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LÉVY PROCESS CONDITIONED BY ITS HEIGHT PROCESS

June 11, 2011

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ABSTRACT. In the present work, we consider spectrally positive Lévy processes $(X_t, t \geq 0)$ not drifting to $+\infty$ and we are interested in conditioning these processes to reach arbitrarily large heights before hitting 0.

This way we obtain a new conditioning of Lévy processes to stay positive. The (honest) law of this conditioned process is defined as a Doob h -transform via a martingale. For Lévy processes with infinite variation paths, this martingale is $(\int \tilde{\rho}_t(dz)e^{\alpha z} + I_t) \mathbf{1}_{\{t \leq T_0\}}$ for some α and where $(\tilde{\rho}_t, t \geq 0)$ is the so-called *exploration process* defined in [10], T_0 is the hitting time of 0 for X .

When the process X is a compensated compound Poisson process, the previous martingale is defined through the jumps of the future infimum process of X . The computations are easier in this case because X can be viewed as the contour process of a (sub)critical *splitting tree*. We also can give an alternative characterization of our conditioned process in the vein of spine decompositions.

1. INTRODUCTION

In this paper, we consider Lévy processes $(X_t, t \geq 0)$ with no negative jumps (or spectrally positive), not drifting to $+\infty$ and conditioned to reach arbitrarily large heights before hitting 0. Let \mathbb{P}_x be the law of X conditional on $X_0 = x$ and $(\mathcal{F}_t, t \geq 0)$ be its natural filtration.

Many papers deal with conditioning Lévy processes in the literature. In seminal works by L. Chaumont [6, 7] and then in [8], for general Lévy processes, L. Chaumont and R. Doney construct a family of measures $\mathbb{P}_x^\uparrow, x > 0$ of Lévy processes starting from x and conditioned to stay positive defined via a h -transform and it can be obtained as the limit

$$\mathbb{P}_x^\uparrow(\Theta, t < \zeta) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(\Theta, t < \mathbf{e}/\varepsilon | X_s > 0, 0 \leq s \leq \mathbf{e}/\varepsilon)$$

for $t \geq 0$, $\Theta \in \mathcal{F}_t$ and for \mathbf{e} an exponential r.v. with parameter 1 independent from the process X and where ζ is the killing time of X . In the spectrally positive case, when $\mathbb{E}[X_1] < 0$, \mathbb{P}_x^\uparrow is a sub-probability while, if $\mathbb{E}[X_1] = 0$, it is a probability. In [14], K. Hirano considers Lévy processes drifting to $-\infty$ conditioned to stay positive. More precisely, under exponential moment assumption, he is interested in two types of conditioning events: either the process X is conditioned to reach $(-\infty, 0]$ after time s or to reach level $s > 0$ before $(-\infty, 0]$. Then, at the limit $s \rightarrow \infty$, in both cases, the conditioned Lévy processes converge to distributions which can be described via h -transforms. In [5, ch. VII], J. Bertoin considers spectrally negative Lévy processes i.e. with no positive jumps, and also constructs a family of conditioned processes to stay positive via the scale function associated with X .

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Here, we restrict ourselves to study spectrally positive Lévy processes and consider a new way to obtain a Lévy process conditioned to stay positive without additional assumptions and contrary to [8], the law of the conditioned process is honest. The process X is conditioned to reach arbitrarily large heights before $T_0 := \inf\{t \geq 0; X_t = 0\}$ using its associated height process H defined below.

More precisely, for $t \geq 0, \Theta \in \mathcal{F}_t$, we are interested in the limit

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \left| \sup_{0 \leq t \leq T_0} H_t \geq a \right. \right). \quad (1)$$

In fact, since the process X can be viewed as a contour process of a random tree, this kind of conditioning is equivalent to conditioning a tree to reach large heights.

In the following, we will consider three different cases for the Lévy process X : a Lévy process with finite variation and infinite Lévy measure, a Lévy process with finite variation and finite Lévy measure and finally a Lévy process with infinite variation.

In the first case, as it is stated in Theorem 2.3, the conditioning in (1) is trivial because $\mathbb{P}_x(\sup_{0 \leq t \leq T_0} H_t \geq a) = 1$ for all positive a .

In the second case, X is simply a compensated compound Poisson process whose Laplace exponent can be written as

$$\psi(\lambda) = \lambda - \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr)$$

where Λ is a finite measure on $(0, \infty)$ such that $m := \int_{(0, \infty)} r \Lambda(dr) \leq 1$. Thus, X is either recurrent or drifts to $-\infty$ and its hitting time T_0 of 0 is finite a.s. In this finite variation case, the height H_t at time t is the (finite) number of records of the future infimum, that is, the number of times s such that

$$X_{s-} < \inf_{[s, t]} X.$$

The process X is then conditioned to reach height a before T_0 . In the limit $a \rightarrow \infty$ in (1), one obtains a h -transform defined via a martingale which depends on the jumps of the future infimum. In the particular case $m = 1$, this martingale is $X_{\cdot \wedge T_0}$ and we recover the same h -transform as the one obtained in [8].

The key result used in our proof is due to A. Lambert [19]. Indeed, the process X can be seen as a contour process of a splitting tree [13]. These random trees are genealogical trees where each individual lives independently of other individuals, gives birth at rate $\Lambda(\mathbb{R}^+)$ to individuals whose life-lengths are distributed as $\Lambda(\cdot)/\Lambda(\mathbb{R}^+)$. Then, to consider X conditioned to reach height n before T_0 is equivalent to look at a splitting tree conditional on having alive descendance at generation n .

Notice that we only consider the case when the drift α of X equals 1. However, the case $\alpha \neq 1$ can be treated in the same way because X is still the contour process of a splitting tree but visited at speed α .

We also obtain a more precise result about conditional subcritical and critical splitting trees. For $n \in \mathbb{N}$, set \mathbb{P}^n the law of a splitting tree conditional on $\{\mathcal{Z}_n \neq 0\}$ where \mathcal{Z}_n denotes the number of extant individuals in the splitting tree belonging to generation n . In fact, $(\mathcal{Z}_n, n \geq 0)$ is a Galton-Watson process. We are interested in the law of the tree under \mathbb{P}^n as $n \rightarrow \infty$. We obtain that under a $x \log x$ -condition on the measure Λ , the limiting tree has a unique infinite spine where individuals have the size-biased lifelength distribution $m^{-1} z \Lambda(dz)$ and typical finite subtrees are grafted on this spine.

The spine decomposition with a size-biased spine that we obtain is similar to the construction of size-biased splitting trees marked with a uniformly chosen point in [11] where all individuals on the line of descent between root and this marked individual have size-biased lifelengths. It is also analogous to the construction of size-biased Galton-Watson trees in Lyons et al. [23]. These trees arise by conditioning subcritical or critical GW-trees on non-extinction. See also [2, 12, 17]. In [9], T. Duquesne studied the so-called sin-trees that were introduced by D. Aldous in [2]. These trees are infinite trees with a unique infinite line of descent. He also considers the analogous problem for continuous trees and continuous state branching processes as made by other authors in [16, 17, 21].

We finally consider the case where X has paths with infinite variation. Its associated Laplace exponent is specified by the Lévy-Khinchin formula

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \Lambda(dr)(e^{-\lambda r} - 1 + \lambda r)$$

where $\alpha \geq 0$, $\int_{(0,\infty)} \Lambda(dr)(r \wedge r^2) < \infty$ and either $\beta > 0$ or $\int_{(0,1)} \Lambda(dr)r = \infty$. In order to compute the limit (1) in that case, we use the height process $(H_t, t \geq 0)$ defined in [10, 20] which is the analogue of the discrete-space height process in the finite variation case. We set $S_t := \sup_{[0,t]} X$. Then, since 0 is regular for itself for $S - X$, H is defined through local time. Indeed, for $t \geq 0$, H_t is the value at time t of the local time at level 0 of $S^{(t)} - X^{(t)}$ where $X^{(t)}$ is the time-reversed process of X at t

$$X_s^{(t)} := X_{t-s} - X_{(t-s)-} \quad s \in [0, t]$$

(with the convention $X_{0-} = X_0$) and $S^{(t)} := \sup_{0 \leq r \leq s} X_r^{(t)}$ is its past supremum. The result that we obtain is similar to the finite variation case: the limit in (1) allows us to define a family of (honest) probabilities $(\mathbb{P}_x^\uparrow, x > 0)$ of Lévy processes conditioned to stay positive via a h -transform and the martingale

$$\int_0^{H_t} \rho_t(dz)e^{\alpha z} \mathbf{1}_{\{t \leq T_0\}}$$

where $\rho_t(\cdot)$ is a random positive measure on \mathbb{R}^+ which is a slight modification of the *exploration process* defined in [10, 20] and $\alpha = \psi'(0) \geq 0$ (since X does not drift to $+\infty$).

Again, in the recurrent case (i.e. if $\alpha = 0$), we observe that the previous quantity equals $X_{t \wedge T_0}$ and we recover the h -transform $h(x) = x$ of [8] in the spectrally positive case. Indeed, for general Lévy processes, the authors consider the law of the Lévy process conditioned to stay positive which is defined via the h -transform

$$h(x) = \mathbb{E} \left[\int_{[0,\infty)} \mathbf{1}_{\{I_t \geq -x\}} dL_t \right]$$

where I is the past infimum process and L is a local time at 0 for $X - I$. In the particular spectrally positive case, $L = -I$ and h is the identity.

Under \mathbb{P}_x^\uparrow , the height process $(H_t, t \geq 0)$ can be compared to the left height process \overleftarrow{H} studied in [9]. In that paper, T. Duquesne gives a genealogical interpretation of a continuous-state branching process with immigration by defining two continuous contour processes \overleftarrow{H} and \overrightarrow{H} that code the left and right parts of the infinite line of descent.

The paper is organized as follows. In Section 2, we treat the finite variation case and investigate the limiting process after stating some properties about splitting trees. Section 3 is devoted to studying the conditioned splitting tree and Section 4 to considering Lévy processes with infinite variation.

2. FINITE VARIATION CASE

2.1. Definitions and statement of result. Let Λ be a positive measure on $(0, \infty)$ such that $\Lambda \neq 0$ and

$$\int_{(0, \infty)} (x \wedge 1) \Lambda(dx) < \infty$$

and let $(X_t, t \geq 0)$ be a spectrally positive Lévy process with Lévy measure Λ and such that

$$\mathbb{E}_0 \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad \lambda > 0$$

where \mathbb{P}_x is the law of X conditioned to $X_0 = x$ and

$$\psi(\lambda) = \lambda - \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr).$$

We denote by $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$ the natural filtration of X . We will suppose that $m := \int_{(0, \infty)} r \Lambda(dr) \leq 1$ that is, X is recurrent ($m = 1$) or drifts to $-\infty$ ($m < 1$). Then the hitting time $T_0 := \inf\{t \geq 0; X_t = 0\}$ is finite almost surely. Observe that since X is spectrally positive, the first hitting time of $(-\infty, 0]$ is T_0 .

Definition 2.1. *The height process H associated with X is defined by*

$$H_t := \#\{0 \leq s \leq t; X_{s-} < \inf_{s \leq r \leq t} X_r\}.$$

We set

$$\{s_t^1 < \dots < s_t^{H_t}\} := \{0 \leq s \leq t < T_0; X_{s-} < \inf_{s \leq r \leq t} X_r\},$$

$$I_s^t := \inf_{s \leq r \leq t} X_r, \quad 0 \leq s \leq t$$

and we denote the records of $(I_s^t, s \leq t)$ by $\rho_i^t := \inf_{s_i^t \leq r \leq t} X_r - X_{s_i^t-}$ for $1 \leq i \leq H_t$ and $\rho_0^t = \inf_{0 \leq r \leq t} X_r$ (see Figure 1).

The assumption $\int_{(0, \infty)} (x \wedge 1) \Lambda(dx) < \infty$ implies that the paths of X have finite variation and then for all positive t , H_t is finite a.s. (Lemma 3.1 in [20]).

Remark 2.2. *The process X can be seen as a LIFO (last in-first out) queue [20, 25]. Indeed, a jump of X at time t corresponds to the entrance in the system of a new customer who requires a service $\Delta X_t := X_t - X_{t-}$. This customer is served in priority at rate 1 until a new customer enters the system. Then, the ρ_i^t 's are the remaining service times of the H_t customers present in the system at time t .*

The sequence $(\rho_i^t, i \leq H_t)$ can be seen as a random positive measure on non-negative integers which puts weight ρ_i^t on $\{i\}$. Its total mass is X_t and its support is $\{0, \dots, H_t\}$. We denote by S the set of measures on \mathbb{N} with compact support. For ν in S , set $\nu_i := \nu(\{i\})$, $i \geq 0$ and

$$H(\nu) := \max\{i \geq 0; \nu_i \neq 0\}.$$

Then, according to [25, p.200], the process $(\rho^{t \wedge T_0}, t \geq 0)$ is a S -valued Markov process. Its infinitesimal generator \mathcal{A} is defined by

$$\mathcal{A}(f)(\nu) = \left[\int_{(0, \infty)} (f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, r, 0, \dots) - f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, 0, \dots)) \Lambda(dr) - \frac{\partial f}{\partial x_{H(\nu)}}(\nu_0, \dots, \nu_{H(\nu)}, 0, \dots) \right] \mathbf{1}_{\{\nu \neq 0\}}. \quad (2)$$

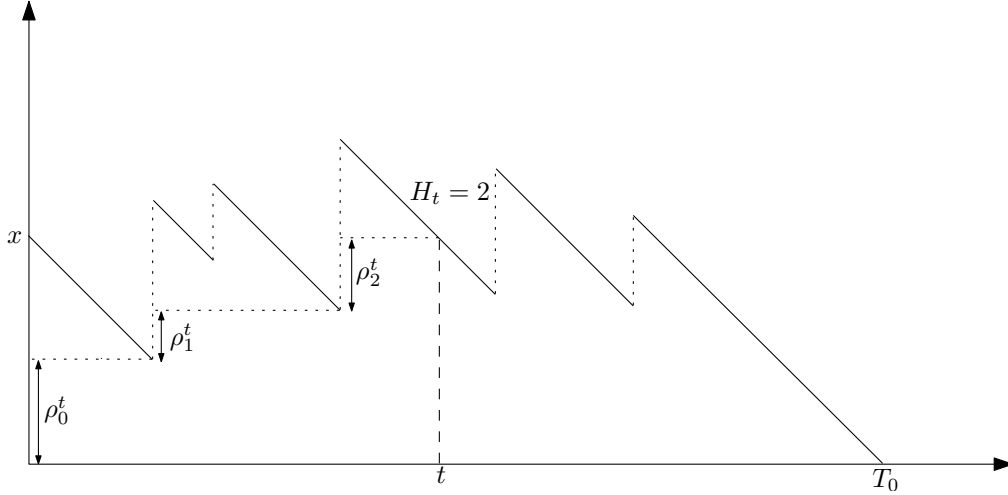


FIGURE 1. A trajectory of the process X started at x and killed when it reaches 0 and the remaining service times ρ_i^t for $i \in \{0, \dots, H_t\}$.

In the following proposition, we condition the Lévy process X to reach arbitrarily large heights before T_0 .

Theorem 2.3. (i) Assume that $b := \Lambda(\mathbb{R}^+)$ is finite and either $m = 1$ and $\int_{(0, \infty)} z^2 \Lambda(dz) < \infty$ or $m < 1$ and $\int_{[1, \infty)} z \log z \Lambda(dz) < \infty$. Then, for $t \geq 0$ and $\Theta \in \mathcal{F}_t$,

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta]$$

where

$$M_t = \sum_{i=0}^{H_t} \rho_i^t m^{-i}.$$

In particular, if $m = 1$, then $M_t = X_t$. Moreover, the process $(M_{t \wedge T_0}, t \geq 0)$ is a (\mathcal{F}_t) -martingale under \mathbb{P}_x .

(ii) If $b = \infty$, the conditioning with respect to $\{\sup_{s \in (0, T_0)} H_s \geq a\}$ is trivial in the sense that for all $a \geq 0$, $\mathbb{P}_x(\sup_{s \in (0, T_0)} H_s \geq a) = 1$.

Observe that if $b < \infty$, the process X is simply a compensated compound Poisson process whose jumps occur at rate b and are distributed as $\Lambda(\cdot)/b$.

The proof of this result will be made in Section 2.3. It uses the fact that X can be viewed as the contour process of a splitting tree visited at speed 1. Notice that the case where X is a Lévy process with Laplace exponent $\psi(\lambda) = \alpha\lambda - \int_{(0,\infty)}(1 - e^{-\lambda r})\Lambda(dr)$ and $b < \infty$ can be treated in a same way if $m \leq \alpha$. Indeed, in that case, X is still the contour process of a splitting tree but it is visited at speed α . Theorem 2.3 is still valid but the martingale becomes

$$M_{t \wedge T_0} = \sum_{i=0}^{H_t} \rho_i^t \left(\frac{\alpha}{m}\right)^i \mathbf{1}_{\{t \leq T_0\}}.$$

Before the proof, we define the splitting trees and recall some of their properties.

2.2. Splitting Trees. Most of what follows is taken from [19]. We denote the set of finite sequences of positive integers by

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

where $(\mathbb{N}^*)^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_n)$, then its generation is $|u| := n$, its ancestor at generation i is denoted by $u|i$ and if $v = (v_1, \dots, v_m)$ we denote by uv the concatenation of u and v

$$uv := (u_1, \dots, u_n, v_1, \dots, v_m).$$

In chronological trees, each individual has a birth level α and a death level ω . Let p_1, p_2 be the two canonical projections of $\mathbb{U} := \mathcal{U} \times [0, +\infty)$ on \mathcal{U} and $[0, \infty)$. We will denote by \mathcal{T} the projection of $\mathbb{T} \subset \mathbb{U}$ on \mathcal{U}

$$\mathcal{T} := p_1(\mathbb{T}) = \{u : \exists \sigma \geq 0, (u, \sigma) \in \mathbb{T}\}.$$

Definition 2.4. A subset \mathbb{T} of \mathbb{U} is a chronological tree if

- (i) $\rho := (\emptyset, 0) \in \mathbb{T}$ (the root)
- (ii) \mathcal{T} is a discrete tree
- (iii) $\forall u \in \mathcal{T}, \exists 0 \leq \alpha(u) < \omega(u) \leq \infty$ such that $(u, \sigma) \in \mathbb{T}$ if and only if $\alpha(u) < \sigma \leq \omega(u)$. $\alpha(u)$ (resp. $\omega(u)$) is the birth (resp. death) level of u
- (iv) if $ui \in \mathcal{T}$, then $\alpha(u) < \alpha(ui) < \omega(u)$ (an individual has only children during its life)
- (v) if $ui, uj \in \mathcal{T}$ then $i \neq j$ implies $\alpha(ui) \neq \alpha(uj)$ (no simultaneous births).

For $u \in \mathcal{T}$, we denote by $\zeta(u) := \omega(u) - \alpha(u)$ its *lifetime duration*. For two chronological trees \mathbb{T}, \mathbb{T}' and $x = (u, \sigma) \in \mathbb{T}$ such that $\sigma \neq \omega(u)$ (not a death point) and $\sigma \neq \alpha(ui)$ (not a birth point), we denote by $G(\mathbb{T}', \mathbb{T}, x)$ the *graft* of \mathbb{T}' on \mathbb{T} at x

$$G(\mathbb{T}', \mathbb{T}, x) := \mathbb{T} \cup \{(vw, \sigma + \tau) : (w, \tau) \in \mathbb{T}'\}.$$

Recall that Λ is a σ -finite measure on $(0, \infty]$ such that $\int_{(0,\infty)}(r \wedge 1)\Lambda(dr) < \infty$. A *splitting tree* [11, 13] is a random chronological tree defined as follows. For $x \geq 0$, we denote by \mathbf{P}_x the law of a splitting tree starting from an ancestor individual \emptyset with lifetime $(0, x]$. We define recursively the family of probabilities $\mathbf{P} = (\mathbf{P}_x)_{x \geq 0}$. Let $(\alpha_i, \zeta_i)_{i \geq 1}$ be the atoms of a Poisson measure on $(0, x) \times (0, +\infty]$ with intensity measure $\text{Leb} \otimes \Lambda$ where Leb is the Lebesgue measure. Then \mathbf{P} is the unique family of probabilities on chronological trees \mathbb{T} such that

$$\mathbb{T} = \bigcup_{n \geq 1} G(\mathbb{T}_n, \emptyset \times (0, x), \alpha_n)$$

where, conditional on the Poisson measure, the (\mathbb{T}_n) are independent splitting trees and for $n \geq 1$, conditional on $\zeta_n = \zeta$, \mathbb{T}_n has law \mathbf{P}_ζ .

The measure Λ is called the *lifespan measure* of the splitting tree and when it has a finite mass b , there is an equivalent definition of a splitting tree:

- individuals behave independently from one another and have i.i.d. lifetime durations distributed as $\frac{\Lambda(\cdot)}{b}$,
- conditional on her birthdate α and her lifespan ζ , each individual reproduces according to a Poisson point process on $(\alpha, \alpha + \zeta)$ with intensity b ,
- births arrive singly.

Because of the additional hypothesis $m < 1$ (resp. $m = 1$), the splitting trees that we consider are subcritical (resp. critical).

We now display a branching process included in a splitting tree and which will be useful in the following. According to [19], in the case $b < \infty$, if for $n \in \mathbb{N}$, \mathcal{Z}_n is the number of alive individuals of generation n

$$\mathcal{Z}_n := \#\{v \in \mathcal{T} : |v| = n\}, \quad (3)$$

then $(\mathcal{Z}_n, n \geq 0)$ is a Bienaymé-Galton-Watson (BGW) process starting at 1 and with offspring distribution defined by

$$p_k := \int_{(0, \infty)} \frac{\Lambda(dz)}{b} \frac{(bz)^k}{k!} e^{-bz}, \quad k \geq 0. \quad (4)$$

The condition $m \leq 1$ implies that the splitting trees that we consider have finite lengths a.s. and we can consider their associated JCCP (for jumping chronological contour process) $(Y_t, t \geq 0)$ as it is done in [19].

It is a càdlàg piecewise linear function which visits once each point of the tree at constant speed -1 . The visit of the tree begins at the death level of the ancestor. When the visit of an individual v of the tree begins, the value of the process is her death level $\omega(v)$. Then, it visits v backwards in time. If she has no child, her visit is interrupted after a time $\zeta(v)$; otherwise the visit stops when the birth level of her youngest child (call it w) is reached. Then, the contour process jumps from $\alpha(w)$ to $\omega(w)$ and starts the visit of w in the same way. When the visits of w and all her descendance will be complete, the visit of v can continue (at this point, the value of the JCCP is $\alpha(w)$) until another birth event occurs. When the visit of v is finished, the visit of her mother can resume (at level $\alpha(v)$). This procedure then goes on recursively until level 0 is encountered ($0 = \alpha(\emptyset) =$ birth level of the root) and after that the value of the process is 0 (see Figure 2). For a more formal definition of this process, read Section 3 in [19].

Moreover, the splitting tree can be fully recovered from its JCCP and we will use this correspondence to prove Theorem 2.3. It enables us to link the genealogical height in the chronological tree and the height process of the JCCP.

Proposition 2.5 (Lambert,[19]). *The process $(Y_t, t \geq 0)$ has the law of the Lévy process $(X_t, 0 \leq t \leq T_0)$ under \mathbb{P}_x .*

Moreover, if, as in Definition 2.1, for $t \geq 0$, we consider

$$h(t) := \#\{0 \leq s \leq t; Y_{s-} < \inf_{s \leq r \leq t} Y_r\},$$

then $h(t)$ is exactly the genealogical height in \mathcal{T} of the individual visited at time t by the contour process.

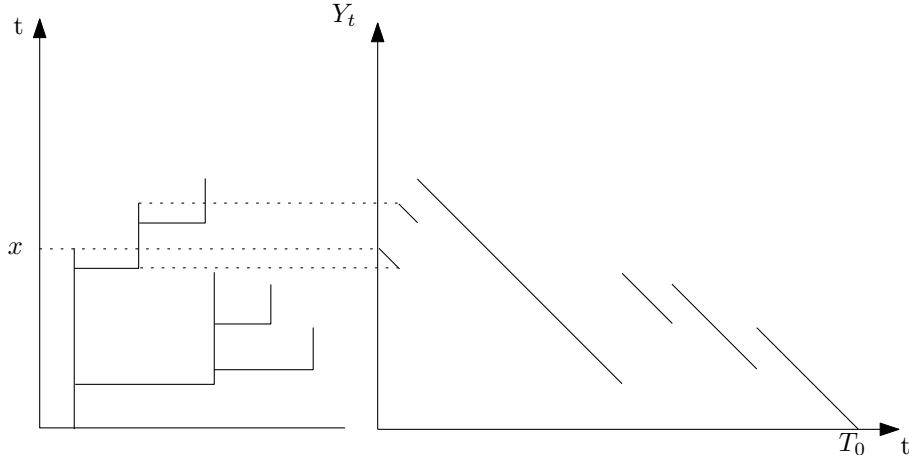


FIGURE 2. On the left panel, a splitting tree whose ancestor has lifespan duration x (vertical axis is time and horizontal axis shows filiation) and its associated Jumping Chronological Contour Process $(Y_t, t \geq 0)$ on the right panel.

2.3. Proof of Theorem 2.3. Thanks to Proposition 2.5, the process X is the JCCP of a splitting tree with lifespan measure Λ . For $a \geq 1$, let $\tau_a := \inf\{t \geq 0; H_t \geq a\}$. Then, $\{\sup_{s \leq T_0} H_s \geq a\} = \{\tau_a < T_0\}$. Furthermore, according to the second part of Proposition 2.5, the events $\{X \text{ reaches height } a \text{ before } T_0\}$ and $\{\text{the splitting tree is alive at generation } a\}$ coincide.

We first prove the simpler point (ii) of Theorem 2.3 where $b = \infty$. According to [19], if for $n \geq 0$, Z_n denotes the sum of lifespans of individuals of generation n in the splitting tree

$$Z_n := \sum_{v \in \mathcal{T}, |v|=n} \zeta(v),$$

then under \mathbb{P}_x , the process $(Z_n, n \geq 0)$ is a Jirina process starting at x and with branching mechanism

$$F(\lambda) := \int_{(0, \infty)} (1 - e^{-\lambda r}) \Lambda(dr) = \lambda - \psi(\lambda),$$

that is, $(Z_n, n \geq 0)$ is a time-homogeneous Markov chain with values in \mathbb{R}^+ , satisfying the branching property with respect to initial condition and such that

$$\mathbb{E}_x \left[e^{-\lambda Z_n} \right] = e^{-x F_n(\lambda)}$$

where $F_n := F \circ \dots \circ F$ is the n -th iterate of F . Hence, since H is the genealogical height in the splitting tree ,

$$\mathbb{P}_x(\tau_a < T_0) = \mathbb{P}_x(Z_a \neq 0) = 1 - \exp \left(-x \lim_{\lambda \rightarrow \infty} F_a(\lambda) \right) = 1$$

by monotone convergence and because the mass of Λ is infinite.

We now make the proof of Theorem 2.3(i) and suppose that b is finite.

$$\mathbb{P}_x \left(\Theta, t < T_0 \left| \sup_{s \leq T_0} H_s \geq a \right. \right) = \frac{\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) + \mathbb{P}_x(\Theta, t < \tau_a < T_0)}{\mathbb{P}_x(\tau_a < T_0)} \quad (5)$$

We will investigate the asymptotic behaviors of the three probabilities in the last display as $a \rightarrow \infty$.

As previously, H is the genealogical height in the splitting tree but in this case, we can use the process $(\mathcal{Z}_a, a \geq 1)$ defined by (3). This process $(\mathcal{Z}_a, a \geq 1)$ is a BGW process with offspring generating function defined by (4) and such that under \mathbb{P}_x , \mathcal{Z}_1 has a Poisson distribution with parameter bx . With an easy computation, one sees that the mean offspring number equals $m = \int_{(0, \infty)} r \Lambda(dr) \leq 1$. Hence the BGW-process is critical or subcritical. We have

$$\mathbb{P}_x(\tau_a < T_0) = \mathbb{P}_x(\mathcal{Z}_a \neq 0) = \sum_{k \geq 0} \mathbb{P}_x(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k) \mathbb{P}_x(\mathcal{Z}_1 = k)$$

and by the branching property,

$$\mathbb{P}_x(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k) = 1 - (1 - \mathbb{P}(\mathcal{Z}_{a-1} \neq 0))^k \underset{a \rightarrow \infty}{\sim} k \mathbb{P}(\mathcal{Z}_{a-1} \neq 0).$$

We first treat the subcritical case. According to Yaglom [26], if $(\mathcal{Z}_n, n \geq 0)$ is subcritical ($m < 1$) and if $\sum_{k \geq 1} p_k(k \log k) < \infty$, then there exists $c > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(\mathcal{Z}_n \neq 0)}{m^n} = c. \quad (6)$$

In the following lemma, we show that this log-condition holds with assumptions of the Theorem 2.3.

Lemma 2.6. *If $\int_{[1, \infty)} z \log z \Lambda(dz) < \infty$, then $\sum_{k \geq 1} p_k(k \log k) < \infty$.*

Proof. According to (4),

$$\begin{aligned} \sum_{k \geq 2} p_k k \log k &= \sum_{k \geq 2} k \log k b^{-1} \int_{(0, \infty)} \Lambda(dz) \frac{(bz)^k}{k!} e^{-bz} \\ &= b^{-1} \int_{(0, \infty)} \Lambda(dz) e^{-bz} \sum_{k \geq 2} k \log k \frac{(bz)^k}{k!} \end{aligned}$$

by Fubini-Tonelli theorem. Since we have

$$\log k \leq \frac{k-z}{z} + \log z, \quad k \geq 2, z > 0,$$

$$\begin{aligned} \sum_{k \geq 2} k \log k \frac{(bz)^k}{k!} &\leq \sum_{k \geq 2} k \left(\frac{k}{z} - 1 + \log z \right) \frac{(bz)^k}{k!} \\ &\leq \sum_{k \geq 2} k \frac{z^{k-1} b^k}{(k-1)!} + z \log z \sum_{k \geq 2} \frac{z^{k-1} b^k}{(k-1)!} \\ &\leq b(z+1)e^{bz} + bz \log z e^{bz} \end{aligned}$$

and

$$\sum_{k \geq 2} p_k k \log k \leq \int_{(0, \infty)} \Lambda(dz)(z + 1 + z \log z) < \infty.$$

□

Then if $\int_{[1, \infty)} r \log r \Lambda(dr) < \infty$, the log-condition of (6) is fulfilled and there exists a constant c such that

$$\lim_{a \rightarrow \infty} \mathbb{P}(\mathcal{Z}_{a-1} \neq 0 | \mathcal{Z}_1 = 1) / m^{a-1} = c.$$

Then,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k)}{m^{a-1}} = kc.$$

Moreover,

$$\frac{\mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = k)}{m^{a-1}} \leq \frac{k \mathbb{P}(\mathcal{Z}_a \neq 0 | \mathcal{Z}_1 = 1)}{m^{a-1}} \leq Ck$$

and

$$\sum_{k \geq 0} Ck \mathbb{P}_x(\mathcal{Z}_1 = k) = C \mathbb{E}_x[\mathcal{Z}_1] = Cbx < \infty$$

where C is some positive constant. Hence, using the dominated convergence theorem,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}_x(\tau_a < T_0)}{m^{a-1}} = c \mathbb{E}_x[\mathcal{Z}_1] = cbx \quad (7)$$

Similarly, if $(\mathcal{Z}_n, n \geq 0)$ is critical ($m = 1$), since the variance of its reproduction law

$$\sigma^2 = \sum_{k \geq 1} k^2 p_k - m^2 = b \int_{(0, \infty)} z^2 \Lambda(dz) - m + m^2 = b \int_{(0, \infty)} z^2 \Lambda(dz)$$

is finite, one also knows [3] the asymptotic behavior of $\mathbb{P}(\mathcal{Z}_n \neq 0)$. Indeed, we have the Kolmogorov's estimate

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\mathcal{Z}_n \neq 0) = \frac{2}{\sigma^2}. \quad (8)$$

Then

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}_x(\tau_a < T_0)}{a-1} = \frac{2}{\sigma^2} bx. \quad (9)$$

We are now interested in the behavior of $\mathbb{P}_x(\Theta, \tau_a \leq t < T_0)$ when a goes to infinity. In fact, we will show that it goes to 0 faster than m^{a-1} (resp. $1/a$) if $m < 1$ (resp. $m = 1$). Since $b = \Lambda(\mathbb{R}^+) < \infty$, the total number N_t of jumps of X before t has a Poisson distribution with parameter bt . Hence, since $\{\tau_a \leq t\} \subset \{N_t \geq a\}$,

$$\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) \leq \mathbb{P}(N_t \geq a) = \sum_{i \geq a} e^{-bt} \frac{(bt)^i}{i!} \leq \frac{(bt)^a}{a!}.$$

Thus, using the last equation and equation (7) or (9), the first term of the r.h.s. of (5) vanishes as $a \rightarrow \infty$ for $m \leq 1$.

We finally study the term $\mathbb{P}_x(\Theta, t < \tau_a < T_0)$. For a word u , $i \in \mathbb{N}$ and $x \in \mathbb{R}$, we denote by $A(u, i, x)$ the event $\{u$ gives birth before age x to a daughter which has alive descendance at generation $|u| + i\}$, that is, if $|u| = j$,

$$A(u, i, x) := \{\exists v \in \mathcal{U}; |v| = i, uv \in \mathcal{T} \text{ and } \alpha(uv|j+1) - \alpha(u) \leq x\}.$$

Let v_t be the individual visited at time t . Hence, using the Markov property at time t ,

$$\begin{aligned} \mathbb{P}_x(\Theta, t < \tau_a < T_0) &= \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_{\{t < \tau_a\}} \mathbf{1}_\Theta \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \right] \\ &= \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_{\{t < \tau_a\}} \mathbf{1}_\Theta \mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \right] \end{aligned}$$

and by the branching property,

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) = 1 - \prod_{i=0}^{H_t} \left(1 - \mathbb{P}_{\rho_i^t}(\mathcal{Z}_{a-i} \neq 0) \right) \quad \text{a.s.}$$

As previously, since computations for subcritical and critical cases are equivalent, we only detail the first one. We have, with another use of (6),

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \underset{a \rightarrow \infty}{\sim} \sum_{i=0}^{H_t} m^{a-i-1} \rho_i^t cb \quad \text{a.s.}$$

We want to use the dominated convergence theorem to prove that

$$\lim_{a \rightarrow \infty} m^{1-a} \mathbb{P}_x(\Theta, t < \tau_a < T_0) = cb \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_\Theta \sum_{i=0}^{H_t} \rho_i^t m^{-i} \right]$$

and then, using (7),

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \middle| \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x \left[\mathbf{1}_{\{t < T_0\}} \mathbf{1}_\Theta \sum_{i=0}^{H_t} \rho_i^t m^{-i} \right]$$

so that the proof of the subcritical case would be finished. We have

$$m^{-a} \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \mathbf{1}_{\{t < T_0\}} \leq m^{-a} \sum_{i=0}^{H_t} \mathbb{P}_x(A(v_t | i, a - i, \rho_i^t)) \mathbf{1}_{\{t < T_0\}} \leq C' \sum_{i=0}^{H_t} \rho_i^t m^{-i} \mathbf{1}_{\{t < T_0\}} \quad \text{a.s.}$$

where C' is a positive, deterministic constant. Hence, to obtain an integrable upper bound, since $\mathbb{E}_x[\rho_0^t \mathbf{1}_{\{t < T_0\}}] \leq x$, it is sufficient to prove that

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] < \infty \quad (10)$$

in order to use the dominated convergence theorem. Recall that $X^{(t)}$ denotes the time-reversal of X at time t

$$X_s^{(t)} := X_{t-s} - X_{(t-s)-} \quad s \in [0, t], \quad (X_{0-} = X_0)$$

and $S^{(t)}$ is its associated past supremum process

$$S_s^{(t)} := \sup\{X_r^{(t)} : 0 \leq r \leq s\}.$$

It is known that the process $X^{(t)}$ has the law of X under \mathbb{P}_0 [5, ch.II]. We also have

$$H_t = R_t := \# \left\{ 0 \leq s \leq t; X_s^{(t)} = S_s^{(t)} \right\}$$

which is the number of records of the process $S^{(t)}$ during $[0, t]$ and the ρ_i^t 's are the overshoots of the successive records. More precisely, if we denote by $\tilde{T}_1 < \tilde{T}_2 < \dots$ the record times of $X^{(t)}$, the overshoots are

$$\tilde{\rho}_i := X_{\tilde{T}_i}^{(t)} - \sup_{0 \leq s < \tilde{T}_i} X_s^{(t)}, \quad i \geq 1.$$

We come back to the proof of (10). We have

$$\sum_{i=1}^{H_t} \rho_i^t m^{-i} = \sum_{i=1}^{R_t} \tilde{\rho}_{R_t-i+1} m^{-i} = \sum_{i=1}^{R_t} \tilde{\rho}_i m^{i-R_t-1} = \sum_{i \geq 1} \tilde{\rho}_i m^{i-R_t-1} \mathbf{1}_{\{\tilde{T}_i \leq t\}}.$$

We denote by $(\tilde{\mathcal{F}}_s, s \geq 0)$ the natural filtration of $X^{(t)}$. Thus, using Fubini-Tonelli theorem and the strong Markov property for $X^{(t)}$ applied at time \tilde{T}_i

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] = \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i m^{i-R_t-1} \mathbf{1}_{\{\tilde{T}_i \leq t\}} \right] = m^{-1} \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} f(t - \tilde{T}_i) \right]$$

where

$$f(s) = \mathbb{E} \left[\frac{1}{m^{R_s}} \right], \quad s \geq 0.$$

However, as previously, we have almost surely $R_t \leq N_t$ where N_t is the number of jumps of X or $X^{(t)}$ before t and so has a Poisson distribution with parameter bt . Thus,

$$f(s) \leq \mathbb{E}[m^{-N_s}] = \exp(bs(m^{-1} - 1)) = e^{\kappa s}$$

where κ is some positive constant and

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] \leq m^{-1} e^{\kappa t} \sum_{i \geq 1} \mathbb{E} \left[\tilde{\rho}_i; \tilde{T}_i \leq t \right].$$

Moreover,

$$\begin{aligned} \mathbb{E} \left[\tilde{\rho}_i; \tilde{T}_i \leq t \right] &= \mathbb{E} \left[\tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t - \tilde{T}_{i-1}\}} \mathbf{1}_{\{\tilde{T}_{i-1} \leq t\}} \right] = \int_0^t \mathbb{P}(\tilde{T}_{i-1} \in ds) \mathbb{E} \left[\tilde{\rho}_i; \tilde{T}_i \leq t - s \right] \\ &\leq \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \mathbb{P}(\tilde{T}_{i-1} \leq t) \end{aligned}$$

since $((\tilde{\rho}_i, \tilde{T}_i - \tilde{T}_{i-1}), i \geq 1)$ are i.i.d. random variables. Then

$$\mathbb{E}_x \left[\sum_{i=1}^{H_t} \rho_i^t m^{-i} \right] \leq m^{-1} e^{\kappa t} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \sum_{i \geq 0} \mathbb{P}(\tilde{T}_i \leq t) = m^{-1} e^{\kappa t} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] \sum_{i \geq 0} \mathbb{P}(R_t \geq i).$$

The sum in the r.h.s equals $\mathbb{E}[R_t]$ which is finite since $R_t \leq N_t$ a.s. According to Theorem VII.17 in [5], the joint law of $(X_{\tilde{T}_1}^{(t)}, \Delta X_{\tilde{T}_1}^{(t)})$ is given by

$$\mathbb{E} \left[F \left(X_{\tilde{T}_1}^{(t)}, \Delta X_{\tilde{T}_1}^{(t)} \right); \tilde{T}_1 < \infty \right] = \int_{(0, \infty)} \Lambda(dy) \int_0^y dx F(x, y). \quad (11)$$

Thus, since $\{\tilde{T}_1 \leq t\} \subset \{X_{\tilde{T}_1-}^{(t)} \geq -t\}$,

$$\begin{aligned} \mathbb{E} \left[\tilde{\rho}_1; \tilde{T}_1 \leq t \right] &\leq \mathbb{E} \left[\tilde{\rho}_1 \mathbf{1}_{\left\{ \Delta X_{\tilde{T}_1}^{(t)} - X_{\tilde{T}_1}^{(t)} \leq t \right\}} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \int_{(0, \infty)} \Lambda(dy) \int_0^y x \mathbf{1}_{\{y-x \leq t\}} dx \\ &= \int_{(t, \infty)} \Lambda(dy) (yt - t^2/2) + \int_{(0, t]} \Lambda(dy) y^2/2 < \infty \end{aligned}$$

and the proof of (10) is completed.

Finally, in the critical case $m = 1$, computations are similar. Thanks to (8),

$$\mathbb{P}_x \left(\bigcup_{i=0}^{H_t} A(v_t | i, a - i, \rho_i^t) \middle| \mathcal{F}_t \right) \underset{a \rightarrow \infty}{\sim} \frac{1}{a} \sum_{i=0}^{H_t} \rho_i^t \frac{2}{\sigma^2} b = \frac{1}{a} \frac{2b}{\sigma^2} X_t \quad \text{a.s.}$$

by the definition of the ρ_i^t 's. Moreover, when $m = 1$, the process X is a (\mathcal{F}_t) -martingale and $\mathbb{E}_x[X_t] = \mathbb{E}_x[X_0] = x$. Then, by the dominated convergence theorem,

$$\lim_{a \rightarrow \infty} a \mathbb{P}_x(\Theta, t < \tau_a < T_0) = \frac{2b}{\sigma^2} \mathbb{E}_x[X_{t \wedge T_0} \mathbf{1}_\Theta]$$

and then, using (9),

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \middle| \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x[X_{t \wedge T_0} \mathbf{1}_\Theta].$$

We conclude the proof by showing that $(M_{t \wedge T_0}, t \geq 0)$ is a (\mathcal{F}_t) -martingale. First, if $m = 1$, we have $M_t = X_t$ and as it was stated just before, in this case, the process X is a martingale and so is $X_{\cdot \wedge T_0}$. We now consider the case $m < 1$. Recall from (2) that the infinitesimal generator of $(\rho^{t \wedge T_0}, t \geq 0)$ is

$$\begin{aligned} \mathcal{A}(f)(\nu) &= \left[\int_{(0, \infty)} (f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, r, 0, \dots) - f(\nu_0, \nu_1, \dots, \nu_{H(\nu)}, 0, \dots)) \Lambda(dr) \right. \\ &\quad \left. - \frac{\partial f}{\partial x_{H(\nu)}}(\nu_0, \dots, \nu_{H(\nu)}, 0, \dots) \right] \mathbf{1}_{\{\nu \neq 0\}}, \quad \nu \in S. \end{aligned}$$

Let g be the application from S to \mathbb{R} such that

$$g(\nu) := \sum_{i=0}^{H(\nu)} \nu_i m^{-i}, \quad \nu \in S. \quad (12)$$

Clearly, we have $g(\rho^t) = M_t$ for $t \leq T_0$. According to [24, ch.VII], if $\mathcal{G}_t := \{\sigma(\rho^r), 0 \leq r \leq t\}$, to prove that M is a (\mathcal{G}_t) -martingale, it is sufficient to show that $\mathcal{A}(g) = 0$. For $\nu \in S$, we have

$$\begin{aligned} \mathcal{A}(g)(\nu) &= \left(\int_{(0, \infty)} (g(\nu) + r m^{-H(\nu)-1} - g(\nu)) \Lambda(dr) - m^{-H(\nu)} \right) \mathbf{1}_{\{\nu \neq 0\}} \\ &= \left(\int_{(0, \infty)} r \Lambda(dr) m^{-H(\nu)-1} - m^{-H(\nu)} \right) \mathbf{1}_{\{\nu \neq 0\}} = 0. \end{aligned}$$

Then, $M_{\cdot \wedge T_0}$ is a (\mathcal{G}_t) -martingale. Moreover, since $\langle \rho^t, \mathbf{1} \rangle = X_t$, $\mathcal{F}_t \subset \mathcal{G}_t$ and clearly $\mathcal{G}_t \subset \mathcal{F}_t$. Then, $\mathcal{F}_t = \mathcal{G}_t$ and $M_{\cdot \wedge T_0}$ is a (\mathcal{F}_t) -martingale. \square

2.4. A new probability.

Definition 2.7. For $x > 0$, we define a new probability measure \mathbb{P}_x^\uparrow on (Ω, \mathcal{F}_t) by

$$\mathbb{P}_x^\uparrow(\Theta) = \frac{1}{x} \mathbb{E}_x [M_{t \wedge T_0} \mathbf{1}_\Theta], \quad \Theta \in \mathcal{F}_t.$$

First, notice that since $M_{\cdot \wedge T_0}$ is a martingale, this definition is consistent and that this new measure is honest, that is, it has mass 1. Second, $(X_t, t \geq 0)$ is no more Markovian under \mathbb{P}_x^\uparrow while $(\rho^t, t \geq 0)$ is because \mathbb{P}_x^\uparrow is obtained by a h -transform via the application g defined by (12).

Proposition 2.8. Under \mathbb{P}_x^\uparrow , in probability,

$$X_t \xrightarrow[t \rightarrow \infty]{} +\infty.$$

Proof. It is sufficient to prove that for all positive λ , $\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] \rightarrow 0$ as t tends to ∞ . First, in the critical case $m = 1$,

$$\mathbb{E}_x^\uparrow [e^{-\lambda X_t}] = \frac{1}{x} \mathbb{E}_x [e^{-\lambda X_t} X_{t \wedge T_0}] \xrightarrow[t \rightarrow \infty]{} 0 \quad (13)$$

using dominated convergence theorem, because $(X_{t \wedge T_0}, t \geq 0)$ is a non-negative martingale converging a.s. to 0 and the mapping $x \mapsto e^{-\lambda x} x$ is bounded on \mathbb{R}^+ .

We now suppose that $m < 1$. Then, for all positive a , distinguishing between $H_t > a$ and $H_t \leq a$, we have

$$\begin{aligned} \mathbb{E}_x^\uparrow [e^{-\lambda X_t}] &= \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=0}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \right] \\ &\leq \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right] + \mathbb{P}_x(t \leq T_0) + \frac{m^{-a}}{x} \mathbb{E}_x [e^{-\lambda X_t} X_{t \wedge T_0}] \end{aligned}$$

since $\sum_i \rho_i^t = X_t$. The second term of the r.h.s is an upper bound of the term $i = 0$ of the sum and it goes to 0 as t tends to ∞ since T_0 is finite a.s. So does the third term similarly to the critical case in (13). Then, to finish the proof, it is sufficient to show that the first term tends to 0 uniformly in t as a goes to ∞ . We have

$$A(t, a) := \mathbb{E}_x \left[e^{-\lambda X_t} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right] \leq \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \sum_{i=1}^{H_t} m^{-i} \rho_i^t \mathbf{1}_{\{t \leq T_0\}} \mathbf{1}_{\{H_t > a\}} \right]$$

and as in proof of Theorem 2.3, by time reversing at time t , we have

$$A(t, a) \leq \sum_{i \geq 1} \mathbb{E} \left[e^{-\lambda S_t} m^{-(R_t - i + 1)} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} \mathbf{1}_{\{R_t > a\}} \right] = \sum_{i \geq 1} m^{i-1} B_i(t, a)$$

where the \tilde{T}_i 's are the times of successive records of the supremum process S , the $\tilde{\rho}_i$'s are the associated overshoots and R_t is the number of records up to time t . In particular, $S_t = \sum_{j=1}^{R_t} \tilde{\rho}_j$.

On the one hand, let us treat the case when $i \leq a$. Since $\{R_t > a\} \subset \{\tilde{T}_a \leq t\}$ and by applying the strong Markov property at time \tilde{T}_a ,

$$\begin{aligned} B_i(t, a) &\leq \mathbb{E} \left[\prod_{j=1}^{R_t} e^{-\lambda \tilde{\rho}_j} m^{-R_t} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} \right] \\ &= \mathbb{E} \left[\prod_{j=1}^a e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} \mathbb{E} \left[\prod_{j=a+1}^{R_t} e^{-\lambda \tilde{\rho}_j} m^{-R_t} \middle| \tilde{\mathcal{F}}_{\tilde{T}_a} \right] \right] \\ &= \mathbb{E} \left[\prod_{j=1}^a e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_a \leq t\}} m^{-a} f(t - \tilde{T}_a) \right] \end{aligned} \quad (14)$$

where for $s \geq 0$, $f(s) := \mathbb{E} \left[\prod_{j=1}^{R_s} e^{-\lambda \tilde{\rho}_j} m^{-R_s} \right]$. This function f is bounded. Indeed, for $s \geq 0$,

$$\begin{aligned} f(s) &= \sum_{i \geq 1} m^{-i} \mathbb{E} \left[\prod_{j=1}^i e^{-\lambda \tilde{\rho}_j}; \tilde{T}_i \leq s < \tilde{T}_{i+1} \right] \leq \sum_{i \geq 1} m^{-i} \mathbb{E} \left[\prod_{j=1}^i \left(e^{-\lambda \tilde{\rho}_j} \mathbf{1}_{\{\tilde{T}_j - \tilde{T}_{j-1} \leq s\}} \right) \right] \\ &\leq \sum_{i \geq 1} m^{-i} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1}; \tilde{T}_1 < \infty \right]^i \end{aligned}$$

since the r.v. $(\tilde{\rho}_j, \tilde{T}_j - \tilde{T}_{j-1})_{j \geq 1}$ are i.i.d. Applying equation (11) with the function $F(x, y) = e^{-\lambda x}$, we have

$$\mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \int_{(0, \infty)} \Lambda(dy) \frac{1 - e^{-\lambda y}}{\lambda} = 1 - \frac{\psi(\lambda)}{\lambda}.$$

Then,

$$f(s) \leq \sum_{i \geq 1} m^{-i} \left(1 - \frac{\psi(\lambda)}{\lambda} \right)^i < \infty$$

since $1 - \psi(\lambda)/\lambda < m$ for all positive λ because $m = 1 - \psi'(0)$ and ψ is strictly convex.

We come back to (14). If we denote by C an upper bound for f , for $i \leq a$,

$$\begin{aligned} B_i(t, a) &\leq C m^{-a} \prod_{j=1, j \neq i}^a \mathbb{E} \left[e^{-\lambda \tilde{\rho}_j} \mathbf{1}_{\{\tilde{T}_j - \tilde{T}_{j-1} \leq t\}} \right] \mathbb{E} \left[e^{-\lambda \tilde{\rho}_i} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t\}} \right] \\ &\leq C m^{-a} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right]^{a-1} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_1} \tilde{\rho}_1 \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] \end{aligned}$$

Applying again (11) with $F(x, y) = x e^{-\lambda x}$, we obtain

$$\mathbb{E} \left[\tilde{\rho}_1 e^{-\lambda \tilde{\rho}_1} \mathbf{1}_{\{\tilde{T}_1 < \infty\}} \right] = \frac{\psi'(\lambda)}{\lambda} - \frac{\psi(\lambda)}{\lambda^2} < \infty. \quad (15)$$

Hence,

$$\begin{aligned} \sum_{i=1}^a m^{i-1} B_i(t, a) &\leq C(\lambda) m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda} \right)^{a-1} \sum_{i=1}^a m^i \\ &\leq C(\lambda) a m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda} \right)^{a-1}. \end{aligned} \quad (16)$$

On the other hand, for $i > a$, as in previous computations, we apply the strong Markov property at time \tilde{T}_i to obtain

$$\begin{aligned} \sum_{i \geq a+1} m^{i-1} B_i(t, a) &\leq \sum_{i \geq a+1} \mathbb{E} \left[\prod_{j=1}^i e^{-\lambda \tilde{\rho}_j} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i \leq t\}} m^{-1} f(t - \tilde{T}_i) \right] \\ &\leq C m^{-1} \sum_{i \geq a+1} \mathbb{E} \left[e^{-\lambda \tilde{\rho}_i} \tilde{\rho}_i \mathbf{1}_{\{\tilde{T}_i - \tilde{T}_{i-1} \leq t\}} \mathbf{1}_{\{\tilde{T}_{i-1} \leq t\}} \right] \\ &\leq C m^{-1} \left(\frac{\psi'(\lambda)}{\lambda} - \frac{\psi(\lambda)}{\lambda^2} \right) \sum_{i \geq a+1} \mathbb{P}(R_t \geq i-1) \end{aligned}$$

where we have first used the previous function f and its upper bound C and then that $(\tilde{\rho}_i, \tilde{T}_i - \tilde{T}_{i-1})$ is independent from \tilde{T}_{i-1} and equation (15).

Then

$$\sum_{i \geq a+1} m^{i-1} B_i(t, a) \leq \tilde{C}(\lambda) \sum_{i \geq a} \mathbb{P}(R_\infty \geq i) \quad (17)$$

where $\tilde{C}(\lambda)$ is a finite constant independent from t and a and $R_\infty := \lim_{t \rightarrow +\infty} R_t \in \mathbb{N}$ is the total number of records of S . Moreover, thanks to the strong Markov property, R_∞ follows a geometric distribution with success probability $\mathbb{P}(\tilde{T}_1 = \infty) = 1 - m$ according to (11) with $F \equiv 1$.

Finally, putting together (16) and (17), we obtain

$$A(t, a) \leq \sum_{i \geq 1} m^{i-1} B_i(t, a) \leq \bar{C}(\lambda) \left(a m^{-a+1} \left(1 - \frac{\psi(\lambda)}{\lambda} \right)^{a-1} + \sum_{i \geq a} \mathbb{P}(R_\infty \geq i) \right) \quad (18)$$

where $\bar{C}(\lambda)$ is a finite constant independent from t and a . The first term of the r.h.s. of (18) goes to 0 as a goes to ∞ since $1 - \psi(\lambda)/\lambda < m$. The second term also tends to 0 because $\mathbb{E}[R_\infty] = (1 - m)^{-1}$ is finite. Hence, we have proved that $A(t, a)$ tends to 0 as $a \rightarrow \infty$ uniformly in t and $\mathbb{E}_x^\uparrow[e^{-\lambda X_t}] \rightarrow 0$ which ends the proof. \square

3. CONDITIONED SUBCRITICAL AND CRITICAL SPLITTING TREES

In this section, we are interested in the behavior as $n \rightarrow \infty$ of a (sub)critical splitting tree conditioned to be alive at generation n . More precisely, we want to give a spine decomposition of the limiting tree. We also give the distribution of its contour process which visits the left part of the spine.

We will use notation of Section 4.4 of [18]. For $n \geq 0$ and $u \in \mathcal{T}$, we denote by $E_n(u)$ the event $\{u \text{ has an extant descendance at generation } |u| + n\}$. For simplicity of notation, we will denote by E_n the event $E_n(\emptyset)$. Let \mathbb{P}^n be the law of the splitting tree on this event

$$\mathbb{P}^n := \mathbb{P}(\cdot | E_n) = \mathbb{P}(\cdot | \mathcal{Z}_n > 0).$$

On the event E_n , we define a distinguished lineage $u_0^n u_1^n \cdots u_n^n$ as the first lineage of the tree that reaches generation n as explained below. First, $u_0^n = \emptyset$ and one defines recursively: for $i \geq 1$, $u_0^n u_1^n \cdots u_i^n$ is the youngest daughter of $u_0^n u_1^n \cdots u_{i-1}^n$ that has a descendance at generation n . We set $u^{(n)} := u_0^n u_1^n \cdots u_n^n$ and we denote by

$$\mathbb{B}^n := \{x \in \mathbb{T}; p_1(x) = u^{(n)} | k \text{ for } 0 \leq k \leq n\}$$

this lineage of \mathbb{T} defined on the event E_n . For $0 \leq k \leq n$, the age at which the individual $u^{(n)}|k$ gives birth to individual $u^{(n)}|k+1$ is

$$A_k^n := \alpha(u^{(n)}|k+1) - \alpha(u^{(n)}|k),$$

its residual lifetime is

$$R_k^n := \omega(u^{(n)}|k) - \alpha(u^{(n)}|k+1).$$

and its total lifelength is denoted by $T_k^n = A_k^n + R_k^n$. Observe that modulo labelling, the sequence $((A_k^n, R_k^n), 0 \leq k \leq n)$ characterizes the spine \mathbb{B}^n .

In the following, when we say that a tree (marked with a special lineage) converges for finite truncations, we mean that there is convergence for events that are measurable with respect to the σ -field generated by the truncations $\{x \in \mathbb{T}; p_2(x) \leq \sigma\}$, $\sigma > 0$.

Theorem 3.1. *We suppose that either $m < 1$ and $\int_{[1, \infty)} z \log z \Lambda(dz) < \infty$ or $m = 1$ and $\int_{(0, \infty)} z^2 \Lambda(dz) < \infty$. Then, as $n \rightarrow \infty$, the law of \mathbb{B}^n under \mathbb{P}^n converges for finite truncations to an infinite spine \mathbb{B} whose distribution is characterized by an i.i.d sequence $(A_k, R_k)_{k \geq 0}$ such that*

$$(A_1, R_1) \stackrel{(d)}{=} (UD, (1-U)D)$$

where U is a uniform r.v. on $(0, 1)$ independent from the size-biased r.v. D

$$\mathbb{P}(D \in dz) = \frac{z \Lambda(dz)}{m} \quad z > 0.$$

Moreover, if $(x_i, i \in \mathbb{N})$ are the atoms of a Poisson measure on \mathbb{B} with intensity b , then as $n \rightarrow \infty$, the law of \mathbb{T} under \mathbb{P}^n converges for finite truncations to an infinite tree with a unique infinite branch \mathbb{B} on which are grafted at points (x_i) i.i.d. trees with common law \mathbb{P} .

Proof. First, for $t > 0$ we denote by $K_n(t)$ the number of individuals at time t that have alive descendance at generation n . Then, as $n \rightarrow \infty$, $\mathbb{P}^n(K_n(t)) \rightarrow 1$. Hence, the limiting tree under \mathbb{P}^n has a unique infinite branch.

We now investigate the law of this limiting spine. Let p be a natural integer. We denote by \mathcal{F}^p the σ -field generated by the lifespans of all individuals until generation p and the numbers and birthdates of their daughters.

Under \mathbb{P}^n , we denote by N_i the number of children of $u^{(n)}|i$ and by $\alpha_{ij}, 1 \leq j \leq N_i$ her age at their births. Then, for $n_1, \dots, n_p \in \mathbb{N}^*$, $z_1, \dots, z_p > 0$ and $0 < x_{ij} < z_i$ for $1 \leq i \leq p, 1 \leq j \leq n_i$, we have

$$\begin{aligned} \mathbb{P}^n(\mathcal{E}^p) &:= \mathbb{P}^n(T_i^n \in dz_i, N_i = n_i, \alpha_{ij} \in dx_{ij}, j = 1, \dots, n_i, i = 1, \dots, p) \\ &= \frac{1}{\mathbb{P}(E_n)} \mathbb{E} \left[\prod_{i=1}^p \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \prod_{l=1}^{u_i^n - 1} \mathbf{1}_{(E_{n-i}(\tilde{v}_i^l))^c} \right) \mathbf{1}_{E_{n-p}(u^{(n)}|p)} \right] \end{aligned}$$

where $\tilde{v}_i^l, 1 \leq l \leq u_i^n - 1$ are the younger sisters of $u^{(n)}|i = u_0^n \cdots u_i^n$. Conditioning with respect to the σ -field \mathcal{F}^p and thanks to the branching property, we have

$$\mathbb{P}^n(\mathcal{E}^p) = \prod_{i=1}^p \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \right) \mathbb{E} \left[\prod_{i=1}^p \mathbb{P}(E_{n-i}^c)^{u_i^n - 1} \frac{\mathbb{P}(E_{n-p})}{\mathbb{P}(E_n)} \right].$$

Since we consider subcritical or critical trees, $\mathbb{P}(E_{n-i}^c)$ goes to 1 as $n \rightarrow \infty$. Moreover, for $m < 1$, according to Lemma 2.6, if $\int_{[1,\infty)} z \log z \Lambda(dz) < \infty$, then the log-condition of (6) is fulfilled and there exists $c > 0$ such that

$$\frac{\mathbb{P}(E_n)}{m^n} = \frac{\mathbb{P}(\mathcal{Z}_n \neq 0)}{m^n} \xrightarrow{n \rightarrow \infty} c.$$

Furthermore, in critical case $m = 1$, if $\int_{(0,\infty)} z^2 \Lambda(dz)$ is finite, according to (8), there is $c' > 0$ such that

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\mathcal{Z}_n \neq 0) = c'.$$

In both cases, we obtain by the dominated convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}^n(\mathcal{E}^p) &= \prod_{i=1}^p \left(\frac{\Lambda(dz_i)}{b} e^{-bz_i} \frac{(bz_i)^{n_i}}{n_i!} \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_i} \right) m^{-p} \\ &= \prod_{i=1}^p \frac{z_i \Lambda(dz_i)}{m} \cdot \frac{e^{-bz_i} (bz_i)^{n_i-1}}{(n_i-1)!} \cdot \frac{1}{n_i} \cdot \prod_{j=1}^{n_i} \frac{dx_{ij}}{z_j}. \end{aligned}$$

Thus, we see that in the limit $n \rightarrow \infty$, the individuals of the spine have independent size-biased lifelengths (first term of the r.h.s.), give birth to one plus a Poissonian number of children (second term) whose birth dates are independent uniform during the life of their mother (fourth term) and one of them is marked uniformly among them (third term).

Hence, we obtain a limiting spine \mathbb{B} characterized by a sequence $(A_k + R_k, A_k)$ which are independent r.v. with the same joint distribution as (D, UD) . Moreover, conditional on $T_k = A_k + R_k$, the k -th individual of \mathbb{B} has a Poissonian number (with parameter $T_k b$) of non-marked daughters which are born independently and uniformly on $[0, T_k]$.

Intersecting the previous event \mathcal{E}^p with events involving trees descending from sisters of the marked individuals and applying the branching property leads by similar computations to the last statement of the theorem. \square

We denote by $(X^\uparrow(t), t \geq 0)$ the contour of the limiting tree. In fact, it only visits the left part of the infinite spine of the tree. An immediate consequence of the previous theorem deals with the law of this contour process.

Corollary 3.2. *Let U and D be r.v. as in Theorem 3.1 and X be the Lévy process defined in Section 2 and independent from U and D . We have*

$$\left(X^\uparrow(t), t \geq 0 \right) \stackrel{(d)}{=} \left(\sum_{k \geq 0} X_k(t - \varsigma_k) \mathbf{1}_{\{t \geq \varsigma_k\}} \right)$$

where

- $(X_k, k \geq 0)$ is a sequence of i.i.d. processes distributed as X started at level D and stopped at the hitting time η of level UD .
- $\varsigma_0 = 0$ and $\varsigma_k = \eta_1 + \dots + \eta_k$ where for $k \geq 1$, η_k is the stopping time for the process X_k .

Remark 3.3. *An open problem is to prove that the contour X^\uparrow of the conditioned tree has the same distribution as the conditioned process X under \mathbb{P}^\uparrow defined in Section 2.*

4. INFINITE VARIATION CASE

4.1. Definitions and main results. In this section, we consider the same setting as T. Duquesne and J.F. Le Gall in [10, Ch 1]. We assume that X is a Lévy process with no negative jumps, which does not drift to $+\infty$ (so that X hits 0 a.s.) and has paths with infinite variation. Its law is characterized by its Laplace transform

$$\mathbb{E} \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad \lambda > 0$$

specified by the Lévy-Khinchin formula

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 + \int_{(0, \infty)} \Lambda(dr) (e^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r < 1\}})$$

where $\alpha_0 \in \mathbb{R}$, $\beta \geq 0$ and Λ is a positive measure on $(0, +\infty)$ such that $\int_{(0, \infty)} \Lambda(dr) (1 \wedge r^2) < \infty$. Actually, since X does not drift to $+\infty$, it has finite first moment and then $\int_{(0, \infty)} \Lambda(dr) (r \wedge r^2) < \infty$. We can rewrite ψ as

$$\psi(\lambda) = \alpha \lambda + \beta \lambda^2 + \int_{(0, \infty)} \Lambda(dr) (e^{-\lambda r} - 1 + \lambda r).$$

Then, X does not drift to $+\infty$ iff $\psi'(0) = \alpha \geq 0$. More precisely, if $\alpha = 0$, X is recurrent while if $\alpha < 0$, it drifts to $-\infty$.

According to Corollary VII.5 in [5], X has infinite variation iff $\lim_{\lambda \rightarrow \infty} \psi(\lambda)/\lambda = +\infty$, which is satisfied iff

$$\beta > 0 \text{ or } \int_{(0, 1)} r \Lambda(dr) = \infty. \quad (19)$$

We suppose that

$$\int_{[1, +\infty)} \frac{d\lambda}{\psi(\lambda)} < \infty. \quad (20)$$

which implies (19).

We denote by I the past infimum process of X

$$I_t = \inf_{0 \leq s \leq t} X_s.$$

The process $X - I$ is a strong Markov process and because X has infinite variation, 0 is regular for itself for this process. Thus, we can consider its excursion measure N away from 0 and normalized so that $-I$ is the associated local time.

The point 0 is also regular for itself for the Markov process $X - S$. We denote by L a local time at 0 for this process and by N^* the associated excursion measure normalized so that for every Borel subset B of $(-\infty, 0)$,

$$N^* \left(\int_0^\sigma \mathbf{1}_B(X_s) \right) = m(B) \quad (21)$$

where m denotes the Lebesgue measure on \mathbb{R} (see Lemma 1.1.1 in [10] for more details). This also fixes the normalization of the local time L .

We now define the height process H associated with X which is the counterpart of the height process in the finite variation case. As in Section 2, we denote by $X^{(t)}$ the time-reversed process associated with X at time t and by $S^{(t)}$ its past supremum.

Definition 4.1. *The height process $H = (H_t, t \geq 0)$ associated with the Lévy process X is the local time at level 0 at time t of the process $S^{(t)} - X^{(t)}$ with the normalization fixed in (21) and $H_0 := 0$.*

According to Theorem 1.4.3 in [10], the condition (20) implies that H has a.s. continuous sample paths. There is an alternative definition of H_t : there exists a positive sequence $(\varepsilon_k, k \geq 0)$ such that a.s., for all $t \geq 0$,

$$H_t = \lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_0^t \mathbf{1}_{\{X_t < I_s^t + \varepsilon_k\}} ds < \infty$$

where for $0 \leq s \leq t$, we set by

$$I_s^t := \inf_{s \leq r \leq t} X_r$$

the future infimum process of X at time t . Since this process is non-decreasing, one can define its right-continuous inverse I_t^{-1}

$$I_t^{-1}(u) := \inf\{s \geq 0; I_s^t > u\}, \quad 0 \leq u \leq X_t.$$

Observe that for $0 \leq u \leq I_t$, $I_t^{-1}(u) = 0$.

We now define the equivalent of the sequence $(\rho_i^t, i \leq H_t)$ defined in Section 2. Here, we consider the measure $\tilde{\rho}_t$ defined in [10, p.25] and called exploration process to which we add a weight I_t at 0.

Definition 4.2. *For $t \geq 0$, the random positive measure ρ_t is defined by*

$$\langle \rho_t, f \rangle = \langle \tilde{\rho}_t, f \rangle + I_t f(0) = \int_{[0,t]} d_s I_s^t f(H_s) + I_t f(0)$$

where $d_s I_s^t$ is the Stieljes measure associated with the non-decreasing function $s \mapsto I_s^t$.

Notice that this random measure has support $[0, H_t]$, that $\rho_t(\{0\}) = I_t$ and that its total mass is $\langle \rho_t, \mathbf{1} \rangle = I_t + (X_t - I_t) = X_t$.

As in Section 2, we condition the Lévy process X to reach large heights before hitting 0 and the following theorem is the counterpart of Theorem 2.3.

Theorem 4.3. *Recall that $\alpha = \psi'(0)$. Then, for $t \geq 0$ and $\Theta \in \mathcal{F}_t$,*

$$\lim_{a \rightarrow \infty} \mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{1}{x} \mathbb{E}_x \left[\int_0^{H_t} \rho_t(dz) e^{\alpha z}; t \leq T_0, \Theta \right].$$

Moreover, if

$$M_t := \int_0^{H_t} \rho_t(dz) e^{\alpha z},$$

$(M_{t \wedge T_0}, t \geq 0)$ is a martingale under \mathbb{P}_x . In particular, similarly to the finite variation case, if $\alpha = 0$ (recurrent case), then $M_t = X_t$.

This theorem is proved in Section 4.2.

Remark 4.4. *In the particular case $\Lambda \equiv 0$ (so we assume $\beta > 0$), we have $\psi(\lambda) = \alpha\lambda + \beta\lambda^2$ and X is a Brownian motion with drift $-\alpha$ and variance 2β . Then, as it is noticed in [22], the local time process at 0 for $S - X$ is S/β . Also, for $t \geq 0$, $H_t = (X_t - I_t)/\beta$ and $\tilde{\rho}_t/\beta$ is the Lebesgue measure on $[0, H_t]$. Finally, in that particular case, if $\alpha > 0$, the process*

$$\left(I_{t \wedge T_0} + \frac{\beta}{\alpha} \left(e^{\frac{\alpha}{\beta}(X_t - I_t)} - 1 \right) \mathbf{1}_{\{t \leq T_0\}}, t \geq 0 \right)$$

is a martingale. In fact, this process belongs to a larger class of martingales called Kennedy's martingales [15] and studied in [4].

As in the finite variation case, we define a new probability \mathbb{P}_x^\dagger as a h -transform via the martingale $M_{\cdot \wedge T_0}$. The following proposition is the counterpart of Proposition 2.8 in the finite variation case.

Proposition 4.5. *Under \mathbb{P}_x^\dagger , in probability*

$$X_t \xrightarrow[t \rightarrow \infty]{} +\infty.$$

We give a proof of this proposition in Section 4.3.

4.2. Proof of Theorem 4.3. Before the proof, we give two technical lemmas concerning integrability of the exploration process and the height process.

Recall that $\tilde{\rho}$ is the exploration measure process defined in [10]

$$\langle \tilde{\rho}_t, f \rangle = \int_{[0,t]} d_s I_s^t f(H_s), \quad t \geq 0.$$

We denote by e_α the mapping

$$e_\alpha : x \mapsto e^{\alpha x}.$$

Lemma 4.6. *For $t > 0$,*

$$\mathbb{E}[\langle \tilde{\rho}_t, e_\alpha \rangle] = \mathbb{E} \left[\int_0^{H_t} \tilde{\rho}_t(dz) e^{\alpha z} \right] < \infty.$$

Proof. We have

$$\mathbb{E} \left[\int_0^{H_t} \tilde{\rho}_t(dz) e^{\alpha z} \right] = \mathbb{E} \left[\int_0^t d_s I_s^t e^{\alpha H_s} \right] \leq \mathbb{E} \left[\int_0^t d_s I_s^t e^{\alpha H_t} \right] = \mathbb{E}[(X_t - I_t) e^{\alpha H_t}] = \mathbb{E}[S_t^{(t)} e^{\alpha H_t}].$$

By time-reversion at t , we have $\mathbb{E}[S_t^{(t)} e^{\alpha H_t}] = \mathbb{E}[S_t e^{\alpha L_t}]$ where $(L_t, t \geq 0)$ is a local time at 0 for $S - X$. First, if $\alpha = 0$, $\mathbb{E}[S_t e^{\alpha L_t}] = \mathbb{E}[S_t]$ is finite according to [5, Thm VII.4]. Moreover, if $\alpha > 0$,

$$\mathbb{E}[S_t e^{\alpha L_t}] = \int_0^\infty da \mathbb{E}[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}] = \mathbb{E}[S_t] + \int_1^\infty da \mathbb{E}[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}]. \quad (22)$$

Since $\mathbb{E}[S_t]$ is finite, in order to prove the lemma, it is sufficient to show that $a \mapsto \mathbb{E}[S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}]$ is integrable on $[1, +\infty)$. For $b > 0$, if $L^{-1}(\cdot) := \inf\{s \geq 0; L(s) > \cdot\}$ denotes the right-continuous inverse of L ,

$$\mathbb{E}[S_t \mathbf{1}_{\{L_t \geq b\}}] = \mathbb{E}[S_{L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}] + \mathbb{E}[(S_t - S_{L^{-1}(b)}) \mathbf{1}_{\{L^{-1}(b) \leq t\}}] \quad (23)$$

On the one hand, if we denote by φ the bijective inverse mapping of ψ , the process $((L^{-1}(b), S_{L^{-1}(b)}), b < L(\infty))$ (called ascending ladder process) is a bivariate Lévy process killed at rate $\psi'(0) = \alpha$ and whose bivariate Laplace exponent κ is

$$\kappa(\mu, \lambda) := c \frac{\mu - \psi(\lambda)}{\varphi(\mu) - \lambda} \quad (24)$$

in the sense

$$\mathbb{E} \left[e^{-\mu L^{-1}(b) - \lambda S_{L^{-1}(b)}} \right] = e^{-b\kappa(\mu, \lambda)}.$$

Moreover, c is a constant depending on the normalization of the local time L (see Section VI.1 and Theorem VII.4 in [5]). In fact, as it is proved in the proof of Lemma 1.1.2 in [10], with the normalization defined by (21), we have $c = 1$. Then, for $\mu > 0$, using Markov's inequality and (24), the first term of the r.h.s. of (23) satisfies

$$\begin{aligned} \mathbb{E} [S_{L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}] &= \mathbb{E} [S_{L^{-1}(b)} \mathbf{1}_{\{e^{-\mu L^{-1}(b)} \geq e^{-\mu t}\}}] \leq e^{\mu t} \mathbb{E} [S_{L^{-1}(b)} e^{-\mu L^{-1}(b)}] \\ &\leq b e^{\mu t} \frac{\partial}{\partial \lambda} \kappa(\mu, 0) e^{-b\kappa(\mu, 0)} = b e^{\mu t} \frac{\mu - \alpha \varphi(\mu)}{\varphi(\mu)^2} e^{-b\mu/\varphi(\mu)}. \end{aligned} \quad (25)$$

On the other hand, by the strong Markov property applied at the stopping time $L^{-1}(b)$, the second term of r.h.s. of (23) is

$$\mathbb{E} [(S_t - S_{L^{-1}(b)}) \mathbf{1}_{\{L^{-1}(b) \leq t\}}] = \mathbb{E} [S'_{t-L^{-1}(b)} \mathbf{1}_{\{L^{-1}(b) \leq t\}}]$$

where S' has same distribution as process S and is independent from $L^{-1}(b)$. Then,

$$\mathbb{E} [(S_t - S_{L^{-1}(b)}) \mathbf{1}_{\{L^{-1}(b) \leq t\}}] \leq \mathbb{E}[S'_t] \cdot \mathbb{P}(L^{-1}(b) \leq t).$$

According to [5, Thm VII.4], $\mathbb{E}[S'_t]$ is finite and by similar computations as before, for $\mu > 0$,

$$\mathbb{P}(L^{-1}(b) \leq t) \leq e^{\mu t} e^{-b\mu/\varphi(\mu)}. \quad (26)$$

Then, using (25), (26) and choosing $b = \ln a/\alpha$, we have

$$\mathbb{E} [S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}] \leq C(\mu, t) \left(\frac{\ln a}{\alpha} + 1 \right) a^{-\frac{\mu}{\varphi(\mu)\alpha}}$$

where $C(\mu, t)$ is a constant only depending on μ and t . Since $\varphi = \psi^{-1}$ is concave,

$$\frac{\varphi(\mu)}{\mu} \leq \varphi'(0) = \alpha^{-1}$$

and since by hypothesis $\psi(\lambda)/\lambda \rightarrow \infty$, for μ large enough, $\mu/(\varphi(\mu)\alpha) > 1$. Hence, the application $a \mapsto \mathbb{E} [S_t \mathbf{1}_{\{L_t \geq \ln a/\alpha\}}]$ is integrable on $[1, +\infty)$ and the proof is complete. \square

Lemma 4.7. For any $\xi > 0$ and $t \geq 0$,

$$\int_0^t \mathbb{E} [e^{\xi H_s}] ds < \infty.$$

Proof. By time reversibility at time s , $\mathbb{E} [e^{\xi H_s}] = \mathbb{E} [e^{\xi L_s}]$. Using (26) and similar techniques as in the previous proof, we have for all $\mu > 0$

$$\mathbb{E} [e^{\xi H_s}] \leq e^{\mu s} \int_1^\infty a^{-\mu/(\varphi(\mu)\xi)} da + 1.$$

Then, $\int_0^t \mathbb{E} [e^{\xi H_s}] ds$ is finite for μ large enough. \square

Proof of Theorem 4.3.

Step 1: We begin the proof with showing that $(M_{t \wedge T_0}, t \geq 0)$ is a martingale. In the case $\alpha = 0$, $M = X$ and it is known that $(X_t, t \geq 0)$ is a martingale.

We now suppose that $\alpha > 0$. For $t \geq 0$, we set

$$\nu_t := (I_t + x) \mathbf{1}_{\{I_t > -x\}} \delta_0 + \tilde{\rho}_t$$

and

$$\theta := \inf\{t \geq 0; \nu_t = 0\}.$$

If we set $\tilde{\mathcal{G}}_t := \sigma(\tilde{\rho}_s, 0 \leq s \leq t)$ and $\mathcal{H}_t := \sigma(\nu_s, 0 \leq s \leq t)$, we have the inclusions $\tilde{\mathcal{G}}_t \subset \mathcal{H}_t \subset \mathcal{F}_t$. The second one is trivial and the first one holds because ν_t is the sum of the measure $\tilde{\rho}_t$ (which satisfies $\tilde{\rho}_t(\{0\}) = 0$) and a measure whose support is $\{0\}$. Moreover, $\tilde{\mathcal{G}}_t = \mathcal{F}_t$ because $\langle \tilde{\rho}_t, \mathbf{1} \rangle = X_t - I_t$ and because $-I_t$ is the local time at time t at level 0 of the process $\tilde{\rho}_t$. Hence, the three filtrations are equal and it is sufficient to prove that $M_{\cdot \wedge T_0}$ is a \mathcal{H} -martingale.

According to Corollary 2.5 and Remark 2.7 in [1], if f is a bounded non-negative function on $[0, \infty)$ of class C^1 such that its first derivative f' is bounded and such that f has a finite limit at $+\infty$, then, under \mathbb{P}_0 , the process $(M_t^f, t \geq 0)$ defined by

$$M_t^f := e^{-\langle \nu_{t \wedge \theta}, f \rangle} + \int_0^{t \wedge \theta} e^{-\langle \nu_s, f \rangle} (f'(H_s) - \psi(f(H_s))) ds \quad (27)$$

is a martingale with respect to the filtration \mathcal{H} . In fact, in [1], the authors only proved this result in the case $\beta = 0$ and $\int_{(0,1]} \Lambda(dr)r = \infty$. However, their proof can easily be adapted in the general setting using results of [10, Chap. 3].

Recall that $e_\alpha : z \mapsto e^{\alpha z}$ and let $(h_K)_{K \geq 1}$ be a sequence of functions of class C^1 such that for all $K \geq 1$, $0 \leq h_K \leq K \wedge e_\alpha$, $0 \leq h'_K \leq e'_\alpha$, h'_K is bounded and $(h_K)_K$ (resp. $(h'_K)_K$) is an increasing sequence which converges pointwise to e_α (resp. e'_α).

Then, for $a > 0, K \geq 1$, applying (27) with $f = ah_K$,

$$M_t^{a,K} := \frac{1 - e^{-a\langle \nu_{t \wedge \theta}, h_K \rangle}}{a} + \int_0^{t \wedge \theta} e^{-a\langle \nu_t, h_K \rangle} \left(\frac{\psi(ah_K(H_s))}{a} - h'_K(H_s) \right) ds$$

defines a martingale. We first have

$$M_t^{a,K} \xrightarrow{a \rightarrow 0} M_t^K := \langle \nu_{t \wedge \theta}, h_K \rangle + \int_0^{t \wedge \theta} (\psi'(0)h_K(H_s) - h'_K(H_s)) ds \quad \text{a.s.}$$

By Beppo Levi's theorem, since $(h_K)_K$ and $(h'_K)_K$ are non-negative and non-decreasing, we have with probability 1

$$\begin{aligned} M_t^\infty &:= \lim_{K \rightarrow \infty} M_t^K = \langle \nu_{t \wedge \theta}, e_\alpha \rangle + \int_0^{t \wedge \theta} (\alpha e^{\alpha H_s} - \alpha e^{\alpha H_s}) ds \\ &= \langle \nu_{t \wedge \theta}, e_\alpha \rangle. \end{aligned}$$

We want to show that $(M_t^\infty)_t$, that is the limit as $a \rightarrow 0$ and then as $K \rightarrow \infty$ of the martingales $M^{a,K}$, is still a martingale. To do this, we use the dominated convergence theorem and we have to find an upper bound for $M_t^{a,K}$ independent from a and K . We have

$$|M_t^{a,K}| \leq \langle \nu_t, e_\alpha \rangle + \int_0^t \alpha e^{\alpha H_s} ds + \int_0^t \frac{\psi(ah_K(H_s))}{ah_K(H_s)} h_K(H_s) ds.$$

Moreover, there exists $C_1, C_2 > 0$ such that $\frac{\psi(\lambda)}{\lambda} \leq C_1 + C_2 \lambda$. Hence,

$$\begin{aligned} |M_t^{a,K}| &\leq (I_t + x) \mathbf{1}_{\{I_t > -x\}} + \langle \tilde{\rho}_t, e_\alpha \rangle + \int_0^t \alpha e^{\alpha H_s} ds + \int_0^t (C_1 e^{\alpha H_s} + C_2 a (e^{\alpha H_s} - 1)^2) ds \\ &\leq x + \langle \tilde{\rho}_t, e_\alpha \rangle + \int_0^t (AC_2 + (\alpha + C_1)e^{\alpha H_s} + AC_2 e^{2\alpha H_s}) ds \end{aligned} \quad (28)$$

if a belongs to the compact set $[0, A]$.

Thus, according to Lemmas 4.6 and 4.7, the r.h.s. of (28) is integrable. Then, applying the dominated convergence theorem, we have proved that $(\langle \nu_{t \wedge \theta}, e_\alpha \rangle, t \geq 0)$ is a martingale under \mathbb{P}_0 with respect to \mathcal{H} .

Furthermore,

$$\langle \nu_{t \wedge \theta}, e_\alpha \rangle = \left((I_t + x) \mathbf{1}_{\{I_t > -x\}} + \langle \tilde{\rho}_t, e_\alpha \rangle \right) \mathbf{1}_{\{t \leq \theta\}} = (x + I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_{-x}\}}$$

where T_{-x} is the hitting time of $(-\infty, -x]$ by X under \mathbb{P}_0 .

Then, since $(x + I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_{-x}\}}$ under \mathbb{P}_0 has the same law as

$$(I_t + \langle \tilde{\rho}_t, e_\alpha \rangle) \mathbf{1}_{\{t \leq T_0\}} = M_{t \wedge T_0}$$

under \mathbb{P}_x , $M_{\cdot \wedge T_0}$ is a martingale w.r.t. the filtration $\mathcal{H} = \mathcal{F}$.

Step 2: We are now able to prove the main part of Theorem 2.3. For $t > 0$ and $\Theta \in \mathcal{F}_t$, we have

$$\mathbb{P}_x \left(\Theta, t < T_0 \mid \sup_{s \leq T_0} H_s \geq a \right) = \frac{\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) + \mathbb{P}_x(\Theta, t < \tau_a < T_0)}{\mathbb{P}_x(\sup_{s \leq T_0} H_s \geq a)} \quad (29)$$

and we will investigate the asymptotic behaviors of the three probabilities in the last equation. First, the assumption (20) enables us to define the bijective mapping $\phi : (0, \infty) \rightarrow (0, \infty)$ such that

$$\phi(t) := \int_t^\infty \frac{d\lambda}{\psi(\lambda)}, \quad t > 0.$$

We denote by v its bijective inverse. Trivially, $v(a) \rightarrow 0$ as a goes to ∞ and thanks to Lemma 2.1 in [17], for $u \geq 0$, we have

$$\lim_{a \rightarrow \infty} \frac{v(a-u)}{v(a)} = e^{\alpha u}. \quad (30)$$

According to [10, Cor. 1.4.2], the mapping v can be linked to the excursion measure N :

$$v(a) = N(\sup H > a), \quad a > 0.$$

Then, by excursion theory and the exponential formula,

$$\mathbb{P}_x \left(\sup_{s \leq T_0} H_s \geq a \right) = 1 - \exp(-xN(\sup H > a)) = 1 - e^{-xv(a)}.$$

Hence,

$$\mathbb{P}_x \left(\sup_{s \leq T_0} H_s \geq a \right) \underset{a \rightarrow \infty}{\sim} xv(a). \quad (31)$$

Then, we consider the probability $\mathbb{P}_x(\Theta, \tau_a \leq t < T_0) \leq \mathbb{P}_x(\tau_a \leq t)$ and we want to prove that it goes to 0 faster than $v(a)$ as $a \rightarrow \infty$. We define

$$\sigma_a := \sup\{t < \tau_a, X_t = I_t\}$$

the left-end point of the first excursion of $X - I$ which reaches height a and set $\varepsilon_a := \tau_a - \sigma_a$.

Let $(e(t), t \geq 0)$ be the excursion point process of $X - I$ at level 0, that is, for $t \geq 0$

$$e(t) = \begin{cases} ((X - I)_{s+I^{-1}(t-)}, 0 \leq s \leq I^{-1}(t) - I^{-1}(t-)) & \text{if } I^{-1}(t-) < I^{-1}(t) \\ \partial & \text{otherwise} \end{cases}$$

where ∂ is a cemetery point.

For a generic excursion ε with duration $\sigma = \sigma(\varepsilon)$, we denote by

$$h^* := \sup_{[0, \sigma]} H(\varepsilon)$$

its maximum height. For $t \geq 0$, we denote by Δ_t the length of the excursion $e(t)$ of $X - I$ and we set $h_t^* := h^*(e(t))$. Then, since $(\Delta_t, h_t^*)_{t \geq 0}$ is the image by a measurable application of the Poisson point process $(e(t), t \geq 0)$, it is a Poisson point process on $(0, \infty) \times (0, \infty)$. Distinguishing its atoms (δ, h) between $h > a$ and $h < a$, we obtain that

$$Y_t^a := \sum_{s \leq t} \Delta_s \mathbf{1}_{\{h_s^* < a\}}$$

and

$$\tilde{Y}_t^a := \sum_{s \leq t} \Delta_s \mathbf{1}_{\{h_s^* > a\}}$$

are independent. Moreover, $(Y_t^a, t \geq 0)$ is a subordinator with Lévy measure $N(\Delta \in dr; \sup H < a)$. If φ^a denotes its Laplace exponent, as a tends to ∞ , $\varphi^a \rightarrow \varphi$ where $\varphi = \psi^{-1}$ is the Laplace exponent of the subordinator $\sum_{s \leq t} \Delta_s$ (see Theorem VII.1 in [5]).

Furthermore, $\gamma_a := \inf\{s \geq 0; \tilde{Y}_s^a \neq 0\}$ is independent from Y^a and follows an exponential distribution with parameter $N(\sup H > a) = v(a)$. Hence,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda \sigma_a} \right] &= \mathbb{E} \left[e^{-\lambda Y^a(\gamma_a)} \right] = v(a) \int_0^\infty dt e^{-v(a)t} \mathbb{E} \left[e^{-\lambda Y_t^a} \right] = v(a) \int_0^\infty dt e^{-v(a)t} e^{-\varphi^a(\lambda)t} \\ &= \frac{v(a)}{v(a) + \varphi^a(\lambda)}. \end{aligned}$$

Hence, since σ_a and ε_a are independent,

$$\mathbb{P}_x(\tau_a \leq t) = \mathbb{P}_x(\sigma_a + \varepsilon_a \leq t) = \mathbb{E}_x \left[\mathbb{P}_x(\sigma_a \leq t - \varepsilon_a | \varepsilon_a) \mathbf{1}_{\{\varepsilon_a \leq t\}} \right]$$

and by Markov inequality, for all $\lambda > 0$,

$$\mathbb{P}_x(\tau_a \leq t) \leq \mathbb{E}_x \left[e^{\lambda(t - \varepsilon_a)} \frac{v(a)}{v(a) + \varphi^a(\lambda)} \mathbf{1}_{\{\varepsilon_a \leq t\}} \right] = \frac{v(a)e^{\lambda t}}{v(a) + \varphi^a(\lambda)} \mathbb{E}_x \left[e^{-\lambda \varepsilon_a} \mathbf{1}_{\{\varepsilon_a \leq t\}} \right].$$

Then,

$$\frac{\mathbb{P}_x(\tau_a \leq t)}{v(a)} \leq \frac{e^{\lambda t} \mathbb{P}_x(\varepsilon_a \leq t)}{v(a) + \varphi^a(\lambda)} \xrightarrow{a \rightarrow \infty} 0 \quad (32)$$

since $\varphi^a \rightarrow \varphi$ and the r.v. ε_a is the time to reach height a for an excursion of $X - I$ conditioned on $\sup H > a$, so that $\mathbb{P}_x(\varepsilon_a \leq t)$ vanishes in the limit $a \rightarrow \infty$.

We finally study the asymptotic behavior of the last term of (29). We have

$$\mathbb{P}_x(\Theta, t < \tau_a < T_0) = \mathbb{E}_x \left[\mathbf{1}_\Theta \mathbf{1}_{\{t \leq T_0 \wedge \tau_a\}} \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) \right]$$

and

$$1 - \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) = \mathbb{E}_x \left[\exp \left(- \sum_{0 \leq u \leq X_t} \chi(u, e(I_t^{-1}(u))) \right) \middle| \mathcal{F}_t \right]$$

where for $u > 0$ and for a generic excursion ε of $X - I$ of duration $\sigma = \sigma(\varepsilon)$,

$$\chi(u, \varepsilon) = \begin{cases} \infty & \text{if } \sup_{[0, \sigma]} H(\varepsilon) > a - H_{I_t^{-1}(u)} \\ 0 & \text{otherwise} \end{cases}$$

where we remind the reader that $I_t^{-1}(\cdot)$ is the right-continuous inverse of $(I_s^t, s \leq t)$. Then, by the exponential formula for Poisson point processes,

$$\begin{aligned} 1 - \mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t) &= \exp\left(-\int_0^{X_t} du \int N(d\varepsilon) \left(1 - e^{-\chi(u, \varepsilon)}\right)\right) \\ &= \exp\left(-\int_0^{X_t} du N\left(h^* > a - H_{I_t^{-1}(u)}\right)\right) \\ &= \exp\left(-\int_0^t (d_s I_s^t + I_t \delta_0(s)) N(h^* > a - H_s)\right) \\ &= \exp(-\langle \rho_t, v(a - \cdot) \rangle). \end{aligned}$$

Hence, since $\langle \rho_t, v(a - \cdot) \rangle \leq X_t v(a - H_t)$ goes to 0 as a goes to ∞ and according to (30),

$$\frac{\mathbb{P}_x(\tau_a < T_0 | \mathcal{F}_t)}{v(a)} = \frac{1 - \exp(-\langle \rho_t, v(a - \cdot) \rangle)}{v(a)} \xrightarrow{a \rightarrow \infty} \langle \rho_t, e_\alpha \rangle.$$

Then, using Fatou's Lemma,

$$\liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \geq \mathbb{E}_x[\mathbf{1}_\Theta \mathbf{1}_{\{t \leq T_0\}} \langle \rho_t, e_\alpha \rangle] = \mathbb{E}_x[\mathbf{1}_\Theta M_{t \wedge T_0}].$$

Replacing Θ by Θ^c in the latter display, we have

$$\liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(t < \tau_a < T_0)}{v(a)} - \limsup_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \geq \mathbb{E}_x[M_{t \wedge T_0}] - \mathbb{E}_x[\mathbf{1}_\Theta M_{t \wedge T_0}].$$

Since $M_{\cdot \wedge T_0}$ is a martingale, $\mathbb{E}_x[M_{t \wedge T_0}] = x$. Moreover,

$$\frac{\mathbb{P}_x(t < \tau_a < T_0)}{v(a)} = \frac{\mathbb{P}_x(\tau_a < T_0)}{v(a)} - \frac{\mathbb{P}_x(\tau_a < T_0, \tau_a \leq t)}{v(a)}$$

and in the limit $a \rightarrow \infty$, the first term of the r.h.s. tends to x according to (31) and the second vanishes thanks to (32). Finally, we have

$$\limsup_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \leq \mathbb{E}_x[\mathbf{1}_{\Theta^c} M_{t \wedge T_0}] \leq \liminf_{a \rightarrow \infty} \frac{\mathbb{P}_x(\Theta, t < \tau_a < T_0)}{v(a)} \quad (33)$$

and putting together (31), (32) and (33), the proof is completed. \square

4.3. Proof of the Proposition 4.5. We want to prove that under \mathbb{P}_x^\uparrow , X_t goes to infinity in probability as $t \rightarrow \infty$. We make a similar proof as that of Proposition 2.8, that is, we want to prove that for $\lambda > 0$, $\mathbb{E}_x^\uparrow[e^{-\lambda X_t}] \rightarrow 0$ as t tends to ∞ .

First, in the case $\alpha = 0$,

$$\mathbb{E}_x^\uparrow[e^{-\lambda X_t}] = \frac{1}{x} \mathbb{E}_x[e^{-\lambda X_t} X_{t \wedge T_0}] \xrightarrow{t \rightarrow \infty} 0$$

using dominated convergence theorem.

We now suppose that $\alpha > 0$. Then, for all positive a ,

$$\begin{aligned} \mathbb{E}_x^\uparrow \left[e^{-\lambda X_t} \right] &= \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \left(I_t + \int_0^t d_s I_s^t e^{\alpha H_s} \right); t \leq T_0 \right] \\ &\leq \mathbb{P}_x(t \leq T_0) + \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda X_t} \int_0^t d_s I_s^t e^{\alpha H_s} \left(\mathbf{1}_{\{H_t \leq a\}} + \mathbf{1}_{\{H_t > a\}} \right) \mathbf{1}_{\{t \leq T_0\}} \right] \\ &\leq \mathbb{P}_x(t \leq T_0) + \frac{e^{\alpha a}}{x} \mathbb{E}_x \left[e^{-\lambda X_t} (X_t - I_t); t \leq T_0 \right] \\ &\quad + \frac{1}{x} \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \int_0^t d_s I_s^t e^{\alpha H_s}; H_t > a \right]. \end{aligned} \quad (34)$$

The first and the second terms of the r.h.s. vanish as t goes to ∞ thanks to the dominated convergence theorem and because T_0 is finite a.s. We now want to show that the third term of (34) vanishes as a goes to infinity uniformly in t . By time-reversing at time t , we have

$$B(t, a) := \mathbb{E}_x \left[e^{-\lambda(X_t - I_t)} \int_0^t d_s I_s^t e^{\alpha H_s}; H_t > a \right] = \mathbb{E} \left[e^{-\lambda S_t} \int_0^t d_r S_r e^{\alpha(L_t - L_r)}; L_t > a \right].$$

Then, observing that $\{L_t > a\} = \{L^{-1}(a) < t\}$ and applying the strong Markov property at the stopping time $L^{-1}(a)$,

$$\begin{aligned} B(t, a) &= \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} e^{\alpha a} \mathbf{1}_{\{L_t > a\}} e^{-\lambda(S_t - S_{L^{-1}(a)})} e^{\alpha(L_t - a)} \right. \\ &\quad \left. \cdot \left(\int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} + \int_{L^{-1}(a)}^t d_r S_r e^{-\alpha L_r} \right) \right] \\ &= \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} e^{\alpha a} \mathbf{1}_{\{L_t > a\}} \left(f_1(t - L^{-1}(a)) \int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} \right. \right. \\ &\quad \left. \left. + e^{-\alpha a} f_2(t - L^{-1}(a)) \right) \right] \end{aligned} \quad (35)$$

where for $u \geq 0$,

$$f_1(u) = \mathbb{E} \left[e^{-\lambda S_u} e^{\alpha L_u} \right]$$

and

$$f_2(u) = \mathbb{E} \left[e^{-\lambda S_u} e^{\alpha L_u} \int_0^u d_r S_r e^{-\alpha L_r} \right] \leq \mathbb{E} \left[e^{-\lambda S_u} S_u e^{\alpha L_u} \right]$$

We first show that these two functions are bounded on $(0, \infty)$. As in (22), we have

$$f_1(u) = \mathbb{E} \left[e^{-\lambda S_u} e^{\alpha L_u} \right] = \int_1^\infty da \mathbb{E} \left[e^{-\lambda S_u} \mathbf{1}_{\{L_u > \ln a / \alpha\}} \right] + \mathbb{E} \left[e^{-\lambda S_u} \right].$$

For $b > 0$, using (24),

$$\mathbb{E} \left[e^{-\lambda S_u} \mathbf{1}_{\{L_u > b\}} \right] \leq \mathbb{E} \left[e^{-\lambda S_{L^{-1}(b)}} \mathbf{1}_{\{b < L(\infty)\}} \right] = \exp \left(-b \frac{\psi(\lambda)}{\lambda} \right).$$

Hence, choosing $b = \ln a / \alpha$,

$$f_1(s) \leq \int_1^\infty a^{-\psi(\lambda)/(\lambda\alpha)} da + 1 < \infty$$

because ψ is convex and $\psi(\lambda)/\lambda > \psi'(0) = \alpha$ as soon as $\lambda > 0$.

We do the same with f_2 . For $b > 0$, if M denotes an upper bound of the mapping $x \mapsto xe^{-\lambda x}$,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda S_s} S_s \mathbf{1}_{\{L_s > b\}} \right] &= \mathbb{E} \left[e^{-\lambda S_{L^{-1}(b)}} e^{-\lambda(S_s - S_{L^{-1}(b)})} (S_s - S_{L^{-1}(b)} + S_{L^{-1}(b)}) \mathbf{1}_{\{L_s > b\}} \right] \\ &\leq M \mathbb{E} \left[e^{-\lambda S_{L^{-1}(b)}} \mathbf{1}_{\{L_s > b\}} \right] + \mathbb{E} \left[e^{-\lambda S_{L^{-1}(b)}} S_{L^{-1}(b)} \mathbf{1}_{\{L_s > b\}} \right] \\ &\leq \exp \left(-b \frac{\psi(\lambda)}{\lambda} \right) \left(M + b \frac{\psi'(\lambda)\lambda - \psi(\lambda)}{\lambda^2} \right) \end{aligned}$$

using that $(S_{L^{-1}(b)}, b < L(\infty))$ is a subordinator with Laplace exponent $\psi(\lambda)/\lambda$ according to (24). Then, taking $b = \ln a/\alpha$,

$$f_2(s) \leq C(\lambda) \int_1^\infty (1 + \ln a) a^{-\alpha^{-1}\psi(\lambda)/\lambda} da + M < \infty.$$

We come back to (35). If C_1 (resp. C_2) denotes an upper bound of f_1 (resp. f_2) and observing that

$$\int_0^{L^{-1}(a)} d_r S_r e^{-\alpha L_r} \leq S_{L^{-1}(a)} \quad \text{a.s.},$$

we have

$$\begin{aligned} B(t, a) &\leq C_1 e^{\alpha a} \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} S_{L^{-1}(a)} \mathbf{1}_{\{L_t > a\}} \right] + C_2 \mathbb{E} \left[e^{-\lambda S_{L^{-1}(a)}} \mathbf{1}_{\{L_t > a\}} \right] \\ &\leq C_1 \frac{\psi'(\lambda)\lambda - \psi(\lambda)}{\lambda^2} a e^{\alpha a} e^{-a \frac{\psi(\lambda)}{\lambda}} + C_2 e^{-a \frac{\psi(\lambda)}{\lambda}}. \end{aligned}$$

Thus, $B(t, a)$ goes to 0 uniformly in t as a tends to $+\infty$ and the proof is completed.

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