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Serge Nicaise, Cristina Pignotti, Julie Valein

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## EXPONENTIAL STABILITY OF THE WAVE EQUATION WITH BOUNDARY TIME-VARYING DELAY

SERGE NICAISE

Université de Valenciennes et du Hainaut Cambrésis  
LAMAV, Institut des Sciences et Techniques de Valenciennes  
59313 Valenciennes Cedex 9, France

CRISTINA PIGNOTTI

Dipartimento di Matematica Pura e Applicata  
Università di L'Aquila  
Via Vetoio, Loc. Coppito, 67010 L'Aquila, Italy

JULIE VALEIN

Institut Elie Cartan de Nancy  
Université Henri Poincaré  
B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France

**ABSTRACT.** We consider the wave equation with a time-varying delay term in the boundary condition in a bounded and smooth domain  $\Omega \subset \mathbb{R}^n$ . Under suitable assumptions, we prove exponential stability of the solution. These results are obtained by introducing suitable energies and suitable Lyapunov functionals. Such analysis is also extended to a nonlinear version of the model.

**1. Introduction.** We are interested in the effect of a time-varying delay in boundary stabilization of the wave equation in domains of  $\mathbb{R}^n$ . Delay effects arise in many practical problems and it is well known that they can induce some unstabilities, see [5, 6, 7, 23, 28].

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with a boundary  $\Gamma$  of class  $C^2$ . We assume that  $\Gamma$  is divided into two parts  $\Gamma_D$  and  $\Gamma_N$ , i.e.  $\Gamma = \Gamma_D \cup \Gamma_N$ , with  $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$ ,  $\Gamma_D \neq \emptyset$  and  $\Gamma_N \neq \emptyset$ .

In this domain  $\Omega$ , we consider the initial boundary value problem

$$u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (1)$$

$$u(x, t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \quad (2)$$

$$\frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau(t)) \quad \text{on } \Gamma_N \times (0, +\infty) \quad (3)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \quad (4)$$

$$u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)) \quad \text{in } \Gamma_N \times (0, \tau(0)), \quad (5)$$

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where  $\nu(x)$  denotes the outer unit normal vector to the point  $x \in \Gamma$  and  $\frac{\partial u}{\partial \nu}$  is the normal derivative. Moreover,  $\tau(t) > 0$  is the time-varying delay,  $\mu_1$  and  $\mu_2$  are positive real numbers and the initial datum  $(u_0, u_1, f_0)$  belongs to a suitable space.

On the function  $\tau$  we assume that there exists a positive constant  $\bar{\tau}$  such that

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0. \quad (6)$$

Moreover, we assume

$$\tau'(t) < 1 \quad \forall t > 0, \quad (7)$$

and

$$\tau \in W^{2,\infty}([0, T]), \quad \forall T > 0. \quad (8)$$

Note that, if  $t < \tau(t)$ , then  $u_t(x, t - \tau(t))$  is in the past and we need an initial value in the past. Moreover, by (7) and the mean value theorem, we have

$$\tau(t) - \tau(0) < t,$$

which implies

$$t - \tau(t) > -\tau(0).$$

We thus obtain the initial condition (5).

We are interested in giving an exponential stability result for such a problem.

Let us denote by  $\langle v, w \rangle$  or, equivalently, by  $v \cdot w$  the euclidean inner product between two vectors  $v, w \in \mathbb{R}^n$ .

We assume that there exists  $x_0 \in \mathbb{R}^n$  such that denoting by  $m$  the standard multiplier

$$m(x) := x - x_0, \quad (9)$$

we have

$$m(x) \cdot \nu(x) \leq 0 \quad \text{on } \Gamma_D \quad (10)$$

and, for some positive constant  $\delta$ ,

$$m(x) \cdot \nu(x) \geq \delta \quad \text{on } \Gamma_N. \quad (11)$$

It is well-known that if  $\mu_2 = 0$ , that is in absence of delay, the energy of problem (1) – (4) is exponentially decaying to zero. See for instance Chen [2, 3], Lagnese [16, 17], Lasiecka and Triggiani [18], Komornik and Zuazua [15], Komornik [13, 14], Zuazua [29]. On the contrary, if  $\mu_1 = 0$ , that is if we have only the delay part in the boundary condition on  $\Gamma_N$ , system (1) – (4) becomes unstable. See, for instance Datko, Lagnese and Polis [7].

The above problem, with both  $\mu_1, \mu_2 > 0$  and a constant delay  $\tau$ , has been studied in one space dimension by Xu, Yung and Li [28], on networks by Nicaise and Valein [24] and in higher space dimension by Nicaise and Pignotti [23]. Assuming that

$$\mu_2 < \mu_1, \quad (12)$$

in [23] a stabilization result in general space dimension is given, by using a suitable observability inequality obtained combining Carleman estimates for the wave equation by Lasiecka, Triggiani and Yao [19] and compactness-uniqueness arguments.

The case of time-varying delay has been studied by Nicaise, Valein and Fridman [25] in one space dimension. In [25] an exponential stability result is given, under the condition

$$\mu_2 < \sqrt{1-d} \mu_1 \quad (13)$$

where  $d$  is a constant such that

$$\tau'(t) \leq d < 1, \quad \forall t > 0. \quad (14)$$

Here, under the same conditions (13) – (14), we extend this result to general space dimension. Moreover, we remove the hypothesis

$$\tau(t) \geq \tau_0 > 0, \quad \forall t > 0, \tag{15}$$

assumed in [25], that is the delay may degenerate.

We will study also a nonlinear version of the above model. Consider the system

$$u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \tag{16}$$

$$u(x, t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \tag{17}$$

$$\frac{\partial u}{\partial \nu}(x, t) = -\beta_1(u_t(x, t)) - \beta_2(u_t(x, t - \tau(t))) \quad \text{on } \Gamma_N \times (0, +\infty) \tag{18}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \tag{19}$$

$$u_t(x, t - \tau(0)) = g_0(x, t - \tau(0)) \quad \text{in } \Gamma_N \times (0, \tau(0)), \tag{20}$$

where  $\beta_j : \mathbb{R} \rightarrow \mathbb{R}, j = 1, 2$ , satisfy suitable growth assumptions. In particular we assume

$$|\beta_j(s)| \leq c_j |s|, \quad \forall s \in \mathbb{R}, \quad j = 1, 2, \tag{21}$$

for some positive constants  $c_1, c_2$  and

$$\beta_2(s) \cdot s \geq 0, \quad \forall s \in \mathbb{R}. \tag{22}$$

From now on we fix  $c_1, c_2$  as the smallest constants such that (21) holds. Moreover we assume that  $\beta_1$  and  $\beta_2$  are continuous on  $\mathbb{R}$  and that they satisfy

$$\exists \gamma_1 > 0, \forall x, y \in \mathbb{R}, (\beta_1(x) - \beta_1(y))(x - y) \geq \gamma_1(x - y)^2, \tag{23}$$

$$\exists \gamma_2 > 0, \forall x, y \in \mathbb{R}, |\beta_2(x) - \beta_2(y)| \leq \gamma_2|x - y|. \tag{24}$$

Note that (21) and (24) imply  $c_2 \leq \gamma_2$  and from (21) and (23) we deduce

$$\beta_1(s) \cdot s \geq \gamma_1 s^2, \quad \forall s \in \mathbb{R}. \tag{25}$$

Under a suitable relation between the above coefficients we can give a well-posedness result and an exponential stability estimate for problem (16) – (20). To prove the well-posedness of the nonlinear model we need to assume (15). In our opinion, this is only a technical assumption but at the moment we are not able to remove it.

The paper is organized as follows. Well-posedness of the problems is analysed in section 2 using semigroup theory. In subsection 2.1 we study the well-posedness of problem (1) – (4), while in subsection 2.2 we concentrate on problem (16) – (20). In section 3 and section 4 we prove the exponential stability of the linear and nonlinear problems respectively.

**2. Well-posedness of the problems.** Using semigroup theory we can give the well-posedness of problem (1) – (4) and problem (16) – (20).

**2.1. Linear problem.** Let us set

$$z(x, \rho, t) = u_t(x, t - \tau(t)\rho), \quad x \in \Gamma_N, \quad \rho \in (0, 1), \quad t > 0. \tag{26}$$

Then, problem (1) – (4) is equivalent to

$$u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \tag{27}$$

$$\tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_\rho(x, \rho, t) = 0 \quad \text{in } \Gamma_N \times (0, 1) \times (0, +\infty) \tag{28}$$

$$u(x, t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \tag{29}$$

$$\frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 z(x, 1, t) \quad \text{on } \Gamma_N \times (0, +\infty) \tag{30}$$

$$z(x, 0, t) = u_t(x, t) \quad \text{on } \Gamma_N \times (0, \infty) \tag{31}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \tag{32}$$

$$z(x, \rho, 0) = f_0(x, -\rho\tau(0)) \quad \text{in } \Gamma_N \times (0, 1). \tag{33}$$

To prove the well-posedness of (27) – (33) we require some assumptions between  $\tau$ ,  $\mu_1$  and  $\mu_2$  (stated precisely below) and we have to distinguish two cases. First, we assume also (15), i.e. we assume

$$0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t > 0. \tag{34}$$

In the second case we assume only (6). but we also require that

$$\tau \in W^{3,\infty}([0, T]), \quad \forall T > 0. \tag{35}$$

2.1.1. *First case.* Assume for the moment that (34) holds.

If we denote by

$$U := (u, u_t, z)^T,$$

then

$$U' = (u_t, u_{tt}, z_t)^T = \left( u_t, \Delta u, \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \right)^T.$$

Therefore, problem (27) – (33) can be rewritten as

$$\begin{cases} U' = \mathcal{A}(t)U \\ U(0) = (u_0, u_1, f_0(\cdot, -\cdot\tau(0)))^T, \end{cases} \tag{36}$$

in the Hilbert space  $\mathcal{H}$  defined by

$$\mathcal{H} := V \times L^2(\Omega) \times L^2(\Gamma_N \times (0, 1)) \tag{37}$$

equipped with the usual inner product

$$\begin{aligned} & \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \int_{\Omega} \{ \nabla u(x) \nabla \tilde{u}(x) + v(x) \tilde{v}(x) \} dx + \int_{\Gamma_N} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma. \end{aligned} \tag{38}$$

The operator  $\mathcal{A}(t)$  is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}(t)) := & \left\{ (u, v, z)^T \in (E(\Delta, L^2(\Omega)) \cap V) \times V \times L^2(\Gamma_N; H^1(0, 1)) : \right. \\ & \left. \frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \text{ on } \Gamma_N; v = z(\cdot, 0) \text{ on } \Gamma_N \right\}, \end{aligned} \tag{39}$$

where, as usual,

$$V = H^1_{\Gamma_D}(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D \},$$

and

$$E(\Delta, L^2(\Omega)) = \{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \}.$$

Notice that the domain of the operator  $\mathcal{A}(t)$  is independent of the time  $t$ , i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)), \quad \forall t > 0. \tag{40}$$

Recall that for a function  $u \in E(\Delta, L^2(\Omega))$ , then  $\partial u / \partial \nu$  belongs to  $H^{-1/2}(\Gamma_N)$  and the next Green formula is valid (see section 1.5 of [9])

$$\int_{\Omega} \nabla u \nabla w dx = - \int_{\Omega} \Delta u w dx + \langle \frac{\partial u}{\partial \nu}; w \rangle_{\Gamma_N}, \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{41}$$

where  $\langle \cdot; \cdot \rangle_{\Gamma_N}$  means the duality pairing between  $H^{-1/2}(\Gamma_N)$  and  $H^{1/2}(\Gamma_N)$ .

Note further that for  $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ ,  $\frac{\partial u}{\partial \nu}$  belongs to  $L^2(\Gamma_N)$ , since  $z(\cdot, 1)$  is in  $L^2(\Gamma_N)$ .

A general theory for equations of type (36) has been developed using semigroup theory [11, 12, 26]. The simplest way to prove existence and uniqueness results is to show that the triplet  $\{\mathcal{A}, \mathcal{H}, Y\}$ , with  $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$  for some fixed  $T > 0$  and  $Y = \mathcal{D}(\mathcal{A}(0))$ , forms a CD-system (or constant domain system, see [11, 12]). More precisely, the following theorem gives some existence and uniqueness results and is proved in Theorem 1.9 of [11] (see also Theorem 2.13 of [12] or [1])

**Theorem 2.1.** *Assume that*

- (i)  $Y = \mathcal{D}(\mathcal{A}(0))$  is a dense subset of  $\mathcal{H}$ ,
- (ii) (40) holds,
- (iii) for all  $t \in [0, T]$ ,  $\mathcal{A}(t)$  generates a strongly continuous semigroup on  $\mathcal{H}$  and the family  $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$  is stable with stability constants  $C$  and  $m$  independent of  $t$  (i.e. the semigroup  $(S_t(s))_{s \geq 0}$  generated by  $\mathcal{A}(t)$  satisfies  $\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms}\|u\|_{\mathcal{H}}$ , for all  $u \in \mathcal{H}$  and  $s \geq 0$ ),
- (iv)  $\partial_t \mathcal{A}$  belongs to  $L^{\infty}_*([0, T], B(Y, \mathcal{H}))$ , which is the space of equivalent classes of essentially bounded, strongly measurable functions from  $[0, T]$  into the set  $B(Y, \mathcal{H})$  of bounded operators from  $Y$  into  $\mathcal{H}$ .

Then, problem (36) has a unique solution  $U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H})$  for any initial datum in  $Y$ .

Our goal is then to check the above assumptions for problem (36).

**Lemma 2.2.**  $D(\mathcal{A}(0))$  is dense in  $\mathcal{H}$ .

*Proof.* The proof is the same as the one of Lemma 2.1 of [25], we give it for the sake of completeness. Let  $(f, g, h)^T \in \mathcal{H}$  be orthogonal to all elements of  $D(\mathcal{A}(0))$ , namely

$$0 = \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_{\Omega} \{ \nabla u(x) \nabla f(x) + v(x)g(x) \} dx + \int_{\Gamma_N} \int_0^1 z(x, \rho)h(x, \rho) d\rho d\Gamma,$$

for all  $(u, v, z)^T \in D(\mathcal{A}(0))$ .

We first take  $u = 0$  and  $v = 0$  and  $z \in \mathcal{D}(\Gamma_N \times (0, 1))$ . As  $(0, 0, z)^\top \in D(\mathcal{A}(0))$ , we get

$$\int_{\Gamma_N} \int_0^1 z(x, \rho)h(x, \rho)d\rho d\Gamma = 0.$$

Since  $\mathcal{D}(\Gamma_N \times (0, 1))$  is dense in  $L^2(\Gamma_N \times (0, 1))$ , we deduce that  $h = 0$ .

In the same manner, by taking  $u = 0, z = 0$  and  $v \in \mathcal{D}(\Omega)$  we see that  $g = 0$ .

The above orthogonality condition is then reduced to

$$0 = \int_{\Omega} \nabla u \nabla f dx, \forall (u, v, z)^\top \in D(\mathcal{A}(0)).$$

By restricting ourselves to  $v = 0$  and  $z = 0$ , we obtain

$$\int_{\Omega} \nabla u(x) \nabla f(x) dx = 0, \forall (u, 0, 0)^\top \in D(\mathcal{A}(0)).$$

But we easily check that  $(u, 0, 0)^\top \in D(\mathcal{A}(0))$  if and only if  $u \in D(\Delta) = \{u \in E(\Delta, L^2(\Omega)) \cap V : \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N\}$ , the domain of the Laplace operator with mixed boundary conditions. Since it is well known that  $D(\Delta)$  is dense in  $V$  (equipped with the inner product  $\langle \cdot, \cdot \rangle_V$ ), we conclude that  $f = 0$ . □

Assuming

$$\mu_2 \leq \sqrt{1-d} \mu_1, \tag{42}$$

we will show that  $\mathcal{A}(t)$  generates a  $C_0$  semigroup on  $\mathcal{H}$  and using the variable norm technique of Kato from [11] and Theorem 2.1, that problem (36) has a unique solution.

Let  $\xi$  be a positive real number such that

$$\frac{\mu_2}{\sqrt{1-d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{43}$$

Note that, from (42), such a constant  $\xi$  exists.

Let us define on the Hilbert space  $\mathcal{H}$  the following time-dependent inner product

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t &:= \int_{\Omega} \{ \nabla u(x) \nabla \tilde{u}(x) + v(x) \tilde{v}(x) \} dx \\ &+ \xi \tau(t) \int_{\Gamma_N} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma. \end{aligned} \tag{44}$$

Using this time-dependent inner product and Theorem 2.1 we obtain the following existence and uniqueness result:

**Theorem 2.3.** *Assume (8), (14), (34), (42) and fix  $\xi$  such that (43) holds. Then for any initial datum  $U_0 \in \mathcal{D}(\mathcal{A}(0))$  there exists a unique solution*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$$

of system (36).

*Proof.* We first notice that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t, s \in [0, T], \tag{45}$$

where  $\phi = (u, v, z)^\top$  and  $c$  is a positive constant. Indeed, for all  $s, t \in [0, T]$ , we have

$$\begin{aligned} \|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right) \int_{\Omega} (|\nabla u(x)|^2 + v^2) dx \\ &\quad + \xi \left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right) \int_{\Gamma_N} \int_0^1 z(x, \rho)^2 d\rho d\Gamma. \end{aligned}$$

We notice that  $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$ . Moreover  $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$  for some  $c > 0$ . Indeed,

$$\tau(t) = \tau(s) + \tau'(a)(t - s), \quad \text{where } a \in (s, t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau'(a)|}{\tau(s)} |t - s|.$$

By (8),  $\tau'$  is bounded and therefore,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0} |t - s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

by (34), which proves (45).

Now we calculate  $\langle \mathcal{A}(t)U, U \rangle_t$  for a fixed  $t$ . Take  $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ . Then,

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \left\langle \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho-1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle \\ &= \int_{\Omega} \{\nabla v(x)\nabla u(x) + v(x)\Delta u(x)\} dx - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_\rho(x, \rho) z(x, \rho) d\rho d\Gamma. \end{aligned}$$

So, by Green's formula,

$$\langle \mathcal{A}(t)U, U \rangle_t = \int_{\Gamma_N} \frac{\partial u}{\partial \nu}(x)v(x) d\Gamma - \xi \int_{\Gamma_N} \int_0^1 (1 - \tau'(t)\rho) z_\rho(x, \rho) z(x, \rho) d\rho d\Gamma. \quad (46)$$

Integrating by parts in  $\rho$ , we get

$$\begin{aligned} \int_{\Gamma_N} \int_0^1 z_\rho(x, \rho) z(x, \rho) (1 - \tau'(t)\rho) d\rho d\Gamma &= \int_{\Gamma_N} \int_0^1 \frac{1}{2} \frac{\partial}{\partial \rho} z^2(x, \rho) (1 - \tau'(t)\rho) d\rho d\Gamma \\ &= \frac{\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x, \rho) d\rho d\Gamma + \frac{1}{2} \int_{\Gamma_N} \{z^2(x, 1)(1 - \tau'(t)) - z^2(x, 0)\} d\Gamma. \end{aligned} \quad (47)$$

Therefore, from (46) and (47),

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &= \int_{\Gamma_N} \frac{\partial u}{\partial \nu}(x)v(x) d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} \{z^2(x, 1)(1 - \tau'(t)) - z^2(x, 0)\} d\Gamma \\ &\quad - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x, \rho) d\rho d\Gamma \\ &= - \int_{\Gamma_N} (\mu_1 v(x) + \mu_2 z(x, 1))v(x) d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} \{z^2(x, 1)(1 - \tau'(t)) - z^2(x, 0)\} d\Gamma \\ &\quad - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x, \rho) d\rho d\Gamma \\ &= -\mu_1 \int_{\Gamma_N} v^2(x) d\Gamma - \mu_2 \int_{\Gamma_N} z(x, 1)v(x) d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} z^2(x, 1)(1 - \tau'(t)) d\Gamma \\ &\quad + \frac{\xi}{2} \int_{\Gamma_N} v^2(x) d\Gamma - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 z^2(x, \rho) d\rho d\Gamma, \end{aligned}$$



from which follows, using Cauchy-Schwarz's inequality and (14),

$$\begin{aligned} \langle \mathcal{A}(t)U, U \rangle_t &\leq \left( -\mu_1 + \frac{\mu_2}{2\sqrt{1-d}} + \frac{\xi}{2} \right) \int_{\Gamma_N} v^2(x) d\Gamma \\ &\quad + \left( \frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2} \right) \int_{\Gamma_N} z^2(x, 1) d\Gamma + \kappa(t) \langle U, U \rangle_t, \end{aligned} \quad (48)$$

where

$$\kappa(t) = \frac{(\tau'(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)}. \quad (49)$$

Now, observe that from (43),

$$-\mu_1 + \frac{\mu_2}{2\sqrt{1-d}} + \frac{\xi}{2} \leq 0, \quad \frac{\mu_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2} \leq 0.$$

Then,

$$\langle \mathcal{A}(t)U, U \rangle_t - \kappa(t) \langle U, U \rangle_t \leq 0, \quad (50)$$

which means that the operator  $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$  is dissipative.

Moreover  $\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2+1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)^2}$  is bounded on  $[0, T]$  for all  $T > 0$  (by (8) and (34)) and we have

$$\frac{d}{dt} \mathcal{A}(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2} z_\rho \end{pmatrix}$$

with  $\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho-1)}{\tau(t)^2}$  bounded on  $[0, T]$  by (8) and (34). Thus

$$\frac{d}{dt} \tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), \mathcal{H})), \quad (51)$$

the space of equivalence classes of essentially bounded, strongly measurable functions from  $[0, T]$  into  $B(D(\mathcal{A}(0)), \mathcal{H})$ .

Now, we will show that  $\lambda I - \mathcal{A}(t)$  is surjective for fixed  $t > 0$  and  $\lambda > 0$ . Given  $(f, g, h)^T \in \mathcal{H}$ , we seek  $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$  solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

that is verifying

$$\begin{cases} \lambda u - v = f \\ \lambda v - \Delta u = g \\ \lambda z + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \quad (52)$$

Suppose that we have found  $u$  with the appropriated regularity. Then,

$$v := \lambda u - f \in V \quad (53)$$

and we can determine  $z$ . Indeed, by (39),

$$z(x, 0) = v(x), \quad \text{for } x \in \Gamma_N, \quad (54)$$

and, from (52),

$$\lambda z(x, \rho) + \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho(x, \rho) = h(x, \rho), \quad \text{for } x \in \Gamma_N, \rho \in (0, 1). \quad (55)$$

Then, by (54) and (55), we obtain

$$z(x, \rho) = v(x)e^{-\lambda\rho\tau(t)} + \tau(t)e^{-\lambda\rho\tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda\sigma\tau(t)} d\sigma,$$

if  $\tau'(t) = 0$ , and

$$z(x, \rho) = v(x)e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} + e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x, \sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma,$$

otherwise. So, from (53),

$$z(x, \rho) = \lambda u(x)e^{-\lambda\rho\tau(t)} - f(x)e^{-\lambda\rho\tau(t)} + \tau(t)e^{-\lambda\rho\tau(t)} \int_0^\rho h(x, \sigma)e^{\lambda\sigma\tau(t)} d\sigma, \quad \text{on } \Gamma_N \times (0, 1), \tag{56}$$

if  $\tau'(t) = 0$ , and

$$z(x, \rho) = \lambda u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} - f(x)e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} + e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x, \sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma, \quad \text{on } \Gamma_N \times (0, 1) \tag{57}$$

otherwise.

In particular, if  $\tau'(t) = 0$

$$z(x, 1) = \lambda u(x)e^{-\lambda\tau(t)} + z_0(x), \quad x \in \Gamma_N, \tag{58}$$

with  $z_0 \in L^2(\Gamma_N)$  defined by

$$z_0(x) = -f(x)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)} \int_0^1 h(x, \sigma)e^{\lambda\sigma\tau(t)} d\sigma, \quad x \in \Gamma_N, \tag{59}$$

and, if  $\tau'(t) \neq 0$

$$z(x, 1) = \lambda u(x)e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0(x), \quad x \in \Gamma_N, \tag{60}$$

with  $z_0 \in L^2(\Gamma_N)$  defined by

$$z_0(x) = -f(x)e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + e^{\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \int_0^1 \frac{h(x, \sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\lambda\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma, \quad x \in \Gamma_N. \tag{61}$$

It remains to find  $u$ . By (53) and (52), the function  $u$  satisfies

$$\lambda(\lambda u - f) - \Delta u = g,$$

that is

$$\lambda^2 u - \Delta u = g + \lambda f. \tag{62}$$

Problem (62) can be reformulated as

$$\int_\Omega (\lambda^2 u - \Delta u) w dx = \int_\Omega (g + \lambda f) w dx, \quad \forall w \in H_{\Gamma_D}^1(\Omega). \tag{63}$$

Integrating by parts,

$$\begin{aligned} \int_\Omega (\lambda^2 u - \Delta u) w dx &= \int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx - \int_{\Gamma_N} \frac{\partial u}{\partial \nu} w d\Gamma \\ &= \int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx + \int_{\Gamma_N} (\mu_1 v + \mu_2 z(x, 1)) w d\Gamma. \end{aligned}$$

If  $\tau'(t) = 0$ , by (53) and (58), we have

$$\begin{aligned} \int_{\Omega} (\lambda^2 u - \Delta u) w dx &= \int_{\Omega} (\lambda^2 u w + \nabla u \nabla w) dx \\ &+ \int_{\Gamma_N} \{ \mu_1 (\lambda u - f) w + \mu_2 (\lambda u e^{-\lambda \tau(t)} + z_0) w \} d\Gamma, \end{aligned}$$

and if  $\tau'(t) \neq 0$ , by (53) and (60),

$$\begin{aligned} \int_{\Omega} (\lambda^2 u - \Delta u) w dx &= \int_{\Omega} (\lambda^2 u w + \nabla u \nabla w) dx \\ &+ \int_{\Gamma_N} \{ \mu_1 (\lambda u - f) w + \mu_2 (\lambda u e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0) w \} d\Gamma. \end{aligned}$$

Therefore, (63) can be rewritten as

$$\begin{aligned} \int_{\Omega} (\lambda^2 u w + \nabla u \nabla w) dx &+ \int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda \tau(t)}) \lambda u w d\Gamma \\ &= \int_{\Omega} (g + \lambda f) w dx + \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma, \quad \forall w \in H_{\Gamma_D}^1(\Omega), \end{aligned} \tag{64}$$

if  $\tau'(t) = 0$ , and

$$\begin{aligned} \int_{\Omega} (\lambda^2 u w + \nabla u \nabla w) dx &+ \int_{\Gamma_N} (\mu_1 + \mu_2 e^{\lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))}) \lambda u w d\Gamma \\ &= \int_{\Omega} (g + \lambda f) w dx + \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma, \quad \forall w \in H_{\Gamma_D}^1(\Omega), \end{aligned} \tag{65}$$

otherwise. As the left-hand side of (64) or (65) is coercive on  $H_{\Gamma_D}^1(\Omega)$ , the Lax-Milgram lemma guarantees the existence and uniqueness of a solution  $u \in H_{\Gamma_D}^1(\Omega)$  of (64) or (65).

If we consider  $w \in \mathcal{D}(\Omega)$  in (64) or (65),  $u$  solves in  $\mathcal{D}'(\Omega)$

$$\lambda^2 u - \Delta u = g + \lambda f, \tag{66}$$

and thus  $u \in E(\Delta, L^2(\Omega))$ .

Using Green's formula (41) in (64) and using (66), we obtain, if  $\tau'(t) = 0$

$$\int_{\Gamma_N} (\mu_1 + \mu_2 e^{-\lambda \tau(t)}) \lambda u w d\Gamma + \left\langle \frac{\partial u}{\partial \nu}; w \right\rangle_{\Gamma_N} = \mu_1 \int_{\Gamma_N} f w d\Gamma - \mu_2 \int_{\Gamma_N} z_0 w d\Gamma,$$

from which follows

$$\frac{\partial u}{\partial \nu} + (\mu_1 + \mu_2 e^{-\lambda \tau(t)}) \lambda u = \mu_1 f - \mu_2 z_0 \quad \text{on } \Gamma_N. \tag{67}$$

Therefore, from (67),

$$\frac{\partial u}{\partial \nu} = -\mu_1 v - \mu_2 z(\cdot, 1) \quad \text{on } \Gamma_N,$$

where we have used (53) and (58).

We find the same result if  $\tau'(t) \neq 0$ .

So, we have found  $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$  which verifies (52), and thus  $\lambda I - \mathcal{A}(t)$  is surjective for some  $\lambda > 0$  and  $t > 0$ . Again as  $\kappa(t) > 0$ , this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \tag{68}$$

for any  $\lambda > 0$  and  $t > 0$ .

Then, (45), (50) and (68) imply that the family  $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$  is a stable family of generators in  $\mathcal{H}$  with stability constants independent of  $t$ , by Proposition 1.1 from [11]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (40), (45), (50), (51), (68) and Lemma 2.2, and thus, the problem

$$\begin{cases} \tilde{U}' = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0 \end{cases}$$

has a unique solution  $\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{H})$  for  $U_0 \in D(\mathcal{A}(0))$ . The requested solution of (36) is then given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with  $\beta(t) = \int_0^t \kappa(s)ds$ , because

$$\begin{aligned} U'(t) &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{U}'(t) \\ &= \kappa(t)e^{\beta(t)}\tilde{U}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t) \\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t) + \tilde{\mathcal{A}}(t)\tilde{U}(t)) \\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t) = \mathcal{A}(t)e^{\beta(t)}\tilde{U}(t) \\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. □

2.1.2. *The general case.* In this subsection (6) only holds, so  $\tau$  may be also degenerate, i.e.  $\tau(t) = 0$  for some times  $t$ . Moreover we assume (35). Taking

$$\tau_\epsilon(t) = \tau(t) + \epsilon, \quad \forall 0 < \epsilon < \epsilon_0$$

then

$$0 < \epsilon \leq \tau_\epsilon(t) \leq \bar{\tau} + \epsilon, \tag{69}$$

i.e.  $\tau_\epsilon$  satisfies (34). Therefore, by Theorem 2.3, there exists a unique solution

$$U_\epsilon = (u^\epsilon, v^\epsilon, z^\epsilon)^T \in C([0, +\infty), \mathcal{D}(\mathcal{A}_\epsilon(t))) \cap C^1([0, \infty), \mathcal{H})$$

for  $U_{\epsilon,0} \in \mathcal{D}(\mathcal{A}_\epsilon(0))$ , of problem

$$\begin{cases} U'_\epsilon = \mathcal{A}_\epsilon(t)U_\epsilon \\ U_\epsilon(0) = (u_0, u_1, f_0(\cdot, - \cdot \tau_\epsilon(0)))^T = U_{\epsilon,0}, \end{cases} \tag{70}$$

where the operator  $\mathcal{A}_\epsilon(t)$  is defined by

$$\mathcal{A}_\epsilon(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'_\epsilon(t)\rho-1}{\tau_\epsilon(t)}z_\rho \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho-1}{\tau(t)+\epsilon}z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}_\epsilon(t)) = \mathcal{D}(\mathcal{A}(t)).$$

The aim is then to take the limit of  $(u_\epsilon)_{0 < \epsilon < \epsilon_0}$  when  $\epsilon$  tends to 0.

To pass at the limit, we need to have more regularity on the solution and, for that purpose, we use Theorem 2.13 of [11] (see also Theorem 3.2.3 of [1]).

We now fix  $0 < \epsilon < \epsilon_0$ . We consider the family of Hilbert spaces

$$X = X_0 = \mathcal{H}, \quad X_1 = \left( V \cap H^{3/2}(\Omega) \right) \times V \times L^2(\Gamma_N; H^1(0, 1)),$$

$$X_2 = \left( V \cap H^{3/2}(\Omega) \right) \times \left( V \cap H^{3/2}(\Omega) \right) \times L^2(\Gamma_N; H^2(0, 1)),$$

with the usual norms

$$\|\cdot\|_0 = \|\cdot\|_{\mathcal{H}}, \quad \|\cdot\|_1 = \|\cdot\|_{X_1}, \quad \|\cdot\|_2 = \|\cdot\|_{X_2}.$$

We can easily check that

$$X_2 \hookrightarrow X_1 \hookrightarrow X_0 = X$$

and

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2.$$

Let  $Y = \mathcal{D}(\mathcal{A}_\epsilon(t))$ .  $Y$  is a dense subset of  $X = X_0 = \mathcal{H}$  and a subset of  $X_1$ . Indeed, by a result of Lions and Magenes [21], if  $u \in H_{\Gamma_D}^1(\Omega)$ ,  $\Delta u \in L^2(\Omega)$  and  $\partial u / \partial \nu \in L^2(\Gamma_N)$ , then  $u \in H^{3/2}(\Omega)$ . Consequently

$$\mathcal{D}(\mathcal{A}_\epsilon(t)) \cap X_1 = \mathcal{D}(\mathcal{A}_\epsilon(t)) = Y, \quad \forall t \in [0, T].$$

The family of operators  $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(t) : t \in [0, T]\}$  is a stable family of generators in  $X = \mathcal{H}$  with stability constants independent of  $t$  (see the previous subsection).

We have that

$$\frac{d}{dt} \mathcal{A}_\epsilon(t) \in L_*^\infty([0, T], B(D(\mathcal{A}_\epsilon(0)), X)) \cap L_*^\infty([0, T], B(D(\mathcal{A}_\epsilon(0)) \cap X_2, X_1)),$$

$$\frac{d^2}{dt^2} \mathcal{A}_\epsilon(t) \in L_*^\infty([0, T], B(D(\mathcal{A}_\epsilon(0)) \cap X_2, X)),$$

because

$$\frac{d}{dt} \mathcal{A}_\epsilon(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{\tau''(t)(\tau(t)+\epsilon)\rho - \tau'(t)(\tau'(t)\rho - 1)}{(\tau(t)+\epsilon)^2} z_\rho \end{pmatrix},$$

and

$$\frac{d^2}{dt^2} \mathcal{A}_\epsilon(t)U = \begin{pmatrix} 0 \\ 0 \\ \frac{[\tau'''(\tau+\epsilon)\rho + \tau''\tau'\rho - \tau''(\tau'\rho - 1) - \tau'(\tau''\rho)](\tau+\epsilon)^2 + 2\tau'(\tau+\epsilon)[\tau''(\tau+\epsilon)\rho - \tau'(\tau'\rho - 1)]}{(\tau+\epsilon)^4} z_\rho \end{pmatrix},$$

and by (6) and (8).

Finally, again with a result of [21] and as  $\frac{\tau'(t)\rho - 1}{\tau(t) + \epsilon}$  is bounded on  $[0, T]$  by (6) and (8), if  $\phi \in \mathcal{D}(\mathcal{A}_\epsilon(t))$  and  $\mathcal{A}_\epsilon(t)\phi \in X$ , then  $\phi \in X_1$  with

$$\|\phi\|_1 \leq \nu(\|\mathcal{A}_\epsilon(t)\phi\|_0 + \|\phi\|_0),$$

and, if  $\phi \in \mathcal{D}(\mathcal{A}_\epsilon(t))$  and  $\mathcal{A}_\epsilon(t)\phi \in X_1$ , then  $\phi \in X_2$  with

$$\|\phi\|_2 \leq \nu(\|\mathcal{A}_\epsilon(t)\phi\|_1 + \|\phi\|_0).$$

Introduce now the space  $D^2(0)$  defined by

$$D^2(0) = \{\phi \in \mathcal{D}(\mathcal{A}(0)) \cap X_2 : -\mathcal{A}(0)\phi \in \mathcal{D}(\mathcal{A}(0))\}.$$

Therefore, by the result of [11] (see also [1]), for all initial data  $U_{\epsilon,0} \in D^2(0)$ , there exists a unique solution  $U_\epsilon \in C^1([0, T], \mathcal{H}) \cap C([0, T], \mathcal{D}(\mathcal{A}_\epsilon(0)))$  of (70) which satisfies, moreover,

$$\frac{d^2}{dt^2} U_\epsilon \in C([0, T], \mathcal{H}).$$

We then have more regularity of the solution with more regular initial data. Therefore, we can give a sense to the derivative of the stronger energy  $\tilde{E}_\epsilon$  defined as follows:

$$\tilde{E}_\epsilon(t) = \frac{1}{2} \int_\Omega \left( (\Delta u_\epsilon)^2 + (\nabla u_{\epsilon,t})^2 \right) dx + \frac{q\tau_\epsilon(t)}{2} \int_{\Gamma_N} \int_0^1 u_{\epsilon,tt}^2(x, t - \tau_\epsilon(t)\rho) d\rho d\Gamma, \quad (71)$$

for  $(u_0, u_1, f_0(\cdot, -\cdot \tau_\epsilon(0)))^T \in D^2(0)$ , where  $q$  is a suitable positive constant. Then the derivative of  $\tilde{E}_\epsilon$  gives

$$\begin{aligned} \tilde{E}'_\epsilon(t) &= \int_\Omega (u_{\epsilon,ttt}u_{\epsilon,tt} + \nabla u_{\epsilon,t} \nabla u_{\epsilon,tt}) dx + \frac{q\tau'_\epsilon(t)}{2} \int_{\Gamma_N} \int_0^1 u_{\epsilon,tt}^2(x, t - \tau_\epsilon(t)\rho) d\rho d\Gamma \\ &+ q\tau_\epsilon(t) \int_{\Gamma_N} \int_0^1 u_{\epsilon,tt}(x, t - \tau_\epsilon(t)\rho) u_{\epsilon,ttt}(x, t - \tau_\epsilon(t)\rho) (1 - \tau'_\epsilon(t)\rho) d\rho d\Gamma. \end{aligned}$$

By Green's formula and integrating by parts in  $\rho$ , we obtain

$$\tilde{E}'_\epsilon(t) = \int_{\Gamma_N} \frac{\partial u_{\epsilon,t}}{\partial \nu} u_{\epsilon,tt} d\Gamma - \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,tt}^2(x, t - \tau_\epsilon(t)) (1 - \tau'_\epsilon(t)) d\Gamma + \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,tt}^2(x, t) d\Gamma.$$

Since  $u_\epsilon$  satisfies (70),

$$\frac{\partial u_{\epsilon,t}}{\partial \nu} = -\mu_1 u_{\epsilon,tt}(t) - \mu_2 u_{\epsilon,tt}(t - \tau_\epsilon(t)) (1 - \tau'_\epsilon(t)),$$

and we obtain

$$\begin{aligned} \tilde{E}'_\epsilon(t) &= \left(\frac{q}{2} - \mu_1\right) \int_{\Gamma_N} u_{\epsilon,tt}^2(t) d\Gamma - \mu_2 (1 - \tau'_\epsilon(t)) \int_{\Gamma_N} u_{\epsilon,tt}(x, t - \tau_\epsilon(t)) u_{\epsilon,tt}(x, t) d\Gamma \\ &\quad - \frac{q}{2} \int_{\Gamma_N} u_{\epsilon,tt}^2(x, t - \tau_\epsilon(t)) (1 - \tau'_\epsilon(t)) d\Gamma. \end{aligned}$$

By Cauchy-Schwarz's inequality, we get, for  $\alpha > 0$ ,

$$\begin{aligned} \tilde{E}'_\epsilon(t) &\leq \left(\frac{q}{2} - \mu_1 + \frac{\alpha\mu_2(1 - \tau'_\epsilon(t))}{2}\right) \int_{\Gamma_N} u_{\epsilon,tt}^2(t) d\Gamma \\ &+ \left(\frac{\mu_2(1 - \tau'_\epsilon(t))}{2\alpha} - \frac{q(1 - \tau'_\epsilon(t))}{2}\right) \int_{\Gamma_N} u_{\epsilon,tt}^2(x, t - \tau_\epsilon(t)) d\Gamma. \end{aligned}$$

Assume now that there exists  $d < 1$  such that

$$-\frac{d}{1-d} \leq \tau'(t) \leq d < 1, \quad \forall t > 0, \tag{72}$$

and that (42) holds. If we set  $\tau'_{min} = \min_{t \in [0, T]} \tau'(t)$ , then (72) implies

$$1 - d \leq \frac{1}{1 - \tau'_{min}}. \tag{73}$$

By (42) and (73), we have

$$\mu_2 \leq \frac{1}{\sqrt{1 - \tau'_{min}}} \mu_1,$$

which implies

$$\frac{\mu_2}{\alpha} \leq 2\mu_1 + \mu_2\alpha(\tau'_{min} - 1),$$

with  $\alpha = \frac{1}{\sqrt{1 - \tau'_{min}}}$ . Consequently we can choose  $q > 0$  such that

$$\frac{\mu_2}{\alpha} \leq q \leq 2\mu_1 + \mu_2\alpha(\tau'_{min} - 1),$$

and thus

$$\tilde{E}'_\epsilon(t) \leq 0.$$

Under the assumption (72), we have

$$\tilde{E}_\epsilon(t) \leq \tilde{E}_\epsilon(0), \quad \forall t > 0,$$

i.e., for all  $0 < \epsilon < \epsilon_0$  and  $t > 0$ ,

$$\begin{aligned} & \int_{\Omega} \left( (u_{\epsilon,tt})^2 + (\nabla u_{\epsilon,t})^2 \right) dx + q\tau_{\epsilon}(t) \int_{\Gamma_N} \int_0^1 u_{\epsilon,tt}^2(x, t - \tau_{\epsilon}(t)\rho) d\rho d\Gamma \\ & \leq \int_{\Omega} \left( (\Delta u_0)^2 + (\nabla u_1)^2 \right) dx + q(\tau(0) + \epsilon) \int_{\Gamma_N} \int_0^1 f_{0,t}^2(x, -(\tau(0) + \epsilon)\rho) d\rho d\Gamma. \end{aligned} \tag{74}$$

Therefore, assuming that  $(f_{0,t}(x, -(\tau(0) + \epsilon)\rho))_{0 < \epsilon < \epsilon_0}$  is bounded on  $L^2(\Gamma_N \times (0, 1))$ , the sequence  $(u_{\epsilon})_{\epsilon}$  is bounded on  $H^1((0, T); V) \cap H^2((0, T); L^2(\Omega))$ , and thus, there exists  $u \in H^1((0, T); V) \cap H^2((0, T); L^2(\Omega))$  such that, up to a subsequence,

$$u_{\epsilon} \rightharpoonup u \quad \text{in } H^1((0, T); V) \cap H^2((0, T); L^2(\Omega)).$$

The limit  $u$  then satisfies (1) in  $\mathcal{D}'(\Omega \times (0, T))$  and (2), (4). Moreover  $u$  satisfies (3) since  $u_{\epsilon,t}|_{\Gamma_N} \rightarrow u_t|_{\Gamma_N}$  in  $L^2((0, T) \times \Gamma_N)$  and by using Lebesgue’s convergence theorem. In the same manner, we find that  $u$  verifies (5), since, by change of variable and by (74) we have

$$\int_{\Gamma_N} \int_{t-(\tau(t)+\epsilon)}^t u_{\epsilon,tt}^2(x, t) dt d\Gamma \leq C, \quad \forall t \in [0, T],$$

and thus

$$\int_{\Gamma_N} \int_{-\tau(0)}^0 u_{\epsilon,tt}^2(x, t) dt d\Gamma \leq C, \quad \forall t \in [0, T].$$

In conclusion we have proved the next existence result.

**Theorem 2.4.** *Assume (6), (14), (35), (42), (72) and let  $(f_{0,t}(x, -(\tau(0) + \epsilon)\rho))_{0 < \epsilon < \epsilon_0}$  be bounded on  $L^2(\Gamma_N \times (0, 1))$ . Then, for all initial data  $U_0 \in D^2(0)$ , there exists a unique solution  $u \in H^1((0, T); V) \cap H^2((0, T); L^2(\Omega))$  of (70).*

**2.2. Nonlinear problem.** Here we restrict ourselves to the case where (34) holds.

As previously, if we set  $z(x, \rho, t)$  as in (26), problem (16) – (20) is equivalent to

$$u_{tt}(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (0, +\infty) \tag{75}$$

$$\tau(t)z_t(x, \rho, t) + (1 - \tau'(t)\rho)z_{\rho}(x, \rho, t) = 0 \quad \text{in } \Gamma_N \times (0, 1) \times (0, +\infty) \tag{76}$$

$$u(x, t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \tag{77}$$

$$\frac{\partial u}{\partial \nu}(x, t) = -\beta_1(u_t(x, t)) - \beta_2(z(x, 1, t)) \quad \text{on } \Gamma_N \times (0, +\infty) \tag{78}$$

$$z(x, 0, t) = u_t(x, t) \quad \text{on } \Gamma_N \times (0, \infty) \tag{79}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \tag{80}$$

$$z(x, \rho, 0) = g_0(x, -\rho\tau(0)) \quad \text{in } \Gamma_N \times (0, 1). \tag{81}$$

Then problem (75) – (81) can be rewritten as

$$\begin{cases} U' = \mathcal{A}(t)U \\ U(0) = (u_0, u_1, g_0(\cdot, -\cdot\tau(0)))^T \end{cases} \tag{82}$$

where the operator  $\mathcal{A}$  is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ \frac{\tau'(t)\rho-1}{\tau(t)}z_{\rho} \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \left\{ (u, v, z)^T \in (E(\Delta, L^2(\Omega)) \cap V) \times V \times L^2(\Gamma_N; H^1(0, 1)) : \right. \\ \left. \frac{\partial u}{\partial \nu} = -\beta_1(v) - \beta_2(z(\cdot, 1)) \text{ on } \Gamma_N; v = z(\cdot, 0) \text{ on } \Gamma_N \right\}. \tag{83}$$

Notice that the domain of the operator  $\mathcal{A}(t)$  is independent of the time  $t$ , i.e. (40) holds. Note further that for  $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ ,  $\partial u / \partial \nu$  belongs to  $L^2(\Gamma_N)$ , by (21) and since  $z(\cdot, 1)$  is in  $L^2(\Gamma_N)$ .

We observe that the operator  $\mathcal{A}(t)$  defined before is nonlinear (due to the domain (83) of the operator  $\mathcal{A}(t)$ ) and therefore the technique developed in Section 2 can not be applied here. For nonlinear operators  $\mathcal{A}(t)$  similar results exist (see [4, 8, 10, 20]) but for maximal operators  $\mathcal{A}(t)$  with one inner product independent of  $t$ . For our system we need a variant of such results for maximal monotone operators  $\mathcal{A}(t)$  for a time-dependent inner product depending “smoothly” on  $t$ .

We have the following result from [10] (see also [22]):

**Theorem 2.5.** *Let  $X$  be a real separable Hilbert space. For a fixed  $T > 0$  and any time  $t \in [0, T]$  we assume that there exists an inner product  $\langle \cdot, \cdot \rangle_t$  on  $X$  depending “smoothly” on  $t$  in the following sense: there exists  $c > 0$  such that*

$$\frac{\|u\|_t}{\|u\|_s} \leq e^{c|t-s|}, \quad \forall u \in X, \forall t, s \in [0, T]. \tag{84}$$

Assume furthermore that:

- (i) for all  $t \in [0, T]$ ,  $A(t)$  is a maximal monotone operator for the inner product  $\langle \cdot, \cdot \rangle_t$ ;
- (ii) the domain  $\mathcal{D}(A(t))$  of  $A(t)$  is independent of  $t$ , for all  $t \in [0, T]$ ;
- (iii) there exists a positive constant  $K$  such that

$$\|A(t)u - A(s)u\|_0 \leq K|t - s|(1 + \|u\|_0 + \|A(s)u\|_0), \quad \forall u \in \mathcal{D}(A(t)), \forall s, t \in [0, T], \tag{85}$$

where here  $\|\cdot\|_0 = \|\cdot\|_{t=0}$ . Then for all  $v \in \mathcal{D}(A(t))$  the evolution equation

$$\begin{cases} u' + A(t)u = 0 & \text{for } 0 \leq t \leq T \\ u(0) = v \end{cases} \tag{86}$$

has a unique solution  $u \in C([0, T]; X)$  such that  $u(t)$  belongs to  $\mathcal{D}(A(t))$  for all  $t \in [0, T]$ , its strong derivative  $u'(t) = -A(t)u(t)$  exists and is continuous except at a countable numbers of values  $t$ .

Therefore to prove the existence and uniqueness of the solution of (82), we define an inner product depending “smoothly” on  $t$ .

For that, we assume that

$$\gamma_2 \leq \gamma_1 \sqrt{1 - d} \tag{87}$$

holds, where  $\gamma_1, \gamma_2$  is defined by (23) and (24).

Let  $\xi$  be a positive real number such that

$$\frac{\gamma_2}{\sqrt{1 - d}} \leq \xi \leq 2\gamma_1 - \frac{\gamma_2}{\sqrt{1 - d}}. \tag{88}$$

Note that, from (87), such a constant  $\xi$  exists.



Let us define on the Hilbert space  $\mathcal{H}$  the following time-dependent inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_t := \int_{\Omega} \{ \nabla u(x) \nabla \tilde{u}(x) + v(x) \tilde{v}(x) \} dx + \xi \tau(t) \int_{\Gamma_N} \int_0^1 z(x, \rho) \tilde{z}(x, \rho) d\rho d\Gamma, \quad (89)$$

where  $\xi$  is defined by (88).

Note that if  $\beta_1$  and  $\beta_2$  are linear, i.e.  $\beta_i(s) = \mu_i s$  with  $\mu_i > 0$ , the assumptions (42) and (87), and (43) and (88) are exactly the same.

The aim of this section is then to prove the following theorem:

**Theorem 2.6.** *Assume (8), (14), (21), (22), (23), (24), (34), (87) hold. Moreover assume that  $\beta_2$  is nondecreasing. For any initial datum  $U_0 \in \mathcal{D}(\mathcal{A}(0))$ , then there exists a unique solution*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}(t))) \cap C^1([0, +\infty), \mathcal{H})$$

of problem (82).

To prove Theorem 2.6, we thus check that (84) holds and that

$$\tilde{\mathcal{A}}_-(t) = -\tilde{\mathcal{A}}(t) = -\mathcal{A}(t) + \kappa(t)I \quad (90)$$

satisfies the assumptions (i) to (iii) of Theorem 2.5, where  $\kappa$  is defined by (49).

The proof of (84) is the same that in Theorem 2.3, so we omit it.

We clearly have (ii) for  $\tilde{\mathcal{A}}_-(t)$  since  $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\tilde{\mathcal{A}}_-(t))$ .

Therefore, it remains to show (i) and (iii), which is the aim of the three following lemmas.

**Lemma 2.7.** *Assume (14), (23), (24), (87) and (88) hold. Then  $\tilde{\mathcal{A}}_-(t)$  is a monotone operator in  $\mathcal{H}$  for the inner product  $\langle \cdot, \cdot \rangle_t$  for any fixed  $t \geq 0$ , i.e.:*

$$\left\langle \tilde{\mathcal{A}}_-(t)\phi_1 - \tilde{\mathcal{A}}_-(t)\phi_2, \phi_1 - \phi_2 \right\rangle_t \geq 0, \quad \forall \phi_1, \phi_2 \in \mathcal{D}(\tilde{\mathcal{A}}_-(t)). \quad (91)$$

*Proof.* First, from the definition of  $\mathcal{A}(t)$ , for  $\phi_i = (u_i, v_i, z_i)^T \in \mathcal{D}(\mathcal{A}(t))$ ,

$$\begin{aligned} \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t &= \\ & \int_{\Omega} \{ (\nabla v_1 - \nabla v_2)(\nabla u_1 - \nabla u_2) + (\Delta u_1 - \Delta u_2)(v_1 - v_2) \} dx \\ & + \xi \int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma. \end{aligned}$$

So, by Green's formula,

$$\begin{aligned} \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t &= \int_{\Gamma_N} (v_1 - v_2) \frac{\partial}{\partial \nu} (u_1 - u_2) d\Gamma \\ & + \xi \int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma. \end{aligned}$$

Integrating by parts in  $\rho$ , we get

$$\begin{aligned} & \int_{\Gamma_N} \int_0^1 \left( \frac{\partial z_1}{\partial \rho} - \frac{\partial z_2}{\partial \rho} \right) (z_1 - z_2) (\tau'(t)\rho - 1) d\rho d\Gamma \\ &= \frac{1}{2} \int_{\Gamma_N} \int_0^1 (\tau'(t)\rho - 1) \frac{\partial}{\partial \rho} (z_1 - z_2)^2 d\rho d\Gamma \\ &= -\frac{\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma + \frac{\tau'(t) - 1}{2} \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma \\ & \quad + \frac{1}{2} \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma. \end{aligned}$$

Therefore

$$\begin{aligned} & \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t = \\ & \int_{\Gamma_N} (v_1 - v_2) \frac{\partial}{\partial \nu} (u_1 - u_2) d\Gamma - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \\ & - \frac{\xi(1 - \tau'(t))}{2} \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma. \end{aligned}$$

As  $\phi_i \in \mathcal{D}(\mathcal{A}(t))$  for  $i = 1, 2$ , we obtain

$$\begin{aligned} & \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t = \\ & - \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0)) (\beta_1(z_1(x, 0)) - \beta_1(z_2(x, 0))) d\Gamma \\ & - \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0)) (\beta_2(z_1(x, 1)) - \beta_2(z_2(x, 1))) d\Gamma \\ & - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \\ & - \frac{\xi(1 - \tau'(t))}{2} \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma. \end{aligned}$$

From (23), (24) and Cauchy-Schwarz's inequality

$$\begin{aligned} \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t &\leq \left( \frac{\xi}{2} - \gamma_1 + \frac{\gamma_2}{2\sqrt{1-d}} \right) \int_{\Gamma_N} (z_1(x, 0) - z_2(x, 0))^2 d\Gamma \\ & + \left( \frac{\gamma_2\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2} \right) \int_{\Gamma_N} (z_1(x, 1) - z_2(x, 1))^2 d\Gamma \\ & - \frac{\xi\tau'(t)}{2} \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma. \end{aligned}$$

By (88) and the definition (49) of  $\kappa$ , we get

$$\begin{aligned} & \langle \mathcal{A}(t)\phi_1 - \mathcal{A}(t)\phi_2, \phi_1 - \phi_2 \rangle_t \\ & \leq \kappa(t)\xi\tau(t) \int_{\Gamma_N} \int_0^1 (z_1 - z_2)^2 d\rho d\Gamma \leq \kappa(t) \langle \phi_1 - \phi_2, \phi_1 - \phi_2 \rangle_t. \end{aligned}$$

By the definition (90) of  $\tilde{\mathcal{A}}_-(t)$ , we obtain (91). □

**Lemma 2.8.** *Assume that (8), (21), (22), (23) and (34) hold. Moreover assume that  $\beta_2$  is nondecreasing. Then  $\tilde{\mathcal{A}}_-(t)$  is a maximal operator in  $\mathcal{H}$ , i.e. for all  $(f, g, h)^T \in \mathcal{H}$ , there exists  $(u, v, z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$  such that*

$$(I + \tilde{\mathcal{A}}_-(t))(u, v, z)^T = (f, g, h)^T. \tag{92}$$

*Proof.* Given  $(f, g, h)^T \in \mathcal{H}$ , we seek  $U = (u, v, z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$  solution of

$$\begin{cases} (1 + \kappa(t))u - v = f \\ (1 + \kappa(t))v - \Delta u = g \\ (1 + \kappa(t))z + \frac{1 - \tau'(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \quad (93)$$

In the beginning of this proof we follow the proof of Theorem 2.3. Suppose that we have found  $u$  with the appropriated regularity. Then  $v$  is given by

$$v := (1 + \kappa(t))u - f \in V, \quad (94)$$

and  $z$  by

$$\begin{aligned} z(x, \rho) &= (1 + \kappa(t))u(x)e^{-(1+\kappa(t))\rho\tau(t)} - f(x)e^{-(1+\kappa(t))\rho\tau(t)} \\ &\quad + \tau(t)e^{-(1+\kappa(t))\rho\tau(t)} \int_0^\rho h(x, \sigma)e^{(1+\kappa(t))\sigma\tau(t)} d\sigma \quad \text{on } \Gamma_N \times (0, 1), \end{aligned} \quad (95)$$

if  $\tau'(t) = 0$ , and

$$\begin{aligned} z(x, \rho) &= (1 + \kappa(t))u(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} - f(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \\ &\quad + e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)} \int_0^\rho \frac{h(x, \sigma)\tau(t)}{1 - \tau'(t)\sigma} e^{-(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma \end{aligned} \quad (96)$$

on  $\Gamma_N \times (0, 1)$  otherwise.

In particular, if  $\tau'(t) = 0$

$$z(x, 1) = (1 + \kappa(t))u(x)e^{-(1+\kappa(t))\tau(t)} + z_0(x), \quad x \in \Gamma_N, \quad (97)$$

with  $z_0 \in L^2(\Gamma_N)$  defined by

$$z_0(x) = -f(x)e^{-(1+\kappa(t))\tau(t)} + \tau(t)e^{-(1+\kappa(t))\tau(t)} \int_0^1 h(x, \sigma)e^{(1+\kappa(t))\sigma\tau(t)} d\sigma, \quad x \in \Gamma_N, \quad (98)$$

and, if  $\tau'(t) \neq 0$

$$z(x, 1) = (1 + \kappa(t))u(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} + z_0(x), \quad x \in \Gamma_N, \quad (99)$$

with  $z_0 \in L^2(\Gamma_N)$  defined by

$$\begin{aligned} z_0(x) &= -f(x)e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \\ &\quad + e^{(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t))} \int_0^1 \frac{h(x, \sigma)\tau(t)}{1 - \tau'(t)\sigma} e^{-(1+\kappa(t))\frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\sigma)} d\sigma, \quad x \in \Gamma_N. \end{aligned} \quad (100)$$

By (93), as in the proof of Theorem 2.3, the function  $u$  satisfies

$$(1 + \kappa(t))^2 u - \Delta u = g + (1 + \kappa(t))f, \quad (101)$$

which can be reformulated as

$$\int_\Omega ((1 + \kappa(t))^2 u - \Delta u) w dx = \int_\Omega (g + (1 + \kappa(t))f) w dx, \quad \forall w \in H_{\Gamma_D}^1(\Omega). \quad (102)$$

Integrating by parts and since  $(u, v, z)^T \in \mathcal{D}(\mathcal{A}(t))$ , we obtain

$$\begin{aligned} &\int_\Omega ((1 + \kappa(t))^2 u w + \nabla u \nabla w) dx + \int_{\Gamma_N} (\beta_1(v) + \beta_2(z(x, 1))) w d\Gamma \\ &= \int_\Omega (g + (1 + \kappa(t))f) w dx, \end{aligned}$$

for all  $w \in H_{\Gamma_D}^1(\Omega)$ .

Assume that  $\tau'(t) = 0$ . Using (94) and (97), we get

$$\gamma(u, w) = F(w), \quad \forall w \in H_{\Gamma_D}^1(\Omega), \tag{103}$$

where the form  $\gamma$  (linear on  $w$  but not on  $u$ ) is defined by

$$\begin{aligned} \gamma(u, w) &= \int_{\Omega} ((1 + \kappa(t))^2 uw + \nabla u \nabla w) dx \\ &\quad + \int_{\Gamma_N} \left( \beta_1((1 + \kappa(t))u - f) + \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) \right) w d\Gamma, \end{aligned}$$

and the linear form  $F$  is defined by

$$F(w) = \int_{\Omega} (g + (1 + \kappa(t))f) w dx.$$

Introducing the (nonlinear) mapping

$$\mathcal{B} : V \rightarrow V' : u \rightarrow \mathcal{B}u,$$

where  $\mathcal{B}u(w) = \gamma(u, w)$ , we see that (103) is equivalent to

$$\mathcal{B}u = F,$$

since  $F$  clearly belongs to  $V'$ . This means that the solvability of (103) is equivalent to the surjectivity of  $\mathcal{B}$ . This surjectivity is obtained using Corollary II.2.2 of [27], which states that  $\mathcal{B}$  is surjective if  $\mathcal{B}$  is monotone, hemicontinuous, bounded and coercive. Let us then check these properties.

We first prove that  $\mathcal{B}$  is monotone, i.e.

$$[\mathcal{B}u - \mathcal{B}v](u - v) \geq 0, \quad \forall u, v \in V. \tag{104}$$

In view of the definition of  $\mathcal{B}$ ,

$$\begin{aligned} [\mathcal{B}u - \mathcal{B}v](u - v) &= \int_{\Omega} ((1 + \kappa(t))^2(u - v)^2 + |\nabla(u - v)|^2) dx \\ &\quad + \int_{\Gamma_N} (\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)) (u - v) d\Gamma \\ &\quad + \int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) \right. \\ &\quad \quad \left. - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0) \right) (u - v) d\Gamma. \end{aligned}$$

By (23),

$$\begin{aligned} &\int_{\Gamma_N} (\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)) (u - v) d\Gamma \\ &= \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} (\beta_1((1 + \kappa(t))u - f) - \beta_1((1 + \kappa(t))v - f)) \\ &\quad \cdot ((1 + \kappa(t))u - f) - ((1 + \kappa(t))v - f) d\Gamma \geq 0, \end{aligned}$$

and as  $\beta_2$  is nondecreasing,

$$\begin{aligned} &\int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) \right. \\ &\quad \left. - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0) \right) (u - v) d\Gamma \\ &= \frac{e^{(1+\kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \left( \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) \right. \\ &\quad \left. - \beta_2((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0) \right) \\ &\quad \cdot \left( ((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0) - ((1 + \kappa(t))ve^{-(1+\kappa(t))\tau(t)} + z_0) \right) d\Gamma \geq 0. \end{aligned}$$

This two estimates clearly imply (104).

The boundedness of  $\mathcal{B}$  follows from the properties (21) satisfied by  $\beta_1$  and  $\beta_2$ , the fact that  $1 + \kappa(t)$  is bounded by (34) and (8), Cauchy-Schwarz’s inequality and a trace theorem (reminding that  $f$  and  $z_0$  are fixed).

The hemicontinuity of  $\mathcal{B}$  means that the function  $s \rightarrow \mathcal{B}(u + sw)(w)$  is continuous for each  $u, w \in V$ . As

$$\begin{aligned} \mathcal{B}(u + sw)(w) &= \int_{\Omega} ((1 + \kappa(t))^2 (u + sw)w + \nabla(u + sw)\nabla w) dx \\ &+ \int_{\Gamma_N} (\beta_1((1 + \kappa(t))(u + sw) - f) + \beta_2((1 + \kappa(t))(u + sw)e^{-(1+\kappa(t))\tau(t)} + z_0))w d\Gamma, \end{aligned}$$

this follows from the continuity of  $\beta_1$  and  $\beta_2$ .

It remains to check the coerciveness of  $\mathcal{B}$ , i.e.

$$\frac{\mathcal{B}u(u)}{\|u\|_V} \rightarrow \infty \quad \text{if } \|u\|_V \rightarrow +\infty. \tag{105}$$

From the definition of  $\mathcal{B}$ , we have, since  $1 + \kappa(t) > 1$ ,

$$\mathcal{B}u(u) \geq \|u\|_V^2 + \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f)u d\Gamma + \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0)u d\Gamma.$$

We deduce, by (25) and (22)

$$\begin{aligned} \mathcal{B}u(u) &\geq \|u\|_V^2 + \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f)f d\Gamma \\ &\quad - \frac{e^{(1+\kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0)z_0 d\Gamma. \end{aligned}$$

By Cauchy-Schwarz’s inequality, (34), (21), a trace theorem and the fact that  $1 + \kappa(t)$  is bounded, we obtain that there exists  $C > 0$  such that

$$\left| \frac{1}{1 + \kappa(t)} \int_{\Gamma_N} \beta_1((1 + \kappa(t))u - f)f d\Gamma \right| \leq C \left( \|u\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)} \right) \|f\|_{H^1(\Omega)}$$

and

$$\begin{aligned} &\left| \frac{e^{(1+\kappa(t))\tau(t)}}{1 + \kappa(t)} \int_{\Gamma_N} \beta_2((1 + \kappa(t))ue^{-(1+\kappa(t))\tau(t)} + z_0)z_0 d\Gamma \right| \\ &\leq C \left( \|u\|_{H^1(\Omega)} + \|z_0\|_{H^1(\Omega)} \right) \|z_0\|_{H^1(\Omega)}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{B}u(u) &\geq \|u\|_V^2 - C \left( \left( \|u\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)} \right) \|f\|_{H^1(\Omega)} \right. \\ &\quad \left. + \left( \|u\|_{H^1(\Omega)} + \|z_0\|_{H^1(\Omega)} \right) \|z_0\|_{H^1(\Omega)} \right), \end{aligned}$$

which implies (105).

Therefore, by Corollary II.2.2 of [27], there exists  $u \in V$  solution of (103). If  $\tau'(t) \neq 0$ , we obtain the same result by similar arguments. This function  $u$  satisfies (101) by choosing test function in  $\mathcal{D}(\Omega)$  and then satisfies

$$\frac{\partial u}{\partial \nu} = -\beta_1(v) - \beta_2(z(\cdot, 1)) \quad \text{on } \Gamma_N$$

by Green’s formula.

So we have found  $(u, v, z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$  which verifies (92).  $\square$

It remains to show (iii) of Theorem 2.5 to finish the proof of Theorem 2.6. This is the aim of the following lemma.

**Lemma 2.9.** *Assume that (8), (14) and (34) hold. Then (85) holds.*

*Proof.* Let  $\phi = (u, v, z)^T \in \mathcal{D}(\tilde{\mathcal{A}}_-(t))$ . By definition (90) of  $\tilde{\mathcal{A}}_-(t)$ , we have

$$\begin{aligned} \left\| \tilde{\mathcal{A}}_-(t)\phi - \tilde{\mathcal{A}}_-(s)\phi \right\|_0 &= \|(\kappa(t) - \kappa(s))\phi - (\mathcal{A}(t)\phi - \mathcal{A}(s)\phi)\|_0 \\ &\leq |\kappa(t) - \kappa(s)| \|\phi\|_0 + \|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0. \end{aligned} \tag{106}$$

As

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)(\tau'(t)^2 + 1)^{\frac{1}{2}}} - \frac{\tau'(t)(\tau'(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)^2}$$

is bounded on  $[0, T]$  for all  $T > 0$  (by (34) and (8)), by the mean value theorem there exists  $K > 0$  such that

$$|\kappa(t) - \kappa(s)| \|\phi\|_0 \leq K |t - s| \|\phi\|_0. \tag{107}$$

Moreover

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 = \xi\tau(0) \int_{\Gamma_N} \int_0^1 \left( \frac{\tau'(t)\rho - 1}{\tau(t)} - \frac{\tau'(s)\rho - 1}{\tau(s)} \right)^2 z_\rho^2 d\rho d\Gamma.$$

In addition

$$\left( \frac{\tau'(t)\rho - 1}{\tau(t)} \right)' = \frac{\tau''(t)\tau(t)\rho - \tau'(t)^2\rho + \tau'(t)}{\tau(t)^2}$$

is bounded on  $[0, T]$  for all  $T > 0$  by (34) and (8). By the mean value theorem, we then obtain that there exists  $K > 0$  such that

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 \leq K^2 |t - s|^2 \xi\tau(0) \int_{\Gamma_N} \int_0^1 z_\rho^2(x, \rho) d\rho d\Gamma.$$

Moreover by (34) and (14)

$$\left( \frac{\tau(t)}{\tau'(t)\rho - 1} \right)^2 \leq \frac{\bar{\tau}^2}{(1 - d)^2}.$$

Therefore

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 \leq \left( \frac{K\bar{\tau}}{1 - d} \right)^2 |t - s|^2 \xi\tau(0) \int_{\Gamma_N} \int_0^1 \left( \frac{\tau'(t)\rho - 1}{\tau(t)} z_\rho(x, \rho) \right)^2 d\rho d\Gamma.$$

This leads to

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0^2 \leq \left( \frac{K\bar{\tau}}{1 - d} \right)^2 |t - s|^2 \|\mathcal{A}(t)\phi\|_0^2,$$

and thus to

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0 \leq \left( \frac{K\bar{\tau}}{1 - d} \right) |t - s| \|\mathcal{A}(t)\phi\|_0.$$

By definition (90) of  $\tilde{\mathcal{A}}_-(t)$ , we have

$$\|\mathcal{A}(t)\phi\|_0 \leq \left\| \tilde{\mathcal{A}}_-(t)\phi \right\|_0 + \kappa(t) \|\phi\|_0,$$

with  $\kappa(t)$  bounded on  $[0, T]$ . Consequently there exists  $C > 0$  such that

$$\|\mathcal{A}(t)\phi - \mathcal{A}(s)\phi\|_0 \leq C |t - s| \left( \left\| \tilde{\mathcal{A}}_-(t)\phi \right\|_0 + \|\phi\|_0 \right). \tag{108}$$

Therefore (106), (107) and (108) imply (85).  $\square$

*Proof of Theorem 2.6.* The assumptions of Theorem 2.5 were verified for  $\tilde{\mathcal{A}}_-(t)$  and the inner product  $\langle \cdot, \cdot \rangle_t$  defined by (89). Consequently the evolution equation

$$\begin{cases} \tilde{U}' + \tilde{\mathcal{A}}_-(t)\tilde{U} = 0 \\ \tilde{U}(0) = U_0 \in \mathcal{D}(\mathcal{A}(t)), \end{cases} \quad (109)$$

has a unique solution  $\tilde{U} \in C([0, T]; \mathcal{H})$  such that  $\tilde{U}(t)$  belongs to  $\mathcal{D}(\tilde{\mathcal{A}}_-(t))$  for all  $t \in [0, T]$ , its strong derivative  $\tilde{U}'(t) = -\tilde{\mathcal{A}}_-(t)\tilde{U}(t)$  exists and is continuous except at a countable numbers of values  $t$ .

The requested solution of (82) is then given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with  $\beta(t) = \int_0^t \kappa(s)ds$ . □

**3. Stability result for the linear problem .** In this section, we will give an exponential stability result for problem (1) – (4) under the assumption (13). We define the energy of system (1) – (4) as

$$E(t) := \frac{1}{2} \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx + \frac{\xi}{2} \tau(t) \int_0^1 \int_{\Gamma_N} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma, \quad (110)$$

where  $\xi$  is a positive constant such that

$$2\mu_1 - \frac{\mu_2}{\sqrt{1-d}} - \xi > 0, \quad \text{and} \quad \xi - \frac{\mu_2}{\sqrt{1-d}} > 0. \quad (111)$$

Note that from (13) such a constant  $\xi$  exists. We have the following identity.

**Proposition 1.** *For any regular solution of problem (1) – (4) we have*

$$\begin{aligned} E'(t) = & -\mu_1 \int_{\Gamma_N} u_t^2(x, t) d\Gamma - \int_{\Gamma_N} \mu_2 u_t(x, t) u_t(x, t - \tau(t)) d\Gamma \\ & - \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t - \tau(t))(1 - \tau'(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t) d\Gamma. \end{aligned} \quad (112)$$

*Proof.* Differentiating (110) we obtain

$$\begin{aligned} E'(t) = & \int_{\Omega} \{u_t u_{tt} + \nabla u \nabla u_t\} dx + \frac{\xi}{2} \tau'(t) \int_0^1 \int_{\Gamma_N} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\ & + \xi \tau(t) \int_{\Gamma_N} \int_0^1 u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho d\Gamma, \end{aligned}$$

and then, applying Green's formula,

$$\begin{aligned} E'(t) = & \int_{\Gamma_N} u_t \frac{\partial u}{\partial \nu} d\Gamma + \frac{\xi}{2} \tau'(t) \int_0^1 \int_{\Gamma_N} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\ & + \xi \tau(t) \int_{\Gamma_N} \int_0^1 u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho d\Gamma. \end{aligned} \quad (113)$$

Now, observe that, if  $\tau(t) \neq 0$ ,

$$u_t(x, t - \tau(t)\rho) = -\tau^{-1}(t) \partial_{\rho}(u(x, t - \tau(t)\rho)),$$

and

$$u_{tt}(x, t - \tau(t)\rho) = \tau^{-2}(t) \partial_{\rho\rho}(u(x, t - \tau(t)\rho)).$$

Therefore,

$$\begin{aligned} & \int_0^1 u_t(x, t - \tau(t)\rho)u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\ &= -\tau^{-3}(t) \int_0^1 \partial_\rho(u(x, t - \tau(t)\rho))\partial_{\rho\rho}(u(x, t - \tau(t)\rho))(1 - \tau'(t)\rho)d\rho. \end{aligned} \tag{114}$$

By integration by parts, (114) becomes

$$\begin{aligned} & \int_0^1 u_t(x, t - \tau(t)\rho)u_{tt}(x, t - \tau(t)\rho)(1 - \tau'(t)\rho)d\rho \\ &= -\frac{1}{2}\tau'(t)\tau^{-3}(t) \int_0^1 (\partial_\rho(u(t - \tau(t)\rho)))^2d\rho \\ &\quad -\frac{\tau^{-1}(t)}{2}u_t^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2}u_t^2(x, t) \\ &= -\frac{1}{2}\tau'(t)\tau^{-1}(t) \int_0^1 u_t^2(x, t - \tau(t)\rho)d\rho \\ &\quad -\frac{\tau^{-1}(t)}{2}u_t^2(x, t - \tau(t))(1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2}u_t^2(x, t). \end{aligned} \tag{115}$$

Using (113), (115) and the boundary condition (3) on  $\Gamma_N$ , we have

$$\begin{aligned} E'(t) &= -\mu_1 \int_{\Gamma_N} u_t^2(x, t)d\Gamma - \mu_2 \int_{\Gamma_N} u_t(x, t)u_t(x, t - \tau(t))d\Gamma \\ &\quad + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t)d\Gamma - \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t - \tau(t))(1 - \tau'(t))d\Gamma. \end{aligned} \tag{116}$$

So, for any time  $t$  such that  $\tau(t) \neq 0$ , the identity (112) is proved.

Now, let  $t$  be such that  $\tau(t) = 0$ . Then, from (113) we have

$$E'(t) = -(\mu_1 + \mu_2) \int_{\Gamma_N} u_t^2(x, t)d\Gamma + \frac{\xi}{2}\tau'(t) \int_{\Gamma_N} u_t^2(x, t)d\Gamma. \tag{117}$$

Therefore, identity (112) is proved for all times  $t > 0$ . □

**Proposition 2.** *Assuming (14), for any regular solution of problem (1) – (4) the energy decays and there exists a positive constant  $C$  such that*

$$E'(t) \leq -C \int_{\Gamma_N} \{u_t^2(x, t) + u_t^2(x, t - \tau(t))\}d\Gamma. \tag{118}$$

*Proof.* In the case of  $\tau(t) \neq 0$ , from (116), applying Cauchy-Schwarz’s inequality, we obtain

$$\begin{aligned} E'(t) &\leq -\mu_1 \int_{\Gamma_N} u_t^2(x, t)d\Gamma + \frac{1}{\sqrt{1-d}} \frac{\mu_2}{2} \int_{\Gamma_N} u_t^2(x, t)d\Gamma \\ &\quad + \sqrt{1-d} \frac{\mu_2}{2} \int_{\Gamma_N} u_t^2(x, t - \tau(t))d\Gamma \\ &\quad - \frac{\xi}{2}(1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x, t - \tau(t))d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t)d\Gamma, \end{aligned}$$

from which easily follows (118) recalling (111). In the case of  $\tau(t) = 0$ , from (117) easily follows (118) observing that by (111)



$$\xi < 2\mu_1 < \frac{2(\mu_1 + \mu_2)}{d}.$$

□

**Remark 1.** The choice to apply Cauchy-Schwarz's inequality with a factor  $\sqrt{1-d}$  in the proof of the above proposition is made in order to give the stability result under the best assumption between  $\mu_1$  and  $\mu_2$ .

Now, let us introduce the Lyapunov functional

$$\hat{E}(t) = E(t) + \gamma \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u]u_t dx + \mathcal{E}(t) \right\}, \quad (119)$$

where  $\gamma$  is a positive small constant that we will choose later on,  $m$ , defined by (9), satisfies (10) and (11), and  $\mathcal{E}(t)$  is defined by

$$\mathcal{E}(t) := \xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma. \quad (120)$$

Note that, from Poincaré's Theorem, the functional  $\hat{E}$  is equivalent to the energy  $E$ , that is there exist two positive constant  $d_1, d_2$  such that

$$d_1 \hat{E}(t) \leq E(t) \leq d_2 \hat{E}(t), \quad \forall t \geq 0. \quad (121)$$

Moreover, we denote by  $E_S(\cdot)$  the standard energy for wave equation without delay, that is

$$E_S(t) := \frac{1}{2} \int_{\Omega} (u_t^2(x, t) + |\nabla u(x, t)|^2) dx. \quad (122)$$

The following estimate holds true.

**Lemma 3.1.** *For any regular solution of problem (1) – (4),*

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u]u_t dx \right\} \\ \leq -C_0 E_S(t) + C \left\{ \int_{\Gamma_N} [u_t^2(x, t) + u_t^2(x, t - \tau(t))] d\Gamma \right\}, \end{aligned} \quad (123)$$

for suitable positive constants  $C_0, C$ .

*Proof.* The standard multiplier identity gives

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u]u_t dx \right\} &= - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ &+ \int_{\Gamma} (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma + \int_{\Gamma} [2m \cdot \nabla u + (n-1)u] \frac{\partial u}{\partial \nu} d\Gamma. \end{aligned} \quad (124)$$

Therefore, using (2) and (10), we obtain

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u]u_t dx \right\} &\leq - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ &+ \int_{\Gamma_N} (m \cdot \nu)(u_t^2 - |\nabla u|^2) d\Gamma + \int_{\Gamma_N} [2m \cdot \nabla u + (n-1)u] \frac{\partial u}{\partial \nu} d\Gamma. \end{aligned} \quad (125)$$

From (125) and Young's inequality, recalling that by (11)  $m \cdot \nu \geq \delta$  on  $\Gamma_N$ , we have

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &\leq - \int_{\Omega} \{u_t^2 + |\nabla u|^2\} dx \\ &+ \int_{\Gamma_N} (m \cdot \nu) u_t^2 d\Gamma - \delta \int_{\Gamma_N} |\nabla u|^2 d\Gamma + \frac{c}{\varepsilon} \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma \\ &+ \varepsilon \int_{\Gamma_N} (|\nabla u|^2 + u^2) d\Gamma, \end{aligned} \quad (126)$$

for some positive constants  $\varepsilon, c$ . From (126), using the trace's inequality and Poincaré's Theorem, for  $\varepsilon$  small enough we deduce

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{\Omega} [2m \cdot \nabla u + (n-1)u] u_t dx \right\} &\leq -C_0 E_S(t) \\ &+ C \int_{\Gamma_N} u_t^2 d\Gamma + C \int_{\Gamma_N} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma, \end{aligned} \quad (127)$$

for suitable positive constants  $C_0, C$ . Therefore, using the boundary condition (3) and Cauchy-Schwarz's inequality in (127), we obtain (123).  $\square$

We can also estimate the component  $\mathcal{E}(\cdot)$  in the Lyapunov functional (119).

**Lemma 3.2.** *For any regular solution of problem (1) – (4),*

$$\frac{d}{dt} \mathcal{E}(t) \leq -2\mathcal{E}(t) + \xi \int_{\Gamma_N} u_t^2 d\Gamma. \quad (128)$$

*Proof.* Differentiating (120) we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &= \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\ &- 2\xi \tau'(t) \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} \rho u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\ &+ 2\xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho d\Gamma. \end{aligned} \quad (129)$$

Now, let us suppose  $\tau(t) \neq 0$  and integrate by parts the last term in (129). As

$$\begin{aligned} &\int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho \\ &= -\tau^{-3}(t) \int_0^1 e^{-2\tau(t)\rho} \partial_{\rho}(u(x, t - \tau(t)\rho)) \partial_{\rho\rho}(u(x, t - \tau(t)\rho)) (1 - \tau'(t)\rho) d\rho, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho \\ &= -\frac{\tau'(t)\tau^{-3}(t)}{2} \int_0^1 e^{-2\tau(t)\rho} (\partial_{\rho}(u(x, t - \tau(t)\rho)))^2 d\rho \\ &- \tau^{-2}(t) \int_0^1 e^{-2\tau(t)\rho} (\partial_{\rho}(u(x, t - \tau(t)\rho)))^2 (1 - \tau'(t)\rho) d\rho \\ &- \frac{\tau^{-3}(t)}{2} \left[ e^{-2\tau(t)\rho} (\partial_{\rho}(u(x, t - \tau(t)\rho)))^2 (1 - \tau'(t)\rho) \right]_0^1 \end{aligned} \quad (130)$$

and then

$$\begin{aligned}
& \int_0^1 e^{-2\tau(t)\rho} u_t(x, t - \tau(t)\rho) u_{tt}(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho \\
&= -\frac{\tau'(t)\tau^{-1}(t)}{2} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho \\
&\quad - \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho \\
&\quad - \frac{\tau^{-1}(t)}{2} e^{-2\tau(t)} u_t^2(x, t - \tau(t)) (1 - \tau'(t)) + \frac{\tau^{-1}(t)}{2} u_t^2(x, t).
\end{aligned} \tag{131}$$

Now, substituting identity (131) in (129), we obtain

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}(t) &= \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
&\quad - 2\xi \tau'(t) \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} \rho u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
&\quad - \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
&\quad - 2\xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) (1 - \tau'(t)\rho) d\rho d\Gamma \\
&\quad - \xi e^{-2\tau(t)} \int_{\Gamma_N} u_t^2(x, t - \tau(t)) (1 - \tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_t^2(x, t) d\Gamma,
\end{aligned} \tag{132}$$

and so

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}(t) &= -2\xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
&\quad - \xi e^{-2\tau(t)} \int_{\Gamma_N} u_t^2(x, t - \tau(t)) (1 - \tau'(t)) d\Gamma + \xi \int_{\Gamma_N} u_t^2(x, t) d\Gamma
\end{aligned} \tag{133}$$

from which immediately follows estimate (128) for  $t$  such that  $\tau(t) \neq 0$ . In the case of  $\tau(t) = 0$ , note that from (129) we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}(t) &= \xi \tau'(t) \int_{\Gamma_N} \int_0^1 e^{-2\tau(t)\rho} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\
&\leq \xi d \int_{\Gamma_N} \int_0^1 u_t^2(x, t) d\rho d\Gamma = \xi d \int_{\Gamma_N} u_t^2(x, t) d\Gamma \\
&= \xi d \int_{\Gamma_N} u_t^2(x, t) d\Gamma - 2\mathcal{E}(t).
\end{aligned} \tag{134}$$

Then, even in this case, we obtain (128).  $\square$

Now, we can deduce the exponential stability estimate for problem (1) – (5).

**Theorem 3.3.** *Assume (13) and (14). There exist positive constants  $C_1, C_2$  such that for any solution of problem (1) – (5),*

$$E(t) \leq C_1 E(0) e^{-C_2 t}, \quad \forall t \geq 0. \tag{135}$$

*Proof.* From Proposition 2, Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} \frac{d}{dt} \hat{E}(t) &\leq -C \left\{ \int_{\Gamma_N} [u_t^2(x, t) + u_t^2(x, t - \tau(t))] d\Gamma \right\} \\ &\quad + \gamma \left( -C_0 E_S(t) + \tilde{C} \int_{\Gamma_N} [u_t^2(x, t) + u_t^2(x, t - \tau(t))] d\Gamma - 2\mathcal{E}(t) \right). \end{aligned} \tag{136}$$

Then, for  $\gamma$  sufficiently small, we can estimate

$$\frac{d}{dt} \hat{E}(t) \leq -\gamma C_0 E_S(t) - 2\gamma \mathcal{E}(t). \tag{137}$$

Now, observe that by assumption (6) on  $\tau(t)$ , we can deduce

$$\begin{aligned} \mathcal{E}(t) &\geq \xi \tau(t) \int_{\Gamma_N} \int_0^1 e^{-2\bar{\tau}} u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma \\ &\geq \frac{c\xi \tau(t)}{2} \int_{\Gamma_N} \int_0^1 u_t^2(x, t - \tau(t)\rho) d\rho d\Gamma, \end{aligned} \tag{138}$$

for some positive constant  $c$ .

Therefore, from (137) and (138),

$$\frac{d}{dt} \hat{E}(t) \leq -\gamma C_0 E_S(t) - 2\gamma \mathcal{E}(t) \leq -cE(t) \leq -C\hat{E}(t), \tag{139}$$

for suitable positive constants  $c, C$ , where we used also the first inequality in (121).

This clearly implies

$$\hat{E}(t) \leq e^{-Ct} \hat{E}(0),$$

and so, using (121),

$$E(t) \leq C_1 e^{-C_2 t} E(0),$$

for suitable constants  $C_1, C_2 > 0$ . □

**4. Nonlinear stability result.** In this section we consider the problem (16)–(20) with  $\beta_1, \beta_2$  satisfying (21), (25). Moreover we assume

$$\gamma_1 > \frac{c_2}{\sqrt{1-d}}, \tag{140}$$

where  $\gamma_1, c_2$  are the constants in (21) and (25) (which comes from (21) and (23)) and  $d$  is as in (14).

We define the energy associated to the problem as in (110) with the constant  $\xi$  such that

$$2\gamma_1 - \frac{c_2}{\sqrt{1-d}} - \xi > 0 \quad \text{and} \quad \xi - \frac{c_2}{\sqrt{1-d}} > 0. \tag{141}$$

Note that, from assumption (140), such a constant  $\xi$  exists.

Notice that (87) implies (140), since  $c_2 \leq \gamma_2$ . Moreover the existence of  $\xi$  verifying (88) guarantees that  $\xi$  verifies (141), since again  $c_2 \leq \gamma_2$ .

The following identity holds true.

**Proposition 3.** *For any regular solution of problem (16)–(19) we have*

$$\begin{aligned} E'(t) &= - \int_{\Gamma_N} u_t(x, t) \beta_1(u_t(x, t)) d\Gamma - \int_{\Gamma_N} u_t(x, t) \beta_2(u_t(x, t - \tau(t))) d\Gamma \\ &\quad - \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t - \tau(t)) (1 - \tau'(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t) d\Gamma. \end{aligned} \tag{142}$$

*Proof.* The proof is analogous to the one of Proposition 1, so we omit the details. □

**Proposition 4.** *For any regular solution of problem (16) – (19) the energy decays and there exists a positive constant  $C$  such that*

$$E'(t) \leq -C \int_{\Gamma_N} \{u_t^2(x, t) + u_t^2(x, t - \tau(t))\} d\Gamma. \quad (143)$$

*Proof.* In the case of  $\tau(t) \neq 0$ , from (142), we obtain, by (21) and (25),

$$\begin{aligned} E'(t) \leq & -\gamma_1 \int_{\Gamma_N} u_t^2(x, t) d\Gamma + \int_{\Gamma_N} c_2 |u_t(x, t)| |u_t(x, t - \tau(t))| d\Gamma \\ & - \frac{\xi}{2} (1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x, t - \tau(t)) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t) d\Gamma. \end{aligned}$$

Then, applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned} E'(t) \leq & -\gamma_1 \int_{\Gamma_N} u_t^2(x, t) d\Gamma + \frac{\xi}{2} \int_{\Gamma_N} u_t^2(x, t) d\Gamma \\ & - \frac{\xi}{2} (1 - \tau'(t)) \int_{\Gamma_N} u_t^2(x, t - \tau(t)) d\Gamma + \frac{c_2}{2\sqrt{1-d}} \int_{\Gamma_N} u_t^2(x, t) d\Gamma \\ & + \sqrt{1-d} \frac{c_2}{2} \int_{\Gamma_N} u_t^2(x, t - \tau(t)) d\Gamma. \end{aligned} \quad (144)$$

From (144) estimate (143) easily follows recalling that  $\xi$  satisfies (141).

If  $t$  is such that  $\tau(t) = 0$ , then from (142) we deduce

$$\begin{aligned} E'(t) = & - \int_{\Gamma_N} u_t(x, t) \beta_1(u_t(x, t)) d\Gamma - \int_{\Gamma_N} u_t(x, t) \beta_2(u_t(x, t)) d\Gamma \\ & + \frac{\xi}{2} \tau'(t) \int_{\Gamma_N} u_t^2(x, t) d\Gamma. \end{aligned}$$

Then, from (25) and (22)

$$E'(t) \leq -(\gamma_1 - \frac{\xi}{2}d) \int_{\Gamma_N} u_t^2(x, t) d\Gamma,$$

and this clearly gives (143) observing that by (141)

$$\gamma_1 > \frac{\xi}{2} + \frac{c_2}{2\sqrt{1-d}} > \frac{\xi}{2} > \frac{\xi}{2}d.$$

□

Now, let  $\hat{E}$  be the Lyapunov functional introduced in (119) with a small enough positive constant  $\gamma$  and let  $\mathcal{E}$  be defined as in (120).

Even in this case Lemma 3.1 holds true. Indeed inequality (127) is obtained without using the boundary condition (3) on  $\Gamma_N$ . From (127) we easily deduce estimate (123) for suitable positive constants  $C_0, C$ , using the boundary condition (18) and the assumptions (21) on the functions  $\beta_1, \beta_2$ .

We can estimate also the component  $\mathcal{E}(\cdot)$  in the Lyapunov functional (119) as in the previous case, and so analogously to the linear case, we can deduce an exponential stability estimate for problem (16) – (20).

**Theorem 4.1.** *Assume (140). There exist positive constants  $D_1, D_2$  such that for any solution of problem (16) – (20),*

$$E(t) \leq D_1 E(0) e^{-D_2 t}, \quad \forall t \geq 0. \quad (145)$$

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*E-mail address:* serge.nicaise@univ-valenciennes.fr

*E-mail address:* pignotti@univaq.it

*E-mail address:* Julie.Valein@iecn.u-nancy.fr