The Vector Addition System Reversible Reachability Problem

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Abstract. The reachability problem for vector addition systems is a central problem of net theory. This problem is known to be decidable but the complexity is still unknown. Whereas the problem is EXPSPACE-hard, no elementary upper bounds complexity are known. In this paper we consider the reversible reachability problem. This problem consists to decide if two configurations are reachable one from each other. We show that this problem is EXPSPACE-complete. As an application of the introduced materials we characterize the reversibility domains of a vector addition system.

1 Introduction

Vector addition systems (VASs) or equivalently Petri nets are one of the most popular formal methods for the representation and the analysis of parallel processes [EN94]. Their reachability problem is central since many computational problems (even outside the realm of parallel processes) reduce to the reachability problem. Sacerdote and Tenney provided in [ST77] a partial proof of decidability of this problem. The proof was completed in 1981 by Mayr [May81] and simplified by Kosaraju [Kos82] from [ST77,May81]. Ten years later [Lam92], Lambert provided a further simplified version based on [Kos82]. This last proof still remains difficult and the upper-bound complexity of the corresponding algorithm is just known to be non-primitive recursive. Nowadays, the exact complexity of the reachability problem for VASs is still an open-problem. The problem is known to be EXPSPACE-hard [CLM76] but even the existence of an elementary upper-bound complexity is open.

Recently, in [Ler11] we provided a new proof of the reachability problem based on the notion of *production relations* inspired by Hauschildt [Hau90]. That proof shows that reachability sets are *almost semilinear*, a class of sets introduced in that paper that extends the class of Presburger sets. An application of that result was provided; we proved that a final configuration is not reachable from an initial one if and only if there exists a forward inductive invariant definable in the Presburger arithmetic that contains the initial configuration but not the final one. Since we can decide if a Presburger formula denotes a forward inductive invariant, we deduce that there exist checkable certificates of non-reachability in the Presburger arithmetic. In particular, there exists a simple algorithm for deciding the general VAS reachability problem based on two semi-algorithms. A first one that tries to prove the reachability by enumerating finite sequences of actions and a second one that tries to prove the non-reachability by enumerating Presburger formulas. The Presburger inductive invariants presented in that

paper are obtained by over approximating production relations thanks to strongly connected subreachability graphs (called *witness graph* and recalled in Section 6). As a direct consequence, configurations in these graphs are reachable one from each other.

In this paper we consider the reversible reachability problem that consists to decide if two configurations are reachable one from each other. We prove that this problem is EXPSPACE-complete. This result extends known result for the subclasses of reversible and cyclic vector addition systems [BF97]. We also prove that the general *coverability problem* reduces to the reversible reachability problem (see Section 3). As an application of the introduced materials we characterize the reversibility domains of a vector addition system in the last Section 11.

2 Projected Vectors

In this paper, some components of vectors in $\mathbb{Z}^d$ are projected away. In order to avoid multiple dimensions, we introduce an additional element $\star \notin \mathbb{Z}$, the set $\mathbb{Z}_\star = \mathbb{Z} \cup \{\star\}$, and the set $\mathbb{Z}_I^d$ of vectors $\boldsymbol{z} \in \mathbb{Z}_\star^d$ such that $I = \{i \mid \boldsymbol{z}(i) = \star\}$. Operations on $\mathbb{Z}$ are extended component-wise into operations on $\mathbb{Z}_I^d$ by interpreting $\star$ as a projected component. More formally we denote by $\boldsymbol{z}_1 + \boldsymbol{z}_2$ where $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{Z}_I^d$ the vector $\boldsymbol{z} \in \mathbb{Z}_I^d$ defined by $\boldsymbol{z}(i) = \boldsymbol{z}_1(i) + \boldsymbol{z}_2(i)$ for every $i \notin I$. Symmetrically given $\boldsymbol{z} \in \mathbb{Z}_I^d$ and an integer $k \in \mathbb{Z}$, we denote by $k\boldsymbol{z}$ the vector in $\mathbb{Z}_I^d$ defined by $(k\boldsymbol{z})(i) = k(\boldsymbol{z}(i))$ for every $i \notin I$. The relation $\leq$ is extended over $\mathbb{Z}_*^d$ component-wise by $\boldsymbol{z}_1 \leq \boldsymbol{z}_2$ if $\boldsymbol{z}_2(i) \neq \star$ then $\boldsymbol{z}_1(i) \neq \star$ and in this case $\boldsymbol{z}_1(i) \leq \boldsymbol{z}_2(i)$.

Example 2.1. We have $k(\star, 1) = (\star, k)$ even if $k = 0$. We also have $(\star, 5) - (\star, 2) = (\star, 3)$ and $(\star, 1) + (\star, 2) = (\star, 3)$. We have $\cdots \leq -1 \leq 0 \leq 1 \leq \cdots \leq \star$.

The *projection* of a vector $\boldsymbol{z} \in \mathbb{Z}_I^d$ over a set $L \subseteq \{1, \ldots, d\}$ of indexes is the vector in $\mathbb{Z}_{I \cup L}^d$ defined by $\pi_L(\boldsymbol{z})(i) = \boldsymbol{z}(i)$ for every $i \notin L$. The projection of a set $\boldsymbol{Z} \subseteq \mathbb{Z}_I^d$ over L is defined as expected by $\pi_L(\boldsymbol{Z}) = \{\pi_L(\boldsymbol{z}) \mid \boldsymbol{z} \in \boldsymbol{Z}\}$.

Example 2.2. Let $L = \{1\}$. We have $\pi_L(1000, 1) = (\star, 1)$ and $\pi_L(4, \star) = (\star, \star)$. We also have $\pi_L(\{(2, 0), (1, 1), (2, 0)\}) = \{(\star, 0), (\star, 1), (\star, 2)\}$.

Let $\boldsymbol{z} \in \mathbb{Z}_I^d$. We denote by $||\boldsymbol{z}||_\infty$ the natural number equals to 0 if $I = \{1, \ldots, d\}$ and equals to $\max_{i \notin I} |\boldsymbol{z}(i)|$ otherwise. Given a finite set $\boldsymbol{Z} \subseteq \mathbb{Z}_I^d$ we denote by $||\boldsymbol{Z}||_\infty$ the natural number $\max_{\boldsymbol{z} \in \boldsymbol{Z}} ||\boldsymbol{z}||_\infty$ if $\boldsymbol{Z}$ is non empty and 0 is $\boldsymbol{Z}$ is empty.

3 Vector Addition Systems

A *Vector Addition System* (*VAS*) is a finite set $\boldsymbol{A} \subseteq \mathbb{Z}^d$. Vectors $\boldsymbol{a} \in \boldsymbol{A}$ are called *actions* and vectors $\boldsymbol{c} \in \mathbb{N}_\star^d$ with $\mathbb{N}_\star = \mathbb{N} \cup \{\star\}$ are called *configurations*. A configuration in $\mathbb{N}^d$ is said to be *standard* and we denote by $\mathbb{N}_I^d$ the set of configurations $\boldsymbol{c} \in \mathbb{N}_\star^d$ such that $I = \{i \mid \boldsymbol{c}(i) = \star\}$. Given a word $\sigma = \boldsymbol{a}_1 \ldots \boldsymbol{a}_k$ of actions $\boldsymbol{a}_j \in \boldsymbol{A}$ we denote by $\Delta(\sigma)$ the vector in $\mathbb{Z}^d$ defined by $\Delta(\sigma) = \sum_{j=1}^k \boldsymbol{a}_j$. This vector is called the *displacement* of σ. We also introduce the vector $\Delta_I(\sigma) = \pi_I(\Delta(\sigma))$. A *run* ρ from a configuration

$\boldsymbol{x} \in \mathbb{N}_I^d$ to a configuration $\boldsymbol{y} \in \mathbb{N}_I^d$ *labelled* by a word $\sigma = \boldsymbol{a}_1 \dots \boldsymbol{a}_k$ of actions $\boldsymbol{a}_j \in \boldsymbol{A}$ is a non-empty word $\rho = \boldsymbol{c}_0 \dots \boldsymbol{c}_k$ of configurations $\boldsymbol{c}_j \in \mathbb{N}_I^d$ such that $\boldsymbol{c}_0 = \boldsymbol{x}$, $\boldsymbol{c}_k = \boldsymbol{y}$ and such that $\boldsymbol{c}_j = \boldsymbol{c}_{j-1} + \pi_I(\boldsymbol{a}_j)$ for every $j \in \{1, \dots, k\}$. Note that in this case ρ is unique and $\boldsymbol{y} - \boldsymbol{x} = \Delta_I(\sigma)$. This run is denoted by $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$. The set I is called the set of *projected components* of ρ. The *projection* $\pi_L(\rho)$ of a run $\rho = \boldsymbol{c}_0 \dots \boldsymbol{c}_k$ over a set of indexes $L \subseteq \{1, \dots, d\}$ is defined as expected as the run $\pi_L(\rho) = \pi_L(\boldsymbol{c}_0) \dots \pi_L(\boldsymbol{c}_k)$. Observe that if ρ is the run $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$ then $\pi_L(\rho)$ is the run $\pi_L(\boldsymbol{x}) \xrightarrow{\sigma} \pi_L(\boldsymbol{y})$. The following lemma provides a simple way to deduce a converse result.

Lemma 3.1. *Let L be a set of indexes and $\boldsymbol{c}$ be a configuration such that there exists a run from $\pi_L(\boldsymbol{c})$ labelled by a word σ. If $\boldsymbol{c}(i) \geq |\sigma| \, ||\boldsymbol{A}||_\infty$ for every $i \in L$ then there exists a run from $\boldsymbol{c}$ labelled by σ.*

Proof. Let $\boldsymbol{c} \in \mathbb{N}_I^d$ be a configuration such that there exists a path from $\pi_L(\boldsymbol{c})$ labelled by a word $\sigma = \boldsymbol{a}_1 \dots \boldsymbol{a}_k$ where $\boldsymbol{a}_j \in \boldsymbol{A}$. Let us introduce the vector $\boldsymbol{c}_j = \boldsymbol{c} + \pi_I(\boldsymbol{a}_1 + \dots + \boldsymbol{a}_j)$. Since there exists a run from $\pi_L(\boldsymbol{c})$ labelled by σ we deduce that $\pi_L(\boldsymbol{c}_j) \in \mathbb{N}_{I \cup L}^d$. Observe that for every $j \in \{0, \dots, k\}$ and for every $i \notin I$ we have $\boldsymbol{c}_j(i) \geq \boldsymbol{c}(i) - |\sigma| \, ||\boldsymbol{A}||_\infty$. In particular if $\boldsymbol{c}(i) \geq |\sigma| \, ||\boldsymbol{A}||_\infty$ for every $i \in L \backslash I$ we deduce that $\boldsymbol{c}_j \in \mathbb{N}_I^d$. Therefore $\rho = \boldsymbol{c}_0 \dots \boldsymbol{c}_k$ is a run from $\boldsymbol{c}$ labelled by σ. □

Example 3.2. $\rho = (2,0)(1,1)(0,2)$ is the run $(2,0) \xrightarrow{(-1,1)(-1,1)} (0,2)$. Let $L = \{1\}$ and observe that $\pi_L(\rho) = (\star,0)(\star,1)(\star,2)$ is the run $(\star,0) \xrightarrow{(-1,1)(-1,1)} (\star,2)$.

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two standard configurations. When there exists a run from $\boldsymbol{x}$ to $\boldsymbol{y}$ we say that $\boldsymbol{y}$ is *reachable* from $\boldsymbol{x}$ and if there also exists a run from $\boldsymbol{y}$ to $\boldsymbol{x}$ we say that $(\boldsymbol{x}, \boldsymbol{y})$ is in the *reversible reachability relation*. The problem of deciding this last property is called the *reversible reachability problem*. This problem is shown to be EXPSPACE-hard by a reduction of the *coverability problem* as follows. Given two standard configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ we say that $\boldsymbol{y}$ is *coverable* by $\boldsymbol{x}$ if there exists a standard configuration in $\boldsymbol{y} + \mathbb{N}^d$ reachable from $\boldsymbol{x}$. The coverability problem is known to be EXPSPACE-complete [CLM76,Rac78]. We reduce the coverability problem as follows. We first observe that we can add to a vector addition system $\boldsymbol{A}$ additional actions of the form $(0, \dots, 0, -1, 0, \dots, 0)$ without modifying the coverability problem. Thanks to this transformation a standard configuration $\boldsymbol{y}$ is coverable from a standard configuration $\boldsymbol{x}$ if and only if $\boldsymbol{y}$ is reachable from $\boldsymbol{x}$. We introduce the VAS $\boldsymbol{V}$ in dimension $d+2$ defined by $\boldsymbol{V} = ((0,0) \times \boldsymbol{A}) \cup \{(-1, 1, -\boldsymbol{y}), (1, -1, \boldsymbol{x})\}$. Observe that $(1, 0, \boldsymbol{x})$ and $(0, 1, \boldsymbol{0})$ are in reversible reachability relation of $\boldsymbol{V}$ if and only if $\boldsymbol{y}$ is coverable from $\boldsymbol{x}$ in $\boldsymbol{A}$. As a direct consequence, the reversible reachability problem is EXPSPACE-hard.

4 Subreachability Graphs

A *subreachability graph* is a graph $G = (\boldsymbol{Q}, T)$ where $\boldsymbol{Q} \subseteq \mathbb{N}_I^d$ is a non empty finite set of configurations called *states* and $T \subseteq \boldsymbol{Q} \times \boldsymbol{A} \times \boldsymbol{Q}$ is a finite set of triples $(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y}) \in \boldsymbol{Q} \times \boldsymbol{A} \times \boldsymbol{Q}$ satisfying $\boldsymbol{x} \xrightarrow{\boldsymbol{a}} \boldsymbol{y}$ called *transitions*. The set I is called the set of *projected components* of G and the subreachability graph is said to be *standard* if I is empty. A

witness graph is a strongly connected subreachability graph (see Fig. 1 for examples). The *projection* $\pi_L(t)$ of a transition $t = (\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y})$ over a set of indexes $L \subseteq \{1, \ldots, d\}$ is defined by $\pi_L(t) = (\pi_L(\boldsymbol{x}), \boldsymbol{a}, \pi_L(\boldsymbol{y}))$ and the projection of the set of transitions T is defined by $\pi_L(T) = \{\pi_L(t) \mid t \in T\}$. The projection $\pi_L(G)$ of a subreachability graph $G = (\boldsymbol{Q}, T)$ is the subreachability graph $\pi_L(G) = (\pi_L(\boldsymbol{Q}), \boldsymbol{A}, \pi_L(T))$.

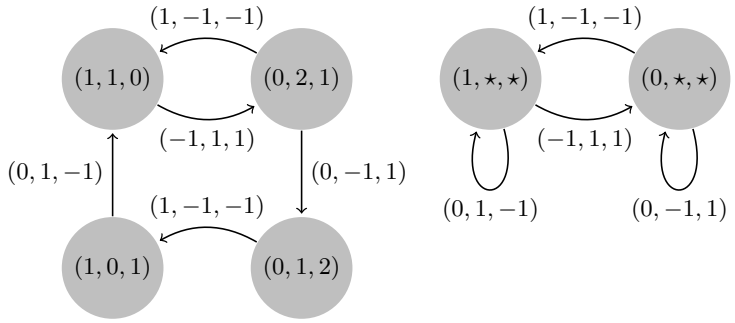

Fig. 1. A subreachability graph G and the subreachability graph $\pi_L(G)$ with $L = \{2, 3\}$.

Example 4.1. A standard subreachability graph $G = (\boldsymbol{Q}, T)$ and the subreachability graph $\pi_L(G)$ projected over the set $L = \{2, 3\}$ are depicted in Fig. 1.

A *path* in a subreachability graph G from a configuration $\boldsymbol{x} \in \boldsymbol{Q}$ to a configuration $\boldsymbol{y} \in \boldsymbol{Q}$ labelled by a word $\sigma = \boldsymbol{a}_1 \ldots \boldsymbol{a}_k$ of actions $\boldsymbol{a}_j \in \boldsymbol{A}$ is a word $p = t_1 \ldots t_k$ of transitions $t_j \in T$ of the form $t_j = (\boldsymbol{c}_{j-1}, \boldsymbol{a}_j, \boldsymbol{c}_j)$ with $\boldsymbol{c}_0 = \boldsymbol{x}$ and $\boldsymbol{c}_k = \boldsymbol{y}$. We observe that the word p is unique. This path is denoted by $\boldsymbol{x} \xrightarrow{\sigma}_G \boldsymbol{y}$. Let us observe that in this case $\rho = \boldsymbol{c}_0 \ldots \boldsymbol{c}_k$ is the unique run $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$. In particular if a path $\boldsymbol{x} \xrightarrow{\sigma}_G \boldsymbol{y}$ exists then the run $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$ also exists. Note that conversely if there exists a run $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$ then there exists a subreachability G such that $\boldsymbol{x} \xrightarrow{\sigma}_G \boldsymbol{y}$. Such a G is obtained by introducing the set of states $\boldsymbol{Q} = \{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_k\}$ and the set of transitions $T = \{t_1, \ldots, t_k\}$ where $t_j = (\boldsymbol{c}_{j-1}, \boldsymbol{a}_j, \boldsymbol{c}_j)$. A path $\boldsymbol{x} \xrightarrow{\sigma}_G \boldsymbol{y}$ is called a *cycle* if $\boldsymbol{x} = \boldsymbol{y}$. The cycle is said to be *simple* if $\boldsymbol{c}_{j_1} = \boldsymbol{c}_{j_2}$ with $j_1 < j_2$ implies $j_1 = 0$ and $j_2 = k$. The projection $\pi_L(p)$ of a path $p = t_1 \ldots t_k$ in G over a set of indexes $L \subseteq \{1, \ldots, d\}$ is the path $\pi_L(p) = \pi_L(t_1) \ldots \pi_L(t_k)$ in $\pi_L(G)$. Observe that the projection of a path $\boldsymbol{x} \xrightarrow{\sigma}_G \boldsymbol{y}$ over L is the path $\pi_L(\boldsymbol{x}) \xrightarrow{\sigma}_{\pi_L(G)} \pi_L(\boldsymbol{y})$. The *Parikh image* of a path is the function $\mu : T \to \mathbb{N}$ defined by $\mu(t)$ is the number of occurrences of t in this path. A cycle is said to be *total* if its Parikh image μ satisfies $\mu(t) \geq 1$ for every $t \in T$.

Example 4.2. Let us come back to the standard witness graph G depicted in Fig. 1. Let us consider the cycle $(1, 1, 0) \xrightarrow{(-1,1,1)(1,-1,-1)}_G (1, 1, 0)$ in G. Its projection over $L = \{2, 3\}$ is the cycle $(1, \star, \star) \xrightarrow{(-1,1,1)(1,-1,-1)}_{\pi_L(G)} (1, \star, \star)$ in the witness graph $\pi_L(G)$ also depicted in Fig. 1.

A word $\sigma \in \boldsymbol{A}^*$ is said to be *forward iterable* from a configuration $\boldsymbol{c}$ if there exists a run $\boldsymbol{c} \xrightarrow{\sigma} \boldsymbol{y}$ such that $\boldsymbol{c} \leq \boldsymbol{y}$. In this case the configuration $\boldsymbol{c}_\star = \pi_L(\boldsymbol{c})$ where $L = \{i \mid \boldsymbol{c}(i) \neq \boldsymbol{y}(i)\}$ is called the *forward limit* of σ from $\boldsymbol{c}$. We observe that σ is forward iterable from $\boldsymbol{c}$ if and only if for every $n \in \mathbb{N}$ there exists a run $\boldsymbol{c} \xrightarrow{\sigma^n} \boldsymbol{y}_n$. In that case L is the minimal set of indexes such that $\pi_L(\boldsymbol{y}_n)$ does not depend on n. Symmetrically σ is said to be *backward iterable* from a configuration $\boldsymbol{c}$ if there exists a run $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{c}$ such that $\boldsymbol{c} \leq \boldsymbol{x}$. In this case the configuration $\boldsymbol{c}_\star = \pi_L(\boldsymbol{c})$ where $L = \{i \mid \boldsymbol{c}(i) \neq \boldsymbol{x}(i)\}$ is called the *backward limit* of σ from $\boldsymbol{c}$.

Example 4.3. The action $\boldsymbol{a} = (0, -1, 1)$ is forward iterable from $\boldsymbol{x} = (0, \star, 0)$ since $(0, \star, 0) \xrightarrow{\boldsymbol{a}} (0, \star, 1)$. Observe that in this case $(0, \star, 0) \xrightarrow{\boldsymbol{a}^n} (0, \star, n)$ for every $n \in \mathbb{N}$. The forward limit of $\boldsymbol{a}$ from $(0, \star, 0)$ is $(0, \star, \star)$.

A configurations $\boldsymbol{c}$ is said to be *forward pumpable* by a cycle $\boldsymbol{q} \xrightarrow{\sigma}_G \boldsymbol{q}$ if σ is forward iterable from $\boldsymbol{c}$ with a forward limit equals to $\boldsymbol{q}$. Note that in this case $\boldsymbol{q}$ is unique since it satisfies $\boldsymbol{q} = \pi_I(\boldsymbol{c})$ where I is the set of projected components of G. Symmetrically a configuration $\boldsymbol{c}$ is said to be *backward pumpable* by a cycle $\boldsymbol{q} \xrightarrow{\sigma}_G \boldsymbol{q}$ if σ is backward iterable from $\boldsymbol{c}$ with a backward limit equals to $\boldsymbol{q}$.

Example 4.4. Let us come back to the witness graph $\pi_L(G)$ depicted in Fig. 1. Observe that $(0, \star, 0)$ is forward pumpable by $(0, \star, \star) \xrightarrow{(0,-1,1)}_{\pi_L(G)} (0, \star, \star)$.

5 Outline

The reminder of this paper is a proof that the reversible reachability problem is in EXPSPACE. We prove that if a pair $(\boldsymbol{x}, \boldsymbol{y})$ of standard configurations are in the reversible reachability relation then there exist runs from $\boldsymbol{x}$ to $\boldsymbol{y}$ and from $\boldsymbol{y}$ to $\boldsymbol{x}$ with lengths bounded by a number double exponential in the size of $(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{y})$. Using the fact that NEXPSPACE=EXPSPACE, and that double exponential numbers can be stored in exponential space, one obtain the EXPSPACE upper bound. These "short" runs are obtained as follows.

Theorem 6.3 gives a bound on the size of the Parikh image of a cycle in a witness graph to achieve a particular displacement vector, using a result of Pottier [Pot91]. This result is used in Section 7, which considers the special case of reversible witness graphs in which each path can be followed by another path such that the total displacement is zero. In Theorem 7.3 it is shown that a reversible witness graph possesses a "short" total cycle that has a zero displacement.

Section 9 takes an arbitrary witness graph G and asserts the existence of a set of indexes J such that the witness graph $\pi_J(G)$ has a "small" number of states and such that states of G that are not "too" large are pumpable by "short" cycles in $\pi_J(G)$.

The development culminates with the main result in Section 10. There, it is shown that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two states of a reversible witness graph G. One then uses the result from Section 9 to generate a reversible witness graph $\pi_J(G)$. Most of the work involves showing how to replace arbitrary path between $\boldsymbol{x}$ and $\boldsymbol{y}$ by "short" paths by exploiting the fact that $\boldsymbol{x}$ and $\boldsymbol{y}$ are pumpable to move from $\pi_J(G)$ back to G.

6 Displacement Vectors

A *displacement vector* of a witness graph G is a finite sum of vectors of the form $\Delta(\sigma) = \sum_{j=1}^{k} \boldsymbol{a}_j$ where $\sigma = \boldsymbol{a}_1 \dots \boldsymbol{a}_k$ is a word labelling a cycle in G. We denote by $\boldsymbol{Z}_G$ the set of *displacement vectors*. Observe that $\boldsymbol{Z}_G$ is a submonoid of $(\mathbb{Z}^d, +)$. Displacement vectors are related to *Kirchhoff functions* as follows. A *Kirchhoff function* for a witness graph $G = (\boldsymbol{Q}, T)$ is a function $\mu : T \to \mathbb{N}$ such that the functions $\mathrm{in}(\mu), \mathrm{out}(\mu) : \boldsymbol{Q} \to \mathbb{N}$ defined bellow are equal.

$$\mathrm{in}(\mu)(\boldsymbol{x}) = \sum_{t \in T \cap (\boldsymbol{Q} \times \boldsymbol{A} \times \{\boldsymbol{x}\})} \mu(t) \qquad \mathrm{out}(\mu)(\boldsymbol{x}) = \sum_{t \in T \cap (\{\boldsymbol{x}\} \times \boldsymbol{A} \times \boldsymbol{Q})} \mu(t)$$

A Kirchhoff function $\mu : T \to \mathbb{N}$ is said to be *total* if $\mu(t) \geq 1$ for every $t \in T$.

Lemma 6.1 (Euler's Lemma). *A function μ is a Kirchhoff function for a witness graph G if and only if μ is a finite sum of Parikh images of cycles in G. In particular a function μ is a total Kirchhoff function if and only if μ is the Parikh image of a total cycle.*

As a direct consequence of the Euler's Lemma, we deduce that a vector $\boldsymbol{z} \in \mathbb{Z}^d$ is a displacement vector of G if and only if there exists a Kirchhoff function μ for G satisfying the following equality:

$$\boldsymbol{z} = \sum_{t = (\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y}) \in T} \mu(t) \boldsymbol{a}$$

In this case $\boldsymbol{z}$ is called the *displacement* of μ.

Example 6.2. Let us come back to the witness graph $\pi_L(G)$ depicted in Fig. 1. A function $\mu : \pi_L(T) \to \mathbb{N}$ is a Kirchhoff function for $\pi_L(G)$ if and only if $\mu(t_1) = \mu(t_2)$ where $t_1 = ((1, \star, \star), (-1, 1, 1), (0, \star, \star))$ and $t_2 = ((0, \star, \star), (1, -1, -1), (1, \star, \star))$. In particular the set of displacement vectors of $\pi_L(G)$ satisfies $\boldsymbol{Z}_{\pi_L(G)} = \{\boldsymbol{z} \in \mathbb{Z}^3 \mid \boldsymbol{z}(1) = 0 \wedge \boldsymbol{z}(2) + \boldsymbol{z}(3) = 0\}$.

The following theorem shows that the displacement vectors $\boldsymbol{z} \in \boldsymbol{Z}_G$ are displacement of Kirchhoff functions μ for G such that $||\mu||_\infty = \max_{t \in T} \mu(t)$ is bounded by a polynomial in $|\boldsymbol{Q}|$, $||\boldsymbol{A}||_\infty$, and $||\boldsymbol{z}||_\infty$.

Theorem 6.3. *Vectors $\boldsymbol{z} \in \boldsymbol{Z}_G$ are displacement of Kirchhoff functions μ such that the following inequality holds where $q = |\boldsymbol{Q}|$, $a = ||\boldsymbol{A}||_\infty$, and $m = ||\boldsymbol{z}||_\infty$:*

$$||\mu||_\infty \leq (q^{d+1} a (1 + 2a)^d + m)^d$$

Proof. We first recall a "Frobenius theorem" proved in [Pot91]. Let $H \in \mathbb{Z}^{d \times n}$ be a matrix and let us denote by $h_{i,j}$ for each $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$ the element of H at position (i, j). We denote by $||H||_{1,\infty}$ the natural number $\max_{1 \leq i \leq d} \sum_{j=1}^{n} |h_{i,j}|$. Given a vector $\boldsymbol{v} \in \mathbb{N}^n$, we introduce the natural number $||\boldsymbol{v}||_1 = \sum_{j=1}^{n} \boldsymbol{v}(j)$. Let $\boldsymbol{V}$ be the set of vectors $\boldsymbol{v} \in \mathbb{N}^n$ such that $H\boldsymbol{v} = \boldsymbol{0}$. Recall that $\boldsymbol{V}$ is a submonoid of $(\mathbb{N}^n, +)$ generated by the finite set $\min(\boldsymbol{V} \backslash \{\boldsymbol{0}\})$ of minimal elements for $\leq$. From [Pot91] we

deduce that vectors $\boldsymbol{v} \in \min(\boldsymbol{V}\backslash\{\mathbf{0}\})$ satisfy the following inequality where r is the rank of H:

$$||\boldsymbol{v}||_1 \leq (1 + ||H||_{1,\infty})^r$$

Observe that if $a = 0$ then $\boldsymbol{z} = \mathbf{0}$ and the theorem is proved with the Kirchhoff function μ defined by $\mu(t) = 0$ for every $t \in T$. So we can assume that $a \geq 1$. Since every cycle labelled by a word σ can be decomposed into a finite sequence of simple cycles labelled by words $\sigma_1, \ldots, \sigma_k$ such that $\Delta(\sigma) = \sum_{j=1}^{k} \Delta(\sigma_j)$ we deduce that the set of displacement vectors $\boldsymbol{Z}_G$ is the submonoid of $(\mathbb{Z}^d, +)$ generated by the set $\boldsymbol{Z}$ of non-zero vectors $\boldsymbol{z} = \Delta(\sigma)$ where σ is the label of a simple cycle. Since the length of a simple cycle is bounded by the cardinal q of $\boldsymbol{Q}$, we get $||\boldsymbol{Z}||_\infty \leq qa$. As a corollary we deduce that the cardinal k of $\boldsymbol{Z}$ is bounded by $k \leq (1 + 2qa)^d - 1$ (the -1 comes from the fact that vectors in $\boldsymbol{Z}$ are non-zero).

Let us consider a vector $\boldsymbol{z} \in \boldsymbol{Z}_G$ and let us introduce a whole enumeration $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k$ of the vectors in $\boldsymbol{Z}$ and the following set $\boldsymbol{V}$ where $n = k + 1$:

$$\boldsymbol{V} = \{\boldsymbol{v} \in \mathbb{N}^n \mid \bigwedge_{i=1}^{d} \sum_{j=1}^{k} \boldsymbol{v}(j)\boldsymbol{z}_j(i) - \boldsymbol{v}(n)\boldsymbol{z}(i) = 0\}$$

We observe that $\boldsymbol{V}$ is associated to a matrix $H \in \mathbb{Z}^{d \times n}$. The rank of H is bounded by d and $||H||_{1,\infty} \leq kqa + m$. We deduce from the Frobenius theorem that vectors $\boldsymbol{v} \in \min(\boldsymbol{V}\backslash\{\mathbf{0}\})$ satisfy the following inequality:

$$||\boldsymbol{v}||_1 \leq (1 + kqa + m)^d \leq (q^{d+1}a(1 + 2a)^d + m)^d$$

Since $\boldsymbol{z} \in \boldsymbol{Z}_G$, there exists a vector $\boldsymbol{v} \in \boldsymbol{V}$ such that $\boldsymbol{v}(n) = 1$. In particular there exists another vector $\boldsymbol{v} \in \min(\boldsymbol{V}\backslash\{\mathbf{0}\})$ such that $\boldsymbol{v}(n) = 1$. Observe that for every $j \in \{1, \ldots, k\}$ there exists a function λ_j that is the Parikh image of a simple cycle such that $\boldsymbol{z}_j$ is the displacement of λ_j. We introduce the Kirchhoff function $\mu = \sum_{j=1}^{k} \boldsymbol{v}(j)\lambda_j$. Since $\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{v}(n) = 1$ we deduce that the displacement of μ is $\boldsymbol{z}$. The theorem is proved by observing that $\mu(t) = \sum_{j=1}^{k} \boldsymbol{v}(j)\lambda_j(t) \leq ||\boldsymbol{v}||_1$ since $\lambda_j(t) \in \{0, 1\}$. □

7 Reversible Witness Graphs

A witness graph G is said to be *reversible* if for every path $\boldsymbol{x} \xrightarrow{u}_G \boldsymbol{y}$ there exists a path $\boldsymbol{y} \xrightarrow{v}_G \boldsymbol{x}$ such that $\Delta(u) + \Delta(v) = \mathbf{0}$. Observe that standard witness graphs are reversible since the condition $\Delta(u) + \Delta(v) = \mathbf{0}$ is implied by the two paths.

Example 7.1. Witness graphs depicted in Fig. 1 are reversible. The witness graph $G = (\{\star\}, \{(\star, 1, \star)\})$ is not reversible.

Let us recall that a submonoid $\boldsymbol{Z}$ of $(\mathbb{Z}^d, +)$ is said to be a *subgroup* if $-\boldsymbol{z} \in \boldsymbol{Z}$ for every $\boldsymbol{z} \in \boldsymbol{Z}$. The following lemma provides two characterizations of the reversible witness graphs.

Lemma 7.2. *A witness graph G is reversible if and only if $\boldsymbol{Z}_G$ is a subgroup of $(\mathbb{Z}^d, +)$ if and only if the zero vector is the displacement of a total Kirchhoff function.*

Proof. Assume first that G is reversible and let us prove that $\boldsymbol{Z}_G$ is a subgroup of $(\mathbb{Z}^d, +)$. Let us consider a cycle $\boldsymbol{x} \xrightarrow{u}_G \boldsymbol{x}$. Since G is reversible, there exists a cycle $\boldsymbol{x} \xrightarrow{v}_G \boldsymbol{x}$ such that $\Delta(u) + \Delta(v) = \mathbf{0}$. We deduce that $-\boldsymbol{Z}_G = \boldsymbol{Z}_G$ since vectors in $\boldsymbol{Z}_G$ are finite sums of vectors $\Delta(u)$ where u is the label of a cycle in G. Therefore $\boldsymbol{Z}_G$ is a subgroup of $\mathbb{Z}^d$.

Now let us assume that $\boldsymbol{Z}_G$ is a subgroup of $(\mathbb{Z}^d, +)$ and let us prove that the zero vector is the displacement of a total Kirchhoff function. Since G is strongly connected, there exists a total cycle $\boldsymbol{x} \xrightarrow{u}_G \boldsymbol{x}$. Observe that $\boldsymbol{z} = \Delta(u)$ is in $\boldsymbol{Z}_G$. Since $\boldsymbol{Z}_G$ is a subgroup we deduce that $-\boldsymbol{z} \in \boldsymbol{Z}_G$. Hence $-\boldsymbol{z}$ is the displacement of a Kirchhoff function λ. Let λ' be Parikh image of $\boldsymbol{x} \xrightarrow{u}_G \boldsymbol{x}$ and observe that $\mu = \lambda + \lambda'$ is a total Kirchhoff function. Moreover the displacement of μ is $-\boldsymbol{z} + \boldsymbol{z} = \mathbf{0}$.

Finally, let us assume that the zero vector is the displacement of a total Kirchhoff function μ and let us prove that G is reversible. Let us consider a path $\boldsymbol{x} \xrightarrow{u}_G \boldsymbol{y}$. Since G is strongly connected, there exists a path $\boldsymbol{y} \xrightarrow{\alpha}_G \boldsymbol{x}$. Let us consider the Parikh image λ of the cycle $\boldsymbol{x} \xrightarrow{u\alpha}_G \boldsymbol{x}$ and let $m = 1 + ||\lambda||_\infty$. We observe that $\mu' = m\mu - \lambda$ is a total Kirchhoff function and the Euler's Lemma shows that μ' is the Parikh image of a cycle $\boldsymbol{x} \xrightarrow{\beta}_G \boldsymbol{x}$. From $\mu' = m\mu - \lambda$ we deduce that $\Delta(\beta) = m\mathbf{0} - \Delta(u\alpha)$. Let us consider $v = \alpha\beta$ and observe that $\boldsymbol{y} \xrightarrow{v}_G \boldsymbol{x}$ and $\Delta(u) + \Delta(v) = \mathbf{0}$. Thus G is reversible. □

The following theorem shows that if G is a reversible witness graph then the zero vector is the displacement of a total Kirchhoff function μ such $||\mu||_\infty$ can be bounded by a polynomial in $|\boldsymbol{Q}|$ and $||\boldsymbol{A}||_\infty$.

Theorem 7.3. *Let G be a reversible witness graph. The zero vector is the displacement of a total Kirchhoff function μ such that the following inequality holds where $q = |\boldsymbol{Q}|$ and $a = ||\boldsymbol{A}||_\infty$:*

$$||\mu||_\infty \leq (q(1+2a))^{d(d+1)}$$

Proof. Since G is strongly connected, every transition $t \in T$ occurs in at least one simple cycle. We denote by λ_t the Parikh image of such a simple cycle and we introduce the Kirchhoff function $\lambda = \sum_{t \in T} \lambda_t$. We have $\lambda(t) \in \{1, \ldots, |T|\}$ for every $t \in T$. We introduce the displacement $\boldsymbol{z}$ of λ. Since G is reversible, we deduce that $-\boldsymbol{z}$ is the displacement vector of a Kirchhoff function for G. As $||\boldsymbol{z}||_\infty \leq |T|qa$, $|T| \leq q|\boldsymbol{A}|$, and $|\boldsymbol{A}| \leq (1+2a)^d$ we deduce that $||\boldsymbol{z}||_\infty \leq q^2 a(1+2a)^d$. Theorem 6.3 shows that $-\boldsymbol{z}$ is the displacement of a Kirchhoff function λ' satisfying the following inequalities:

$$||\lambda'||_\infty \leq (q^d a(1+2a)^d + q^2 a(1+2a)^d)^d \leq (q^{d+1} 2a(1+2a)^d)^d$$

Let us consider the total Kirchhoff function $\mu = \lambda + \lambda'$. Observe that the displacement of μ is the zero vector and since $||\lambda||_\infty \leq |T| \leq q(1+2a)^d \leq (q^{d+1}(1+2a)^d)^d$ we get the theorem with:

$$||\mu||_\infty \leq (q^{d+1} 2a(1+2a)^d)^d + (q^{d+1}(1+2a)^d)^d \leq (q(1+2a))^{d(d+1)}$$

□

8 Extractors

In this section we introduce a way for extracting "large" components of configurations. An *extractor* is a non increasing sequence $\lambda = (\lambda_n)_{1 \leq n \leq d}$ of natural numbers $\lambda_n \in \mathbb{N}$. Let $\boldsymbol{X} \subseteq \mathbb{N}_I^d$. An *excluding* set for $(\lambda, \boldsymbol{X})$ is a set of indexes J such that $\boldsymbol{x}(i) < \lambda_{|J|+1}$ for every $i \notin J$ and for every $\boldsymbol{x} \in \boldsymbol{X}$. Since λ is non increasing we deduce that the class of excluding sets for a couple $(\lambda, \boldsymbol{X})$ is stable by intersection. As this class contains $\{1, \ldots, d\}$ we deduce that there exists a *unique minimal excluding set* J for $(\lambda, \boldsymbol{X})$. By minimality of this set we deduce that for every $i \in J$ there exists $\boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{x}(i) \geq \lambda_{|J|}$. We denote $\lambda(\boldsymbol{X})$ the set $\pi_J(\boldsymbol{X})$ where J is the minimal excluding set for $(\lambda, \boldsymbol{X})$. A set $\boldsymbol{X} \subseteq \mathbb{N}_I^d$ is said to be *normalized* for λ if $\lambda(\boldsymbol{X}) = \boldsymbol{X}$. As a direct consequence of the following lemma we deduce that $\lambda(\boldsymbol{X})$ is normalized for λ for every set $\boldsymbol{X} \subseteq \mathbb{N}_I^d$.

Lemma 8.1. *Let $\boldsymbol{X} \subseteq \mathbb{N}_I^d$ and let L be a set of indexes included in the minimal excluding set of $(\lambda, \boldsymbol{X})$. Then $\lambda(\boldsymbol{X}) = \lambda(\pi_L(\boldsymbol{X}))$.*

Proof. Note that if $\boldsymbol{X}$ is empty the result is immediate so we can assume that $\boldsymbol{X}$ is non empty. Let J be the minimal excluding set of $(\lambda, \boldsymbol{X})$ and observe that J is an excluding set for $\boldsymbol{X}' = \pi_L(\boldsymbol{X})$. In particular the minimal excluding set J' for $\boldsymbol{X}'$ satisfies $J' \subseteq J$. Since J' is an excluding set of $(\lambda, \boldsymbol{X}')$ we deduce that $\boldsymbol{x}'(i) < \lambda_{|J'|+1}$ for every $i \notin J'$. Hence $\pi_L(\boldsymbol{x})(i) < \lambda_{|J'|+1}$ for every $\boldsymbol{x} \in \boldsymbol{X}$. As $\boldsymbol{x} \leq \pi_L(\boldsymbol{x})$ we deduce that J' is an excluding set of $(\lambda, \boldsymbol{X})$. By minimality of J we get the other inclusion $J \subseteq J'$. Thus $J = J'$ and we have proved that $\lambda(\boldsymbol{X}) = \lambda(\pi_L(\boldsymbol{X}))$. □

Example 8.2. Let $\lambda = (5, 3, 2)$ be an extractor. We have $\lambda(\{(1, 8, 1)\}) = \{(1, \star, 1)\}$, $\lambda(\{(1, 8, 1), (3, 1, 1)\}) = \{(\star, \star, 1)\}$.

9 Pumpable Configurations

In this section we show that for arbitrary witness graph G, there exists a set J of indexes such that the number of states of $\pi_J(G)$ is "small" and such that states with "small" size of G are pumpable by "short" cycles of $\pi_J(G)$. The proof of this result is inspired by the Rackoff ideas [Rac78]. All other results or definitions introduced in this section are not used in the sequel.

Theorem 9.1. *Let G be a witness graph and let $s \in \mathbb{N}_{>0}$ be a positive integer. We introduce the positive integer $x = (1 + ||\boldsymbol{A}||_\infty)s$. There exists a set of indexes J such that the number of states of $\pi_J(G)$ is bounded by x^{d^d} and such that every state $\boldsymbol{q} \in \boldsymbol{Q}$ such that $||\boldsymbol{q}||_\infty < s$ is forward and backward pumpable by cycles of $\pi_J(G)$ with lengths bounded by dx^{d^d}.*

Such a set J is obtained by introducing the class of *adapted extractors*. An extractor λ is said to be *adapted* if the following inequality holds for every $n \in \{2, \ldots, d\}$:

$$\lambda_{n-1} \geq \lambda_n^{d-n+1} ||\boldsymbol{A}||_\infty + \lambda_n$$

Lemma 9.2. *Let λ be an adapted extractor, G be a witness graph with a set of states $\boldsymbol{Q} \subseteq \mathbb{N}_I^d$, and let J be the minimal excluding set for $(\lambda, \boldsymbol{Q})$. For every state $\boldsymbol{q} \in \boldsymbol{Q}$ there exists a run $\boldsymbol{q} \xrightarrow{u} \boldsymbol{y}$ such that $\pi_J(\boldsymbol{q}) \xrightarrow{u}_{\pi_J(G)} \pi_J(\boldsymbol{y})$ and such that the bounds $|u| \leq \sum_{|I|<n\leq|J|} \lambda_n^{d+1-n}$, and $\boldsymbol{y}(j) \geq \lambda_{|J|}$ for every $j \in J$ hold.*

Proof. Since $\boldsymbol{Q} \subseteq \mathbb{N}_I^d$ we deduce that $I \subseteq J$. We introduce a parameter $k \in \mathbb{N}$ and we prove the lemma by induction over k under the constraint $|J| - |I| \leq k$. Observe that if $k = 0$ then $I = J$ and the property is proved with $u = \epsilon$ and $\boldsymbol{y} = \boldsymbol{q}$. Assume the property proved for a natural number $k \in \mathbb{N}$ and let us consider a witness graph G with a set of projected components I such that $|J| - |I| \leq k + 1$ where J is the minimal excluding set for $(\lambda, \boldsymbol{Q})$. We consider a state $\boldsymbol{q} \in \boldsymbol{Q}$.

We say that a state $\boldsymbol{p} \in \boldsymbol{Q}$ is normalized if $\{\boldsymbol{p}\}$ is normalized for λ, i.e $\lambda(\{\boldsymbol{p}\}) = \{\boldsymbol{p}\}$ or equivalently $\boldsymbol{p}(i) < \lambda_{|I|+1}$ for every $i \notin I$. Observe that if every state $\boldsymbol{p} \in \boldsymbol{Q}$ is normalized then $\lambda(\boldsymbol{Q}) = \boldsymbol{Q}$ and in particular $J = I$ and the property is proved. So we can assume that there exists a state in $\boldsymbol{p} \in \boldsymbol{Q}$ that is not normalized. Since G is strongly connected, there exists a path $\boldsymbol{q} \xrightarrow{\sigma}_G \boldsymbol{p}$ with a minimal length such that $\boldsymbol{p}$ is not normalized. Let us observe that the number of states in $\boldsymbol{Q}$ that are normalized is bounded by $\lambda_{|I|+1}^{d-|I|}$. By minimality of the length of σ we deduce that $|\sigma| \leq \lambda_{|I|+1}^{d-|I|}$.

We introduce the minimal excluding set K for $(\lambda, \{\boldsymbol{p}\})$. Observe that I is strictly included in K since $\boldsymbol{p}$ is not normalized. Moreover K is included in J since J is an excluding set for $(\lambda, \{\boldsymbol{p}\})$. Lemma 8.1 shows that J is the minimal excluding set of $(\lambda, \pi_K(\boldsymbol{Q}))$. Observe that $|J| - |K| < |J| - |I| \leq k + 1$. By applying the induction on the witness graph $\pi_K(G)$ and the state $\pi_K(\boldsymbol{p})$, we deduce that there exists a run $\pi_K(\boldsymbol{p}) \xrightarrow{u} \boldsymbol{y}$ such that $\pi_J(\boldsymbol{p}) \xrightarrow{u}_{\pi_J(G)} \pi_J(\boldsymbol{y})$ with $|u| \leq \sum_{|K|<n\leq|J|} \lambda_n^{d+1-n}$ and such that $\boldsymbol{y}(j) \geq \lambda_{|J|}$ for every $j \in J$. We introduce the word $v = \sigma u$. Since λ is an adapted extractor we deduce that $\lambda_{|K|} \geq ||\boldsymbol{A}||_\infty \sum_{|K|<n\leq|J|} \lambda_n^{d+1-n}$. From $\boldsymbol{p}(k) \geq \lambda_{|K|}$ for every $k \in K$ we deduce that $\boldsymbol{p}(k) \geq ||\boldsymbol{A}||_\infty |u|$. Since there exists a run from $\pi_K(\boldsymbol{p})$ labelled by u, Lemma 3.1 shows that there exists a run $\boldsymbol{p} \xrightarrow{u} \boldsymbol{z}$. For every $k \in K$ we have $\boldsymbol{z}(k) \geq \lambda_{|J|}$ if $\boldsymbol{p}(k) = \star$ and $\boldsymbol{z}(k) \geq \boldsymbol{p}(k) - ||\boldsymbol{A}||_\infty |u| \geq \lambda_{|J|}$ otherwise since λ is an adapted extractor. As $\boldsymbol{p} \xrightarrow{u} \boldsymbol{z}$ we deduce that $\pi_K(\boldsymbol{p}) \xrightarrow{u} \pi_K(\boldsymbol{z})$. In particular $\pi_K(\boldsymbol{z}) = \boldsymbol{y}$. Let $j \in J \backslash K$. From the previous equality we get $\boldsymbol{z}(j) = \boldsymbol{y}(j)$. Moreover since $\boldsymbol{y}(j) \geq \lambda_{|J|}$ we get $\boldsymbol{z}(j) \geq \lambda_{|J|}$. We have proved that $\boldsymbol{z}(j) \geq \lambda_{|J|}$ for every $j \in J$. Hence the induction is proved. □

Now let us prove Theorem 9.1. We consider a witness graph G with a set of states $\boldsymbol{Q} \subseteq \mathbb{N}_I^d$. We also consider a positive integer $s \in \mathbb{N}_{>0}$ and we introduce the positive integers $a = ||\boldsymbol{A}||_\infty$ and $x = (1+a)s$. Let λ be the adapted extractor defined by $\lambda_d = s$ and the following induction for every $n \in \{2, \ldots, d\}$:

$$\lambda_{n-1} = \lambda_n^d (1 + ||\boldsymbol{A}||_\infty)$$

An immediate induction provides $\lambda_n^{d+1-n} \leq x^{d^d}$ for every $n \in \{1, \ldots, d\}$. We introduce the minimal excluding set J for $(\lambda, \boldsymbol{Q})$. Observe that the number of states in $\pi_J(\boldsymbol{Q})$ is bounded by $\lambda_{|J|+1}^{d-|J|}$. Hence $|\pi_J(\boldsymbol{Q})| \leq x^{d^d}$. Let us consider $\boldsymbol{q} \in \boldsymbol{Q}$ such that $||\boldsymbol{q}||_\infty < s$. Lemma 9.2 shows that there exists a run $\boldsymbol{q} \xrightarrow{\sigma} \boldsymbol{x}$ with $\boldsymbol{x}(j) \geq \lambda_{|J|}$ for every

$j \in J$ such that $\pi_J(\boldsymbol{q}) \xrightarrow{\sigma}_{\pi_J(G)} \pi_J(\boldsymbol{x})$ and such that:

$$|\sigma| \leq \sum_{n=1}^{|J|} \lambda_n^{d+1-n}$$

Since $\pi_J(G)$ is strongly connected there exists a path $\pi_J(\boldsymbol{x}) \xrightarrow{u}_{\pi_J(G)} \pi_J(\boldsymbol{q})$. We can assume that the length of u is minimal. In particular $u = \epsilon$ if $J = \{1, \ldots, d\}$ and $|u| \leq \lambda_{|J|+1}^{d-|J|}$ otherwise. In both case $|\sigma u| \leq dx^{d^d}$. Since λ is an adapted extractor we deduce that $\boldsymbol{x}(j) \geq |u|\, ||\boldsymbol{A}||_\infty$ for every $j \in J$ and by applying Lemma 3.1 we deduce that there exists a run $\boldsymbol{x} \xrightarrow{u} \boldsymbol{y}$. Since $\pi_J(\boldsymbol{x}) \xrightarrow{u} \pi_J(\boldsymbol{q})$ we deduce that $\boldsymbol{y}(j) = \boldsymbol{q}(j)$ for every $j \notin J$. Moreover if $j \in J \backslash I$ since $\boldsymbol{y}(j) \geq s$ and $s > ||\boldsymbol{q}||_\infty$ we get $\boldsymbol{y}(j) > \boldsymbol{q}(j)$. We deduce that $\boldsymbol{q} \leq \boldsymbol{y}$ and $J \backslash I = \{i \mid \boldsymbol{q}(i) \neq \boldsymbol{y}(i)\}$. Therefore $\boldsymbol{q}$ is forward pumpable by the cycle $\pi_J(\boldsymbol{q}) \xrightarrow{\sigma u}_{\pi_J(G)} \pi_J(\boldsymbol{q})$.

Symmetrically we prove the backward case. We have proved Theorem 9.1.

10 Deciding The Reversibility Problem

In this section, the reversible reachability problem is proved to be EXPSPACE-complete. The proof is inspired by the Kosaraju ideas [Kos82]. A word $\alpha \in \boldsymbol{A}^*$ is said to be *reversible* on a configuration $\boldsymbol{c}$ if there exists a word $\beta \in \boldsymbol{A}^*$ such that $\boldsymbol{c} \xrightarrow{\alpha\beta} \boldsymbol{c}$ and $\Delta(\alpha) + \Delta(\beta) = \boldsymbol{0}$. Note that if $\boldsymbol{c}$ is a standard configuration the last condition is implied by the first one.

Theorem 10.1. *Let $\alpha \in \boldsymbol{A}^*$ be a reversible word on a configuration $\boldsymbol{c}$. There exists another word $\alpha' \in \boldsymbol{A}^*$ reversible on $\boldsymbol{c}$ such that $\Delta(\alpha) = \Delta(\alpha')$ and such that:*

$$|\alpha'| \leq 17d^2 x^{15d^{d+2}}$$

where $x = (1 + 2||\boldsymbol{A}||_\infty)\,(1 + ||\boldsymbol{p}||_\infty + ||\Delta(\alpha)||_\infty)$.

Let us assume that $\alpha \in \boldsymbol{A}^*$ is a reversible word on a configuration $\boldsymbol{c}$. There exists a word $\beta \in \boldsymbol{A}^*$ such that the run $\boldsymbol{p} \xrightarrow{\alpha\beta} \boldsymbol{p}$ satisfies $\Delta(\alpha) + \Delta(\beta) = \boldsymbol{0}$. From this run we extract a unique witness graph $G = (\boldsymbol{Q}, T)$ such that $\boldsymbol{p} \xrightarrow{\alpha\beta}_G \boldsymbol{p}$ is a total cycle. In particular the Parikh image of this cycle is a total Kirchhoff function proving that G is reversible by Lemma 7.2.

Let I be the set of projected components of G. We introduce $a = ||\boldsymbol{A}||_\infty$ and $s = 1 + ||\boldsymbol{p}||_\infty + ||\Delta(\alpha)||_\infty$. Let $\boldsymbol{q} = \boldsymbol{p} + \Delta_I(\alpha)$. We have $||\boldsymbol{q}||_\infty \leq ||\boldsymbol{p}||_\infty + ||\Delta(\alpha)||_\infty < s$. Let us introduce $x = (1 + 2a)s$. Theorem 9.1 shows that there exists a set of indexes J such that $\pi_J(G)$ has at most x^{d^d} states and such that $\boldsymbol{p}$ forward pumpable by a cycle $\pi_J(\boldsymbol{p}) \xrightarrow{v}_{\pi_J(G)} \pi_J(\boldsymbol{p})$ and $\boldsymbol{q}$ is backward pumpable by a cycle $\pi_J(\boldsymbol{q}) \xrightarrow{w}_{\pi_J(G)} \pi_J(\boldsymbol{q})$ such that $|v|, |w| \leq dx^{d^d}$. In particular $\Delta_I(v)$ and $-\Delta_I(w)$ are two vectors in $\{\boldsymbol{c} \in \mathbb{N}_I^d \mid \boldsymbol{c}(i) \neq 0 \Leftrightarrow i \in J\}$. We deduce that for every $n \in \mathbb{N}$ we have:

$$\boldsymbol{p} \xrightarrow{v^n} \boldsymbol{p} + n\Delta_I(v) \qquad \boldsymbol{q} - n\Delta_I(w) \xrightarrow{w^n} \boldsymbol{q}$$

Since the witness graph G is reversible, Lemma 7.2 shows that $\pi_J(G)$ is reversible. From Theorem 7.3 we deduce that the zero vector is the displacement of a total Kirchhoff function μ for $\pi_J(G)$ satisfying:

$$||\mu||_\infty \leq (x^{d^d}(1+2a))^{d(d+1)} \leq x^{4d^{d+2}}$$

Note that $|\pi_J(T)| \leq |\pi_J(\boldsymbol{Q})|\,|A| \leq x^{d^d}x^d \leq x^{2d^d}$

Lemma 10.2. *There exists a cycle $\pi_J(\boldsymbol{q}) \xrightarrow{u}_{\pi_J(G)} \pi_J(\boldsymbol{q})$ such that $\Delta(v) + \Delta(u) + \Delta(w) = \mathbf{0}$ and:*

$$|u| \leq 3d\, x^{7d^{d+2}}$$

Proof. Let μ_v, μ_w be the Parikh images of $\pi_J(\boldsymbol{p}) \xrightarrow{v}_{\pi_J(G)} \pi_J(\boldsymbol{p})$ and $\pi_J(\boldsymbol{q}) \xrightarrow{w}_{\pi_J(G)} \pi_J(\boldsymbol{q})$. We introduce the function $\lambda = (1 + 2dx^{d^d})\mu - (\mu_v + \mu_w)$. Observe that λ is a Kirchhoff function for $\pi_J(G)$ satisfying $\lambda(t) \geq (1 + 2dx^{d^d}) - 2dx^{d^d} \geq 1$ for every $t \in \pi_J(T)$. The Euler's Lemma shows that λ is the Parikh image of a total cycle $\pi_J(\boldsymbol{q}) \xrightarrow{u}_{\pi_J(G)} \pi_J(\boldsymbol{q})$. Observe that $\Delta(u) = (1 + 2dx^{d^d})\mathbf{0} - (\Delta(v) + \Delta(w))$. Hence $\Delta(v) + \Delta(u) + \Delta(w) = \mathbf{0}$. The length of u is bounded by:

$$|u| = \sum_{t \in \pi_J(T)} (1 + 2dx^{d^d})\mu(t) - (\mu_v(t) + \mu_w(t)) \leq 3dx^{d^d}||\mu||_\infty|\pi_J(T)| \leq 3dx^{7d^{d+2}}$$

□

Lemma 10.3. *There exists a path $\pi_J(\boldsymbol{p}) \xrightarrow{\tilde{\alpha}}_{\pi_J(G)} \pi_J(\boldsymbol{q})$ such that $\Delta(\tilde{\alpha}) = \Delta(\alpha)$ and:*

$$|\tilde{\alpha}| \leq 2x^{7d^{d+2}}$$

Proof. Since $\pi_J(G)$ is strongly connected, there exists a path $\pi_J(\boldsymbol{q}) \xrightarrow{\tilde{\beta}}_{\pi_J(G)} \pi_J(\boldsymbol{p})$. We can assume that $|\tilde{\beta}|$ is minimal. In particular $|\tilde{\beta}| < x^{d^d}$. Moreover, we know that $\pi_J(\boldsymbol{p}) \xrightarrow{\alpha}_{\pi_J(G)} \pi_J(\boldsymbol{q})$. Observe that $\alpha\tilde{\beta}$ is the label of a cycle in $\pi_J(G)$. Hence $\boldsymbol{z} = \Delta(\alpha) + \Delta(\tilde{\beta})$ is the displacement of a Kirchhoff function for G. We have $||\boldsymbol{z}||_\infty \leq ||\Delta(\alpha)||_\infty + ||\Delta(\tilde{\beta})||_\infty \leq s + |\tilde{\beta}|a$ we get $||\boldsymbol{z}||_\infty \leq s + x^{d^d}a \leq x^{d^d}(1+a)$. Theorem 6.3 shows that $\boldsymbol{z}$ is the displacement of a Kirchhoff function θ for G such that:

$$||\theta||_\infty \leq ((x^{d^d})^{d+1}a(1+2a)^d + x^{d^d}(1+a))^d \leq x^{4d^{d+2}}$$

We introduce the Parikh image f of the path $\pi_J(\boldsymbol{q}) \xrightarrow{\tilde{\beta}}_{\pi_J(G)} \pi_J(\boldsymbol{p})$. Let us add to the strongly connected graph $\pi_J(G)$ an additional transition $t_\bullet$ from $\pi_J(\boldsymbol{q})$ to $\pi_J(\boldsymbol{p})$ and let $G_\bullet$ be this new graph and $T_\bullet = \pi_J(T) \cup \{t_\bullet\}$ be its set of transitions. Functions θ, μ and f are extended over $T_\bullet$ by $\theta(t_\bullet) = \mu(t_\bullet) = f(t_\bullet) = 0$. We also introduce the Parikh image $f_\bullet$ of $t_\bullet$, i.e. $f_\bullet(t_\bullet) = 1$ and $f_\bullet(t) = 0$ for every $t \in \pi_J(T)$. Let us observe that $g = \theta + x^{d^d}\mu - f + f_\bullet$ is a Kirchhoff function for $G_\bullet$. Since $f(t) < x^{d^d}$ we deduce

that $g(t) \geq 1$ for every $t \in \pi_J(T)$. Euler's Lemma shows that g is the Parikh image of a total cycle. Since $g(t_\bullet) = 1$ we deduce that g is the Parikh image of a cycle of the form $(\pi_J(\boldsymbol{p}) \xrightarrow{\tilde{\alpha}} \pi_J(\boldsymbol{q}))\, t_\bullet$. By definition of g we get $\Delta(\tilde{\alpha}) = \boldsymbol{z} + x^{d^d}\mathbf{0} - \Delta(\tilde{\beta})$. Hence $\Delta(\tilde{\alpha}) = \boldsymbol{z} - \Delta(\tilde{\beta})$. Since $\boldsymbol{z} = \Delta(\alpha) + \Delta(\tilde{\beta})$ we get $\Delta(\tilde{\alpha}) = \Delta(\alpha)$. The following inequalities provide the lemma:

$$|\tilde{\alpha}| \leq |\pi_J(T)|\, ||\theta||_\infty + x^{d^d} |\pi_J(T)|\, ||\mu||_\infty \leq 2x^{7d^{d+2}}$$

□

Lemma 10.4. *For every $n \geq |u|a$ we have:*

$$\boldsymbol{q} + n\Delta_I(v) \xrightarrow{u^n} \boldsymbol{q} - n\Delta_I(w)$$

Proof. Let $n \geq |u|a$. We introduce the sequence $(\boldsymbol{x}_k)_{0 \leq k \leq n}$ of configurations $\boldsymbol{x}_k = \boldsymbol{q} + (n-k)\Delta_I(v) - k\Delta_I(w)$. Since $\pi_J(\boldsymbol{x}_k) = \pi_J(\boldsymbol{q})$ we deduce that there exists a run from $\pi_J(\boldsymbol{x}_k)$ labelled by u. Moreover as $\Delta_I(v)(j) \geq 1$ and $-\Delta_I(w)(j) \geq 1$ for every $j \in J$, we deduce that $\boldsymbol{x}_k(j) \geq n \geq |u_p|a$ for every $j \in J$. Lemma 3.1 shows that there exists a run from $\boldsymbol{x}_k$ labelled by u. Since $\Delta(v) + \Delta(u) + \Delta(w) = \mathbf{0}$ we get $\boldsymbol{x}_k \xrightarrow{u} \boldsymbol{x}_{k+1}$. □

Lemma 10.5. *For every $n \geq |\tilde{\alpha}|a$ we have:*

$$\boldsymbol{p} + n\Delta_I(v) \xrightarrow{\tilde{\alpha}} \boldsymbol{q} + n\Delta_I(v)$$

Proof. Observe that $\pi_J(\boldsymbol{p} + n\Delta_I(v)) = \pi_J(\boldsymbol{p})$ and $\pi_J(\boldsymbol{p}) \xrightarrow{\tilde{\alpha}} \pi_J(\boldsymbol{q})$. Moreover for every $j \in J$ we have $(\boldsymbol{p} + n\Delta_I(v))(j) \geq n \geq |\tilde{\alpha}|a$. From Lemma 3.1 we deduce that there exists a run from $\boldsymbol{p} + n\Delta_I(v)$ labelled by $\tilde{\alpha}$. From $\boldsymbol{p} \xrightarrow{\alpha} \boldsymbol{q}$ we deduce that $\boldsymbol{p} + \Delta_I(\alpha) = \boldsymbol{q}$. Since $\Delta(\alpha) = \Delta(\tilde{\alpha})$ we deduce that $\boldsymbol{p} + \Delta_I(\tilde{\alpha}) = \boldsymbol{q}$. We deduce the run $\boldsymbol{p} + n\Delta_I(v) \xrightarrow{\tilde{\alpha}} \boldsymbol{q} + n\Delta_I(v)$. □

Finally, let $n = a \max\{|\tilde{\alpha}|, |u|\}$. We have proved that $\boldsymbol{p} \xrightarrow{\alpha'} \boldsymbol{q}$ where $\alpha' = v^n \tilde{\alpha} u^n w^n$. Note that $\Delta(\alpha') = \Delta(\alpha)$ since $\Delta(\tilde{\alpha}) = \Delta(\alpha)$ and $\Delta(v) + \Delta(u) + \Delta(w) = \mathbf{0}$. We deduce that $\Delta(\alpha') = \Delta(\alpha)$. As $\boldsymbol{q} \xrightarrow{\beta} \boldsymbol{p}$ with $\Delta(\alpha) + \Delta(\beta) = \mathbf{0}$ we deduce that α' is reversible on $\boldsymbol{p}$. Note that $n \leq a3dx^{7d^{d+2}} \leq 3dx^{8d^{d+2}}$. Hence we have:

$$|\alpha'| \leq 2x^{7d^{d+2}} + 3dx^{8d^{d+2}}(2dx^{d^d} + 3dx^{7d^{d+2}}) \leq 17d^2x^{15d^{d+2}}$$

We have proved Theorem 10.1.

Corollary 10.6. *Two standard configurations $\boldsymbol{p}, \boldsymbol{q}$ are is the same strongly connected component of a standard subreachability graph if and only if there exist runs $\boldsymbol{p} \xrightarrow{\alpha} \boldsymbol{q}$ and $\boldsymbol{q} \xrightarrow{\beta} \boldsymbol{p}$ such that:*

$$|\alpha|, |\beta| \leq 17d^2x^{15d^{d+2}}$$

where $x = (1 + 2||\boldsymbol{A}||_\infty)(1 + 2\max\{||\boldsymbol{p}||_\infty, ||\boldsymbol{q}||_\infty\})$.

Theorem 10.7. *The reversible reachability problem is EXPSPACE-complete.*

11 Application : Reversibility Domains

During the execution of a VAS some actions are reversible and some not. More precisely, let $\boldsymbol{D_a}$ be the set of standard configurations $\boldsymbol{c}$ such that there exists a word α satisfying $\boldsymbol{c} \xrightarrow{a} \boldsymbol{c}+\boldsymbol{a} \xrightarrow{\alpha} \boldsymbol{c}$. We observe that the set $\boldsymbol{D_a}$ is an upward closed set for the order $\leq$. In fact $\boldsymbol{c} \xrightarrow{a} \boldsymbol{c}+\boldsymbol{a} \xrightarrow{\alpha} \boldsymbol{c}$ implies the same thing by replacing $\boldsymbol{c}$ with a standard configuration $\boldsymbol{x} \in \boldsymbol{c}+\mathbb{N}^d$. So $\boldsymbol{D_a}$ is characterized by its finite set of minimal elements $\min(\boldsymbol{D_a})$ for $\leq$. As an application of Theorem 10.1, we obtain the following result.

Theorem 11.1. *Configurations $\boldsymbol{c} \in \min(\boldsymbol{D_a})$ satisfy the following inequality where $a = ||\boldsymbol{A}||_\infty$.*

$$||\boldsymbol{c}||_\infty \leq (102d^2a^2)^{(15d^{d+2})^{d+2}}$$

Proof. Observe that if $a = 0$ we are done since in this case $\boldsymbol{c} = \boldsymbol{0}$. So we can assume that $a \geq 1$. We introduce the extractor $\lambda = (\lambda_1, \ldots, \lambda_d)$ defined by $\lambda_{d+1} = a$ and the following induction for $n \in \{1, \ldots, d+1\}$:

$$\lambda_{n-1} = 17d^2(6a\lambda_n)^{15d^{d+2}}$$

Let $\boldsymbol{c} \in \min(\boldsymbol{D_a})$ and let $\boldsymbol{d} = \boldsymbol{c}+\boldsymbol{a}$. Let us consider the minimal excluding set I for $(\lambda, \{\boldsymbol{d}\})$. By minimality of I we have $\boldsymbol{d}(i) < \lambda_{|I|+1}$ for every $i \notin I$ and $\boldsymbol{d}(i) \geq \lambda_{|I|}$ for every $i \in I$. We consider the standard configuration $\boldsymbol{y}$ defined by $\boldsymbol{y}(i) = \lambda_{|I|}$ if $i \in I$ and $\boldsymbol{y}(i) = \boldsymbol{d}(i)$ if $i \notin I$. Let us consider $\boldsymbol{q} = \pi_I(\boldsymbol{c})$ and $\boldsymbol{p} = \pi_I(\boldsymbol{d})$. Since $\boldsymbol{c} \in \boldsymbol{D_a}$ there exists a run $\boldsymbol{d} \xrightarrow{\alpha} \boldsymbol{c}$. In particular $\boldsymbol{p} \xrightarrow{\alpha} \boldsymbol{q} \xrightarrow{\boldsymbol{a}} \boldsymbol{p}$ with $\Delta(\alpha)+\Delta(\boldsymbol{a}) = \boldsymbol{0}$. We deduce that α is reversible on $\boldsymbol{p}$ and Theorem 10.1 shows that there exists a word α' such that $\boldsymbol{p} \xrightarrow{\alpha'} \boldsymbol{q}$, $\Delta(\alpha') = \Delta(\alpha)$ and:

$$|\alpha'| \leq 17d^2x^{15d^{d+2}}$$

where $x = (1+2a)(1+||\boldsymbol{p}||_\infty + ||\boldsymbol{a}||_\infty)$. Note that $||\boldsymbol{p}||_\infty \leq \lambda_{|I|+1} - 1$. We deduce that $x \leq (1+2a)(\lambda_{|I|+1}+a) \leq 6a\lambda_{|I|+1}$ since $1 \leq a$ and $a \leq \lambda_{|I|+1}$. Hence $a|\alpha'| \leq \lambda_{|I|}$ thanks to the induction defining λ. Since $\pi_I(\boldsymbol{y}) = \boldsymbol{p}$ we deduce that there exists a run from $\pi_I(\boldsymbol{y})$ labelled by α'. As $\boldsymbol{y}(i) \geq \lambda_{|I|} \geq a|\alpha'|$ for every $i \in I$, Lemma 3.1 shows that there exists a run $\boldsymbol{y} \xrightarrow{\alpha'} \boldsymbol{x}$. Since $\Delta(\alpha') = \Delta(\alpha) = -\boldsymbol{a}$ we deduce that $\boldsymbol{x} = \boldsymbol{y} - \boldsymbol{a}$. From $\boldsymbol{y} \leq \boldsymbol{d}$ we get $\boldsymbol{x} \leq \boldsymbol{c}$ by subtracting $\boldsymbol{a}$. Moreover as $\boldsymbol{x} \xrightarrow{\boldsymbol{a}} \boldsymbol{y} \xrightarrow{\alpha'} \boldsymbol{x}$ we deduce that $\boldsymbol{x} \in \boldsymbol{D_a}$. By minimality of $\boldsymbol{c}$ we get $\boldsymbol{c} = \boldsymbol{x}$. Hence $\boldsymbol{c} = \boldsymbol{y} - \boldsymbol{a}$. In particular $||\boldsymbol{c}||_\infty \leq \lambda_{|I|} + a \leq \lambda_0 + a$. Finally let us get a bound on λ_0. We get the equality $\lambda_{n-1} = c\lambda_n^e$ by introducing $e = 15d^{d+2}$ and $c = 17d^2(6a)^e$. Hence $\lambda_0 \leq (ca)^{e^{d+1}} \leq (102d^2a^2)^{e^{d+2}}$ and from $e^{d+2} \leq (15d^{d+2})^{d+2}$ we are done. □

Conclusion

The reversible reachability problem is proved to be EXPSPACE-complete in this paper. The proof is inspired by the Rackoff and Kosaraju ideas [Rac78,Kos82]. We have introduced the domain of reversibility $\boldsymbol{D_a}$ of every action $\boldsymbol{a} \in \boldsymbol{A}$. Observe that the reflexive

and transitive closure of the following relation R is a congruence and from [BF97] we deduce that this relation is definable in the Presburger arithmetic. That means there exist a Preburger formula ϕ that exactly denotes the pair $(\boldsymbol{x}, \boldsymbol{y})$ of standard configurations in the reversible reachability relation. As a future work we are interested in characterizing precisely the size of such a formula (we already derive an elementary bound from [BF97] and Theorem 11.1).

$$R = \bigcup_{\boldsymbol{a} \in \boldsymbol{A}} \{(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{a}) \mid \boldsymbol{x} \in \boldsymbol{D_a}\}$$

Such a formula will provide a first hint on the structure of the *production relations* introduced in [Ler11] for solving the general vector addition system reachability problem.

References

[BF97] Zakaria Bouziane and Alain Finkel. Cyclic petri net reachability sets are semi-linear effectively constructible. *Electr. Notes Theor. Comput. Sci.*, 9, 1997.

[CLM76] E. Cardoza, Richard J. Lipton, and Albert R. Meyer. Exponential space complete problems for petri nets and commutative semigroups: Preliminary report. In *STOC'76*, pages 50–54. ACM, 1976.

[EN94] Javier Esparza and Marcus Nielsen. Decidability issues for petri nets - a survey. *Bulletin of the European Association for Theoretical Computer Science*, 52:245–262, 1994.

[Hau90] Dirk Hauschildt. *Semilinearity of the Reachability Set is Decidable for Petri Nets.* PhD thesis, University of Hamburg, 1990.

[Kos82] S. Rao Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing, (STOC 1982), 5-7 May 1982, San Francisco, California, USA*, pages 267–281. ACM, 1982.

[Lam92] Jean Luc Lambert. A structure to decide reachability in petri nets. *Theoretical Computer Science*, 99(1):79–104, 1992.

[Ler11] Jérôme Leroux. Vector addition system reachability problem: a short self-contained proof. In Thomas Ball and Mooly Sagiv, editors, *Proceedings of the 38th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2011, Austin, TX, USA, January 26-28, 2011*, pages 307–316. ACM, 2011.

[May81] Ernst W. Mayr. An algorithm for the general petri net reachability problem. In *Conference Proceedings of the Thirteenth Annual ACM Symposium on Theory of Computation, (STOC 1981), 11-13 May 1981, Milwaukee, Wisconsin, USA*, pages 238–246. ACM, 1981.

[Pot91] Loic Pottier. Minimal solutions of linear diophantine systems: Bounds and algorithms. In R. V. Book, editor, *Proceedings 4th Conference on Rewriting Techniques and Applications, Como (Italy)*, volume 488 of *Lecture Notes in Computer Science*, pages 162–173. Springer, 1991.

[Rac78] Charles Rackoff. The covering and boundedness problems for vector addition systems. *Theoretical Computer Science*, 6(2), 1978.

[ST77] George S. Sacerdote and Richard L. Tenney. The decidability of the reachability problem for vector addition systems (preliminary version). In *Conference Record of the Ninth Annual ACM Symposium on Theory of Computing, 2-4 May 1977, Boulder, Colorado, USA*, pages 61–76. ACM, 1977.