



**HAL**  
open science

# A global existence result for the anisotropic magnetohydrodynamical systems

Van-Sang Ngo

► **To cite this version:**

Van-Sang Ngo. A global existence result for the anisotropic magnetohydrodynamical systems. 2011.  
hal-00599675v1

**HAL Id: hal-00599675**

**<https://hal.science/hal-00599675v1>**

Preprint submitted on 10 Jun 2011 (v1), last revised 2 Oct 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A GLOBAL EXISTENCE RESULT FOR THE ANISOTROPIC MAGNETOHYDRODYNAMICAL SYSTEMS

VAN-SANG NGO

ABSTRACT. We study an anisotropic system arising in magnetohydrodynamics (MHD) in the whole space  $\mathbb{R}^3$ , in the case where there are no diffusivity in the vertical direction and only a small diffusivity in the horizontal direction (of size  $\varepsilon^\alpha$  with  $0 < \alpha \leq \alpha_0$ , for some  $\alpha_0 > 0$ ). We prove the local existence and uniqueness of a strong solution. Using Strichartz-type estimates, we prove that this solution exists globally in time for large initial data, when the rotation is fast enough.

## 1. INTRODUCTION

The fluid core of the Earth is often considered as an enormous dynamo, which generates the Earth's magnetic field due to the motion of the liquid iron. In a moving conductive fluid, magnetic fields can induce currents, which create forces on the fluid, and also change the magnetic field itself. The set of equations which then describe the MHD phenomena are a combination of the Navier-Stokes equations with Maxwell's equations.

In this paper, we consider a MHD model, which describes the motion of an incompressible conducting fluid of density  $\rho$ , kinematic viscosity  $\nu$ , conductivity  $\sigma$ , magnetic diffusivity  $\eta$  and permeability  $\mu_0$ . We suppose that the fluid is fast rotating with angular velocity  $\Omega_0$  around the axis  $e_3$ . We also suppose that the fluid moves with a typical velocity  $U$  in a domain of typical size  $L$  and generates a typical magnetic field  $B$ . We introduce following dimensionless parameters

$$E = \nu\Omega_0^{-1}L^{-2}; \quad \varepsilon = U\Omega_0^{-1}L^{-1}; \quad \Lambda = B^2\rho^{-1}\Omega_0^{-1}\mu_0^{-1}\eta^{-1}; \quad \theta = UL\eta^{-1},$$

which describe the Ekman, Rossby, Elsasser and Reynolds numbers respectively.

In geophysics, since the Earth is fast rotating, the Earth's core is believed to be in the asymptotic regime of small Ekman numbers ( $E \sim 10^{-15}$ ) and small Rossby numbers ( $\varepsilon \sim 10^{-7}$ ). In [13], using these considerations, Desjardins, Dormy and Grenier introduced the following MHD system

$$(1.1) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \frac{\nabla p}{\varepsilon} - \frac{E}{\varepsilon} \Delta u + \frac{u \wedge e_3}{\varepsilon} = -\frac{\Lambda}{\varepsilon} (\operatorname{curl} b) \wedge e_3 + \frac{\Lambda \theta}{\varepsilon} (\operatorname{curl} b) \wedge b \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u - \frac{\operatorname{curl} (u \wedge e_3)}{\theta} + \frac{\Delta b}{\theta} \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, \end{cases}$$

with the following asymptotic conditions

$$(1.2) \quad \varepsilon \rightarrow 0, \quad \Lambda = \mathcal{O}(1), \quad \varepsilon \theta \rightarrow 0 \quad \text{et} \quad E \sim \varepsilon^2,$$

---

1991 *Mathematics Subject Classification.* 76D03; 76D05; 76U05, 76W05.

*Key words and phrases.* MHD systems; rotating fluids; global existence; Strichartz estimates.

and where  $u$ ,  $p$  and  $b$  denote the velocity, pressure and magnetic field of the fluid.

In the case of large scale fluids in fast rotation, the Coriolis force is a dominant factor and leads to an anisotropy between the horizontal and the vertical directions (see the Taylor-Proudman theorem, [12], [17] or [28]). Taking into account this anisotropy, we suppose that the diffusion term in the vertical direction is negligible, compared with the one in the horizontal direction. Thus, we replace the Laplacian operator  $\Delta$  in all direction by the ‘‘horizontal’’ Laplacian operator  $\Delta_h$ . Taking into account the classical vectorial identities,

$$(\operatorname{curl} b) \wedge b = b \cdot \nabla b - \frac{1}{2} \nabla |b|^2, \quad (\operatorname{curl} b) \wedge e_3 = \partial_3 b - \nabla b_3,$$

and  $\operatorname{curl} (u \wedge e_3) = \partial_3 u$  (since  $\operatorname{div} u = 0$ ), we are led to the following MHD system in  $\mathbb{R}^3$ :

$$(1.3) \quad \begin{cases} \partial_t u - \frac{E}{\varepsilon} \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{u \wedge e_3}{\varepsilon} + \frac{\Lambda}{\varepsilon} \partial_3 b = -\nabla \tilde{p} \\ \partial_t b - \frac{1}{\theta} \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \frac{1}{\theta} \partial_3 u = 0 \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0. \end{cases}$$

To the best of my knowledge, there are only few known results, concerning local existence and uniqueness of solution, and global existence of solutions in the case of small initial data of systems of the same type as the MHD system, in both isotropic ([13], [14], [30], [3]) and anisotropic cases ([1], [4]).

The first goal of this paper is to establish Fujita-Kato type results about the local existence and uniqueness of a strong solution (global for small data) of the anisotropic system (1.3). One can already find a proof of the local existence in [4]. Here, we give another proof in this local existence result. Several difficulties arise in proving such results. In the (1.3) system, beside the problem due to the lack of the vertical diffusion terms, we encounter problems coming from the terms  $\frac{\Lambda}{\varepsilon} \partial_3 b$  and  $\frac{1}{\theta} \partial_3 u$ . These terms require more regularity in the vertical direction. However, if  $\frac{\Lambda}{\varepsilon} = \frac{1}{\theta}$ , a simple integration by parts shows that the sum of these two terms cancels, and these terms disappear in the energy estimates. We also remark that similar difficulties arise in estimating the terms  $b \cdot \nabla b$  in the first equation and  $b \cdot \nabla u$  in the second equation.

In our first theorem below, we prove the local existence and uniqueness of solutions of the system (1.3) with more general coefficients and we only suppose that  $\frac{\Lambda}{\varepsilon} = \frac{1}{\theta} = \mu > 0$ . More precisely, we consider the system

$$(1.4) \quad \begin{cases} \partial_t u - \nu \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{u \wedge e_3}{\varepsilon} + \mu \partial_3 b = -\nabla \tilde{p} \\ \partial_t b - \nu' \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \mu \partial_3 u = 0 \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, \end{cases}$$

where  $\nu, \nu' > 0$  are diffusion coefficients in the horizontal direction. In order to state these results, we introduce the anisotropic Sobolev spaces<sup>1</sup>, defined by

$$\mathbf{H}^{\sigma,s}(\mathbb{R}^3) = \left\{ u \in (\mathcal{S}')^3 \mid \|u\|_{\mathbf{H}^{\sigma,s}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} (1 + |\xi_h|^2)^\sigma (1 + |\xi_3|^2)^s |\widehat{u}|^2 d\xi < +\infty \right\},$$

where  $\widehat{u}$  is the Fourier transform of  $u$ . We prove the following theorem

**Theorem 1.1 (Local existence and uniqueness).** *Let  $s > \frac{1}{2}$ . For any divergence free vector fields  $u_0, b_0$  in  $\mathbf{H}^{0,s}(\mathbb{R}^3)$ , there exists a time  $T > 0$  such that the system (1.4) has a unique solution*

$$(u, b) \in (\mathbf{L}^\infty([0, T], \mathbf{H}^{0,s}(\mathbb{R}^3)) \cap \mathbf{C}([0, T], \mathbf{H}^{0,s}(\mathbb{R}^3)))^2,$$

with

$$(\nabla_h u, \nabla_h b) \in (\mathbf{L}^2([0, T], \mathbf{H}^{0,s}(\mathbb{R}^3)))^2.$$

Moreover, there exists a constant  $c > 0$  such that if  $\|u_0\|_{\mathbf{H}^{0,s}} + \|b_0\|_{\mathbf{H}^{0,s}} \leq c \min\{\nu, \nu'\}$ , then the solution exists globally in time, that is

$$(u, b) \in (\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3)) \cap \mathbf{C}(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3)))^2,$$

with

$$(\nabla_h u, \nabla_h b) \in (\mathbf{L}^2(\mathbb{R}_+, \mathbf{H}^{0,s}(\mathbb{R}^3)))^2.$$

The goal of the second and main part of this paper consists in proving that this local strong solution is actually global in time, even if the initial data are large, provided that the rotation is fast enough. Following the ideas of [9], [24] and [6], we wish to show that the large Coriolis force implies global existence of the solutions for large initial data. Notice however that the MHD system is much more complex than the system of general rotating fluids ([9], [24]) or the system of primitive equations ([6]). Here, we address the case where the parameters are “well chosen”:

$$(1.5) \quad \Lambda = 1, \theta = \varepsilon \text{ and } E = \varepsilon^{1+\alpha}, \text{ with } \alpha > 0.$$

To simplify the calculations, we consider a particular case of the system (1.4) where we choose

$$(1.6) \quad \nu = \nu' = \varepsilon^\alpha, \alpha > 0, \text{ and } \mu = \frac{1}{\varepsilon}.$$

These considerations lead us to the following system

$$(1.7) \quad \begin{cases} \partial_t u - \varepsilon^\alpha \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{u \wedge e}{\varepsilon} + \frac{\partial_3 b}{\varepsilon} = -\nabla \tilde{p} \\ \partial_t b - \varepsilon^\alpha \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \frac{\partial_3 u}{\varepsilon} = 0 \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u_0, b|_{t=0} = b_0. \end{cases}$$

To state the second result, we need to introduce the anisotropic Lebesgue spaces  $\mathbf{L}_h^p \mathbf{L}_v^q(\mathbb{R}^3)$  with  $p, q \geq 1$ , which are defined as

$$\begin{aligned} \mathbf{L}_h^p \mathbf{L}_v^q(\mathbb{R}^3) &= \mathbf{L}^p(\mathbb{R}_h^2; \mathbf{L}_v^q(\mathbb{R})) \\ &= \left\{ u \in (\mathcal{S}')^3 \mid \|u\|_{\mathbf{L}_h^p \mathbf{L}_v^q} = \left[ \int_{\mathbb{R}_h^2} \left| \int_{\mathbb{R}_v} |u(x_h, x_3)|^q dx_3 \right|^{\frac{p}{q}} dx_h \right]^{\frac{1}{p}} < +\infty \right\}. \end{aligned}$$

<sup>1</sup>We remark that throughout this paper, we use bold letters, *e.g.*  $\mathbf{H}^{\sigma,s}(\mathbb{R}^3)$ ,  $\mathbf{L}^p(\mathbb{R}^3)$ ,  $\dots$ , to describe the spaces of vector fields, each component of which belongs to  $H^{\sigma,s}(\mathbb{R}^3)$ ,  $L^p(\mathbb{R}^3)$ ,  $\dots$



## 2. CLASSICAL FUJITA-KATO-TYPE RESULTS

In this section, we prove Theorem 1.1 about local existence and uniqueness results of strong solutions (the solutions are global for small data) for the system (1.4). To simplify the proof of Theorem 1.1, we can suppose, without any loss of generality, that  $\nu = \nu' > 0$ . We also remark that, using the divergence free property of  $u$  and  $b$ , we can calculate the pressure term as follows

$$(2.1) \quad \tilde{p} = -\Delta^{-1} \left[ \sum_{i,j} [\partial_i \partial_j (u^i u^j) - \partial_i \partial_j (b^i b^j)] + \frac{1}{\varepsilon} (\partial_2 u_1 - \partial_1 u_2) \right].$$

**2.1. Global existence for small initial data.** In order to prove the global existence of a strong solution for small initial data, we apply the method of approximation of Friedrichs. For any  $n \in \mathbb{N}$ , let

$$J_n u = \mathcal{F}^{-1} (\mathbf{1}_{B(0,n)} \hat{u}),$$

where  $\mathcal{F}^{-1}$  is the inverse Fourier transform. Then, we consider the following approximate system

$$\left\{ \begin{array}{l} \partial_t u_n - \nu J_n \Delta_h u_n + J_n (J_n u_n \cdot \nabla J_n u_n) - J_n (J_n b_n \cdot \nabla J_n b_n) + \frac{J_n (J_n u_n \wedge e_3)}{\varepsilon} + \mu \partial_3 J_n b_n \\ \quad = J_n \nabla \Delta^{-1} \left[ \sum_{i,j} \partial_i \partial_j (J_n u_n^i J_n u_n^j - J_n b_n^i J_n b_n^j) + \frac{1}{\varepsilon} (\partial_2 J_n u_1 - \partial_1 J_n u_2) \right] \\ \partial_t b_n - \nu J_n \Delta_h b_n - J_n (J_n b_n \cdot \nabla J_n u_n) + J_n (J_n u_n \cdot \nabla J_n b_n) + \mu \partial_3 J_n u_n = 0 \\ \operatorname{div} u_n = \operatorname{div} b_n = 0, \\ u_n|_{t=0} = J_n u_0; \quad b_n|_{t=0} = J_n b_0. \end{array} \right.$$

Since this system is an ODE in  $\mathbf{L}^2(\mathbb{R}^3)$ , the Cauchy-Lipschitz theorem implies the local existence in time of a unique solution  $\mathcal{U}_n = \begin{pmatrix} u_n \\ b_n \end{pmatrix}$  in  $\mathbf{L}^2(\mathbb{R}^3)$ , the maximal lifespan of which is  $T_n = T_n(u_0, b_0)$ . Since  $J_n^2 = J_n$ ,  $J_n \mathcal{U}_n = \begin{pmatrix} J_n u_n \\ J_n b_n \end{pmatrix}$  is also a solution of this system. Then, the uniqueness implies that  $J_n \mathcal{U}_n = \mathcal{U}_n$ . Thus,  $\mathcal{U}_n$  is a solution of the following system

$$(2.2) \quad \left\{ \begin{array}{l} \partial_t u_n - \nu \Delta_h u_n + J_n (u_n \cdot \nabla u_n) - J_n (b_n \cdot \nabla b_n) + \frac{u_n \wedge e_3}{\varepsilon} + \mu \partial_3 b_n \\ \quad = J_n \nabla \Delta^{-1} \left[ \sum_{i,j} \partial_i \partial_j (u_n^i u_n^j - b_n^i b_n^j) + \frac{1}{\varepsilon} (\partial_2 u_1 - \partial_1 u_2) \right] \\ \partial_t b_n - \nu \Delta_h b_n - J_n (b_n \cdot \nabla u_n) + J_n (u_n \cdot \nabla b_n) + \mu \partial_3 u_n = 0 \\ \operatorname{div} u_n = \operatorname{div} b_n = 0, \\ u_n|_{t=0} = J_n u_0; \quad b_n|_{t=0} = J_n b_0. \end{array} \right.$$

Now, we take the  $\mathbf{L}^2$  product of the first equation of (2.2) with  $u_n$  and of the second equation with  $b_n$ . Classical algebraic properties of the non-linear terms and integrations by parts give

$$\langle J_n (u_n \cdot \nabla u_n) | u_n \rangle_{\mathbf{L}^2} = \langle J_n (u_n \cdot \nabla b_n) | b_n \rangle_{\mathbf{L}^2} = 0,$$

and

$$\langle J_n (b_n \cdot \nabla u_n) | b_n \rangle_{\mathbf{L}^2} + \langle J_n (b_n \cdot \nabla b_n) | u_n \rangle_{\mathbf{L}^2} = 0.$$

Thus, summing the two obtained equations, we have

$$(2.3) \quad \frac{1}{2} \frac{d}{dt} \|\mathcal{U}_n(t)\|_{\mathbf{L}^2}^2 + \nu \|\nabla_h \mathcal{U}_n(t)\|_{\mathbf{L}^2}^2 = 0, \quad \forall t > 0.$$

Since the solution  $\mathcal{U}_n(t)$  remains bounded in  $\mathbf{L}^2$ -norm on its whole maximal interval of existence  $[0, T_n[$ , we deduce that  $T_n = +\infty$ .

To be able to take the limit in the non-linear term as  $n \rightarrow +\infty$ , we will need more regularity for the approximate solutions. For that, let us estimate their norm in  $\mathbf{H}^{0,s}$ ,  $s > \frac{1}{2}$ . Let  $\psi$  be an even, smooth function in  $C_0^\infty(\mathbb{R})$  such that the support of which is contained in  $B(0, \frac{4}{3})$  and  $\psi$  is equal to 1 on a neighborhood of  $B(0, \frac{3}{4})$ . For any  $z \in \mathbb{R}$ , let

$$\varphi(z) = \psi\left(\frac{z}{2}\right) - \psi(z).$$

We recall the definitions of vertical frequency cut-off operators  $\Delta_q^v$  (for more details, see Appendix A). For any  $q \in \mathbb{Z}$  and for any tempered distribution  $u$ , we set:

$$\begin{aligned} \Delta_q^v u &= \mathcal{F}^{-1} \left( \varphi \left( \frac{|\xi_3|}{2^q} \right) \widehat{u}(\xi) \right), & \forall q \in \mathbb{N}, \\ \Delta_{-1}^v u &= \mathcal{F}^{-1} (\psi(|\xi_3|) \widehat{u}(\xi)), \\ \Delta_q^v u &= 0, & \forall q \leq -2. \end{aligned}$$

Then, we can rewrite the  $\mathbf{H}^{0,s}$ -norm in the following form

$$\|u\|_{\mathbf{H}^{0,s}} = \left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q^v u\|_{\mathbf{L}^2}^2 \right)^{\frac{1}{2}}.$$

In what follows, we will estimate the norm of  $u_n$  in the spaces  $\widetilde{\mathbf{L}}^p([0, t], \mathbf{H}^{0,s})$  ( $p = 2$  or  $p = +\infty$ ), which are defined by

$$\widetilde{\mathbf{L}}^p([0, t], \mathbf{H}^{0,s}) = \left\{ u \in (\mathcal{S}')^3 \mid \|u\|_{\widetilde{\mathbf{L}}^p([0, t], \mathbf{H}^{0,s})} \stackrel{def}{=} \left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q^v u\|_{\mathbf{L}^p([0, t], \mathbf{L}^2)}^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$

For  $p \geq 2$ , we remark that  $\widetilde{\mathbf{L}}^p([0, t], \mathbf{H}^{0,s}) \hookrightarrow \mathbf{L}^p([0, t], \mathbf{H}^{0,s})$ .

By applying the operator  $\Delta_q^v$  to the system (2.2), taking the  $\mathbf{L}^2$ -inner product of the first equation of the obtained system with  $\Delta_q^v u_n$  and of the second equation with  $\Delta_q^v b_n$  and then summing the two obtained equations, we get

$$(2.4) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^2}^2 + \nu \|\Delta_q^v \nabla_h \mathcal{U}_n\|_{\mathbf{L}^2}^2 &\leq |\langle \Delta_q^v (u_n \cdot \nabla u_n) | \Delta_q^v u_n \rangle| + |\langle \Delta_q^v (u_n \cdot \nabla b_n) | \Delta_q^v b_n \rangle| \\ &\quad + |\langle \Delta_q^v (b_n \cdot \nabla u_n) | \Delta_q^v b_n \rangle| + |\langle \Delta_q^v (b_n \cdot \nabla b_n) | \Delta_q^v u_n \rangle| \\ &\quad + \mu |\langle \partial_3 \Delta_q^v b_n | \Delta_q^v u_n \rangle| + |\langle \partial_3 \Delta_q^v u_n | \Delta_q^v b_n \rangle|. \end{aligned}$$

A simple integration by parts proves that

$$(2.5) \quad \mu |\langle \partial_3 \Delta_q^v b_n | \Delta_q^v u_n \rangle| + |\langle \partial_3 \Delta_q^v u_n | \Delta_q^v b_n \rangle| = 0,$$

so, by integrating (2.4) in time, we obtain

$$(2.6) \quad \begin{aligned} & \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,t],\mathbf{L}^2)}^2 + 2\nu \|\Delta_q^v \nabla_h \mathcal{U}_n\|_{\mathbf{L}^2([0,t],\mathbf{L}^2)}^2 - \|\Delta_q^v \mathcal{U}_n(0)\|_{\mathbf{L}^2}^2 \\ & \leq 2 \int_0^t |\langle \Delta_q^v(u_n \cdot \nabla u_n) | \Delta_q^v u_n \rangle| ds + 2 \int_0^t |\langle \Delta_q^v(u_n \cdot \nabla b_n) | \Delta_q^v b_n \rangle| ds \\ & \quad + 2 \int_0^t |\langle \Delta_q^v(b_n \cdot \nabla u_n) | \Delta_q^v b_n \rangle + \langle \Delta_q^v(b_n \cdot \nabla b_n) | \Delta_q^v u_n \rangle| ds. \end{aligned}$$

Inequality (A.10) of Appendix A and Young's inequality imply that

$$(2.7) \quad \begin{aligned} \int_0^t |\langle \Delta_q^v(u_n \cdot \nabla u_n) | \Delta_q^v u_n \rangle| ds & \leq Cd_q 2^{-2qs} \|u_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h u_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2 \\ & \leq Cd_q 2^{-2qs} \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2, \end{aligned}$$

and

$$(2.8) \quad \begin{aligned} & \int_0^t |\langle \Delta_q^v(u_n \cdot \nabla b_n) | \Delta_q^v b_n \rangle| dt \\ & \leq Cd_q 2^{-2qs} \left[ \|\nabla_h u_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})} \|b_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h b_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})} \right. \\ & \quad \left. + \|u_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|\nabla_h u_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|b_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|\nabla_h b_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^{\frac{3}{2}} \right] \\ & \leq Cd_q 2^{-2qs} \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2, \end{aligned}$$

where  $(d_q)$  is a summable sequence of positive constants of sum 1.

In the same way, Inequality (A.11) implies that

$$(2.9) \quad \begin{aligned} & \int_0^t |\langle \Delta_q^v(b_n \cdot \nabla u_n) | \Delta_q^v b_n \rangle + \langle \Delta_q^v(b_n \cdot \nabla b_n) | \Delta_q^v u_n \rangle| dt \\ & \leq Cd_q 2^{-2qs} \left[ \|\nabla_h u_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})} \|b_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h b_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})} \right. \\ & \quad \left. + \|u_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|\nabla_h u_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|b_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})}^{\frac{1}{2}} \|\nabla_h b_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^{\frac{3}{2}} \right] \\ & \leq Cd_q 2^{-2qs} \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2. \end{aligned}$$

From Equations (2.5) to (2.9), we deduce that

$$(2.10) \quad \begin{aligned} & \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,t],\mathbf{L}^2)}^2 + 2\nu \|\Delta_q^v \nabla_h \mathcal{U}_n\|_{\mathbf{L}^2([0,t],\mathbf{L}^2)}^2 \\ & \leq \|\Delta_q^v \mathcal{U}_n(0)\|_{\mathbf{L}^2}^2 + 2Cd_q 2^{-2qs} \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2. \end{aligned}$$

By multiplying (2.10) by  $2^{2qs}$  and then summing with respect to  $q$ , we finally have

$$(2.11) \quad \begin{aligned} & \frac{1}{2} \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})}^2 + 2\nu \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2 \\ & \leq \|\mathcal{U}_n(0)\|_{\mathbf{H}^{0,s}}^2 + 2C \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t],\mathbf{H}^{0,s})}^2. \end{aligned}$$

Let  $c = \frac{1}{4C}$  and suppose that  $\|\mathcal{U}_n(0)\|_{\mathbf{H}^{0,s}} \leq \frac{c\nu}{2}$ . Let  $T_n^* \geq 0$  be the maximal time such that  $\|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \leq (c\nu)^2$ , for any  $0 \leq t \leq T_n^*$ . According to Cauchy-Lipschitz



theorem,  $\mathcal{U}_n(\cdot)$  is continuous in  $\mathbf{L}^2$  with respect to the time variable, so is  $\Delta_q^v \mathcal{U}_n(\cdot)$ . As a consequence,  $\|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,\cdot], \mathbf{L}^2)}$  is continuous in time. Since

$$\sup_{0 < t < t_0} 2^{qs} \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,t], \mathbf{L}^2)} = 2^{qs} \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,t_0], \mathbf{L}^2)},$$

and

$$\sum_{q \geq -1} 2^{qs} \|\Delta_q^v \mathcal{U}_n\|_{\mathbf{L}^\infty([0,t_0], \mathbf{L}^2)} = \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t_0], \mathbf{H}^{0,s})}^2,$$

the application

$$t \mapsto \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t], \mathbf{H}^{0,s})}$$

is continuous on  $[0, t_0]$  for any  $t_0 \geq 0$  such that  $\|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t_0], \mathbf{H}^{0,s})} < +\infty$ . So, we have  $T_n^* > 0$ .

Now, using the fact that  $\|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t], \mathbf{H}^{0,s})}^2 \leq (c\nu)^2$ , from Inequality (2.11), we deduce that, for any  $t$ ,  $0 \leq t < T_n^*$ ,

$$(2.12) \quad \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t], \mathbf{H}^{0,s})}^2 + \nu \|\nabla_h \mathcal{U}_n\|_{\tilde{\mathbf{L}}^2([0,t], \mathbf{H}^{0,s})}^2 \leq \frac{(c\nu)^2}{4} < (c\nu)^2.$$

So, Inequality (2.12) and the continuity of the application  $t \mapsto \|\mathcal{U}_n\|_{\tilde{\mathbf{L}}^\infty([0,t], \mathbf{H}^{0,s})}^2$  imply that  $T_n^* = +\infty$ .

The sequence of solutions  $(\mathcal{U}_n)_n$  is then bounded in  $\tilde{\mathbf{L}}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s})$ , and  $(\nabla_h \mathcal{U}_n)_n$  is bounded in  $\tilde{\mathbf{L}}^2(\mathbb{R}_+, \mathbf{H}^{0,s})$ . In particular,  $(\mathcal{U}_n)_n$  is bounded in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2)$  and so,  $(\partial_t \mathcal{U}_n)_n$  is bounded in  $\mathbf{L}_{loc}^\infty(\mathbb{R}_+, \mathbf{H}^{-N})$  for  $N$  large enough (*e.g.*  $N > 2$ ). Then, the Arzela-Ascoli theorem implies the existence of a subsequence of  $(\mathcal{U}_n)_n$  (which will also be noted by  $(\mathcal{U}_n)_n$ ), which converges to  $\mathcal{U}$  in  $\mathbf{L}_{loc}^\infty(\mathbb{R}_+, \mathbf{H}_{loc}^{-N})$ . Since  $(\mathcal{U}_n)_n$  is bounded in  $\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{L}^2)$ , using an interpolation, we conclude that

$$\mathcal{U}_n(t) \rightarrow \mathcal{U}(t) \text{ in } \mathbf{H}_{loc}^\delta \text{ for any } -N \leq \delta < 0, \text{ and for a.e. } t \geq 0.$$

Since  $(\mathcal{U}_n(t))_n$  is also bounded in  $\mathbf{H}_{loc}^{\delta'}$ , for any  $\delta' < \frac{1}{2}$ , the Sobolev product law implies that

$$\mathcal{U}_n(t) \otimes \mathcal{U}_n(t) \rightarrow \mathcal{U}(t) \otimes \mathcal{U}(t) \text{ in } \dot{\mathbf{H}}^{\delta+\delta'-\frac{3}{2}} \text{ for any } -N \leq \delta < 0, \text{ and for a.e. } t \geq 0,$$

and in particular, that

$$\mathcal{U}_n(t) \otimes \mathcal{U}_n(t) \rightarrow \mathcal{U}(t) \otimes \mathcal{U}(t) \text{ in } \mathcal{D}'.$$

Thus, we can deduce that  $\mathcal{U}(t)$  verifies the system (1.4) in the sense of distributions. Next, we remark that the Banach-Alaoglu theorem implies that  $\mathcal{U}_n \rightharpoonup \mathcal{U}$  weakly-star in  $\tilde{\mathbf{L}}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s})$  and  $\nabla_h \mathcal{U}_n \rightharpoonup \nabla_h \mathcal{U}$  weakly in  $\tilde{\mathbf{L}}^2(\mathbb{R}_+, \mathbf{H}^{0,s})$ . Then, according to the uniform boundedness principle,  $\mathcal{U}$  is bounded in  $\tilde{\mathbf{L}}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s})$  and  $\nabla_h \mathcal{U}$  is bounded in  $\tilde{\mathbf{L}}^2(\mathbb{R}_+, \mathbf{H}^{0,s})$ .

Finally, we prove the continuity in time of the above constructed solution. Applying the operator  $\Delta_q^v$  to the system (1.4), taking the  $\mathbf{L}^2$ -inner product of the first equation of the obtained system with  $\Delta_q^v u$  and of the second equation with  $\Delta_q^v b$  and then summing the two obtained equations, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q^v \mathcal{U}\|_{\mathbf{L}^2}^2 + \nu \|\Delta_q^v \nabla_h \mathcal{U}\|_{\mathbf{L}^2}^2 &= - \langle \Delta_q^v (u \cdot \nabla u) | \Delta_q^v u \rangle - \langle \Delta_q^v (u \cdot \nabla b) | \Delta_q^v b \rangle \\ &\quad + \langle \Delta_q^v (b \cdot \nabla u) | \Delta_q^v b_n \rangle + \langle \Delta_q^v (b \cdot \nabla b) | \Delta_q^v u \rangle, \end{aligned}$$

which implies that

$$(2.13) \quad \frac{1}{2} \left| \frac{d}{dt} \|\Delta_q^v \mathcal{U}\|_{\mathbf{L}^2}^2 \right| \leq \nu \|\Delta_q^v \nabla_h \mathcal{U}\|_{\mathbf{L}^2}^2 + |\langle \Delta_q^v (u \cdot \nabla u) | \Delta_q^v u \rangle| + |\langle \Delta_q^v (u \cdot \nabla b) | \Delta_q^v b \rangle| \\ + |\langle \Delta_q^v (b \cdot \nabla u) | \Delta_q^v b_n \rangle| + \langle \Delta_q^v (b \cdot \nabla b) | \Delta_q^v u \rangle, \\ \leq \nu \|\Delta_q^v \nabla_h \mathcal{U}\|_{\mathbf{L}^2}^2 + Cd_q 2^{-2qs} \|\mathcal{U}\|_{\mathbf{H}^{0,s}} \|\nabla_h \mathcal{U}\|_{\mathbf{H}^{0,s}}^2.$$

We know that,  $\mathcal{U} \in \tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})$  and  $\nabla_h \mathcal{U} \in \tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0,s})$ , for any  $T > 0$  in the existence interval of  $\mathcal{U}$ . So the right-hand side of (2.13) is locally integrable in time. As a consequence,  $\frac{d}{dt} \|\Delta_q^v \mathcal{U}(\cdot)\|_{\mathbf{L}^2}^2$  is locally integrable in time, which means that  $\|\Delta_q^v \mathcal{U}(\cdot)\|_{\mathbf{L}^2}$  is continuous. From the construction of  $\mathcal{U}$ , it is easy to show that  $\Delta_q^v \mathcal{U}(t)$  is weakly continuous in  $\mathbf{L}^2$  with respect to the time variable. Indeed, for any test function  $\phi$ , and for any  $0 \leq a < b$ , we have

$$\sup_{t \in [a, b]} |\langle \Delta_q^v (\mathcal{U}_n(t) - \mathcal{U}(t)) | \phi \rangle| \leq C 2^{2q} \|\mathcal{U}_n - \mathcal{U}\|_{\mathbf{L}^\infty(\mathbf{H}_{loc}^{-2})} \|\phi\|_{\mathbf{H}^2}.$$

Since  $\mathcal{U}_n \rightarrow \mathcal{U}$  strongly in  $\mathbf{L}^\infty(\mathbf{H}_{loc}^{-2})$ , we deduce that  $\Delta_q^v \mathcal{U}_n$  weakly converges towards  $\Delta_q^v \mathcal{U}$ , uniformly in any time interval  $[a, b]$  included in the existence interval of  $\mathcal{U}$ . Thus, the continuity of  $\Delta_q^v \mathcal{U}_n$  implies the weak continuity of  $\Delta_q^v \mathcal{U}$ . Combining this with the continuity of the norm, we conclude that  $\Delta_q^v \mathcal{U}(t)$  is continuous in  $\mathbf{L}^2$  with respect to the time variable.

Now, since  $\mathcal{U} \in \tilde{\mathbf{L}}^\infty([0, T]; \mathbf{H}^{0,s})$ , for all  $T > 0$ , then for all  $\zeta > 0$ , there exists  $N > 0$  such that

$$(2.14) \quad \sum_{|q| > N} 2^{2qs} \|\Delta_q^v \mathcal{U}\|_{\mathbf{L}^\infty([0, T], \mathbf{L}^2)}^2 \leq \frac{\zeta^2}{4}.$$

As  $\Delta_q^v \mathcal{U}$  is continuous in time, there exists  $\delta > 0$  such that if  $|t - t'| < \delta$  then

$$(2.15) \quad \sum_{|q| \leq N} 2^{2qs} \|\Delta_q^v (\mathcal{U}(t) - \mathcal{U}(t'))\|_{\mathbf{L}^\infty([0, T], \mathbf{L}^2)}^2 \leq \frac{\zeta^2}{2}.$$

Combining (2.14) with (2.15), we obtain that  $\|\mathcal{U}(t) - \mathcal{U}(t')\|_{\mathbf{H}^{0,s}} \leq \zeta$ , for all  $|t - t'| \leq \delta$ , which means that  $\mathcal{U}$  is continuous.

**2.2. Local existence for large initial data.** In order to prove the existence of a local strong solution for large initial data, we decompose the system (1.4) into two parts: the linear (smooth) part containing the ‘‘low Fourier frequencies’’ and the non-linear part containing ‘‘high Fourier frequencies’’. First, we write the initial data  $\mathcal{U}_0 = \begin{pmatrix} u_0 \\ b_0 \end{pmatrix}$  as

$$(2.16) \quad \begin{cases} u_0 = J_N u_0 + (\mathbb{I} - J_N) u_0, \\ b_0 = J_N b_0 + (\mathbb{I} - J_N) b_0, \end{cases}$$

where  $N$  is large enough and will be chosen later. We consider the following linear system:

$$(2.17) \quad \begin{cases} \partial_t \bar{u}_N - \nu \Delta_h \bar{u}_N + \mu \partial_3 \bar{b}_N + \frac{1}{\varepsilon} \bar{u}_N \wedge e_3 = -\nabla \bar{p}_N, \\ \partial_t \bar{b}_N - \nu \Delta_h \bar{b}_N + \mu \partial_3 \bar{u}_N = 0, \\ \operatorname{div} \bar{u}_N = \operatorname{div} \bar{b}_N = 0, \\ \bar{u}_N|_{t=0} = J_N u_0, \bar{b}_N|_{t=0} = J_N b_0. \end{cases}$$

This system has a unique global solution  $\bar{\mathcal{U}}_N = \begin{pmatrix} \bar{u}_N \\ \bar{b}_N \end{pmatrix}$ , which satisfies the following energy estimate

$$(2.18) \quad \|\bar{\mathcal{U}}_N\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{H}^{0,s})}^2 + \nu \|\nabla_h \bar{\mathcal{U}}_N\|_{\mathbf{L}^2(\mathbb{R}_+, \mathbf{H}^{0,s})}^2 \leq 2 \|\mathcal{U}_0\|_{\mathbf{H}^{0,s}}^2 = C_0.$$

It is easy to deduce from (2.18) that

$$(2.19) \quad \begin{cases} \|\bar{\mathcal{U}}_N(t)\|_{\mathbf{H}^{0,s}} \leq \sqrt{C_0}, & \forall t > 0, \\ \|\nabla_h \bar{\mathcal{U}}_N(t)\|_{\mathbf{H}^{0,s}} \leq N\sqrt{C_0}, \\ \|\bar{\mathcal{U}}_N\|_{\mathbf{L}^p([0,T], \mathbf{H}^{0,s})} \leq \sqrt{C_0} T^{\frac{1}{p}}, \\ \|\nabla_h \bar{\mathcal{U}}_N\|_{\mathbf{L}^p([0,T], \mathbf{H}^{0,s})} \leq N\sqrt{C_0} T^{\frac{1}{p}}. \end{cases}$$

Now we write:

$$(2.20) \quad \begin{cases} u = \bar{u}_N + \omega, \\ b = \bar{b}_N + \beta, \end{cases}$$

where  $\mathcal{V} = \begin{pmatrix} \omega \\ \beta \end{pmatrix}$  verifies the following system

$$(2.21) \quad \begin{cases} \partial_t \omega - \nu \Delta_h \omega + \omega \cdot \nabla \omega + \omega \cdot \nabla \bar{u}_N + \bar{u}_N \cdot \nabla \omega - \beta \cdot \nabla \beta - \beta \cdot \nabla \bar{b}_N - \bar{b}_N \cdot \nabla \beta \\ \quad + \mu \partial_3 \beta + \frac{1}{\varepsilon} \omega \wedge e_3 = -\nabla \tilde{p} + \bar{b}_N \cdot \nabla \bar{b}_N - \bar{u}_N \cdot \nabla \bar{u}_N \\ \partial_t \beta - \nu \Delta_h \beta - \beta \cdot \nabla \omega - \beta \cdot \nabla \bar{u}_N - \bar{b}_N \cdot \nabla \omega + \omega \cdot \nabla \beta + \omega \cdot \nabla \bar{b}_N + \bar{u}_N \cdot \nabla \beta \\ \quad + \mu \partial_3 \omega = \bar{b}_N \cdot \nabla \bar{u}_N - \bar{u}_N \cdot \nabla \bar{b}_N \\ \operatorname{div} \omega = \operatorname{div} \beta = 0 \\ \omega(0) = (\mathbb{I} - J_N)u_0, \quad \beta(0) = (\mathbb{I} - J_N)b_0. \end{cases}$$

So proving the existence of a local strong solution  $\mathcal{U} = \begin{pmatrix} u \\ b \end{pmatrix}$  of (1.4) is equivalent to proving the existence of a local strong solution of (2.21). We point that all the estimates given below are *a priori* estimates, which can be justified by using the method of Friedrichs as in the previous section and then by passing to the limit.

We apply the operator  $\Delta_q^v$  to the system (2.21). Next, taking the  $\mathbf{L}^2$ -inner product of the first equation of the obtained system with  $\Delta_q^v \omega$  and of the second equation with  $\Delta_q^v \beta$  and summing the two obtained equations, we get

$$(2.22) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Delta_q^v \mathcal{V}\|_{\mathbf{L}^2}^2 + \nu \|\Delta_q^v \nabla_h \mathcal{V}\|_{\mathbf{L}^2}^2 \\ & \leq |\langle \Delta_q^v(\omega \cdot \nabla \omega) | \Delta_q^v \omega \rangle| + |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \omega) | \Delta_q^v \omega \rangle| \\ & \quad + |\langle \Delta_q^v(\omega \cdot \nabla \bar{u}_N) | \Delta_q^v \omega \rangle| + |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \bar{u}_N) | \Delta_q^v \omega \rangle| + |\langle \Delta_q^v(\beta \cdot \nabla \bar{b}_N) | \Delta_q^v \omega \rangle| \\ & \quad + |\langle \Delta_q^v(\bar{b}_N \cdot \nabla \bar{b}_N) | \Delta_q^v \omega \rangle| + |\langle \Delta_q^v(\bar{b}_N \cdot \nabla \beta) | \Delta_q^v \omega \rangle| + \langle \Delta_q^v(\bar{b}_N \cdot \nabla \omega) | \Delta_q^v \beta \rangle| \\ & \quad + |\langle \Delta_q^v(\beta \cdot \nabla \omega) | \Delta_q^v \beta \rangle| + \langle \Delta_q^v(\beta \cdot \nabla \beta) | \Delta_q^v \omega \rangle| + |\langle \Delta_q^v(\omega \cdot \nabla \beta) | \Delta_q^v \beta \rangle| \\ & \quad + |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \beta) | \Delta_q^v \beta \rangle| + |\langle \Delta_q^v(\omega \cdot \nabla \bar{b}_N) | \Delta_q^v \beta \rangle| + |\langle \Delta_q^v(\beta \cdot \nabla \bar{u}_N) | \Delta_q^v \beta \rangle| \\ & \quad + |\langle \Delta_q^v(\bar{b}_N \cdot \nabla \bar{u}_N) | \Delta_q^v \beta \rangle| + |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \bar{b}_N) | \Delta_q^v \beta \rangle|. \end{aligned}$$

Lemma A.9, with  $s_0 = s_1 = s > \frac{1}{2}$ , and Young's inequality give

$$(2.23) \quad \begin{aligned} |\langle \Delta_q^v(\omega \cdot \nabla \omega) | \Delta_q^v \omega \rangle| &\leq Cd_q 2^{-2qs} \|\omega\|_{\mathbf{H}^{0,s}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^2 \\ &\leq Cd_q 2^{-2qs} \|\omega\|_{\mathbf{H}^{0,s}} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2, \end{aligned}$$

and

$$(2.24) \quad \begin{aligned} |\langle \Delta_q^v(\omega \cdot \nabla \beta) | \Delta_q^v \beta \rangle| &\leq Cd_q 2^{-2qs} \left[ \|\nabla_h \omega\|_{\mathbf{H}^{0,s}} \|\beta\|_{\mathbf{H}^{0,s}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}} \right. \\ &\quad \left. + \|\omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \right] \\ &\leq Cd_q 2^{-2qs} (\|\omega\|_{\mathbf{H}^{0,s}} + \|\beta\|_{\mathbf{H}^{0,s}}) \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2. \end{aligned}$$

Using Inequality (A.9) of Appendix A and Young's inequality, we get

$$(2.25) \quad \begin{aligned} &|\langle \Delta_q^v(\beta \cdot \nabla \beta) | \Delta_q^v \omega \rangle + \langle \Delta_q^v(\beta \cdot \nabla \omega) | \Delta_q^v \beta \rangle| \\ &\leq Cd_q 2^{-2qs} \left[ \|\nabla_h \omega\|_{\mathbf{H}^{0,s}} \|\beta\|_{\mathbf{H}^{0,s}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}} + \|\omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}}^{\frac{3}{2}} \right] \\ &\leq Cd_q 2^{-2qs} (\|\omega\|_{\mathbf{H}^{0,s}} + \|\beta\|_{\mathbf{H}^{0,s}}) \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2. \end{aligned}$$

Likewise, using Inequality (A.9) of Appendix A, the estimates (2.19) and Young's inequality, we obtain

$$(2.26) \quad \begin{aligned} &|\langle \Delta_q^v(\bar{b}_N \cdot \nabla \beta) | \Delta_q^v \omega \rangle + \langle \Delta_q^v(\bar{b}_N \cdot \nabla \omega) | \Delta_q^v \beta \rangle| \\ &\leq Cd_q 2^{-2qs} \|\nabla_h \bar{b}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \left[ \|\bar{b}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \right. \\ &\quad \left. + \|\bar{b}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} + \|\nabla_h \bar{b}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\beta\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \right] \\ &\leq d_q 2^{-2qs} \left[ \frac{C}{\nu^2} N^2 C_0^2 (\|\omega\|_{\mathbf{H}^{0,s}}^2 + \|\beta\|_{\mathbf{H}^{0,s}}^2) + \frac{C}{\nu} N^2 C_0 \|\omega\|_{\mathbf{H}^{0,s}} \|\beta\|_{\mathbf{H}^{0,s}} + \frac{\nu}{16} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \right] \\ &\leq d_q 2^{-2qs} \left[ \left( \frac{C}{\nu^2} N^2 C_0^2 + \frac{C}{\nu} N^2 C_0 \right) \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2 + \frac{\nu}{16} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \right]. \end{aligned}$$

Using Inequalities (A.8) of Appendix A, (2.19) and Young's inequality, we also obtain

$$(2.27) \quad \begin{aligned} |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \omega) | \Delta_q^v \omega \rangle| &\leq Cd_q 2^{-2qs} \left[ \|\bar{u}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \bar{u}_N\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\omega\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}}^{\frac{3}{2}} \right. \\ &\quad \left. + \|\nabla_h \bar{u}_N\|_{\mathbf{H}^{0,s}} \|\omega\|_{\mathbf{H}^{0,s}} \|\nabla_h \omega\|_{\mathbf{H}^{0,s}} \right] \\ &\leq d_q 2^{-2qs} \left[ \left( \frac{C}{\nu} N^2 C_0^2 + \frac{C}{\nu} N^2 C_0 \right) \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2 + \frac{\nu}{16} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \right]. \end{aligned}$$

In the same way, we have

$$(2.28) \quad |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \beta) | \Delta_q^v \beta \rangle| \leq d_q 2^{-2qs} \left[ \left( \frac{C}{\nu} N^2 C_0^2 + \frac{C}{\nu} N^2 C_0 \right) \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2 + \frac{\nu}{16} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \right].$$

Finally, Lemma A.14, the Cauchy-Schwarz inequality and the estimates (2.19) imply that

$$(2.29) \quad |\langle \Delta_q^v(\omega \cdot \nabla \bar{u}_N) | \Delta_q^v \omega \rangle| \leq C_N d_q 2^{-2qs} \sqrt{C_0} \|\omega\|_{\mathbf{H}^{0,s}}^2$$

$$(2.30) \quad |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \bar{u}_N) | \Delta_q^v \omega \rangle| \leq C_N d_q 2^{-2qs} C_0 \|\omega\|_{\mathbf{H}^{0,s}}$$

$$(2.31) \quad |\langle \Delta_q^v(\beta \cdot \nabla \bar{b}_N) | \Delta_q^v \omega \rangle| \leq C_N d_q 2^{-2qs} \sqrt{C_0} \|\omega\|_{\mathbf{H}^{0,s}} \|\beta\|_{\mathbf{H}^{0,s}}$$

$$(2.32) \quad |\langle \Delta_q^v(\bar{b}_N \cdot \nabla \bar{b}_N) | \Delta_q^v \omega \rangle| \leq C_N d_q 2^{-2qs} C_0 \|\omega\|_{\mathbf{H}^{0,s}}$$

$$(2.33) \quad |\langle \Delta_q^v(\omega \cdot \nabla \bar{b}_N) | \Delta_q^v \beta \rangle| \leq C_N d_q 2^{-2qs} \sqrt{C_0} \|\omega\|_{\mathbf{H}^{0,s}} \|\beta\|_{\mathbf{H}^{0,s}}$$

$$(2.34) \quad |\langle \Delta_q^v(\bar{u}_N \cdot \nabla \bar{b}_N) | \Delta_q^v \beta \rangle| \leq C_N d_q 2^{-2qs} C_0 \|\beta\|_{\mathbf{H}^{0,s}}$$

$$(2.35) \quad |\langle \Delta_q^v(\beta \cdot \nabla \bar{u}_N) | \Delta_q^v \beta \rangle| \leq C_N d_q 2^{-2qs} \sqrt{C_0} \|\beta\|_{\mathbf{H}^{0,s}}^2$$

$$(2.36) \quad |\langle \Delta_q^v(\bar{b}_N \cdot \nabla \bar{u}_N) | \Delta_q^v \beta \rangle| \leq C_N d_q 2^{-2qs} C_0 \|\beta\|_{\mathbf{H}^{0,s}}.$$

Taking into account the estimates (2.23) to (2.36), we deduce from (2.22), after having summed up in  $q$ , that

$$(2.37) \quad \frac{1}{2} \frac{d}{dt} \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2 + \frac{\nu}{2} \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \leq C (\|\omega\|_{\mathbf{H}^{0,s}} + \|\beta\|_{\mathbf{H}^{0,s}}) \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 \\ + \bar{C}_1(N, \nu, C_0) \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2 + \bar{C}_2(N, C_0) (\|\omega\|_{\mathbf{H}^{0,s}} + \|\beta\|_{\mathbf{H}^{0,s}}).$$

Next, integrating (2.37) in the time variable implies that

$$(2.38) \quad \|\mathcal{V}(T)\|_{\mathbf{H}^{0,s}}^2 + \nu \int_0^T \|\nabla_h \mathcal{V}\|_{\mathbf{H}^{0,s}}^2 dt \\ \leq \|\mathcal{V}(0)\|_{\mathbf{H}^{0,s}}^2 + C_1(N, \nu, C_0) T \|\mathcal{V}\|_{\mathbf{L}^\infty([0,T]; \mathbf{H}^{0,s})}^2 + C_2(N, C_0) T \\ + C \left( \|\omega\|_{\mathbf{L}^\infty([0,T]; \mathbf{H}^{0,s})} + \|\beta\|_{\mathbf{L}^\infty([0,T]; \mathbf{H}^{0,s})} \right) \|\nabla_h \mathcal{V}\|_{\mathbf{L}^2([0,T]; \mathbf{H}^{0,s})}^2.$$

Now, recall that  $\mathcal{V}(0) = \begin{pmatrix} (\mathbb{I} - J_N)u_0 \\ (\mathbb{I} - J_N)b_0 \end{pmatrix}$ . We choose  $N$  large enough such that

$$\sqrt{2} \|\mathcal{V}(0)\|_{\mathbf{H}^{0,s}} \leq \frac{\nu}{2C} = c\nu,$$

and  $T_0 > 0$  small enough, depending of  $C_0$ ,  $N$ , and  $\nu$ , such that

$$\begin{cases} C_1(N, \nu, C_0) T_0 < \frac{1}{2} \\ C_2(N, C_0) T_0 < \frac{c^2 \nu^2}{4}. \end{cases}$$

Let  $T^* = \sup \left\{ 0 < T < T_0 : \|\omega\|_{\mathbf{L}^\infty([0,t]; \mathbf{H}^{0,s})} + \|\beta\|_{\mathbf{L}^\infty([0,t]; \mathbf{H}^{0,s})} \leq 2c\nu, \forall 0 \leq t \leq T \right\}$ . For any  $0 \leq T < T^*$ ,

$$\|\mathcal{V}\|_{\mathbf{L}^\infty([0,T], \mathbf{H}^{0,s})}^2 \leq 2 \left[ \|\mathcal{V}(0)\|_{\mathbf{H}^{0,s}}^2 + C_2(N, C_0) T \right] \leq \frac{3}{2} c^2 \nu^2,$$

so,

$$\|\omega\|_{\mathbf{L}^\infty([0,T], \mathbf{H}^{0,s})} + \|\beta\|_{\mathbf{L}^\infty([0,T], \mathbf{H}^{0,s})} \leq \sqrt{2 \|\mathcal{V}\|_{\mathbf{H}^{0,s}}^2} < 2c\nu.$$

We deduce that  $T^* = T_0$ . Arguing as in Section 2.1, by using the Friedrichs scheme of approximation and then taking the limit of the family of approximate solutions, we obtain the existence on  $[0, T_0]$  of a solution  $(\omega, \beta)$  of (2.21) and so of a solution  $(u, b)$  of (1.4). Finally, we remark that the continuity in the time variable of the solution can be proved in the same way as in Section 2.1.

**2.3. Uniqueness of the strong solution.** This section is devoted to the proof of the uniqueness of the strong solution of (1.4) constructed in the previous sections. As remarked in [20], since the vertical viscosity is zero, we do not have enough regularity properties to estimate the difference of two solutions in  $\mathbf{H}^{0,s}$ . The difficulty comes from terms of the form  $u^3 \partial_3 v$ , for which we need the estimate of  $\partial_3 v$  in  $\mathbf{H}^{0,s}$ . This means that we need  $\mathbf{H}^{0,s+1}$ -regularity results, which we do not know because of the zero vertical viscosity. Here, we follow the ideas of [20], and we will prove the uniqueness of the solution in  $\mathbf{H}^{0,s-1}$ . More precisely, we prove the following theorem:

**Theorem 2.1.** *Let  $s > \frac{1}{2}$  and  $T_0 > 0$ . Let  $(u_1, b_1)$  and  $(u_2, b_2)$  be two solutions of the system (1.4) in  $(\mathbf{L}^\infty([0, T_0], \mathbf{H}^{0,s}) \cap \mathbf{C}([0, T_0], \mathbf{H}^{0,s}))^2$ , corresponding to the same initial data  $(u_0, b_0)$ , such that  $(\nabla_h u_1, \nabla_h b_1)$  and  $(\nabla_h u_2, \nabla_h b_2)$  belong to  $(\mathbf{L}^2([0, T_0], \mathbf{H}^{0,s}))^2$ . Let*

$$\mathcal{U} = \begin{pmatrix} u \\ b \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ b_1 - b_2 \end{pmatrix}.$$

Then, for any  $0 < t \leq T_0$ , we have:

$$(2.39) \quad \frac{d}{dt} \|\mathcal{U}(t)\|_{\mathbf{H}^{0,s-1}}^2 \leq C f(t) \|\mathcal{U}(t)\|_{\mathbf{H}^{0,s-1}}^2$$

where

$$f(t) = \left( 1 + \|u_1\|_{\mathbf{H}^{0,s}}^2 + \|u_2\|_{\mathbf{H}^{0,s}}^2 + \|b_1\|_{\mathbf{H}^{0,s}}^2 + \|b_2\|_{\mathbf{H}^{0,s}}^2 \right) \\ \times \left( 1 + \|\nabla_h u_1\|_{\mathbf{H}^{0,s}}^2 + \|\nabla_h u_2\|_{\mathbf{H}^{0,s}}^2 + \|\nabla_h b_1\|_{\mathbf{H}^{0,s}}^2 + \|\nabla_h b_2\|_{\mathbf{H}^{0,s}}^2 \right).$$

As a consequence of this theorem, by performing an integration in time and then, by using Gronwall lemma, we deduce that, for any  $t \in [0, T]$ ,  $\mathcal{U}_1(t) = \mathcal{U}_2(t)$  in  $\mathbf{H}^{0,s-1}$ . Thus, the solution of (1.4) is unique.

**Proof of Theorem 2.1:** Since  $(u_1, b_1)$  and  $(u_2, b_2)$  are solutions of (1.4),  $(u, b)$  satisfies the following system

$$\begin{cases} \partial_t u - \nu \Delta_h u + u_1 \cdot \nabla u + u \cdot \nabla u_2 - b_1 \cdot \nabla b - b \cdot \nabla b_2 + \frac{1}{\varepsilon} u \wedge e_3 + \mu \partial_3 b = -\nabla \tilde{p} \\ \partial_t b - \nu \Delta_h b - b_1 \cdot \nabla u - b \cdot \nabla u_2 + u_1 \cdot \nabla b + u \cdot \nabla b_2 + \mu \partial_3 u = 0 \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u(0) = b(0) = 0. \end{cases}$$

We apply  $\Delta_q^v$  to the system, then we take the  $\mathbf{L}^2$  scalar product of the first equation with  $\Delta_q^v u$  and of the second equation with  $\Delta_q^v b$ . Summing the two obtained equations, we get

$$(2.40) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q^v \mathcal{U}\|_{\mathbf{L}^2}^2 + \nu \|\Delta_q^v \nabla_h \mathcal{U}\|_{\mathbf{L}^2}^2 \\ \leq |\langle \Delta_q^v (u_1 \cdot \nabla u) | \Delta_q^v u \rangle| + |\langle \Delta_q^v (u_1 \cdot \nabla b) | \Delta_q^v b \rangle| \\ + |\langle \Delta_q^v (b_1 \cdot \nabla b) | \Delta_q^v u \rangle| + |\langle \Delta_q^v (b_1 \cdot \nabla u) | \Delta_q^v b \rangle| + R_h + R_v,$$

where

$$(2.41) \quad R_h = \left| \langle \Delta_q^v (u^h \cdot \nabla_h u_2) | \Delta_q^v u \rangle \right| + \left| \langle \Delta_q^v (b^h \cdot \nabla_h b_2) | \Delta_q^v u \rangle \right| \\ + \left| \langle \Delta_q^v (b^h \cdot \nabla_h u_2) | \Delta_q^v b \rangle \right| + \left| \langle \Delta_q^v (u^h \cdot \nabla_h b_2) | \Delta_q^v b \rangle \right|,$$

and

$$(2.42) \quad R_v = |\langle \Delta_q^v (u^3 \partial_3 u_2) | \Delta_q^v u \rangle| + |\langle \Delta_q^v (b^3 \partial_3 b_2) | \Delta_q^v u \rangle| \\ + |\langle \Delta_q^v (b^3 \partial_3 u_2) | \Delta_q^v b \rangle| + |\langle \Delta_q^v (u^3 \partial_3 b_2) | \Delta_q^v b \rangle|.$$

Using Lemma A.9 of Appendix A and Young's inequality, we obtain

$$(2.43) \quad \left| \langle \Delta_q^v (u_1 \cdot \nabla u) | \Delta_q^v u \rangle_{\mathbf{L}^2} \right| \leq C d_q 2^{-2q(s-1)} \left( \|\nabla_h u_1\|_{\mathbf{H}^{0,s}} \|u\|_{\mathbf{H}^{0,s-1}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}} \right. \\ \left. + \|u_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{3}{2}} \right) \\ \leq d_q 2^{-2q(s-1)} \left[ C f(t) \|u\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{16} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 \right],$$

and

$$(2.44) \quad \left| \langle \Delta_q^v (u_1 \cdot \nabla b) | \Delta_q^v b \rangle_{\mathbf{L}^2} \right| \leq C d_q 2^{-2q(s-1)} \left( \|\nabla_h u_1\|_{\mathbf{H}^{0,s}} \|b\|_{\mathbf{H}^{0,s-1}} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}} \right. \\ \left. + \|u_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|b\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^{\frac{3}{2}} \right) \\ \leq d_q 2^{-2q(s-1)} \left[ C f(t) \|b\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{16} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right].$$

Using Lemma A.10 and Young's inequality, we also have

$$(2.45) \quad \left| \langle \Delta_q^v (b_1 \cdot \nabla b) | \Delta_q^v u \rangle + \langle \Delta_q^v (b_1 \cdot \nabla u) | \Delta_q^v b \rangle \right| \\ \leq C d_q 2^{-2q(s-1)} \|\nabla_h b_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \left[ \|b_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|b\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \right. \\ \left. + \|b_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} + \|\nabla_h b_1\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|b\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \right] \\ \leq d_q 2^{-2q(s-1)} \left[ C f(t) \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right) + \frac{\nu_h}{16} \left( \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 + \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right) \right].$$

The estimate of the ‘‘horizontal term’’  $R_h$  is easy. Indeed, for any divergence free vector fields  $w_1$ ,  $w_2$  and  $w_3$  in  $\mathbf{H}^{0,s}$ , the horizontal gradient of which also belongs to  $\mathbf{H}^{0,s}$ , using Lemma A.5, we get

$$\left| \langle \Delta_q^v (w_1^h \cdot \nabla_h w_2) | \Delta_q^v w_3 \rangle \right| \leq \left\| \Delta_q^v (w_1^h \cdot \nabla_h w_2) \right\|_{\mathbf{L}_v^2(\dot{\mathbf{H}}_h^{-\frac{1}{2}})} \left\| \Delta_q^v w_3 \right\|_{\mathbf{L}_v^2(\dot{\mathbf{H}}_h^{\frac{1}{2}})} \\ \leq C \left\| \Delta_q^v (w_1^h \cdot \nabla_h w_2) \right\|_{\mathbf{L}_v^2(\dot{\mathbf{H}}_h^{-\frac{1}{2}})} \left\| \Delta_q^v w_3 \right\|_{\mathbf{L}^2}^{\frac{1}{2}} \left\| \nabla_h \Delta_q^v w_3 \right\|_{\mathbf{L}^2}^{\frac{1}{2}} \\ \leq C d_q 2^{-2qs} \left\| w_1^h \cdot \nabla_h w_2 \right\|_{\mathbf{H}^{-\frac{1}{2},s}} \left\| w_3 \right\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \left\| \nabla_h w_3 \right\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}}.$$

Here, we use the fact that  $\dot{\mathbf{H}}^{\frac{1}{2}}(\mathbb{R}^2)$  is an interpolant between  $\mathbf{L}^2(\mathbb{R}^2)$  and  $\dot{\mathbf{H}}^1(\mathbb{R}^2)$ . Now, applying Theorem A.8 with  $\sigma = \frac{1}{2}$ ,  $\sigma' = 0$ ,  $s_0 > \frac{1}{2}$ ,  $s_1 = s$ , we see that

$$\left| \langle \Delta_q^v (w_1^h \cdot \nabla_h w_2) | \Delta_q^v w_3 \rangle \right| \\ \leq C d_q 2^{-2qs} \|w_1\|_{\mathbf{H}^{\frac{1}{2},s_0}} \|\nabla_h w_2\|_{\mathbf{H}^{0,s}} \|w_3\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h w_3\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \\ \leq C d_q 2^{-2qs} \|w_1\|_{\mathbf{H}^{\frac{1}{2},s_0}}^{\frac{1}{2}} \|\nabla_h w_1\|_{\mathbf{H}^{0,s_0}}^{\frac{1}{2}} \|\nabla_h w_2\|_{\mathbf{H}^{0,s}} \|w_3\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h w_3\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}}$$

Replacing  $w_1$ ,  $w_2$  and  $w_3$  by the terms appearing in  $R_h$ , we easily get

$$(2.46) \quad R_h \leq d_q 2^{-2q(s-1)} \left[ C f(t) \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right) + \frac{\nu_h}{2} \left( \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 + \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right) \right].$$

The estimate of the ‘‘vertical term’’  $R_v$  is more delicate because of the lack of viscosity in the vertical direction. Let

$$\begin{aligned} A &= |\langle \Delta_q^v (u^3 \partial_3 u_2) | \Delta_q^v u \rangle|, & B &= |\langle \Delta_q^v (b^3 \partial_3 b_2) | \Delta_q^v u \rangle|, \\ C &= |\langle \Delta_q^v (b^3 \partial_3 u_2) | \Delta_q^v b \rangle|, & D &= |\langle \Delta_q^v (u^3 \partial_3 u_2) | \Delta_q^v b \rangle|. \end{aligned}$$

The Bony decomposition in paraproducts and remainders allows us to write

$$A \leq A_1 + A_2 + A_3$$

where

$$\begin{aligned} A_1 &= |\langle \Delta_q^v \sum_{|q'-q| \leq N_0} S_{q'-1}^v u^3 \Delta_{q'}^v (\partial_3 u_2) | \Delta_q^v u \rangle|, \\ A_2 &= |\langle \Delta_q^v \sum_{|q'-q| \leq N_0} S_{q'-1}^v (\partial_3 u_2) \Delta_{q'}^v u^3 | \Delta_q^v u \rangle|, \\ A_3 &= |\langle \Delta_q^v \sum_{q' \geq q - N_0} \sum_{i \in \{0, \pm 1\}} \Delta_{q'-i}^v (\partial_3 u_2) \Delta_{q'}^v u^3 | \Delta_q^v u \rangle|, \end{aligned}$$

where  $N_0$  is an appropriate integer and where the operators  $S_q^v$  are defined by

$$S_q^v = \sum_{j \leq q-1} \Delta_j^v, \quad \forall q \in \mathbb{Z}.$$

Using the Bernstein inequalities (A.1) and (A.2) of Appendix A, the Cauchy-Schwarz inequality and the fact that  $\operatorname{div} u = 0$  and  $s > \frac{1}{2}$ , we can write

$$(2.47) \quad \begin{aligned} \|S_{q'-1}^v u^3\|_{\mathbf{L}_h^2 \mathbf{L}_v^\infty} &\leq C \sum_{-1 \leq p \leq q'-2} 2^{-\frac{p}{2}} \|\Delta_p^v \partial_3 u^3\|_{\mathbf{L}^2} \\ &\leq C \sum_{-1 \leq p \leq q'-2} 2^{-p(s-\frac{1}{2})} 2^{p(s-1)} \|\Delta_p^v \nabla_h u\|_{\mathbf{L}^2} \leq C \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}. \end{aligned}$$

Lemma A.5 implies the existence of square-summable sequences of positive numbers such that

$$\begin{aligned} \|\Delta_{q'}^v u_2\|_{\mathbf{L}^2} &\leq c_{q'}(u_2) 2^{-q's} \|u_2\|_{\mathbf{H}^{0,s}} \\ \|\Delta_{q'}^v \nabla_h u_2\|_{\mathbf{L}^2} &\leq c_{q'}(\nabla_h u_2) 2^{-q's} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}} \\ \|\Delta_q^v u\|_{\mathbf{L}^2} &\leq c_q(u) 2^{-q(s-1)} \|u\|_{\mathbf{H}^{0,s-1}} \\ \|\Delta_q^v \nabla_h u\|_{\mathbf{L}^2} &\leq c_q(\nabla_h u) 2^{-q(s-1)} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}. \end{aligned}$$

Then, using the fact that  $\dot{\mathbf{H}}^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow \mathbf{L}^4(\mathbb{R}^2)$  and that  $\dot{\mathbf{H}}^{\frac{1}{2}}(\mathbb{R}^2)$  is an interpolant between  $\mathbf{L}^2(\mathbb{R}^2)$  and  $\dot{\mathbf{H}}^1(\mathbb{R}^2)$  and the Cauchy-Schwarz inequality, we can define new square-summable sequences of positive numbers

$$c_q^{(1)} = c_q(u)^{\frac{1}{2}} c_q(\nabla_h u)^{\frac{1}{2}} \quad \text{and} \quad c_{q'}^{(2)} = c_{q'}(u_2)^{\frac{1}{2}} c_{q'}(\nabla_h u_2)^{\frac{1}{2}},$$



which satisfy

$$\begin{aligned} \|\Delta_q^v u\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} &\leq \|\Delta_q^v u\|_{\dot{\mathbf{H}}_h^{\frac{1}{2}} \mathbf{L}_v^2} \leq \|\Delta_q^v u\|_{\mathbf{L}^2}^{\frac{1}{2}} \|\Delta_q^v \nabla_h u\|_{\mathbf{L}^2}^{\frac{1}{2}} \leq c_q^{(1)} 2^{-q(s-1)} \|u\|_{\mathbf{H}^{0,s-1}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}} \\ \|\Delta_{q'}^v u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} &\leq \|\Delta_{q'}^v u_2\|_{\dot{\mathbf{H}}_h^{\frac{1}{2}} \mathbf{L}_v^2} \leq \|\Delta_{q'}^v u_2\|_{\mathbf{L}^2}^{\frac{1}{2}} \|\Delta_{q'}^v \nabla_h u_2\|_{\mathbf{L}^2}^{\frac{1}{2}} \leq c_q^{(2)} 2^{-q's} \|u_2\|_{\mathbf{H}^{0,s}} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}} \end{aligned}$$

Using Young's inequality, we deduce that

$$\begin{aligned} (2.48) \quad A_1 &\leq \left\{ \sum_{|q'-q| \leq N_0} 2^{q'} \|S_{q'-1}^v u^3\|_{\mathbf{L}_h^2 \mathbf{L}_v^\infty} \|\Delta_{q'}^v u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} \right\} \|\Delta_q^v u\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} \\ &\leq C 2^{-2q(s-1)} \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{3}{2}} \|u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \left( c_q^{(1)} \sum_{|q'-q| \leq N_0} 2^{(q-q')(s-1)} c_{q'}^{(2)} \right) \\ &\leq d_q 2^{-2q(s-1)} \left[ C f(t) \|u\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{48} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 \right], \end{aligned}$$

where

$$d_q = c_q^{(1)} \sum_{|q'-q| \leq N_0} 2^{(q-q')(s-1)} c_{q'}^{(2)}$$

is a summable sequence of positive numbers.

Similarly, we have:

$$\begin{aligned} \|S_{q'-1}^v \partial_3 u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^\infty} &\leq C \sum_{-1 \leq p \leq q'-2} 2^{\frac{3p}{2}} \|\Delta_p^v u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} \\ &\leq C 2^{q'} \sum_{-1 \leq p \leq q'-2} 2^{-p(s-\frac{1}{2})} \left( 2^{ps} \|\Delta_p^v u_2\|_{\mathbf{L}^2}^{\frac{1}{2}} \right) \left( 2^{ps} \|\Delta_p^v \nabla_h u_2\|_{\mathbf{L}^2}^{\frac{1}{2}} \right) \\ &\leq C 2^{q'} \|u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$\begin{aligned} (2.49) \quad A_2 &\leq C \left\{ \sum_{|q'-q| \leq N_0} \|S_{q'-1}^v \partial_3 u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^\infty} 2^{-q'} \|\Delta_{q'}^v \partial_3 u^3\|_{\mathbf{L}^2} \right\} \|\Delta_q^v u\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} \\ &\leq C 2^{-2q(s-1)} \left\{ c_q^{(1)} \sum_{|q'-q| \leq N_0} 2^{(q-q')(s-1)} c_{q'}^{(2)} \right\} \\ &\quad \times \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{3}{2}} \|u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \\ &\leq d_q 2^{-2q(s-1)} \left[ C f(t) \|u\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{48} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 \right]. \end{aligned}$$

Using again Inequalities (A.1) and (A.2) of Appendix A and Young's inequality, we also get

$$\begin{aligned}
(2.50) \quad A_3 &\leq \left\{ \sum_{q' \geq q - N_0} \sum_{i \in \{0, \pm 1\}} \|\Delta_{q'}^v u^3\|_{\mathbf{L}^2} \|\Delta_{q'-i}^v \partial_3 u_2\|_{\mathbf{L}_h^4 \mathbf{L}_v^2} \right\} \|\Delta_q^v u\|_{\mathbf{L}_h^4 \mathbf{L}_v^\infty} \\
&\leq C 2^q \left\{ \sum_{q' \geq q - N_0} \sum_{i \in \{0, \pm 1\}} \|\Delta_{q'}^v \partial_3 u^3\|_{\mathbf{L}^2} \|\Delta_{q'-i}^v u_2\|_{\mathbf{L}_v^2 \mathbf{L}_h^4} \right\} \|\Delta_q^v u\|_{\mathbf{L}_v^\infty \mathbf{L}_h^4} \\
&\leq C 2^{-2q(s-1)} \left\{ c_q^{(1)} \sum_{|q'-q| \leq N_0} 2^{(q-q')(2s-1)} [c_{q'}^{(2)}]^2 \right\} \\
&\quad \|u\|_{\mathbf{H}^{0,s-1}}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^{\frac{3}{2}} \|u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \|\nabla_h u_2\|_{\mathbf{H}^{0,s}}^{\frac{1}{2}} \\
&\leq d_q 2^{-2q(s-1)} \left[ C f(t) \|u\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{48} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 \right].
\end{aligned}$$

Summing the inequalities (2.48) to (2.50), we obtain

$$(2.51) \quad A \leq d_q 2^{-2q(s-1)} \left[ C f(t) \|u\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{16} \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 \right].$$

Likewise, we estimate the quantities  $B$ ,  $C$  and  $D$  and we obtain

$$(2.52) \quad B \leq d_q 2^{-2q(s-1)} \left[ C f(t) \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right) + \frac{\nu_h}{16} \left( \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 + \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right) \right],$$

$$(2.53) \quad C \leq d_q 2^{-2q(s-1)} \left[ C f(t) \|b\|_{\mathbf{H}^{0,s-1}}^2 + \frac{\nu_h}{16} \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right],$$

$$(2.54) \quad D \leq d_q 2^{-2q(s-1)} \left[ C f(t) \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right) + \frac{\nu_h}{16} \left( \|\nabla_h u\|_{\mathbf{H}^{0,s-1}}^2 + \|\nabla_h b\|_{\mathbf{H}^{0,s-1}}^2 \right) \right].$$

Finally, adding the inequalities (2.43) to (2.46) and (2.51) to (2.54), we deduce from the estimate (2.40), that for any  $0 < t < T_0$ ,

$$\frac{d}{dt} \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right) \leq C f(t) \left( \|u\|_{\mathbf{H}^{0,s-1}}^2 + \|b\|_{\mathbf{H}^{0,s-1}}^2 \right).$$

Theorem 2.1 is then proved.  $\blacksquare$

### 3. GLOBAL EXISTENCE OF STRONG SOLUTIONS WHEN THE ROTATION IS FAST

In this section, we are interested in the MHD system in the case where the rotation is fast and where  $\frac{\Lambda}{\varepsilon} = \frac{1}{\theta}$ . The idea is that, when the rotation is fast, the Coriolis force becomes a dominant factor, and strongly limits the movements of the fluid in the vertical direction (Taylor-Proudman theorem). As a consequence, the limiting behavior of the fluid is two-dimensional. In the case of  $\mathbb{R}^3$ , the fact that the energy is finite implies that this "limit" is in fact zero. This dispersion was proved in the case of rotating fluids ([9], [24]) or in the case of primitive equations ([6]) in the form of Strichartz type estimates, which allows to prove the global existence of strong solutions for large initial data when the rotation is fast enough.

For the MHD system, we already know from Theorem 1.1 that there exists a unique local strong solution for any initial data in  $\mathbf{H}^{0,s}$ ,  $s > \frac{1}{2}$ , and that this solution is global

if the initial data is small enough. The question then arises whether this solution exists globally in time, for large initial data, when the rotation is fast enough.

We remark here that the MHD system is very complex and is composed of six equations (three equations for the velocity and three equations for the magnetic field of the magma). So, in general, it is difficult to obtain Strichartz-type estimates for the linear system. In this section, we show that, in some particular cases, when the different parameters of the MHD system are well chosen, an explicit spectral study of the associated linear system is possible. As a consequence, we can prove Strichartz-type estimates for the linear system, which play a very important role in the proof of the global existence of strong solutions of fast rotating MHD system for large initial data.

We recall that, in this section, we consider the following system of fast rotating MHD equations in  $\mathbb{R}^3$ :

$$(1.7) \quad \begin{cases} \partial_t u - \varepsilon^\alpha \Delta_h u + u \cdot \nabla u - b \cdot \nabla b + \frac{u \wedge e}{\varepsilon} + \frac{\partial_3 b}{\varepsilon} = -\nabla \tilde{p} \\ \partial_t b - \varepsilon^\alpha \Delta_h b + u \cdot \nabla b - b \cdot \nabla u + \frac{\partial_3 u}{\varepsilon} = 0 \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u_0, \quad b|_{t=0} = b_0, \end{cases}$$

where  $\alpha > 0$  is a constant and  $\varepsilon > 0$  is supposed to be very small.

Following the method of [9], we cut off the initial data in Fourier frequencies, and then we decompose (1.7) into two parts: the linear part containing “the medium Fourier frequencies” and the nonlinear part containing the very low or very high Fourier frequencies. Let

$$(3.1) \quad \mathcal{C}_{r,R} = \{ \xi \in \mathbb{R}^3 \mid r \leq |\xi_h| \leq R; r \leq |\xi_3| \leq R; r \leq |\xi| \leq R \}.$$

We define the following cut-off function:

$$(3.2) \quad \psi(\xi) = \chi\left(\frac{|\xi|}{R}\right) \left[ 1 - \chi\left(\frac{2|\xi_h|}{r}\right) \right] \left[ 1 - \chi\left(\frac{2|\xi_3|}{r}\right) \right],$$

where  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^\infty$ -function such that

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq |x| \leq 1 \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

In the introduction, we introduced the linear system

$$(1.8) \quad \begin{cases} \partial_t \bar{u} - \varepsilon^\alpha \Delta_h \bar{u} + \frac{\bar{u} \wedge e_3}{\varepsilon} + \frac{\partial_3 \bar{b}}{\varepsilon} = -\nabla \tilde{p} \\ \partial_t \bar{b} - \varepsilon^\alpha \Delta_h \bar{b} + \frac{\partial_3 \bar{u}}{\varepsilon} = 0 \\ \operatorname{div} \bar{u} = \operatorname{div} \bar{b} = 0 \\ \bar{u}|_{t=0} = \bar{u}_0 \\ \bar{b}|_{t=0} = \bar{b}_0, \end{cases}$$

where  $\bar{u}_0 = \mathcal{F}^{-1}(\psi(\xi)\mathcal{F}(u_0)(\xi))$  and  $\bar{b}_0 = \mathcal{F}^{-1}(\psi(\xi)\mathcal{F}(b_0)(\xi))$ , which are vector fields such that the support of their Fourier transforms is included in the set  $\mathcal{C}_{r,R}$ . Having written

$$u = \bar{u} + \omega \quad \text{and} \quad b = \bar{b} + \beta,$$

we also obtained the following nonlinear system

$$(1.9) \quad \left\{ \begin{array}{l} \partial_t \omega - \varepsilon^\alpha \Delta_h \omega + \omega \cdot \nabla \omega + \omega \cdot \nabla \bar{u} + \bar{u} \cdot \nabla \omega - \beta \cdot \nabla \beta - \beta \cdot \nabla \bar{b} - \bar{b} \cdot \nabla \beta + \frac{\partial_3 \beta}{\varepsilon} + \frac{\omega \wedge e_3}{\varepsilon} \\ \hspace{15em} = -\nabla \tilde{p} + \bar{b} \cdot \nabla \bar{b} - \bar{u} \cdot \nabla \bar{u} \\ \partial_t \beta - \varepsilon^\alpha \Delta_h \beta - \beta \cdot \nabla \omega - \beta \cdot \nabla \bar{u} - \bar{b} \cdot \nabla \omega + \omega \cdot \nabla \beta + \omega \cdot \nabla \bar{b} + \bar{u} \cdot \nabla \beta + \frac{\partial_3 \omega}{\varepsilon} \\ \hspace{15em} = \bar{b} \cdot \nabla \bar{u} - \bar{u} \cdot \nabla \bar{b} \\ \operatorname{div} \omega = \operatorname{div} \beta = 0 \\ \omega|_{t=0} = \omega_0 = u_0 - \bar{u}_0, \quad \beta|_{t=0} = \beta_0 = b_0 - \bar{b}_0. \end{array} \right.$$

**3.1. Strichartz-type estimates for the linear system.** Taking the Leray projection and then the Fourier transformation of the linear system (1.8), we obtain

$$\left\{ \begin{array}{l} \partial_t \widehat{u} + \varepsilon^\alpha |\xi_h|^2 \widehat{u} + \widehat{\mathbb{P}A\bar{u}} + \frac{i\xi_3 \widehat{b}}{\varepsilon} = 0 \\ \partial_t \widehat{b} + \varepsilon^\alpha |\xi_h|^2 \widehat{b} + \frac{i\xi_3 \widehat{u}}{\varepsilon} = 0 \\ \widehat{u}|_{t=0} = \widehat{u}_0 \\ \widehat{b}|_{t=0} = \widehat{b}_0. \end{array} \right.$$

where

$$A = \begin{bmatrix} 0 & -\frac{1}{\varepsilon} & 0 \\ \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let  $\bar{U} = \begin{pmatrix} \bar{u} \\ \bar{b} \end{pmatrix}$ . We can write the obtained system in the following form

$$\partial_t \widehat{U} = \mathbb{B}(\xi, \varepsilon) \widehat{U}$$

where

$$\mathbb{B}(\xi, \varepsilon) = \begin{bmatrix} -\varepsilon^\alpha |\xi_h|^2 + \frac{\xi_1 \xi_2}{\varepsilon |\xi|^2} & \frac{\xi_2^2 + \xi_3^2}{\varepsilon |\xi|^2} & 0 & -\frac{i\xi_3}{\varepsilon} & 0 & 0 \\ -\frac{\xi_1^2 + \xi_3^2}{\varepsilon |\xi|^2} & -\varepsilon^\alpha |\xi_h|^2 - \frac{\xi_1 \xi_2}{\varepsilon |\xi|^2} & 0 & 0 & -\frac{i\xi_3}{\varepsilon} & 0 \\ \frac{\xi_2 \xi_3}{\varepsilon |\xi|^2} & -\frac{\xi_1 \xi_3}{\varepsilon |\xi|^2} & -\varepsilon^\alpha |\xi_h|^2 & 0 & 0 & -\frac{i\xi_3}{\varepsilon} \\ -\frac{i\xi_3}{\varepsilon} & 0 & 0 & -\varepsilon^\alpha |\xi_h|^2 & 0 & 0 \\ 0 & -\frac{i\xi_3}{\varepsilon} & 0 & 0 & -\varepsilon^\alpha |\xi_h|^2 & 0 \\ 0 & 0 & -\frac{i\xi_3}{\varepsilon} & 0 & 0 & -\varepsilon^\alpha |\xi_h|^2 \end{bmatrix}.$$

In what follows, we study the spectral properties of the matrix  $\mathbb{B}(\xi, \varepsilon)$ .

### Eigenvalues:

The characteristic polynomial of  $\mathbb{B}(\xi, \varepsilon)$  is  $P\left(\left(X + \varepsilon^\alpha |\xi_h|^2\right)^2\right)$ , where

$$\begin{aligned} P(Y) &= Y^3 + \frac{\xi_3^2}{\varepsilon^2} \left( \frac{1}{|\xi|^2} + 3 \right) Y^2 + \frac{\xi_3^4}{\varepsilon^4} \left( \frac{1}{|\xi|^2} + 3 \right) Y + \frac{\xi_3^6}{\varepsilon^6} \\ &= \left( Y + \frac{\xi_3^2}{\varepsilon^2} \right) \left[ Y^2 + 2 \frac{\xi_3^2}{\varepsilon^2} \left( 1 + \frac{1}{2|\xi|^2} \right) Y + \frac{\xi_3^4}{\varepsilon^4} \right]. \end{aligned}$$

The roots of  $P(Y)$  are

$$Y = -\frac{\xi_3^2}{\varepsilon^2} \quad \text{and} \quad Y = -\frac{\xi_3^2}{\varepsilon^2} \left(1 + \frac{1}{2|\xi|^2}\right) \pm \frac{\xi_3^2 \sqrt{4|\xi|^2 + 1}}{2\varepsilon^2 |\xi|^2}.$$

Thus, the eigenvalues of the matrix  $\mathbb{B}(\xi, \varepsilon)$  are

$$\begin{aligned} \lambda_{1,2} &= -\varepsilon^\alpha |\xi_h|^2 \pm \frac{i\xi_3}{\varepsilon} \\ \lambda_{3,4} &= -\varepsilon^\alpha |\xi_h|^2 \pm \frac{i\xi_3}{\varepsilon} A \\ \lambda_{5,6} &= -\varepsilon^\alpha |\xi_h|^2 \pm \frac{i\xi_3}{\varepsilon} B \end{aligned}$$

where

$$(3.3) \quad A = \frac{1 + \sqrt{4|\xi|^2 + 1}}{2|\xi|} \quad \text{and} \quad B = \frac{-1 + \sqrt{4|\xi|^2 + 1}}{2|\xi|}.$$

### Eigenvectors:

- Associated to  $\lambda_1 = -\varepsilon^\alpha |\xi_h|^2 + \frac{i\xi_3}{\varepsilon}$ ,

$$W_1 = \begin{pmatrix} W_1^1 \\ W_1^2 \\ W_1^3 \\ W_1^4 \\ W_1^5 \\ W_1^6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

- Associated to  $\lambda_2 = -\varepsilon^\alpha |\xi_h|^2 - \frac{i\xi_3}{\varepsilon}$ ,

$$W_2 = \begin{pmatrix} W_2^1 \\ W_2^2 \\ W_2^3 \\ W_2^4 \\ W_2^5 \\ W_2^6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- Associated to  $\lambda_3 = -\varepsilon^\alpha |\xi_h|^2 + \frac{i\xi_3 A}{\varepsilon}$ ,

$$W_3 = \begin{pmatrix} W_3^1 \\ W_3^2 \\ W_3^3 \\ W_3^4 \\ W_3^5 \\ W_3^6 \end{pmatrix} = \begin{pmatrix} iA\xi_3 |\xi| + A\xi_1 \xi_2 \\ -A\xi_1^2 - A\xi_3^2 \\ -iA\xi_1 |\xi| + A\xi_2 \xi_3 \\ -i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}.$$

- Associated to  $\lambda_4 = -\varepsilon^\alpha |\xi_h|^2 - \frac{i\xi_3 A}{\varepsilon}$ ,

$$W_4 = \begin{pmatrix} W_4^1 \\ W_4^2 \\ W_4^3 \\ W_4^4 \\ W_4^5 \\ W_4^6 \end{pmatrix} = \begin{pmatrix} iA\xi_3 |\xi| - A\xi_1 \xi_2 \\ A\xi_1^2 + A\xi_3^2 \\ -iA\xi_1 |\xi| - A\xi_2 \xi_3 \\ i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ -i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}.$$

- Associated to  $\lambda_5 = -\varepsilon^\alpha |\xi_h|^2 + \frac{i\xi_3 B}{\varepsilon}$ ,

$$W_5 = \begin{pmatrix} W_5^1 \\ W_5^2 \\ W_5^3 \\ W_5^4 \\ W_5^5 \\ W_5^6 \end{pmatrix} = \begin{pmatrix} -iB\xi_3 |\xi| + B\xi_1 \xi_2 \\ -B\xi_1^2 - B\xi_3^2 \\ iB\xi_1 |\xi| + B\xi_2 \xi_3 \\ i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ -i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}.$$

- Associated to  $\lambda_6 = -\varepsilon^\alpha |\xi_h|^2 - \frac{i\xi_3 B}{\varepsilon}$ ,

$$W_6 = \begin{pmatrix} W_6^1 \\ W_6^2 \\ W_6^3 \\ W_6^4 \\ W_6^5 \\ W_6^6 \end{pmatrix} = \begin{pmatrix} -iB\xi_3 |\xi| - B\xi_1 \xi_2 \\ B\xi_1^2 + B\xi_3^2 \\ iB\xi_1 |\xi| - B\xi_2 \xi_3 \\ -i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 \\ i\xi_1 |\xi| - \xi_2 \xi_3 \end{pmatrix}.$$

It is easy to see that, in the domain  $\mathcal{C}_{r,R}$  (see (3.1)),  $W_3$ ,  $W_4$ ,  $W_5$  and  $W_6$  are orthogonal to the vectors

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

and  $W_1$  and  $W_2$  are not. Then, the divergence free property of  $u$  and  $b$  implies that

$$(3.4) \quad \widehat{U}(t) = \sum_{i=3}^6 C_i e^{\lambda_i t} W_i,$$

with

$$(3.5) \quad \widehat{U}(0) = C_3 W_3 + C_4 W_4 + C_5 W_5 + C_6 W_6,$$

where  $C_i$ ,  $i = 3, \dots, 6$  depend on  $\xi$  and  $\bar{U}_0$ . Using the divergence free property of  $\bar{U}_0$ , we can rewrite (3.5) as:

$$\begin{bmatrix} iA\xi_3 |\xi| + A\xi_1 \xi_2 & iA\xi_3 |\xi| - A\xi_1 \xi_2 & -iB\xi_3 |\xi| + B\xi_1 \xi_2 & -iB\xi_3 |\xi| - B\xi_1 \xi_2 \\ -A(\xi_1^2 + \xi_3^2) & A(\xi_1^2 + \xi_3^2) & -B(\xi_1^2 + \xi_3^2) & B(\xi_1^2 + \xi_3^2) \\ -i\xi_3 |\xi| - \xi_1 \xi_2 & i\xi_3 |\xi| - \xi_1 \xi_2 & i\xi_3 |\xi| - \xi_1 \xi_2 & -i\xi_3 |\xi| - \xi_1 \xi_2 \\ \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 \end{bmatrix} \begin{pmatrix} C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = \begin{pmatrix} \widehat{U}_1(0) \\ \widehat{U}_2(0) \\ \widehat{U}_4(0) \\ \widehat{U}_5(0) \end{pmatrix}.$$

Let

$$D = \begin{bmatrix} iA\xi_3 |\xi| + A\xi_1\xi_2 & iA\xi_3 |\xi| - A\xi_1\xi_2 & -iB\xi_3 |\xi| + B\xi_1\xi_2 & -iB\xi_3 |\xi| - B\xi_1\xi_2 \\ -A(\xi_1^2 + \xi_3^2) & A(\xi_1^2 + \xi_3^2) & -B(\xi_1^2 + \xi_3^2) & B(\xi_1^2 + \xi_3^2) \\ -i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & -i\xi_3 |\xi| - \xi_1\xi_2 \\ \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} \widehat{U}_1(0) & iA\xi_3 |\xi| - A\xi_1\xi_2 & -iB\xi_3 |\xi| + B\xi_1\xi_2 & -iB\xi_3 |\xi| - B\xi_1\xi_2 \\ \widehat{U}_2(0) & A(\xi_1^2 + \xi_3^2) & -B(\xi_1^2 + \xi_3^2) & B(\xi_1^2 + \xi_3^2) \\ \widehat{U}_4(0) & i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & -i\xi_3 |\xi| - \xi_1\xi_2 \\ \widehat{U}_5(0) & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 \end{bmatrix},$$

$$D_4 = \begin{bmatrix} iA\xi_3 |\xi| + A\xi_1\xi_2 & \widehat{U}_1(0) & -iB\xi_3 |\xi| + B\xi_1\xi_2 & -iB\xi_3 |\xi| - B\xi_1\xi_2 \\ -A(\xi_1^2 + \xi_3^2) & \widehat{U}_2(0) & -B(\xi_1^2 + \xi_3^2) & B(\xi_1^2 + \xi_3^2) \\ -i\xi_3 |\xi| - \xi_1\xi_2 & \widehat{U}_4(0) & i\xi_3 |\xi| - \xi_1\xi_2 & -i\xi_3 |\xi| - \xi_1\xi_2 \\ \xi_1^2 + \xi_3^2 & \widehat{U}_5(0) & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 \end{bmatrix},$$

$$D_5 = \begin{bmatrix} iA\xi_3 |\xi| + A\xi_1\xi_2 & iA\xi_3 |\xi| - A\xi_1\xi_2 & \widehat{U}_1(0) & -iB\xi_3 |\xi| - B\xi_1\xi_2 \\ -A(\xi_1^2 + \xi_3^2) & A(\xi_1^2 + \xi_3^2) & \widehat{U}_2(0) & B(\xi_1^2 + \xi_3^2) \\ -i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & \widehat{U}_4(0) & -i\xi_3 |\xi| - \xi_1\xi_2 \\ \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \widehat{U}_5(0) & \xi_1^2 + \xi_3^2 \end{bmatrix},$$

and

$$D_6 = \begin{bmatrix} iA\xi_3 |\xi| + A\xi_1\xi_2 & iA\xi_3 |\xi| - A\xi_1\xi_2 & -iB\xi_3 |\xi| + B\xi_1\xi_2 & \widehat{U}_1(0) \\ -A(\xi_1^2 + \xi_3^2) & A(\xi_1^2 + \xi_3^2) & -B(\xi_1^2 + \xi_3^2) & \widehat{U}_2(0) \\ -i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & i\xi_3 |\xi| - \xi_1\xi_2 & \widehat{U}_4(0) \\ \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \xi_1^2 + \xi_3^2 & \widehat{U}_5(0) \end{bmatrix}.$$

Then, we have

$$|\det(D)| = 4\xi_3^2 (\xi_1^2 + \xi_3^2)^2 (4|\xi|^2 + 1),$$

which is strictly positive in the domain  $\mathcal{C}_{r,R}$ , and Cramer's rule gives

$$C_i = \frac{\det(D_i)}{\det(D)}, \quad \text{for any } i = 3, 4, 5, 6.$$

Replacing  $A$  and  $B$  by their values (see (3.3)), we easily get the following estimates

$$|\det(D_i)| \leq C|\widehat{U}(0, \xi)| (\xi_1^2 + \xi_3^2) |\xi|^4.$$

Thus,

$$(3.6) \quad |C_i| = \left| \frac{\det(D_i)}{\det(D)} \right| \leq \frac{C|\widehat{U}(0, \xi)| |\xi|^4}{\xi_3^2 (\xi_1^2 + \xi_3^2) (4|\xi|^2 + 1)} \leq \frac{C|\widehat{U}(0, \xi)| |\xi|^2}{4\xi_3^2 (\xi_1^2 + \xi_3^2)},$$

where  $C$  is a generic positive constant which can change from line to line. By the definition of  $C_i$ , it is also easy to prove that  $C_i = 0$  if  $\xi \notin \mathcal{C}_{r,R}$  (since the support of  $\widehat{U}(0)$  is included in  $\mathcal{C}_{r,R}$ ).

Next, for  $i \in \{3, 4, 5, 6\}$ , we estimate  $C_i e^{\lambda_i t} W_i$ , using the method of duality as in [9] (see also [24] and [25]). That leads us to consider the following operators

$$\begin{aligned} \mathcal{G}_{\pm}^{A,\varepsilon}(t)f(x) &= \mathcal{F}^{-1} \left( e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{A\xi_3}{\varepsilon})} \widehat{f}(\xi) \right) (x) = \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} f(y) e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{A\xi_3}{\varepsilon}) + i(x-y) \cdot \xi} d\xi dy \\ &= \int_{\mathbb{R}_{y_h}^2} \mathcal{F}_{\xi_3}^{-1} \left( \int_{\mathbb{R}_{\xi_h}^2} e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{A\xi_3}{\varepsilon}) + i(x_h - y_h) \xi_h} \mathcal{F}_{x_3}(f)(y_h, \xi_3) d\xi_h \right) dy_h, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_{\pm}^{B,\varepsilon}(t)f(x) &= \mathcal{F}^{-1} \left( e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{B\xi_3}{\varepsilon})} \widehat{f}(\xi) \right) (x) = \int_{\mathbb{R}_y^3 \times \mathbb{R}_\xi^3} f(y) e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{B\xi_3}{\varepsilon}) + i(x-y) \cdot \xi} d\xi dy \\ &= \int_{\mathbb{R}_{y_h}^2} \mathcal{F}_{\xi_3}^{-1} \left( \int_{\mathbb{R}_{\xi_h}^2} e^{-t(\varepsilon^\alpha |\xi_h|^2 \pm i \frac{B\xi_3}{\varepsilon}) + i(x_h - y_h) \xi_h} \mathcal{F}_{x_3}(f)(y_h, \xi_3) d\xi_h \right) dy_h, \end{aligned}$$

where  $f$  is a tempered distribution.

**Theorem 3.1.** *Let  $R$  be large enough,  $r = R^{-\beta}$ ,  $\beta \geq 1$ ,  $p \geq 1$  and  $\psi$  be the cut-off function defined in (3.2). For any  $f \in \mathbf{L}^2(\mathbb{R}^3)$ , we have*

$$(3.7) \quad \left\| \psi(D) \mathcal{G}_{\pm}^{A,\varepsilon}(t)f \right\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^\infty \mathbf{L}_v^2)} \leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|f\|_{\mathbf{L}^2}.$$

and

$$(3.8) \quad \left\| \psi(D) \mathcal{G}_{\pm}^{B,\varepsilon}(t)f \right\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^\infty \mathbf{L}_v^2)} \leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|f\|_{\mathbf{L}^2}.$$

In order to prove this theorem, we introduce the kernel functions:

$$K_{\pm}^A(\theta, \tau, z_h, \xi_3) = \int_{\mathbb{R}_{\xi_h}^2} \psi(\xi) e^{\mp i\theta a(\xi) + iz_h \cdot \xi_h - \tau |\xi_h|^2} d\xi_h,$$

and

$$K_{\pm}^B(\theta, \tau, z_h, \xi_3) = \int_{\mathbb{R}_{\xi_h}^2} \psi(\xi) e^{\mp i\theta b(\xi) + iz_h \cdot \xi_h - \tau |\xi_h|^2} d\xi_h,$$

where

$$a(\xi) = A\xi_3 \quad \text{and} \quad b(\xi) = B\xi_3.$$

Let

$$\gamma_A(\xi) = -\partial_{\xi_2} a(\xi) = \xi_2 \xi_3 \frac{1 + \sqrt{4|\xi|^2 + 1}}{2|\xi|^3 \sqrt{4|\xi|^2 + 1}},$$

and

$$\gamma_B(\xi) = -\partial_{\xi_2} b(\xi) = \xi_2 \xi_3 \frac{1 - \sqrt{4|\xi|^2 + 1}}{2|\xi|^3 \sqrt{4|\xi|^2 + 1}}.$$

In the domain  $\mathcal{C}_{r,R}$ , with  $r = R^{-\beta}$ ,  $\beta \geq 1$  and  $R \gg 1$ , it is easy to prove the following estimates

$$(3.9) \quad 2R^{-3-\beta} |\xi_2| \leq |\gamma_A(\xi)| \leq R^\beta,$$

and

$$(3.10) \quad \frac{1}{24} R^{-3-\beta} |\xi_2| \leq |\gamma_B(\xi)| \leq R^\beta.$$



Similarly, since

$$\partial_{\xi_2} \gamma_A(\xi) = \frac{\gamma_A(\xi)}{\xi_2} - \xi_2^2 \xi_3 \left[ \frac{16|\xi|^2 + 3}{2|\xi|^5 (4|\xi|^2 + 1)^{\frac{3}{2}}} + \frac{3}{2|\xi|^5} \right],$$

and

$$\partial_{\xi_2} \gamma_B(\xi) = \frac{\gamma_B(\xi)}{\xi_2} - \xi_2^2 \xi_3 \left[ \frac{16|\xi|^2 + 3}{2|\xi|^5 (4|\xi|^2 + 1)^{\frac{3}{2}}} - \frac{3}{2|\xi|^5} \right],$$

we deduce that

$$(3.11) \quad |\partial_{\xi_2} \gamma_A(\xi)| \leq \frac{R^\beta}{r} + 8 + \frac{3}{2r^2} + \frac{3}{2r^2} \leq 5R^{2\beta},$$

and

$$(3.12) \quad |\partial_{\xi_2} \gamma_B(\xi)| \leq \frac{R^\beta}{r} + 8 + \frac{3}{2r^2} + \frac{3}{2r^2} \leq 5R^{2\beta}.$$

These estimates allow us to prove the following lemma:

**Lemma 3.2.** *Let  $R$  be large enough,  $r = R^{-\beta}$ ,  $\beta \geq 1$  and  $p \geq 1$ . Then,*

$$(3.13) \quad \|K_{\pm}^A(\theta, \tau, \cdot, \cdot)\|_{\mathbf{L}_{x_h}^\infty \mathbf{L}_{\xi_3}^\infty} \leq CR^{4+3\beta} \theta^{-\frac{1}{2}} e^{-\frac{1}{2}r^2\tau},$$

and

$$(3.14) \quad \|K_{\pm}^B(\theta, \tau, \cdot, \cdot)\|_{\mathbf{L}_{x_h}^\infty \mathbf{L}_{\xi_3}^\infty} \leq CR^{4+3\beta} \theta^{-\frac{1}{2}} e^{-\frac{1}{2}r^2\tau},$$

**Proof:**

We remark that using the invariance by rotation in  $(\xi_1, \xi_2)$ , one can suppose that  $z_2 = 0$ . We introduce the following operators

$$\mathcal{L}_A = \frac{1}{1 + \theta\gamma_A^2} (1 + i\gamma_A \partial_{\xi_2}),$$

and

$$\mathcal{L}_B = \frac{1}{1 + \theta\gamma_B^2} (1 + i\gamma_B \partial_{\xi_2}),$$

which act on the  $\xi_2$  variable and satisfy  $\mathcal{L}_A(e^{\pm i\theta a}) = e^{\pm i\theta a}$  and  $\mathcal{L}_B(e^{\pm i\theta b}) = e^{\pm i\theta b}$ . Then, we can write

$$K_{\pm}^A(\theta, \tau, z_h, \xi_3) = \int_{\mathbb{R}_{\xi_h}^2} e^{\mp i\theta a(\xi) + iz_1 \cdot \xi_1} {}^t \mathcal{L}_A \left( \psi(\xi) e^{-\tau|\xi_h|^2} \right) d\xi_h,$$

and

$$K_{\pm}^B(\theta, \tau, z_h, \xi_3) = \int_{\mathbb{R}_{\xi_h}^2} e^{\mp i\theta b(\xi) + iz_1 \cdot \xi_1} {}^t \mathcal{L}_B \left( \psi(\xi) e^{-\tau|\xi_h|^2} \right) d\xi_h.$$

Simple calculations show that

$${}^t \mathcal{L}_A = \frac{1}{1 + \theta\gamma_A^2} - i(\partial_{\xi_2} \gamma_A) \frac{1 - \theta\gamma_A^2}{(1 + \theta\gamma_A^2)^2} - \frac{i\gamma_A}{1 + \theta\gamma_A^2} \partial_{\xi_2},$$

and

$${}^t \mathcal{L}_B = \frac{1}{1 + \theta\gamma_B^2} - i(\partial_{\xi_2} \gamma_B) \frac{1 - \theta\gamma_B^2}{(1 + \theta\gamma_B^2)^2} - \frac{i\gamma_B}{1 + \theta\gamma_B^2} \partial_{\xi_2}.$$

Thus, Inequalities (3.9) to (3.12) lead us to the estimates

$$\|K_{\pm}^A(\theta, \tau, \dots)\|_{\mathbf{L}_{x_h}^{\infty} \mathbf{L}_{\xi_3}^{\infty}} \leq CR^{4+3\beta} e^{-\frac{1}{2}r^2\tau} \int_{\mathbb{R}} \frac{d\xi_2}{1 + \theta\xi_2^2},$$

and

$$\|K_{\pm}^B(\theta, \tau, \dots)\|_{\mathbf{L}_{x_h}^{\infty} \mathbf{L}_{\xi_3}^{\infty}} \leq CR^{4+3\beta} e^{-\frac{1}{2}r^2\tau} \int_{\mathbb{R}} \frac{d\xi_2}{1 + \theta\xi_2^2},$$

which imply (3.13) and (3.14).  $\blacksquare$

**Proof of Theorem 3.1:** We can directly deduce Theorem 3.1 from Lemma 3.2, by using the duality argument, called the  $TT^*$  method (see [[8], Theorem 3], [[9], Theorem 5], [[24], Theorem 3.1] or [[25], Chapter 3, Theorem 3.3.1] for a complete proof).  $\blacksquare$

From Theorem 3.1, we can deduce the following Strichartz-type estimates for the solution of the linear system.

**Theorem 3.3.** *We keep  $R$ ,  $r$ , and  $\beta$  as given in Theorem 3.1. Let  $\bar{U}$  be the (global) solution of (1.8), where  $\widehat{\bar{U}}(0)$  has its support included in  $\mathcal{C}_{r,R}$ . Then, for any  $p \geq 1$ ,*

$$(3.15) \quad \|\bar{U}\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^{\infty} \mathbf{L}_v^2)} \leq CR^{\frac{1+5p+(3+5p)\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|\bar{U}(0)\|_{\mathbf{L}^2}.$$

**Proof:**

Let us go back to (3.5). Using Theorem 3.1, Plancherel's theorem and the Hölder inequality and the fact that  $C_i = 0$  outside the domain  $\mathcal{C}_{r,R}$ , for any  $i \in \{3, 4, 5, 6\}$ , we get

$$(3.16) \quad \begin{aligned} \|\bar{U}\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^{\infty} \mathbf{L}_v^2)} &\leq \sum_{i=3}^6 \left\| \mathcal{F}^{-1} \left( C_i e^{\lambda_i t} W_i \right) \right\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^{\infty} \mathbf{L}_v^2)} \\ &\leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \sum_{i=3}^6 \|C_i W_i\|_{\mathbf{L}^2(\mathcal{C}_{r,R})} \\ &\leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \sum_{i=3}^6 \|C_i\|_{\mathbf{L}^2(\mathcal{C}_{r,R})} \|W_i\|_{\mathbf{L}^{\infty}(\mathcal{C}_{r,R})}. \end{aligned}$$

Now, we remark that in the domain  $\mathcal{C}_{r,R}$ , with  $r = R^{-\beta}$ ,  $\beta \geq 1$ ,  $R \gg 1$ , it is easy to prove that  $A, B \leq \frac{2}{r} = 2R^{\beta}$ . So,

$$\|W_i\|_{\mathbf{L}^{\infty}(\mathcal{C}_{r,R})} \leq CR^{2+\beta}.$$

Using (3.6) and Hölder inequality, we deduce from Inequality (3.16) that

$$\begin{aligned} \|\bar{U}\|_{\mathbf{L}^p(\mathbb{R}_+, \mathbf{L}_h^{\infty} \mathbf{L}_v^2)} &\leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|\bar{U}(0)\|_{\mathbf{L}^2} \left\| \frac{|\xi|^2}{\xi_3^2 (\xi_1^2 + \xi_3^2)} \right\|_{\mathbf{L}^{\infty}(\mathcal{C}_{r,R})} \|W_i\|_{\mathbf{L}^{\infty}(\mathcal{C}_{r,R})} \\ &\leq CR^{\frac{1+p+3\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|\bar{U}(0)\|_{\mathbf{L}^2} R^{4+5\beta} \\ &\leq CR^{\frac{1+5p+(3+5p)\beta}{p}} \varepsilon^{\frac{1-3\alpha}{4p}} \|\bar{U}(0)\|_{\mathbf{L}^2}, \end{aligned}$$

where  $C$  is a generic positive constant which can change from line to line. Theorem 3.3 is then proved.  $\blacksquare$



and then summing the two obtained quantities, we obtain

$$\begin{aligned}
(3.18) \quad & \frac{1}{2} \|\Delta_q^v \mathcal{V}(T)\|_{\mathbf{L}^2}^2 + \varepsilon^\alpha \int_0^T \|\Delta_q^v \nabla_h \mathcal{V}\|_{\mathbf{L}^2}^2 dt - \frac{1}{2} \|\Delta_q^v \mathcal{V}_0\|_{\mathbf{L}^2}^2 \\
& \leq \int_0^T |\langle \Delta_q^v(\omega \cdot \nabla \omega) | \Delta_q^v \omega \rangle| dt + \int_0^T |\langle \Delta_q^v(\bar{u} \cdot \nabla \omega) | \Delta_q^v \omega \rangle| dt \\
& + \int_0^T |\langle \Delta_q^v(\omega \cdot \nabla \bar{u}) | \Delta_q^v \omega \rangle| dt + \int_0^T |\langle \Delta_q^v(\bar{u} \cdot \nabla \bar{u}) | \Delta_q^v \omega \rangle| dt + \int_0^T |\langle \Delta_q^v(\beta \cdot \nabla \bar{b}) | \Delta_q^v \omega \rangle| dt \\
& + \int_0^T |\langle \Delta_q^v(\bar{b} \cdot \nabla \bar{b}) | \Delta_q^v \omega \rangle| dt + \int_0^T |\langle \Delta_q^v(\bar{b} \cdot \nabla \beta) | \Delta_q^v \omega \rangle + \langle \Delta_q^v(\bar{b} \cdot \nabla \omega) | \Delta_q^v \beta \rangle| dt \\
& + \int_0^T |\langle \Delta_q^v(\beta \cdot \nabla \omega) | \Delta_q^v \beta \rangle + \langle \Delta_q^v(\beta \cdot \nabla \beta) | \Delta_q^v \omega \rangle| dt + \int_0^T |\langle \Delta_q^v(\omega \cdot \nabla \beta) | \Delta_q^v \beta \rangle| dt \\
& + \int_0^T |\langle \Delta_q^v(\bar{u} \cdot \nabla \beta) | \Delta_q^v \beta \rangle| dt + \int_0^T |\langle \Delta_q^v(\omega \cdot \nabla \bar{b}) | \Delta_q^v \beta \rangle| dt + \int_0^T |\langle \Delta_q^v(\beta \cdot \nabla \bar{u}) | \Delta_q^v \beta \rangle| dt \\
& + \int_0^T |\langle \Delta_q^v(\bar{b} \cdot \nabla \bar{u}) | \Delta_q^v \beta \rangle| dt + \int_0^T |\langle \Delta_q^v(\bar{u} \cdot \nabla \bar{b}) | \Delta_q^v \beta \rangle| dt.
\end{aligned}$$

In order to estimate the terms on the right-hand side of (3.18), we use the following inequalities.

**Lemma 3.5.** *Let  $s > \frac{1}{2}$  and  $\bar{U}$  be the solution of (1.8). Then, for any divergence-free vector fields  $V$  and  $W$ , belonging to  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})$ , the horizontal gradients of which belong to  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0,s})$ , there exists a positive constant  $C$  and a summable sequence of positive numbers  $d_q$  such that*

$$\begin{aligned}
(3.19) \quad & \int_0^T |\langle \Delta_q^v(W \cdot \nabla \bar{U}) | \Delta_q^v V \rangle| dt \\
& \leq C d_q 2^{-2qs} R^{7+8\beta+s} \varepsilon^{\frac{1-3\alpha}{4}} \|V\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})} \|W\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})}
\end{aligned}$$

and

$$\begin{aligned}
(3.20) \quad & \int_0^T |\langle \Delta_q^v(\bar{U} \cdot \nabla V) | \Delta_q^v V \rangle| dt \leq d_q 2^{-2qs} \left[ C R^{11+13\beta+2s} \varepsilon^{\frac{1-7\alpha}{4}} \|V\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})}^2 \right. \\
& \left. + \frac{\varepsilon^\alpha}{16} \|\nabla_h V\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0,s})}^2 + C R^{7+8\beta+s} \varepsilon^{\frac{1-3\alpha}{4}} \|V\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})}^2 \right].
\end{aligned}$$

The proof of Lemma 3.5 in the case of rotating fluid systems can be found in [[24] Lemmas 4.4 and 4.5] (see also [[25], Chapter 3, Lemmas 3.4.4 and 3.4.5]). By using the Strichartz estimates proven in Theorem 3.3 and following the lines of the proofs given in [24] or [25], one can easily prove Lemma 3.5 in the case of MHD systems. We also use a general form of Inequality (3.20).

**Lemma 3.6.** *Let  $s > \frac{1}{2}$  and let  $\bar{U}$  be the solution of (1.8). For any divergence-free vector fields  $V$  and  $W$  belonging to  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0,s})$ , the horizontal gradients of which belong to  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0,s})$ , there exists a positive constant  $C$  and a summable sequence of positive*

numbers  $d_q$  such that

$$(3.21) \quad \int_0^T |\langle \Delta_q^v(\bar{U} \cdot \nabla V) | \Delta_q^v W \rangle + \langle \Delta_q^v(\bar{U} \cdot \nabla W) | \Delta_q^v V \rangle| dt \\ \leq d_q 2^{-2qs} \left[ C \left( R^{11+13\beta+2s} \varepsilon^{\frac{1-7\alpha}{4}} + R^{7+8\beta+s} \varepsilon^{\frac{1-3\alpha}{4}} \right) \left( \|V\|_{\tilde{\mathbf{L}}^\infty([0,T],\mathbf{H}^{0,s})}^2 + \|W\|_{\tilde{\mathbf{L}}^\infty([0,T],\mathbf{H}^{0,s})}^2 \right) \right. \\ \left. + \frac{\varepsilon^\alpha}{16} \left( \|\nabla_h V\|_{\tilde{\mathbf{L}}^2([0,T],\mathbf{H}^{0,s})}^2 + \|\nabla_h W\|_{\tilde{\mathbf{L}}^2([0,T],\mathbf{H}^{0,s})}^2 \right) \right].$$

We remark that by taking  $V = W$  in Lemma 3.6, we will obtain Inequality (3.20). The proof of Lemma 3.6 can be derived from the proof of Lemma 3.5, using the same method as in the proof of Lemma A.10 of Appendix A. The details are left for the reader.

Now, using Lemmas 3.5, 3.6, and Lemmas A.12 and A.13 of Appendix A, we obtain

$$(3.22) \quad \|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty([0,T],\mathbf{H}^{0,s})}^2 + \varepsilon^\alpha \|\nabla_h \mathcal{V}\|_{\tilde{\mathbf{L}}^2([0,T],\mathbf{H}^{0,s})}^2 \\ \leq \|\mathcal{V}_0\|_{\mathbf{H}^{0,s}}^2 + C \|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty([0,T],\mathbf{H}^{0,s})} \|\nabla_h \mathcal{V}\|_{\tilde{\mathbf{L}}^2([0,T],\mathbf{H}^{0,s})}^2 \\ + \Phi(\varepsilon) \|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty([0,T],\mathbf{H}^{0,s})}^2 + CR^{7+8\beta+s} \varepsilon^{\frac{1-3\alpha}{4}},$$

where

$$\Phi(\varepsilon) = CR^{7+8\beta+s} \varepsilon^{\frac{1-3\alpha}{4}} + CR^{11+13\beta+2s} \varepsilon^{\frac{1-7\alpha}{4}}.$$

Let

$$R = (8CC_0)^{\frac{1}{\eta_0}} \varepsilon^{-\frac{\alpha}{\eta_0}},$$

where  $C_0$  has been defined in Lemma 3.4. Then

$$\Phi(\varepsilon) = C(8CC_0)^{\frac{7+8\beta+s}{\eta_0}} \varepsilon^{\frac{1}{4}-\alpha\left(\frac{3}{4}+\frac{7+8\beta+s}{\eta_0}\right)} + C(8CC_0)^{\frac{11+13\beta+2s}{\eta_0}} \varepsilon^{\frac{1}{4}-\alpha\left(\frac{7}{4}+\frac{11+13\beta+2s}{\eta_0}\right)}.$$

If

$$(3.23) \quad \alpha \left( \frac{3}{4} + \frac{7+8\beta+s}{\eta_0} \right) < \frac{1}{4} \quad \text{and} \quad \alpha \left( \frac{7}{4} + \frac{11+13\beta+2s}{\eta_0} \right) < \frac{1}{4},$$

then there exists  $\varepsilon_1 > 0$  such that for any  $0 < \varepsilon \leq \varepsilon_1$ ,  $\Phi(\varepsilon) < \frac{1}{2}$ .

Putting  $R = (8CC_0)^{\frac{1}{\eta_0}} \varepsilon^{-\frac{\alpha}{\eta_0}}$  into Inequality (3.17), we get

$$(3.24) \quad \|\mathcal{V}_0\|_{\mathbf{H}^{0,s}} \leq \frac{\varepsilon^\alpha}{8C}.$$

Let  $T^* = \sup \left\{ T > 0 : \|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty([0,t],\mathbf{H}^{0,s})} \leq \frac{\varepsilon^\alpha}{2C}, \forall t < T \right\}$ . Since  $\mathcal{V}$  is continuous in time, Inequality (3.24) implies that  $T^* > 0$ . Using Inequalities (3.22) and (3.24), and the fact that  $\Phi(\varepsilon) < \frac{1}{2}$ , for any  $0 < \varepsilon \leq \varepsilon_1$ , we get

$$(3.25) \quad \|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty([0,T^*],\mathbf{H}^{0,s})}^2 \leq \frac{\varepsilon^{2\alpha}}{32C^2} + 2C(8CC_0)^{\frac{7+8\beta+s}{\eta_0}} \varepsilon^{\frac{1}{4}-\alpha\left(\frac{3}{4}+\frac{7+8\beta+s}{\eta_0}\right)}.$$

So, if

$$(3.26) \quad 2\alpha < \frac{1}{4} - \alpha \left( \frac{3}{4} + \frac{7+8\beta+s}{\eta_0} \right),$$

then, there exists  $\varepsilon_2 > 0$  such that, for any  $0 < \varepsilon \leq \varepsilon_2$ ,

$$2C(8CC_0)^{\frac{7+8\beta+s}{\eta_0}} \varepsilon^{\frac{1}{4}-\alpha\left(\frac{3}{4}+\frac{7+8\beta+s}{\eta_0}\right)} \leq \frac{\varepsilon^{2\alpha}}{32C^2}.$$

Let  $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ . We deduce from (3.25) that for any  $0 < \varepsilon \leq \varepsilon_0$ ,

$$\|\mathcal{V}\|_{\tilde{\mathbf{L}}^\infty((0, T^*), \mathbf{H}^{0,s})} \leq \frac{\varepsilon^\alpha}{4C} < \frac{\varepsilon^\alpha}{2C},$$

which implies that  $T^* = +\infty$ .  $\blacksquare$

## APPENDIX A. DYADIC DECOMPOSITION AND ANISOTROPIC SOBOLEV SPACES

In this appendix, for the reader's convenience, we briefly recall the dyadic decomposition in Fourier frequencies and give some elements of Littlewood-Paley theory. Then, we define the anisotropic Sobolev spaces  $\mathbf{H}^{0,s}$  and state the Bernstein inequalities, in the vertical direction and also in all three directions. Finally, we state important energy-type estimates, which are widely used in this paper.

**A.1. Dyadic decomposition.** For any  $0 < r_1 < r_2$  and  $R > 0$ , let

$$\mathcal{A}_{r_1, r_2} = \{z \in \mathbb{R} : r_1 \leq |z| \leq r_2\} \quad \text{and} \quad B(0, R) = \{z \in \mathbb{R} : |z| \leq R\}.$$

Let  $\psi$  be an even, smooth function in  $C_0^\infty(\mathbb{R})$  such that  $\psi$  is equal to 1 on a neighborhood of  $B(0, \frac{3}{4})$  and that its support is contained in  $B(0, \frac{4}{3})$ . For any  $z \in \mathbb{R}$ , let

$$\varphi(z) = \psi\left(\frac{z}{2}\right) - \psi(z).$$

Then, the support of  $\varphi$  is contained in  $\mathcal{A}_{\frac{3}{4}, \frac{8}{3}}$ , and  $\varphi$  is identically equal to 1 in  $\mathcal{A}_{\frac{4}{3}, \frac{3}{2}}$ .

Let  $\mathcal{F}$  and  $\mathcal{F}^{-1}$  be the Fourier transform and the inverse Fourier transform respectively. We also use the notation  $\widehat{u} = \mathcal{F}u$ . We introduce the following definitions of frequency cut-off operators. We set  $\xi_h = (\xi_1, \xi_2)$ .

**Definition A.1.** For any tempered distribution  $u$ , we set:

$$\begin{aligned} \Delta_q^v u &= \mathcal{F}^{-1} \left( \varphi \left( \frac{|\xi_3|}{2^q} \right) \widehat{u}(\xi) \right), & \Delta_q^h u &= \mathcal{F}^{-1} \left( \varphi \left( \frac{|\xi_h|}{2^q} \right) \widehat{u}(\xi) \right), & \forall q \in \mathbb{N}, \\ \Delta_{-1}^v u &= \mathcal{F}^{-1} (\psi(|\xi_3|) \widehat{u}(\xi)), & \Delta_{-1}^h u &= \mathcal{F}^{-1} (\psi(|\xi_h|) \widehat{u}(\xi)) \\ \Delta_q^v u &= 0, & \Delta_q^h u &= 0, & \forall q \leq -2, \\ S_q^v u &= \sum_{q' \leq q-1} \Delta_{q'}^v u, & S_q^h u &= \sum_{q' \leq q-1} \Delta_{q'}^h u, & \forall q, q \geq 1. \end{aligned}$$

We refer to [5] and [7] for more details. In terms of these operators, we can write any tempered distribution in the following form

$$u = \sum_{j, k \geq -1} \Delta_j^h \Delta_k^v u.$$

Now, we recall the anisotropic Bernstein inequalities (see [7], [19] and [26] for more details). In Section 3, for  $0 < r < R$ , we had introduced the set

$$\mathcal{C}_{r, R} = \{\xi \in \mathbb{R}^3 : |\xi_h| \geq r, |\xi_3| \geq r, |\xi| \leq R\}.$$

**Lemma A.2** (Anisotropic Bernstein Lemma). *Let  $0 < r_1 < r_2$ . There exists a positive constant  $C_0$  such that, for any  $u \in \mathcal{S}'(\mathbb{R}^3)$  for which the support of  $\mathcal{F}(u)$  is contained in the set  $\mathcal{C}_{2^q r_1, 2^q r_2}$ , we have:*

$$(A.1) \quad 2^{qk} C_0^{-k} \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}} \leq 2^{qk} C_0^k \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}}, \quad \forall k \geq 0, \quad \forall p_1, p_2 \geq 1,$$

and

$$(A.2) \quad \|u\|_{\mathbf{L}_h^{p'_1} \mathbf{L}_v^{p'_2}} \leq C_0 2^{q(\frac{2}{p'_1} - \frac{2}{p_1} + \frac{1}{p_2} - \frac{1}{p'_2})} \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}}, \quad \forall 1 \leq p_1 \leq p'_1, \quad \forall 1 \leq p_2 \leq p'_2.$$

We also need a particular case of the Bernstein Lemma in the vertical direction. Let  $\mathcal{F}_v$  be the vertical Fourier transform.

**Lemma A.3** (Vertical Bernstein Lemma). *Let  $0 < r_1 < r_2$ . There exists a positive constant  $C_0$  such that, for any  $u \in \mathcal{S}'(\mathbb{R}^3)$  for which the support of  $\mathcal{F}_v u$  is contained in  $\mathbb{R}_h^2 \times \mathcal{A}_{2^{q r_1}, 2^{q r_2}}$ , we have:*

$$(A.3) \quad 2^{qk} C_0^{-k} \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}} \leq \left\| \partial_3^k u \right\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}} \leq 2^{qk} C_0^k \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}}, \quad \forall k \geq 0, \quad \forall p_1, p_2 \geq 1,$$

and

$$(A.4) \quad \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_3}} \leq C_0 2^{q(\frac{1}{p_2} - \frac{1}{p_3})} \|u\|_{\mathbf{L}_h^{p_1} \mathbf{L}_v^{p_2}}, \quad \forall 1 \leq p_1, \quad \forall 1 \leq p_2 \leq p_3.$$

These inequalities allow us to rewrite the norm of the anisotropic Sobolev spaces  $\mathbf{H}^{\sigma, s}$  in the following equivalent way (see [19] or [9]).

**Lemma A.4.** *Suppose that  $u \in \mathbf{H}^{\sigma, s}$  with  $\sigma, s \in \mathbb{R}$ , then we have*

$$\|u\|_{\mathbf{H}^{\sigma, s}} = \left\{ \sum_{j, k \geq -1} 2^{2(j\sigma + ks)} \left\| \Delta_j^h \Delta_k^v u \right\|_{\mathbf{L}^2}^2 \right\}^{\frac{1}{2}}.$$

We also recall the following relation between each vertical dyadic block and the  $\mathbf{H}^{\sigma, s}$ -norm (see [19] or [9]).

**Lemma A.5.** *Suppose that  $u$  belongs to  $\mathbf{H}^{\sigma, s}$ , then there exists a square-summable sequence of positive numbers  $\{c_q(u, \sigma, s)\}$  with  $\sum_q c_q(u, \sigma, s)^2 = 1$  such that*

$$\left\| \Delta_q^v u \right\|_{\mathbf{L}_v^2(\mathbf{H}_h^\sigma)} \leq c_q(u, \sigma, s) 2^{-qs} \|u\|_{\mathbf{H}^{\sigma, s}}.$$

In the case where  $\sigma = 0$ , the above inequality is simply written as

$$\left\| \Delta_q^v u \right\|_{\mathbf{L}^2} \leq c_q(u, s) 2^{-qs} \|u\|_{\mathbf{H}^{0, s}}.$$

In the case where there is no possible confusion, we will write  $c_q(u)$  (or even  $c_q$ ) instead of  $c_q(u, \sigma, s)$ .

**A.2. Algebraic properties of anisotropic Sobolev spaces  $\mathbf{H}^{\sigma, s}$ .** In this paragraph, we recall the algebraic properties of anisotropic Sobolev spaces  $\mathbf{H}^{\sigma, s}$ . All the main results are contained in [29], [31] or [19]. We first recall the classical product rule, valid in the case of isotropic Sobolev spaces on  $\mathbb{R}^d$ ,  $d \geq 1$ .

**Theorem A.6.** *Let  $s, t < \frac{d}{2}$  and  $s + t > 0$ . For any  $a \in \mathbf{H}^s(\mathbb{R}^d)$  and for any  $b \in \mathbf{H}^t(\mathbb{R}^d)$ ,  $uv$  belongs to  $\mathbf{H}^{s+t-\frac{d}{2}}(\mathbb{R}^d)$  and there exists a constant  $C > 0$  such that*

$$(A.5) \quad \|ab\|_{\mathbf{H}^{s+t-\frac{d}{2}}(\mathbb{R}^d)} \leq C \|a\|_{\mathbf{H}^s(\mathbb{R}^d)} \|b\|_{\mathbf{H}^t(\mathbb{R}^d)}.$$

The following anisotropic version of this product rule was proved in [19].

**Theorem A.7.** *Let  $s, t < 1$ ,  $s + t > 0$ , and  $s', t' < \frac{1}{2}$ ,  $s' + t' > 0$ . For any  $a \in \mathbf{H}^{s, s'}(\mathbb{R}^3)$  and for any  $b \in \mathbf{H}^{t, t'}(\mathbb{R}^3)$ ,  $uv$  belongs to  $\mathbf{H}^{s+t-1, s'+t'-\frac{1}{2}}(\mathbb{R}^3)$  and there exists a constant  $C > 0$  such that*

$$(A.6) \quad \|ab\|_{\mathbf{H}^{s+t-1, s'+t'-\frac{1}{2}}(\mathbb{R}^3)} \leq C \|a\|_{\mathbf{H}^{s, s'}(\mathbb{R}^3)} \|b\|_{\mathbf{H}^{t, t'}(\mathbb{R}^3)}.$$

In Section 2.3, we need more regularity in the vertical direction. For this reason, we recall the following theorem, which is a generalization of Theorem 1.4 of [20] (see also Remark 3 of [20]). One can find a detailed proof of this theorem in [[25], Appendix A.3].

**Theorem A.8.** *Let  $\sigma, \sigma' < 1$  and  $\sigma + \sigma' > 0$ . Let  $s_0 > \frac{1}{2}$  and  $s_1 \leq s_0$  be such that  $s_0 + s_1 > 0$ . For any  $a \in \mathbf{H}^{\sigma, s_0}(\mathbb{R}^3)$  and for any  $b \in \mathbf{H}^{\sigma', s_1}(\mathbb{R}^3)$ ,  $uv$  belongs to  $\mathbf{H}^{\sigma+\sigma'-1, s_1}$  and there exists a constant  $C > 0$  such that*

$$(A.7) \quad \|ab\|_{\mathbf{H}^{\sigma+\sigma'-1, s_1}(\mathbb{R}^3)} \leq C \|a\|_{\mathbf{H}^{\sigma, s_0}(\mathbb{R}^3)} \|b\|_{\mathbf{H}^{\sigma', s_1}(\mathbb{R}^3)}.$$

**A.3. Anisotropic energy-type estimates.** In what follows,  $(c_q)$  (respectively  $(d_q)$ ) denotes a square-summable (respectively summable) sequence, with  $\sum c_q^2 = 1$  (respectively  $\sum d_q = 1$ ), of positive constants (which can depend on several parameters). We also denote  $C$  a generic positive constant which can change from line to line.

Using the dyadic decomposition, J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier have proved the following estimates (see [9]).

**Lemma A.9.** *Let  $s_0 > \frac{1}{2}$  and  $s_1 \leq s_0$  such that  $s_0 + s_1 > 0$ . There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $u \in \mathbf{H}^{0, s_0}(\mathbb{R}^3)$  and  $v \in \mathbf{H}^{0, s_1}(\mathbb{R}^3)$ , the horizontal gradients of which belong to  $\mathbf{H}^{0, s_0}(\mathbb{R}^3)$  and  $\mathbf{H}^{0, s_1}(\mathbb{R}^3)$  respectively,*

$$(A.8) \quad \left| \langle \Delta_q^v(u \cdot \nabla v) | \Delta_q^v v \rangle_{\mathbf{L}^2(\mathbb{R}^3)} \right| \leq C d_q 2^{-2qs_1} \left( \|\nabla_h u\|_{\mathbf{H}^{0, s_0}(\mathbb{R}^3)} \|v\|_{\mathbf{H}^{0, s_1}(\mathbb{R}^3)} \|\nabla_h v\|_{\mathbf{H}^{0, s_1}(\mathbb{R}^3)} \right. \\ \left. + \|u\|_{\mathbf{H}^{0, s_0}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla_h u\|_{\mathbf{H}^{0, s_0}(\mathbb{R}^3)}^{\frac{1}{2}} \|v\|_{\mathbf{H}^{0, s_1}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla_h v\|_{\mathbf{H}^{0, s_1}(\mathbb{R}^3)}^{\frac{3}{2}} \right).$$

In the case of MHD system, we need a more general result (for a proof, see [[25], Appendix A.4]).

**Lemma A.10.** *Let  $s_0 > \frac{1}{2}$  and  $s_1 \leq s_0$  such that  $s_0 + s_1 > 0$ . There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $u \in \mathbf{H}^{0, s_0}(\mathbb{R}^3)$ ,  $v, w \in \mathbf{H}^{0, s_1}(\mathbb{R}^3)$ , with  $\nabla_h u \in \mathbf{H}^{0, s_0}(\mathbb{R}^3)$  and  $\nabla_h v, \nabla_h w \in \mathbf{H}^{0, s_1}(\mathbb{R}^3)$ , we have*

$$(A.9) \quad \left| \langle \Delta_q^v(u \cdot \nabla v) | \Delta_q^v w \rangle + \langle \Delta_q^v(u \cdot \nabla w) | \Delta_q^v v \rangle \right| \\ \leq C d_q 2^{-2qs_1} \|\nabla_h u\|_{\mathbf{H}^{0, s_0}}^{\frac{1}{2}} \|\nabla_h v\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \|\nabla_h w\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \left[ \|u\|_{\mathbf{H}^{0, s_0}}^{\frac{1}{2}} \|v\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \|\nabla_h w\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} + \right. \\ \left. + \|u\|_{\mathbf{H}^{0, s_0}}^{\frac{1}{2}} \|\nabla_h v\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \|w\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} + \|\nabla_h u\|_{\mathbf{H}^{0, s_0}}^{\frac{1}{2}} \|v\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \|w\|_{\mathbf{H}^{0, s_1}}^{\frac{1}{2}} \right].$$

In order to prove the continuity in time of the strong solution, we use the following spaces.

**Definition A.11.** *Let  $p \geq 2$  and  $I \in \mathbb{R}_+$  be a time interval. The spaces  $\widetilde{\mathbf{L}}^p(I, \mathbf{H}^{0, s}(\mathbb{R}^3))$  are defined as the closure of the set of smooth vector fields with respect to the (semi-) norm*

$$\|u\|_{\widetilde{\mathbf{L}}^p(I, \mathbf{H}^{0, s})} = \left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q^v u\|_{\mathbf{L}^p(I, \mathbf{L}^2)}^2 \right)^{\frac{1}{2}}.$$



With the above defined spaces, we get the following integral version of Lemmas A.9 and A.10.

**Lemma A.12.** *Let  $s_0 > \frac{1}{2}$  and  $s_1 \leq s_0$  such that  $s_0 + s_1 > 0$ . There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $u$  and  $v$  in  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_0}(\mathbb{R}^3))$  and  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1}(\mathbb{R}^3))$  respectively, with  $\nabla_h u$  and  $\nabla_h v$  belong to  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0}(\mathbb{R}^3))$  and  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1}(\mathbb{R}^3))$  respectively, we have*

$$(A.10) \quad \int_0^T \left| \langle \Delta_q^v (u \cdot \nabla v) | \Delta_q^v v \rangle_{\mathbf{L}^2(\mathbb{R}^3)} \right| dt \\ \leq Cd_q 2^{-2qs_1} \left( \|\nabla_h u\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0})} \|v\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})} \|\nabla_h v\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})} \right. \\ \left. + \|u\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|\nabla_h u\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|v\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \|\nabla_h v\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})}^{\frac{3}{2}} \right).$$

**Lemma A.13.** *Let  $s_0 > \frac{1}{2}$  and  $s_1 \leq s_0$  such that  $s_0 + s_1 > 0$ . There exists a constant  $C > 0$  such that, for any divergence-free vector fields  $u$  in  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_0}(\mathbb{R}^3))$  and  $v, w$  in  $\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1}(\mathbb{R}^3))$ , with  $\nabla_h u$  in  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0}(\mathbb{R}^3))$  and  $\nabla_h v, \nabla_h w$  in  $\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1}(\mathbb{R}^3))$ , we have*

$$(A.11) \quad \int_0^T \left| \langle \Delta_q^v (u \cdot \nabla v) | \Delta_q^v v \rangle + \langle \Delta_q^v (u \cdot \nabla w) | \Delta_q^v v \rangle \right| dt \\ \leq Cd_q 2^{-2qs_1} \|\nabla_h u\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|\nabla_h v\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \|\nabla_h w\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \\ \times \left[ \|u\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|v\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \|\nabla_h w\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \right. \\ \left. + \|u\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|\nabla_h v\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \|w\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \right. \\ \left. + \|\nabla_h u\|_{\tilde{\mathbf{L}}^2([0, T], \mathbf{H}^{0, s_0})}^{\frac{1}{2}} \|v\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \|w\|_{\tilde{\mathbf{L}}^\infty([0, T], \mathbf{H}^{0, s_1})}^{\frac{1}{2}} \right].$$

Finally, we need the following lemma, which is weaker than Lemmas A.9 and A.10.

**Lemma A.14.** *Let  $s > \frac{1}{2}$ . For any  $R > 0$ , there exists a constant  $C(R) > 0$  such that, for any divergence-free vector fields  $u$  and  $v$  in  $\mathbf{H}^{0, s}(\mathbb{R}^3)$ , with  $\text{supp}(\hat{v}) \subset B(0, R)$ , we have*

$$(A.12) \quad \left\| \Delta_q^v (u \cdot \nabla v) \right\|_{\mathbf{L}^2} \leq C(R) c_q 2^{-qs} \|u\|_{\mathbf{H}^{0, s}} \|v\|_{\mathbf{H}^{0, s}}.$$

## REFERENCES

- [1] H. Abidi and M. Paicu, Global existence for the magnetohydrodynamic system in critical spaces, *Proceedings of the Royal Society of Edinburgh, Section A*, **138**, 2008, no. 3, p.447-476.
- [2] A. Babin, A. Mahalov, and B. Nicolaenko, Regularity and integrability of 3d Euler and Navier-Stokes equations for rotating fluids, *Asymptotic Analysis*, **15**, no. 2, 1997, p.103-150.
- [3] J. Benameur, S. Ibrahim and M. Majdoub, Asymptotic study of a magneto-hydrodynamic system, *Differential Integral Equations*, **18**, 2005, no. 3, p.299-324.
- [4] J. Benameur and R. Selmi, Study of anisotropic MHD system in anisotropic Sobolev spaces, *Annales de la Faculté des Sciences de Toulouse. Mathématiques (6)*, **17**, 2008, no. 1, p.1-22.
- [5] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l'École Normale Supérieure*, **14**, 1981, p.209-246.
- [6] F. Charve and V.-S. Ngo, Global existence for the primitive equations with small anisotropic viscosity (to appear in *Revista Matemática Iberoamericana*).
- [7] J.-Y. Chemin, Fluides parfaits incompressibles, *Astérisque*, **230**, 1995.

- [8] J.-Y. Chemin, B.Desjardins, I. Gallagher and E. Grenier, Anisotropy and dispersion in rotating fluids, *Nonlinear Partial Differential Equations and their application, Collège de France Seminar*, Studies in Mathematics and its Applications, **31**, 2002, p.171-191.
- [9] J.-Y. Chemin, B.Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, Special issue for R. Temam's 60th birthday, *M2AN. Mathematical Modelling and Numerical Analysis*, **34**, no. 2, 2000, p.315-335.
- [10] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non Lipschitziens et équations de Navier-Stokes, *Journal of Differential Equations*, **121**, 1992, p.314-328.
- [11] P. Constantin and C Foias, Navier-Stokes equations, *Chicago University Press*, 1988.
- [12] B. Cushman-Roisin, Introduction to geophysical fluid dynamics, *Prentice-Hall*, 1994.
- [13] B. Desjardins, E. Dormy and E. Grenier, Stability of mixed Ekman-Hartmann boundary layers, *Nonlinearity*, **12**, 1999, no. 2, p.181-199.
- [14] E. Dormy, P. Cardin and D. Jault, MHD flow in a slightly differentially rotating spherical shell, with conducting inner core, in a dipolar magnetic field, *Earth and Planetary Science Letters*, **160**, 1998, p.15-30.
- [15] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archiv for Rational Mechanic Analysis*, **16**, 1964, p.269-315.
- [16] J. Ginibre and G. Velo, Generalised Strichartz inequalities for wave equation, *Journal of Functional Analysis*, **133**, 1995, p.50-68.
- [17] H.P. Greenspan, The theory of rotating fluids, *Cambridge University Press*, 1968.
- [18] L. Hörmander, Linear partial differential operators, *Springer Verlag*, 1969.
- [19] D. Iftimie, The resolution of the Navier-Stokes equations in anisotropic spaces, *Revista Matemática Iberoamericana*, **15**, 1999, p.1-36.
- [20] D. Iftimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, *SIAM Journal on Mathematical Analysis*, **33**, no. 6, 2002, p.1483-1493.
- [21] M. Keel and T. Tao, Endpoint Strichartz estimates, *American Journal of Mathematics*, **121**, 1999, p.629-669.
- [22] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Mathematica*, **63**, 1933, p.193-248.
- [23] P.-L. Lions, Mathematical Topics in Fluid Mechanics, Vol. I, Incompressible Models, *Oxford Science Publications*, 1997.
- [24] V.-S. Ngo, Rotating Fluids with small viscosity, *International Mathematics Research Notices IMRN*, 2009, no. 10, p.1860-1890.
- [25] V.-S. Ngo, Effet dispersif pour les fluides anisotropes avec viscosité évanescence en rotation rapide, Ph.D. thesis, 2009, Université de Paris-Sud (<http://tel.archives-ouvertes.fr/docs/00/46/66/98/PDF/TheseVSNgo.pdf>).
- [26] M. Paicu, Équation périodique de Navier-Stokes sans viscosité dans une direction, *Communications in Partial Differential Equations*, **30**, 2005, no. 7-9, p.1107-1140.
- [27] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Matemática Iberoamericana*, **21**, 2005, no. 1, p.179-235.
- [28] J. Pedlosky, Geophysical fluid dynamics, *Springer*, 1979.
- [29] J. Rauch and M. Reed, Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, *Duke Mathematical Journal*, **49**, 1982, p.397-475.
- [30] F. Rousset, Stability of large amplitude Ekman-Hartmann boundary layers in MHD: the case of ill-prepared data, *Comm. Math. Phys.*, **259**, 2005, no. 1, p.223-256.
- [31] M. Sablé-Tougeron, Régularité microlocale pour des problèmes aux limites non linéaires, *Annales de l'Institut Fourier*, **36**, 1986, p.39-82.
- [32] R. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Mathematical Journal*, **44**, 1977, p.705-714.
- [33] R. Temam, Navier-Stokes Equations, Theory and Numerical Analysis *Amsterdam, North Holland*, 1984.

LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM, UMR 6085 CNRS, UNIVERSITÉ DE ROUEN, 76801 SAINT-ETIENNE DU ROUVRAY CEDEX, FRANCE.

*E-mail address:* van-sang.ngo@univ-rouen.fr