mathbb{E}\{X_1^4\}-\mathbb{E}\{X_1^2\}^2\Big] \to 0 \ .$$

Now, let $\gamma = \mathbb{E}\{X_1^3\}/\mathbb{E}\{X_1^2\}$ and $\sigma_2^2 = \mathbb{E}\big\{\big(X_1^2-\gamma X_1-\mathbb{E}\{X_1^2\}\big)^2\big\}$. Then,

$$n^{-2}\sum_{j_1+j_2+\ldots+j_n=4;\ j_s=2\ or\ 0}\rho_{j_1,j_2,\ldots,j_n} = \mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\sum_{t=s+1}^{n}\frac{P_2(X_s)P_2(X_t)}{n^2}\Big\}$$

$$= \mathbb{E}\Big\{\sum_{s=1}^{n}\sum_{t=s+1}^{n}\frac{\big(X_s^2-\gamma X_s-\mathbb{E}\{X_s^2\}\big)\big(X_t^2-\gamma X_t-\mathbb{E}\{X_t^2\}\big)}{\sigma_2^2n^2}\Big\}$$

$$= \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{\big(X_s^2-\gamma X_s-\mathbb{E}\{X_s^2\}\big)\big(X_t^2-\gamma X_t-\mathbb{E}\{X_t^2\}\big)}{n^2}\Big\}$$

$$= \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\big(X_t^2-\mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} - \frac{\gamma}{\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_s\big(X_t^2-\mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} + \frac{\gamma^2}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_sX_t}{n^2}\Big\}$$

$$= \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s,t}\frac{\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\big(X_t^2-\mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} - \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s}\frac{\big(X_1^2-\mathbb{E}\{X_1^2\}\big)^2}{n^2}\Big\}$$

$$-\frac{\gamma}{\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_s\big(X_t^2-\mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} + \frac{\gamma^2}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_sX_t}{n^2}\Big\}$$

$$= \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\frac{\big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\big]^2}{n^2}\Big\} - \frac{1}{2\sigma_2^2}\mathbb{E}\Big\{\frac{\big(X_1^2 - \mathbb{E}\{X_1^2\}\big)^2}{n}\Big\}$$
$$-\frac{\gamma}{\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_s\big(X_t^2 - \mathbb{E}\{X_t^2\}\big)}{n^2}\Big\} + \frac{\gamma^2}{2\sigma_2^2}\mathbb{E}\Big\{\sum_{s\neq t}\frac{X_sX_t}{n^2}\Big\}. \blacksquare$$

Proposition A.2.2 *We assume that $m_s = m$ for all s. We suppose that $|\mathbb{E}\{X_s^2X_t^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}| \leq \alpha(|t-s|)$, where $\alpha(h) \to 0$ as $h \to \infty$. Then,*

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0 \ .$$

Proof Let $\epsilon > 0$. There exists $k > 0$ such that if $|h| \geq k$, $|\mathbb{E}\{X_s^2X_{s+h}^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_{s+h}^2\}| \leq \epsilon$. We know that

$$\mathbb{E}\Big\{\frac{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2}{n^2}\Big\} = \mathbb{E}\Big\{\frac{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]\Big[\sum_{t=1}^n \big(X_t^2 - \mathbb{E}\{X_t^2\}\big)\Big]}{n^2}\Big\}$$
$$= \frac{\sum_{s,t}\big(\mathbb{E}\{X_s^2X_t^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}\big)}{n^2}$$
$$= \frac{\sum_{s=1}^n\sum_{|t-s|\leq k}\big(\mathbb{E}\{X_s^2X_t^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}\big)}{n^2} + \frac{\sum_{s=1}^n\sum_{|t-s|>k}\big(\mathbb{E}\{X_s^2X_t^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}\big)}{n^2}$$
$$\leq \frac{\sum_{s=1}^n\sum_{|t-s|\leq k}\big(\mathbb{E}\{X_s^2X_t^2\} + \mathbb{E}\{X_1^2\}^2\big)}{n^2} + \frac{\sum_{s=1}^n\sum_{|t-s|>k}\epsilon}{n^2}$$
$$\leq \frac{\sum_{s=1}^n\sum_{|t-s|\leq k}\big(\mathbb{E}\{X_1^4\} + \mathbb{E}\{X_1^2\}^2\big)}{n^2} + \epsilon$$
$$\leq \frac{\sum_{s=1}^n[2k+1]\big(\mathbb{E}\{X_1^4\} + \mathbb{E}\{X_1^2\}^2\big)}{n^2} + \epsilon$$
$$\leq \frac{[2k+1]\big(\mathbb{E}\{X_1^4\} + \mathbb{E}\{X_1^2\}^2\big)}{n} + \epsilon \ .$$

Then, it is enough to choose n such that $(1/n)[2k+1]\big(\mathbb{E}\{X_1^4\} + \mathbb{E}\{X_1^2\}^2\big) \leq \epsilon$. Then, in this case,

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \leq 2\epsilon \ . \blacksquare$$

.

Lemma A.2.1 *We assume that $(1/n)\sum_{s=1}^n \mathbb{E}\{X_s^2\} \to \sigma_0^2$ as $n \to \infty$. Then,*

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0$$

if and only if

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^n \big(X_s^2 - \sigma_0^2\}\big)\Big]^2\Big\} \to 0$$

Proof We have

$$n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}$$

$$=n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2+\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}$$

$$=n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)+\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}$$

$$=n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)\Big]^2+2\Big[\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)\Big]\Big[\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]+\Big[\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\Big\}$$

$$=n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)\Big]^2\Big\}+2n^{-2}\Big[\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]\mathbb{E}\Big\{\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)\Big\}+n^{-2}\Big[\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]^2 .$$

Now,

$$n^{-1}\mathbb{E}\Big\{\sum_{s=1}^{n}\big(X_s^2-\sigma_0^2\big)\Big\}\to 0 ,$$

and therefore,

$$n^{-2}\Big[\sum_{s=1}^{n}\big(\sigma_0^2-\mathbb{E}\{X_s^2\}\big)\Big]^2\to 0 .$$

With these relations, we deduce the lemma. ■

Appendix B

Study of example 2.2.1

We recall that we suppose

$$X_t = \sum_{i=0}^{\infty} C_i(\Theta_{t+i}) f_{i+1}(\Psi_{t+i})$$

where $C_n(x) = \sqrt{2}.cos(4^n x)$ where Θ_i is IID with uniform distribution on $[0, 2\pi]$, where $\{\Psi_i\}$ is a strictly stationary process, independent from $\{\Theta_i\}$ and where $|f_{i+1}(y)| \leq \frac{1}{(i+1)^{1/2+a}}$ with $a > 0$.

B.1 Study of trigonometric functions

We recall $\int_0^1 [\sqrt{2}cos(2\pi nx)]^2 dx = 2\int_0^1 cos(2\pi nx)^2 dx = 2\int_0^1 \frac{1+cos(4\pi nx)}{2} dx = 1$ and $cos(a+b) + cos(a-b) = 2cos(a)cos(b)$.

Lemma B.1.1 *We suppose $i_1 < i_2 < .... < i_p$. Then, $4^{i_1} + 4^{i_2} + ... + 4^{i_p} < (4/3)4^{i_p}$.*

Proof We have $4^{i_1} + 4^{i_2} + ... + 4^{i_p} < 4^{i_p} + 4^{i_p-1} + 4^{i_p-2} + ...$
$\leq 4^{i_p}[1 + 4^{-1} + 4^{-2} + ...] = 4^{i_p}[1/(1-1/4)] = (4/3)4^{i_p}$. ■

Lemma B.1.2 *We suppose $i_1 < i_2 < .... < i_p$. Let a_s be a sequence such that $a_s = 1$ or 2 and where one $a_s = 2$ at more. Then, $a_1 4^{i_1} + a_2 4^{i_2} + ... + a_p 4^{i_p} < (7/3)4^{i_p}$.*

Proof We have $a_1 4^{i_1} + a_2 4^{i_2} + ... + a_p 4^{i_p} \leq 4^{i_1} + 4^{i_2} + ... + 2 * 4^{i_p}$
$< 2 * 4^{i_p} + 4^{i_p-1} + 4^{i_p-2} + ... \leq 2 * 4^{i_p} + (4/3)4^{i_p-1}$
$\leq 4^{i_p}[2 + (4/3)4^{-1}] \leq 4^{i_p}[2 + 1/3]$. ■

Lemma B.1.3 *We suppose $i_1 < i_2 < .... < i_p$. Then, $cos(4^{i_1}x).....cos(4^{i_p}x) = \sum_{s=1}^{B} A_s cos(sx)$ where $B < 4^{i_p+1}$.*

Proof It is true for p=1.
If it is true for p, $cos(4^{i_1}x)cos(4^{i_2}x).....cos(4^{i_p}x) = \sum_{s=1}^{B} A_s cos(sx)$ where $B < 4^{i_p+1}$.
Therefore,

$$cos(4^{i_1}x)cos(4^{i_2}x).....cos(4^{i_p}x)cos(4^{i_{p+1}}x) = \sum_{s=1}^{B} A_s cos(sx)cos(4^{i_{p+1}}x)$$

$$= (1/2) \sum_{s=1}^{4^{i_{p+1}}-1} A_s cos([4^{i_{p+1}} + s]x) + (1/2) \sum_{s=1}^{4^{i_{p+1}}-1} A_s cos([4^{i_{p+1}} - s]x)$$

where $4^{i_{p+1}} - s > 0$ and $4^{i_{p+1}} + s < 4^{i_{p+1}} + 4^{i_p}+1 \leq 2 * 4^{i_{p+1}} < 4^{i_{p+1}+1}$. ■

We deduce the following propositions.

Lemma B.1.4 *Let $i_1 < i_2 < .... < i_p$, $i_s \in \mathbb{N}$. Then,*
$\int_{x=0}^{2\pi} cos(4^{i_1}x)cos(4^{i_2}x).....cos(4^{i_p}x)dx = 0$.

Lemma B.1.5 *Let $i_{s_1} > i_{s_2} > .... > i_{s_q}$. Then, $\mathbb{E}\{C_{i_{s_1}}(\Theta_t)C_{i_{s_2}}(\Theta_t).....C_{i_{s_q}}(\Theta_t)\} = 0$.*

B.2 Some properties

Proposition B.2.1 *Let $t_1 < t_2 < ..... < t_p$. Then, $\mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_p}\} = 0$.*

Proof We have

$$\mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_p}\}$$

$$= \mathbb{E}\Big\{\Big[\sum_{i_1=0}^{\infty} C_{i_1}(\Theta_{t_1+i_1})f_{i_1+1}(\Psi_{t_1+i_1})\Big]\Big[\sum_{i_2=0}^{\infty} C_{i_2}(\Theta_{t_2+i_2})f_{i_2+1}(\Psi_{t_2+i_2})\Big].....\Big[\sum_{i_p=0}^{\infty} C_{i_p}(\Theta_{t_p+i_p})f_{i_p+1}(\Psi_{t_p+i_p})\Big]\Big\}$$

$$= \mathbb{E}\Big\{\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}.....\sum_{i_p=0}^{\infty} C_{i_1}(\Theta_{t_1+i_1})f_{i_1+1}(\Psi_{t_1+i_1})C_{i_2}(\Theta_{t_2+i_2})f_{i_2+1}(\Psi_{t_2+i_2}).....C_{i_p}(\Theta_{t_p+i_p})f_{i_p+1}(\Psi_{t_p+i_p})\Big\}$$

$$= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}.....\sum_{i_p=0}^{\infty} \mathbb{E}\Big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_2+i_2}).....C_{i_p}(\Theta_{t_p+i_p})\Big\}\mathbb{E}\Big\{f_{i_1+1}(\Psi_{t_1+i_1}).....f_{i_p+1}(\Psi_{t_p+i_p})\Big\}\ .$$

If there exists s such that $t_s+i_s \neq t_r+i_r$ when $r \neq s$, $\Theta_{t_s+i_s}$ is independent of the other $\Theta_{t_r+i_r}$. Therefore, in this case, this $C_{i_s}(\Theta_{t_s+i_s})$ is independent of other ones and because $\mathbb{E}\{C_{i_s}(\Theta_{t_s+i_s})\} = 0$, $\mathbb{E}\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_2+i_2}).....C_{i_p}(\Theta_{t_p+i_p})\} = 0$.

Then, we study the $\mathbb{E}\{C_{i_{s_1}}(\Theta_{t_{s_1}+i_{s_1}})C_{i_{s_2}}(\Theta_{t_{s_2}+i_{s_2}}).....C_{i_{s_q}}(\Theta_{t_{s_q}+i_{s_q}})\}$ where $t_{s_1} + i_{s_1} = t_{s_2} + i_{s_2} = ... = t_{s_q} + i_{s_q}$. Indeed, $\mathbb{E}\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_2+i_2}).....C_{i_p}(\Theta_{t_p+i_p})\}$ is thus a product of terms of the form $\mathbb{E}\{C_{i_{s_1}}(\Theta_{t_{s_1}+i_{s_1}})C_{i_{s_2}}(\Theta_{t_{s_2}+i_{s_2}}).....C_{i_{s_q}}(\Theta_{t_{s_q}+i_{s_q}})\}$ where $t_{s_1} + i_{s_1} = t_{s_2} + i_{s_2} = ... = t_{s_q} + i_{s_q}$.

Suppose that $t_{s_1} < t_{s_2} < .... < t_{s_q}$ and $t_{s_1} + i_{s_1} = t_{s_2} + i_{s_2} = ... = t_{s_q} + i_{s_q}$. Then, $i_{s_1} > i_{s_2} > .... > i_{s_q}$. Therefore, by lemma B.1.5,

$$\mathbb{E}\{C_{i_{s_1}}(\Theta_{t_{s_1}+i_{s_1}})C_{i_{s_2}}(\Theta_{t_{s_2}+i_{s_2}}).....C_{i_{s_q}}(\Theta_{t_{s_q}+i_{s_q}})\} = 0\ .\ ■$$

Then, one will use the relation cos(a)cos(b) = [cos(a+b)+cos(a-b)]/2 in order to prove the following proposition.

Proposition B.2.2 *Let $t_1 \neq t_s$ if $s \geq 2$ and $t_2 < t_3 < ..... < t_p$. Then, $\mathbb{E}\{X_{t_1}^2X_{t_2}.....X_{t_p}\} = 0$.*

Proof In the proof of proposition B.2.1, we have proved

$$\mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_p}\}$$

$$= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}.....\sum_{i_p=0}^{\infty} \mathbb{E}\Big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_2+i_2}).....C_{i_p}(\Theta_{t_p+i_p})\Big\}\mathbb{E}\Big\{f_{i_1+1}(\Psi_{t_1+i_1}).....f_{i_p+1}(\Psi_{t_p+i_p})\Big\}\ .$$

Therefore,

$$\mathbb{E}\{X_{t_1}^2X_{t_3}.....X_{t_p}\}$$

$$= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}.....\sum_{i_p=0}^{\infty}\mathbb{E}\big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2})C_{i_3}(\Theta_{t_3+i_3}).....C_{i_p}(\Theta_{t_p+i_p})\big\}$$

$$\mathbb{E}\big\{f_{i_1+1}(\Psi_{t_1+i_1})f_{i_2+1}(\Psi_{t_1+i_2})f_{i_3+1}(\Psi_{t_3+i_3}).....f_{i_p+1}(\Psi_{t_p+i_p})\big\}\ .$$

If there exists $s \geq 3$ such that $t_s + i_s \neq t_r + i_r$ when $r \neq s$, $C_{i_s}(\Theta_{t_s+i_s})$ is independent of other $C_{i_r}(\Theta_{t_r+i_r})$ with $\mathbb{E}\{C_{i_s}(\Theta_{t_s+i_s})\} = 0$. Therefore, if

$$\mathbb{E}\big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2})C_{i_3}(\Theta_{t_3+i_3}).....C_{i_p}(\Theta_{t_p+i_p})\big\} \neq 0\ ,$$

$\mathbb{E}\big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2})C_{i_3}(\Theta_{t_3+i_3}).....C_{i_p}(\Theta_{t_p+i_p})\big\}$ is thus a product of terms of the form $\mathbb{E}\big\{C_{i_{s_1}}(\Theta_{t_{s_1}+i_{s_1}})C_{i_{s_2}}(\Theta_{t_{s_2}+i_{s_2}}).....C_{i_{s_q}}(\Theta_{t_{s_q}+i_{s_q}})\big\}$ where $t_{s_1}+i_{s_1} = t_{s_2}+i_{s_2} = ... = t_{s_q}+i_{s_q}$.

Then, we have to study the products $\mathbb{E}\big\{C_{i_{s_1}}(\Theta_{t_{s_1}+i_{s_1}})C_{i_{s_2}}(\Theta_{t_{s_2}+i_{s_2}}).....C_{i_{s_q}}(\Theta_{t_{s_q}+i_{s_q}})\big\}$ where $t_{s_1} + i_{s_1} = t_{s_2} + i_{s_2} = ... = t_{s_q} + i_{s_q}$.

In order that these terms are not zero, there is a priori only two possible cases

1) There exists h and a sequence s_t such that $t_1 + i_1 = t_1 + i_2 = t_{s_3} + i_{s_3} = ... = t_{s_h} + i_{s_h}$.

2) There exists h and k and two sequences s'_t and $s"_t$ such that $t_1+i_1 = t_{s'_3}+i_{s'_3} = ... = t_{s'_h}+i_{s'_h}$ and $t_1 + i_2 = t_{s"_3} + i_{s"_3} = ... = t_{s"_k} + i_{s"_k}$.

Indeed, by the proof of previous proposition B.2.1, there is no other solutions, which can give a priori a nonzero expectation : if one find a sequence $s"_t \neq 1$ such that $t_{s"_3}+i_{s"_3} = ... = t_{s"_k}+i_{s"_k}$, we shall have, for example, $t_{s"_3} < .... < t_{s"_k}$ and, therefore, $i_{s"_3} > .... > i_{s"_k}$. We shall deduce, by lemma B.1.5, $\mathbb{E}\Big\{C_{i_3}(\Theta_{t_{s"_3}+i_{s"_3}}).....C_{i_p}(\Theta_{t_{s"_k}+i_{s"_k}})\Big\} = 0$. Therefore,

$$\mathbb{E}\big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2})C_{i_3}(\Theta_{t_3+i_3}).....C_{i_p}(\Theta_{t_p+i_p})\big\}$$

$$= \mathbb{E}\Big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2})C_{i_{s'_3}}(\Theta_{t_{s'_3}+i_{s'_3}}).....C_{i_{s'_h}}(\Theta_{t_{s'_h}+i_{s'_h}})\Big\}\mathbb{E}\Big\{C_{i_{s"_3}}(\Theta_{t_{s"_3}+i_{s"_3}}).....C_{i_{s"_k}}(\Theta_{t_{s"_k}+i_{s"_k}})\Big\}$$

$$= 0\ .$$

Therefore there are only two cases where we can find a priori a non-zero expectation

1) $t_1 + i_1 = t_1 + i_2 = t_{s_3} + i_{s_3} = ... = t_{s_p} + i_{s_p}$ with $p \geq 3$ because we are interested in $\mathbb{E}\{X_{t_1}^2 X_{t_2}.....X_{t_p}\} = 0$ with $p \geq 2$.

2) $t_1 + i_1 = t_{s'_3} + i_{s'_3} = ... = t_{s'_h} + i_{s'_h}$ and $t_1 + i_2 = t_{s"_3} + i_{s"_3} = ... = t_{s"_k} + i_{s"_k}$.

At first, suppose $t_1 + i_1 = t_1 + i_2 = t_{s_3} + i_{s_3} = ... = t_{s_p} + i_{s_p}$. Then, $i_1 = i_2$. Then, $C_{i_1}(\Theta_{t_1+i_1})C_{i_2}(\Theta_{t_1+i_2}) = C_{i_1}(\Theta_{t_1+i_1})C_{i_1}(\Theta_{t_1+i_1}) = Cos(2 * 4^{i_1}\Theta_{t_1+i_1}) + 1$. Suppose, for example $t_{s_3} < t_{s_4} < .... < t_{s_h}$. Because $t_{s_3} + i_{s_3} = ... = t_{s_h} + i_{s_h}$, then, $i_{s_3} > i_{s_4} > .... > i_{s_h}$.

If $t_1 + i_1 = t_{s_3} + i_{s_3} = T$ and $t_1 > t_{s_3}$, $i_1 < i_{s_3}$. Therefore, by lemma B.1.2 , $C_{i_3}(\Theta_T)$ is orthogonal to $Cos(2 * 4^{i_1}\Theta_T)C_{i_{s_4}}(\Theta_T)C_{i_{s_5}}(\Theta_T).....C_{i_{s_h}}(\Theta_T)$, and, therefore, orthogonal to $C_{i_1}(\Theta_T)C_{i_2}(\Theta_T)C_{i_{s_4}}(\Theta_T)C_{i_{s_5}}(\Theta_T).....C_{i_{s_h}}(\Theta_T) = [Cos(2*4^{i_1}\Theta_T)+1]C_{i_{s_4}}(\Theta_T)C_{i_{s_5}}(\Theta_T).....$ $C_{i_{s_h}}(\Theta_T)$. Therefore, $\mathbb{E}\big\{C_{i_1}(\Theta_T)C_{i_2}(\Theta_T)C_{i_{s_3}}(\Theta_T)C_{i_{s_4}}(\Theta_T).....C_{i_{s_h}}(\Theta_T)\big\} = 0$

Suppose always $t_{s_3} < t_{s_4} < .... < t_{s_h}$. If $t_1 + i_1 = T$ and $t_1 \leq t_{s_3}$, $i_1 \geq i_{s_3}$. Thus, by lemmas B.1.1 and B.1.3, $Cos(2 * 4^{i_1}\Theta_T)$ is orthogonal to $C_{i_{s_3}}(\Theta_T)C_{i_{s_4}}(\Theta_T)C_{i_{s_5}}(\Theta_T).....C_{i_{s_h}}(\Theta_T)$.

Therefore, $\mathbb{E}\Big\{C_{i_1}(\Theta_T)C_{i_2}(\Theta_T)C_{i_{s_3}}(\Theta_T).....C_{i_{s_h}}(\Theta_T)\Big\} = 0$.

Suppose now that there exists h and k and two sequences s'_t and $s"_t$ such that $t_1 + i_1 = t_{s'_3} + i_{s'_3} = ... = t_{s'_h} + i_{s'_h}$ and $t_1 + i_2 = t_{s"_3} + i_{s"_3} = ... = t_{s"_k} + i_{s"_k}$.

For example, study $\mathbb{E}\Big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_{s'_3}}(\Theta_{t_{s'_3}+i_{s'_3}}).....C_{i_{s'_h}}(\Theta_{t_{s'_h}+i_{s'_h}})\Big\}$ where $t_1+i_1 = t_{s_3}+i_{s_3} = ... = t_{s'_h} + i_{s'_h}$. Because, $t_1 \neq t_{s_3} \neq ..... \neq t_{s_h}$, $i_1 \neq i_{s_3} \neq ..... \neq i_{s_h}$. For example, $i_1 > i_{s_3} > ..... > i_{s_h}$. Then, par lemmas B.1.1 and B.1.3, $\sqrt{2}cos(4^{i_1}\Theta_T) = C_{i_1}(4^{i_1}\Theta_T)$ is orthogonal to $C_{i_{s_3}}(\Theta_{t_{s_3}+i_{s_3}}).....C_{i_{s_h}}(\Theta_{t_{s_h}+i_{s_h}})$. Therefore, $\mathbb{E}\Big\{C_{i_1}(\Theta_{t_1+i_1})C_{i_{s_3}}(\Theta_{t_{s_3}+i_{s_3}}).....C_{i_{s_h}}(\Theta_{t_{s_h}+i_{s_h}})\Big\} = 0$. ■

B.3 Calculation of $\mathbb{E}\{X_t^2 X_{t+h}^2\} - \mathbb{E}\{X_t^2\}\mathbb{E}\{X_{t+h}^2\}$

We suppose

$$\Big|\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}\Big|$$

$$\leq \frac{1}{(1+i)^{1+2a}} \ \frac{1}{(1+r)^{1+2a}} \ \epsilon(t+h+r-(t+i)) \ ,$$

where $1 \geq \epsilon(t) > 0$ is decreasing and converges to 0 as $t \to \infty$.

We shall prove that $|\mathbb{E}\{X_t^2 X_{t+h}^2\} - \mathbb{E}\{X_t^2\}\mathbb{E}\{X_{t+h}^2\}| \leq \alpha(h)$ where $\alpha(h) \to 0$ as $h \to \infty$.

B.3.1 Elementary equalities

Because $X_t = \sum_{i=0}^{\infty} C_i(\Theta_{t+i}) f_{i+1}(\Psi_{t+i})$,

$$X_t^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_i(\Theta_{t+i})C_j(\Theta_{t+j}) f_{i+1}(\Psi_{t+i}) f_{j+1}(\Psi_{t+j}) \ ,$$

and

$$E\{X_t^2\} = \sum_{i=0}^{\infty} E\{C_i(\Theta_{t+i})^2\}E\{f_{i+1}(\Psi_{t+i})^2\} \leq \sum_{i=0}^{\infty} \frac{1}{(1+i)^{1+2a}} < \infty \ .$$

Moreover,

$$X_{t+h}^2 = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s}) f_{r+1}(\Psi_{t+h+r}) f_{s+1}(\Psi_{t+h+s}) \ .$$

Therefore,

$$X_t^2 X_{t+h}^2 =$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s}) \ .$$

B.3.2 Study of $E\{X_t^2 X_{t+h}^2\}$

We have

$$E\{X_t^2 X_{t+h}^2\}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$= \sum_{i=j}\sum_{r=s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$+ \sum_{i=j}\sum_{r\neq s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$+ \sum_{i\neq j}\sum_{r=s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$+ \sum_{i\neq j}\sum_{r\neq s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$= \sum_{i=j}\sum_{r=s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$+ \sum_{i\neq j}\sum_{r\neq s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

$$= \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \mathbb{E}\{C_i(\Theta_{t+i})^2 C_r(\Theta_{t+h+r})^2\}E\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\}$$
$$+ \sum_{i\neq j, r\neq s}\ \sum_{i=h+r, j=h+s} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$
$$+ \sum_{i\neq j, r\neq s}\ \sum_{i=h+s, j=h+r} \mathbb{E}\{C_i(\Theta_{t+i})C_j(\Theta_{t+j})C_r(\Theta_{t+h+r})C_s(\Theta_{t+h+s})\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})f_{j+1}(\Psi_{t+j})f_{r+1}(\Psi_{t+h+r})f_{s+1}(\Psi_{t+h+s})\}$$

(Therefore, if $h \geq 1$)

$$= \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \mathbb{E}\{C_i(\Theta_{t+i})^2 C_r(\Theta_{t+h+r})^2\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\}$$

$$= \sum_{i, i=h+r} \mathbb{E}\{C_i(\Theta_{t+i})^2 C_r(\Theta_{t+h+r})^2\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\}$$
$$+ \sum_{i, r, i\neq h+r} \mathbb{E}\{C_i(\Theta_{t+i})^2\}\mathbb{E}\{C_r(\Theta_{t+h+r})^2\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} \ .$$

B.3.3 Study of $\mathbb{E}\{X_{t+h}^2 X_t^2\} - \mathbb{E}\{X_{t+h}^2\}\mathbb{E}\{X_t^2\}$

We recall

$$\mathbb{E}\{X_t^2\}\mathbb{E}\{X_{t+h}^2\}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \mathbb{E}\{C_i(\Theta_{t+i})^2\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{C_j(\Theta_{t+h+j})^2\}\mathbb{E}\{f_{j+1}(\Psi_{t+h+j})^2\}$$

$$= \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}$$

Therefore,

$$\mathbb{E}\{X_{t+h}^2 X_t^2\} - \mathbb{E}\{X_{t+h}^2\}\mathbb{E}\{X_t^2\}$$

$$= \sum_{i,i=h+r} \Big[\mathbb{E}\{C_i(\Theta_{t+i})^2 C_r(\Theta_{t+h+r})^2\}\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}E\{f_{r+1}(\Psi_{t+h+r})^2\}\Big]$$

$$+ \sum_{i,r,i\neq h+r} \Big[\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - E\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}\Big]$$

$$= \sum_{r} \Big[\mathbb{E}\{C_{r+h}(\Theta_{t+h+r})^2 C_r(\Theta_{t+h+r})^2\}\mathbb{E}\{f_{h+r+1}(\Psi_{t+h+r})^2 f_{r+1}(\Psi_{t+h+r})^2\}$$

$$-\mathbb{E}\{f_{h+r+1}(\Psi_{t+h+r})^2\}E\{f_{r+1}(\Psi_{t+h+r})^2\}\Big]$$

$$+ \sum_{i,r,i\neq h+r} \big[\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}\big] \ .$$

Let $K_t = \sum_{i=t}^{\infty} \frac{1}{(1+i)^{1+2a}}$. Then,

$$\Big|\sum_{i,r,i\neq h+r} \big[\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}\big]\Big|$$

$$\leq \sum_{i=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{(1+i)^{1+2a}} \ \frac{1}{(1+r)^{1+2a}} \ \epsilon(h+r-i)$$

$$\leq \sum_{i=0}^{h/2}\sum_{r=0}^{\infty} \frac{1}{(1+i)^{1+2a}}\frac{1}{(1+r)^{1+2a}}\epsilon(h+r-i) + \sum_{i=h/2+1}^{\infty}\sum_{r=0}^{\infty} \frac{1}{(1+i)^{1+2a}}\frac{1}{(1+r)^{1+2a}}\epsilon(h+r-i) \ .$$

Now, if $i \leq h/2$, $h+r-i \geq h/2$ and $\epsilon(h+r-i) \leq \epsilon(h/2)$. Moreover, $\epsilon(r) \leq 1$. Therefore,

$$\Big|\sum_{i,r,i\neq h+r} \big[\mathbb{E}\{f_{i+1}(\Psi_{t+i})^2 f_{r+1}(\Psi_{t+h+r})^2\} - \mathbb{E}\{f_{i+1}(\Psi_{t+i})^2\}\mathbb{E}\{f_{r+1}(\Psi_{t+h+r})^2\}\big]\Big|$$

$$\leq \sum_{i=0}^{h/2}\sum_{r=0}^{\infty} \frac{1}{(1+i)^{1+2a}}\frac{1}{(1+r)^{1+2a}}\epsilon(h/2) + \sum_{i=h/2+1}^{\infty}\sum_{r=0}^{\infty} \frac{1}{(1+i)^{1+2a}}\frac{1}{(1+r)^{1+2a}}$$

$$\leq (K_0)^2\epsilon(h/2) + K_{h/2}K_0$$

whichi converges to 0 if $h \to \infty$.

Moreover,

$$\sum_r \Big[E\{C_{h+r}(\Theta_{t+h+r})^2 C_r(\Theta_{t+h+r})^2\}E\{f_{h+r+1}(\Psi_{t+h+r})^2 f_{r+1}(\Psi_{t+h+r})^2\}$$

$$-E\{f_{h+r+1}(\Psi_{t+h+r})^2\}E\{f_{r+1}(\Psi_{t+h+r})^2\}\Big]$$

$$\leq \sum_r \Big[4E\{f_{h+r+1}(\Psi_{t+h+r})^2 f_{r+1}(\Psi_{t+h+r})^2\} + E\{f_{h+r+1}(\Psi_{t+h+r})^2\}E\{f_{r+1}(\Psi_{t+h+r})^2\}\Big]$$

$$\leq \sum_r \Big[4\frac{1}{(1+h+r)^{1+2a}}\ \frac{1}{(1+r)^{1+2a}} + \frac{1}{(1+h+r)^{1+2a}}\ \frac{1}{(1+r)^{1+2a}}\Big]$$

$$\leq \frac{1}{(1+h)^{1+2a}} \sum_r \Big[4\ \frac{1}{(1+r)^{1+2a}} + \frac{1}{(1+r)^{1+2a}}\Big]$$

$$\leq \frac{5K_0}{(1+h)^{1+2a}}$$

which converges to 0 if $h \to \infty$.

We deduce

$$|\mathbb{E}\{X_t^2 X_{t+h}^2\} - \mathbb{E}\{X_t^2\}\mathbb{E}\{X_{t+h}^2\}| \leq (K_0)^2\epsilon(h/2) + K_{h/2}K_0 + \frac{5K_0}{(1+h)^{1+2a}}\ .$$

B.4 Conclusion

At first, because $\{\Theta_t\}$ is IID and $\{\Psi_t\}$ is strictly stationary, $X_t = \sum_{i=0}^{\infty} C_i(\Theta_{t+i})f_{i+1}(\Psi_{t+i})$ is also strictly stationary. Then, $(1/n)\sum_{s=1}^n \mathbb{E}\{X_s^2\} = \mathbb{E}\{X_1^2\}$.

Then, $|\mathbb{E}\{X_s^2 X_t^2\} - \mathbb{E}\{X_s^2\}\mathbb{E}\{X_t^2\}| \leq \alpha(|t-s|)$ where $\alpha(h) \to 0$ as $h \to \infty$. Then, by proposition, A.2.2 , $n^{-2}\mathbb{E}\Big\{\Big[\sum_{s=1}^n \big(X_s^2 - \mathbb{E}\{X_s^2\}\big)\Big]^2\Big\} \to 0$. Then, the first condition of proposition 3.8.1 holds.

Moreover, clearly, $\mathbb{E}\{X_1\} = 0$ and $\mathbb{E}\{|X_s|^p\} < \infty$ for all $p \in \mathbb{N}$.

Moreover, by proposition, B.2.1 $\mathbb{E}\{X_{t_1}X_{t_2}.....X_{t_p}\} = 0$. Therefore $\mathcal{P}\{X_{n+1}|X_n, X_{n-1}, ...\} = 0$. Then, the second condition of proposition 3.8.1 holds with $\nu_2 = 0$.

At last, by proposition B.2.2, $\mathbb{E}\{X_{t_1}^2 X_{t_2}.....X_{t_p}\} = 0$. Then, the third condition of corollary 3.8.1 holds. One can also remark that X_1 is bounded and apply corollary 3.10.1.

We deduce that

$$\mathbb{E}\Big\{\frac{X_1 + X_2 + ... + X_n}{\sqrt{n}}\Big\} \overset{M}{\to} N(0, \mathbb{E}\{X_1^2\})\ .$$

Bibliography

[1] IBRAGIMOV I.A. LINNIK Yu V.(1971) Independent and stationary sequences of random variables. Wolters-Noordhoff, Groningen.

[2] BRADLEY R.C. (1984) On a very weak Bernouilli condition. Stochastics, 13, 61-81.

[3] DEHLING H. DENKER M. PHILLIPPS W. (1984) Versik Processes and very weak Bernouilli processes with summable rates are in,dependent. Proc. Amer. Math Soc. 91, 618-624.

[4] WITHERS C. S. (1981) Central limit theorems for dependent random variables. I. Z. Wahrsch. verw. Gebiete, 54, 509-534.

[5] COGBURN R. (1960) Asymptotic properties of stationary sequences. Univ. Calif. Publ. Statist. 3, 99-146.

[6] ROSENBLATT M. (1972) Uniform ergodicity and strong mixing. Z. Wahrsch. Werw. Gebiete. 24, 79-84.

[7] PINSKER M.S. (1964) Information and information stability of random variables and processes. Holden Day, San Francisco.

[8] BERNSTEIN (1939) Quelques remarques sur le thorme limite Liapounoff. Dokl. Akad, Nauk SSSR, 24, 3-8.

[9] BROWN (1970) Characteristics functions, moments and the Central Limit Theorem. Ann. Math. Statist. 41 658-664.

[10] ESSEEN C.G. JANSON S. (1985) On moments conditions for normed sums of independent random variables and martingales differences. Stochastics Processes and their Applications, 19, 173-182.

[11] HERRNDORF N. (1984) A functional Central Limit Theorem for r-mixing sequences. Journal of multivariate analysis. 15, 141-146.

[12] BIRKEL T. (1988) Moment bounds for associated sequences. Ann. Prob. 16-3, 1184-1193.

[13] KRUGOV V.M. (1988) The convergence of moments of random sums. Theory of Probability and its applications. 33-2, 339-342.

[14] MAIROBODA R. E. (1989) The Central limit Theorem for empirical moment generating functions. Theory of Probability and its applications, vol 34-2, 332-335.

[15] YOKOHAMA R. (1980) Moment bounds for stationary mixing sequences. Z Wahrschein-lichkeistheorie, ver. Gebiete, 52, 45-57.

[16] YOKOHAMA R. (1983) The convergence of moments in the Central limit Theorem for stationary f-mixing processes. Analysis mathematica, 9, 79-84.

[17] COX D. KIM T.Y. (1995) Moment bounds for mixing random variables useful in nonparametric function estimation. Stochastics Processes and their Applications. 56, 151-158.

[18] DOUKHAN P, LOUHICHI S. (1999) A new weak dependence condition and application to moments inequalities. Stochastics Processes and their Applications. (84) 313-342.

[19] COULON-PRIEUR C. DOUKHAN P (2000) A triangular central limit theorem under a new weak dependence condition. Stat Prob. Let. 47, 61-68.

[20] HALL P. HEYDE C.C. (1980) Martingale limit theory and its applications. Academic Press. New York.

[21] LANCASTER H. O. (1960) Orthogonal models for contingency tables. Developments in statistics. Academic Press, New York.

[22] BLACHER R. (1993) Higher Order Correlation Coefficients. Statistics 25, 1-15.

[23] BLACHER R. (1995) Central limit theorem by polynomial dependence coefficients. Journal of computational and applied mathematics 57, 45-56.

[24] BLACHER R. (1990) Theoreme de la limite centrale par les moments. Compte rendus de l'Académie des Sciences de Paris. t-311 srie I, p 465-468.

[25] BLACHER R. (2007) Central limit theorem by moments. Stat. Prob. Letters.Vol 77, Issue 17, p 1647-1651.

[26] BLACHER R. (1989) Loi de la somme de N variables aléatoires et théorème de la limite centrale par les coefficients de dépendance polynômiale et les moments. Rapport de Recherche 768-M, laboratoire TIM3, Université de Grenoble, France.

[27] BLACHER R. (2003) Multivariate quadratic forms of random vectors. Journal of Multivariate Analysis 87, 2-23.

[28] IBRAGIMOV I. LIFSHITS M. (1998) On the convergence of generalized moments in almost sure central limit theorem. Stat and Prob Letters 343-351.

[29] SOULIER P. (2001) Moment bounds and the central limit theorem for functions of Gaussian vectors. Stat and Prob Letters 193-203.

[30] ROZOVSKY L.V. . (2002) An estimate of the remainder in the central limit theorem for a sum of independent random variables with infinite moments of a higher order. Theor Proba and its appl, 174-183.