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# Error estimate for Godunov approximation of locally constrained conservation laws

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## Abstract

We consider a model of traffic flow with unilateral constraint on the flux introduced by R. M. COLOMBO and P. GOATIN (J. Differ. Equ. 234(2):654–675, 2007), for which the convergence of numerical approximation using monotone Finite Volume schemes has been performed by B. ANDREIANOV *et al.* (Numer. Math. 115:609–645, 2010). We derive for this problem some new  $BV$ -estimate, and make use of it to provide an error estimate of optimal order  $h^{1/2}$  for the Godunov approximation of the problem. Numerical illustrations are then provided to illustrate the optimality of the result.

**keywords:** Locally constrained scalar conservation laws, monotone finite volume scheme,  $BV$ -estimate, error estimate.

AMS 35L65, 65M15, 76M12, 90B20

## 1 Presentation of the continuous problem

Recently, R. M. COLOMBO and P. GOATIN [12] analyzed the following model of traffic flow, inspired from the so-called LWR model [19, 23], with a constraint on the flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}, \\ f(u)(0, t) \leq F(t) & \text{for } t \in \mathbb{R}_+, \end{cases} \quad (1)$$

where  $f$  is supposed to be Lipschitz continuous on  $[0, 1]$  and bell-shaped, i.e. there exists  $\bar{u} \in (0, 1)$  such that

$$f(0) = f(1) = 0, \quad f \geq 0, \quad f'(u)(\bar{u} - u) > 0 \text{ for a.e. } u \in [0, 1], \quad (2)$$

and where the constraint  $F$  satisfies

$$0 \leq F(t) \leq f(\bar{u}), \text{ for a.e. } t \in \mathbb{R}^+.$$

This constraint  $F$  models toll gates or lights. In the case where the flow is not constrained, i.e.  $F \equiv f(\bar{u})$ , then it is well known that the good notion of solution for the problem (1) is the notion of entropy solution [21, 26, 17]. In the case where the constraint becomes active, i.e.  $F(t) < f(\bar{u})$  on a non negligible set of  $\mathbb{R}_+$ , then it is shown in [12] that a non-classical shock with zero speed can appear at the interface  $\{x = 0\}$ , so that the constraint can be satisfied. It has then be pointed out by B. ANDREIANOV, P. GOATIN and N. SEGUIN [3] that the problem (1) can be immersed in the framework of scalar conservation laws with discontinuous flux functions, that has been widely studied during the last years. Among the numerous references available on this topic, let us mention that ADIMURTHI and G.D. VEERAPPA GOWDA [1] exhibited that such scalar conservation laws with discontinuous flux function admits an infinite number of solutions. More precisely, for the problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (3)$$

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prescribing that the solution  $u$  satisfies the “classical” entropy condition away from  $\{x = 0\}$ , i.e. that for all  $\kappa \in \mathbb{R}$ ,

$$\partial_t |u - \kappa| + \partial_x \Phi_\kappa(u) \leq 0, \text{ in } \mathcal{D}'(\mathbb{R}^* \times \mathbb{R}_+), \quad (4)$$

where

$$\Phi_\kappa(u) = \text{sign}(u - \kappa)(f(u) - f(\kappa)), \quad (5)$$

and is a weak solution, implying that

$$f(u_L) = f(u_R), \quad (6)$$

where  $u_L$  and  $u_R$  respectively denote the traces of  $u$  on  $\{x = 0\}$  from  $\{x < 0\}$  and  $\{x > 0\}$ , i.e.

$$u_L(t) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_{-\epsilon}^0 u(x, t) dx, \quad u_R(t) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^\epsilon u(x, t) dx, \quad (7)$$

then some undercompressive waves can be generated by the interface  $\{x = 0\}$ . An additional criterion is required to ensure the uniqueness of the solution to the problem (3)–(6). This additional condition can be prescribed by requiring that the traces  $(u_L(t), u_R(t))$  belong to a so-called  $L^1$ -dissipative germ  $\mathcal{G}(t)$ . We now give a simple definition of a  $L^1$ -dissipative germ.

**Definition 1.1** *A subset  $\mathcal{G}$  of  $[0, 1]^2$  is said to be a  $L^1$ -dissipative germ corresponding to the function  $f$  if it satisfies*

$$\begin{cases} f(u_L) = f(u_R), & \forall (u_L, u_R) \in \mathcal{G}, \\ \Phi_{\kappa_R}(u_R) - \Phi_{\kappa_L}(u_L) \leq 0, & \forall (u_L, u_R) \in \mathcal{G}, \forall (\kappa_L, \kappa_R) \in \mathcal{G}. \end{cases}$$

We refer to the recent contribution of B. ANDREIANOV, K. H. KARLSEN and N. H. RISEBRO [2] for an extensive discussion on the use of  $L^1$ -dissipative germs in the more general case of scalar conservation laws with discontinuous flux functions. We just describe now the  $L^1$ -dissipative germ involved in the problem (1). For  $t \in \mathbb{R}^+$ , we denote by  $A(t)$  (resp.  $B(t)$ ) the unique value in  $[0, 1]$  such that

$$A(t) \geq \bar{u} \quad (\text{resp. } B(t) \leq \bar{u}), \quad F(t) = f(A(t)) = f(B(t)). \quad (8)$$

Then, following [3], the  $L^1$ -dissipative germ  $\mathcal{G}(t)$  corresponding to the problem (1), represented in Figure 1, is given by

$$\mathcal{G}(t) = \mathcal{G}_1(t) \cup \mathcal{G}_2(t) \cup \mathcal{G}_3(t),$$

where

- $\mathcal{G}_1(t) = (A(t), B(t))$ ,
- $\mathcal{G}_2(t) = \{(c, c) \in [0, 1]^2 \mid f(c) \leq F(t)\}$ ,
- $\mathcal{G}_3(t) = \{(c_L, c_R) \in [0, 1]^2 \mid c_L < c_R \text{ and } f(c_L) = f(c_R) \leq F(t)\}$ .

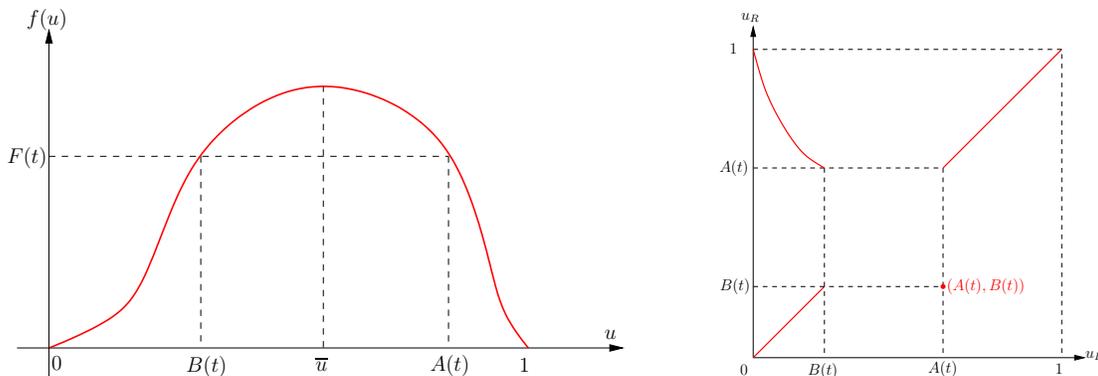


Figure 1: Graphical representation of the values  $A(t)$  and  $B(t)$  corresponding to the constraint  $F(t)$ , and graphical representation of the corresponding  $L^1$ -dissipative germ  $\mathcal{G}(t)$ . An example of  $L^1$ -dissipative germ

**Remark 1.1** Notice that the existence of the traces  $u_L, u_R \in L^\infty(\mathbb{R}_+)$  defined in (7) is a consequence of the work of E. YU. PANOV [22] and of the condition (2).

**Definition 1.2** A function  $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+; [0, 1])$  is said to be a solution of the problem (1) if:

1. for all  $\kappa \in [0, 1]$ , for all  $\psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$  such that  $\psi(0, \cdot) = 0$ ,

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} |u(x, t) - \kappa| \partial_t \psi(x, t) dx dt + \int_{\mathbb{R}} |u_0(x) - \kappa| \psi(x, 0) dx \\ + \int_0^{+\infty} \int_{\mathbb{R}} \Phi_\kappa(u)(x, t) \partial_x \psi(x, t) dx dt \geq 0; \end{aligned} \quad (9)$$

2. for almost every  $t \in \mathbb{R}_+$ ,  $(u_L(t), u_R(t)) \in \mathcal{G}(t)$ .

**Remark 1.2** In [3], three equivalent notions of solutions are proposed for the problem (1). Here, we choose to focus on only one of them, which will be the one that we be used in the sequel.

We now state an existence and uniqueness result for the solution  $u$  of the problem (1). The time continuity of the solution  $u$  prescribed below is a consequence of the result stated in [9].

**Theorem 1** ([12, 3]) Let  $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ , and let  $F \in L^\infty(\mathbb{R}^+; [0, f(\bar{u})])$ , then there exists a unique  $u$  solution to the problem (1) in the sense of the Definition 1.2, which furthermore can be assumed to belong to  $\mathcal{C}(\mathbb{R}^+, L^1_{loc}(\mathbb{R}))$ . Moreover, if  $v$  is another solution corresponding to the initial data  $v_0 \in L^\infty(\mathbb{R}; [0, 1])$  such that  $(u_0 - v_0) \in L^1(\mathbb{R})$ , then one has, for all  $t \in \mathbb{R}^+$ ,

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}.$$

## 2 Godunov approximation of the problem and main result

### 2.1 The Godunov approximation

In this section, we introduce the Godunov approximation of the problem (1). For the sake of simplicity, we choose to deal with uniform discretization of  $\mathbb{R} \times \mathbb{R}_+$ . Nevertheless, note that all the following results can be adapted to the case of non-uniform approximations.

Let  $h > 0$ , we denote, for all  $i \in \mathbb{Z}$ ,

$$x_i = ih, \quad x_{i+1/2} = (i + 1/2)h.$$

For all  $i \in \mathbb{Z}$ , we denote by

$$u_{i+1/2}^0 = \frac{1}{h} \int_{x_i}^{x_{i+1}} u_0(x) dx.$$

Let  $k > 0$ , we denote by  $t^n = nk$  ( $n \in \mathbb{N}$ ), and by

$$F^n = \frac{1}{k} \int_{t^n}^{t^{n+1}} F(t) dt, \quad A^n = \max\{s \in [0, 1] \mid f(s) = F^n\}, \quad B^n = \min\{s \in [0, 1] \mid f(s) = F^n\}.$$

We define the Godunov scheme by

$$\frac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{k} h + G_{i+1}^n(u_{i+1/2}^n, u_{i+3/2}^n) - G_i^n(u_{i-1/2}^n, u_{i+1/2}^n) = 0, \quad (10)$$

where  $G_i^n$  is the Godunov solver through the edge  $x_i$ , given by

- the classical Godunov  $G$  solver if  $i \neq 0$ , i.e.

$$G_i^n(a, b) = G(a, b) = \begin{cases} \min_{s \in [a, b]} f(s) & \text{if } a \leq b, \\ \max_{s \in [b, a]} f(s) & \text{if } b \leq a; \end{cases} \quad (11)$$

- the constrained Godunov  $G_0^n$  solver if  $i = 0$ , i.e.

$$G_0^n(a, b) = \min(F^n, G(a, b)). \quad (12)$$

**Proposition 2.1** *The constrained Godunov solver (12) coincides with the Godunov solver for the constrained problem.*

*Proof:* Consider  $u_0(x) = a\chi_{x<0} + b\chi_{x>0}$ . First, solving the Riemann problem with constraint, i.e.

$$\begin{cases} \partial_t v + \partial_x f(v) = 0 & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ v|_{t=0} = a\chi_{x<0} + b\chi_{x>0} & \text{in } \mathbb{R} \end{cases}$$

provides that either the solution  $v$  is continuous at  $x = 0$  and  $t > 0$ , with  $v(0, t) = c$  that does not depend on  $t$ , or we have a contact discontinuity  $f(a) = f(b)$  and  $a < b$ . In any case,  $t \mapsto f(v)(0, t)$  is constant on  $\mathbb{R}_+^*$ , and is exactly given by the formula (11).

Assume first that  $f(v)(0, t) \leq F^n$ , then, clearly,  $v$  satisfies (9). Moreover,  $(v(0^-, t), v(0^+, t))$  either belongs to  $\mathcal{G}_2(t)$  or  $\mathcal{G}_3(t)$  (that do not depend on time, since  $F \equiv F^n$ ). Therefore,  $v$  is the unique solution of the problem (1) for the constant constraint  $F \equiv F^n$ .

Assume now that  $f(v(0, t)) > F^n$ , then one deduces from the formula (11) and from the fact that  $f$  is bell-shaped (2) that  $a \geq B^n$  and  $b \leq A^n$ . In this case, define  $u$  as the solution of two distinct initial boundary value problems

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{in } \mathbb{R}_- \times \mathbb{R}_+ \\ u|_{t=0} = a & \text{in } \mathbb{R}_- \\ u|_{x=0} = A^n & \text{in } \mathbb{R}_+, \end{cases} \quad \begin{cases} \partial_t u + \partial_x f(u) = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ u|_{t=0} = b & \text{in } \mathbb{R}_+ \\ u|_{x=0} = B^n & \text{in } \mathbb{R}_+. \end{cases}$$

Now, it is easy to check that at both side of the interface, the boundary is characteristic so that the boundary condition is fulfilled in a strong sense. Clearly,  $u$  satisfies (9), and its traces on the interface belong to  $\mathcal{G}_3(t)$ , thus  $u$  is the unique solution to the problem (1) for the constant constraint  $F \equiv F^n$ , and the flux at the interface is exactly given by  $F^n$ .  $\square$

It is well-known (and easy to check) that the function  $(a, b) \mapsto G(a, b)$  is  $L_f$ -Lipschitz continuous with respect to both variables, and that  $G$  is non-decreasing w.r.t. its first variable, and non-increasing w.r.t. its second variable.

All along this paper, we assume that the following CFL condition is fulfilled:

$$\frac{2L_f k}{h} \leq 1. \quad (13)$$

In the following, we also assume that the time step is bounded. Without loss of generality, we impose that

$$k \leq 1. \quad (14)$$

**Definition 2.1** *We denote by  $u_h$  the so-called approximate solution, defined almost everywhere by*

$$u_h(x, t) = u_{i+1/2}^n \quad \text{if } (x, t) \in (x_i, x_{i+1}) \times [t^n, t^{n+1}).$$

We now state the  $L^\infty$  stability of the scheme. We refer to [3, Proposition 4.2] for the proof of Proposition 2.2.

**Proposition 2.2** *Under the CFL condition (13), then*

$$0 \leq u_h \leq 1 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}_+.$$

## 2.2 Convergence of the scheme and error estimate

We state here the following convergence result, which is the main result of [3].

**Theorem 2** ([3]) *Let  $F \in L^\infty(\mathbb{R}_+; [0, f(\bar{u})])$ , and let  $u_0 \in L^\infty(\mathbb{R}; [0, 1])$ , then, under the CFL condition*

$$\frac{2L_f k}{h} \leq 1 - \xi, \quad \text{with } \xi \in (0, 1), \quad (15)$$

*then the discrete solution  $u_h$  defined in Definition 2.1 converges in  $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)$  towards the unique solution  $u$  to the problem in the sense of Definition 1.2 as  $h$  tends to 0.*

**Remark 2.1** *Note that the CFL condition (15) is stronger than (13), so that some numerical diffusion stabilizes the scheme. However, we will deduce from this work that, if the data are sufficiently regular (roughly speaking in  $BV_{loc}$ ), then the scheme still converges under the CFL condition (13).*

**Assumption 1** *The functions  $A$  and  $B$  defined in (8) belong to  $BV_{loc}(\mathbb{R}_+)$ .*

In the sequel, we denote by  $\omega_R$  the subset of  $\mathbb{R} \times \mathbb{R}^+$  given by

$$\omega_R = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ \mid |x| \leq R - L_f t\}. \quad (16)$$

Because the solution  $u$  propagates with finite speed lower or equal to  $L_f$ , for  $(x, t) \in \omega_R$ ,  $u(x, t)$  depends only on the restriction  $(u_0)|_{[-R, R]}$  of  $u_0$  to  $[-R, R]$ .

**Theorem 3** *Let  $u_0 \in BV_{loc}(\mathbb{R})$ , let  $u$  be the unique solution of the problem (1), and let  $u_h$  be the discrete solution given by its Godunov approximation. Then, under the CFL condition (13) and Assumption 1, for all  $R > 0$ , there exists  $C$  depending only  $u_0, f, A, B$  and  $R$  such that*

$$\iint_{\omega_R} |u_h(x, t) - u(x, t)| dx dt \leq Ch^{1/2}. \quad (17)$$

The proof of Theorem 3 is based on the doubling variable technique introduced by S. N. KRUŽKOV [17] for proving the uniqueness of the solution of the entropy solution of a multidimensional scalar conservation law, and then used by N. N. KUZNETSOV [18] in order to obtain some error estimate for the approximation of scalar conservation laws by monotone Finite Differences methods.

In order to obtain the optimal convergence rate 1/2 in (17), it is necessary to show that

- (i) the exact solution  $u$  belongs to  $BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$ ,
- (ii) the approximate solution  $u_h$  is uniformly bounded with respect to the discretization in  $BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$ .

## 2.3 Outline of the paper

In Section 3, we introduce additional unknowns  $u_L^n, u_R^n$  corresponding to artificial traces of the discrete solution on the interface  $\{x = 0\}$ .

We will derive in Section 4 an uniform (with respect to the approximation step  $h$ ) estimate for the local total variation of the approximate solution  $u_h$ . As a direct consequence, this will yield an estimate on the local total variation of the exact solution  $u$ .

In Section 6, we use the doubling variable technique for obtaining the error estimate (17).

## 3 Discrete traces on the interface

In this section, we seek to introduce, for all  $n \in \mathbb{N}$ , two artificial discrete traces  $u_L^n, u_R^n$  such that  $(u_L^n, u_R^n)$  belongs to the approximate germ  $\mathcal{G}^n$ , defined by

$$\mathcal{G}^n = \mathcal{G}_1^n \cup \mathcal{G}_2^n$$

with

$$\mathcal{G}_1^n = (A^n, B^n), \quad \mathcal{G}_2^n = \{(c, c) \in [0, 1]^2 \mid f(c) < F^n\},$$

and then to derive some properties on them.

**Proposition 3.1** For all  $n \in \mathbb{N}$ , there exists  $(u_L^n, u_R^n) \in \mathcal{G}^n$  such that

$$G_0^n(u_{-1/2}^n, u_{1/2}^n) = G(u_{-1/2}^n, u_L^n) = G(u_R^n, u_{-1/2}^n) = f(u_L^n) = f(u_R^n). \quad (18)$$

In the case where  $G(u_{-1/2}^n, u_{1/2}^n) < F^n \leq f(\bar{u})$ , one has either  $u_L^n = u_R^n = u_{-1/2}^n$  or  $u_L^n = u_R^n = u_{1/2}^n$ . Moreover, the case  $u_L^n = A^n$  and  $u_R^n = B^n$  only occurs when  $u_{-1/2}^n \leq A^n$  and  $u_{-1/2}^n \geq B^n$ .

*Proof:* We can prove the above Proposition by a case by case study. For reader's convenience, we drop the index  $n$ . Let  $u_{-1/2}$  and  $u_{1/2}$  belong to  $[0, 1]$ , then we define by  $u_{-1/2}^* \in [\bar{u}, 1]$  and  $u_{1/2}^* \in [0, \bar{u}]$  by

$$f(u_{-1/2}^*) = f(u_{-1/2}), \quad f(u_{1/2}^*) = f(u_{1/2}).$$

1. Assume that  $0 \leq u_{-1/2} < B^n$ .
  - (a) If  $u_{1/2} \leq u_{-1/2}^*$ , then  $u_L = u_R = u_{-1/2}$  satisfies (18).
  - (b) If  $u_{1/2} > u_{-1/2}^*$ , then  $u_L = u_R = u_{1/2}$  satisfies (18).
2. Assume that  $B^n \leq u_{-1/2} \leq 1$ .
  - (a) If  $0 \leq u_{1/2} \leq A^n$ , then  $u_L = A^n$  and  $u_R = B^n$  satisfies (18).
  - (b) If  $A^n < u_{1/2} \leq 1$ , then  $u_L = u_R = u_{1/2}$  satisfies (18).

□

**Remark 3.1** Note that  $u_L^n, u_R^n$  are not in general unique. Indeed, assume that  $u_{-1/2}^n < B^n$ , and that  $u_{1/2}^n = u_{-1/2}^{n,*}$ , then, as  $(u_L^n, u_R^n)$ , one can either choose  $(u_{-1/2}^n, u_{-1/2}^n)$  or  $(u_{1/2}^n, u_{1/2}^n)$ . However, the flux  $f(u_L^n) = f(u_R^n)$  through the interface  $\{x = 0\}$  is unique.

In the sequel, we denote by  $u_{L,h}$  and  $u_{R,h}$  the functions defined by

$$u_{L,h}(t) := u_L^n, \quad u_{R,h}(t) = u_R^n \quad \text{for } t \in [t^n, t^{n+1}). \quad (19)$$

## 4 BV estimates

### 4.1 BV estimate on the approximate constraint

**Lemma 4.1** Under Assumption 1, the functions  $A_h, B_h$  defined respectively by  $A_h(t) = A^n$ ,  $B_h(t) = B^n$  if  $t \in [t^n, t^{n+1})$  belong to  $BV_{loc}(\mathbb{R}_+)$ , and, for all  $T > 0$ ,

$$TV_{[0,T]}(A_h - B_h) = TV_{[0,T]}(A_h) + TV_{[0,T]}(B_h) \leq TV_{[0,T]}(A) + TV_{[0,T]}(B) + 1. \quad (20)$$

*Proof:* Assume first that the functions  $A, B$  are smooth, so  $F = f(A) = f(B)$  is Lipschitz continuous. Then, for all  $n \in \mathbb{N}$ , there exists  $\tilde{t}^n \in (t^n, t^{n+1})$  such that  $F^n = F(\tilde{t}^n)$ . Hence,

$$A^n = A(\tilde{t}^n), \quad B^n = B(\tilde{t}^n).$$

Thus, denoting by  $N$  the index of the time step where  $t^N \leq T < t^{N+1}$ , one gets that

$$\begin{aligned} TV_{[0,T]}(A_h) &= \sum_{n=0}^{N-1} |A^{n+1} - A^n| = \sum_{n=0}^{N-1} |A(\tilde{t}^{n+1}) - A(\tilde{t}^n)| \\ &= \sum_{n=0}^{N-2} |A(\tilde{t}^{n+1}) - A(\tilde{t}^n)| + |A(\tilde{t}^N) - A(\tilde{t}^{N-1})|. \end{aligned}$$

Since

$$|A(\tilde{t}^N) - A(\tilde{t}^{N-1})| \leq 1 - \bar{u},$$

and

$$\sum_{n=0}^{N-2} |A(\tilde{t}^{n+1}) - A(\tilde{t}^n)| \leq TV_{[0,T]}(A),$$

one obtains that

$$TV_{[0,T]}(A_h) \leq TV_{[0,T]}(A) + (1 - \bar{u}).$$

Similarly, we can state that

$$TV_{[0,T]}(B_h) \leq TV_{[0,T]}(B) + \bar{u},$$

so that the estimate (20) holds for smooth functions  $A, B$ . Assume now that  $A, B$  only belong to  $BV_{loc}(\mathbb{R}_+)$ , then there exists some sequences  $(A_\nu)_{\nu \in \mathbb{N}}$  and  $(B_\nu)_{\nu \in \mathbb{N}}$  of smooth functions (obtained for example by convolution with smoothing kernels) such that

$$\begin{aligned} A_\nu &\rightarrow A, & B_\nu &\rightarrow B \quad \text{a.e. in } \mathbb{R}_+ \text{ as } \nu \rightarrow \infty, \\ TV_{[0,T]}(A_\nu) &\leq TV_{[0,T]}(A), & TV_{[0,T]}(B_\nu) &\leq TV_{[0,T]}(B), \end{aligned}$$

and

$$A_{\nu,h} \rightarrow A_h, \quad B_{\nu,h} \rightarrow B_h \quad \text{a.e. in } \mathbb{R}_+ \text{ as } \nu \rightarrow \infty.$$

Then we can pass to the limit and extend (20) to functions  $A, B$  merely in  $BV_{loc}(\mathbb{R}_+)$ .  $\square$

## 4.2 Space $BV$ estimate on the approximate solution

In the sequel, we introduce a modified total variation, that takes the discrete traces into account:

$$\mathcal{TV}(u_h(\cdot, t^n)) = \sum_{i \neq 0} \left| u_{i+1/2}^n - u_{i-1/2}^n \right| + \left| u_{-1/2}^n - u_L^n \right| + \left| u_{1/2}^n - u_R^n \right|,$$

while the classical total variation of  $u_h(\cdot, t^n)$  is given by

$$TV(u_h(\cdot, t^n)) = \sum_{i \in \mathbb{Z}} \left| u_{i+1/2}^n - u_{i-1/2}^n \right|.$$

We first state the following technical lemma.

**Lemma 4.2** *Let  $(c, d) \in [0, 1]$  such that  $c \geq d$ . Then, for all  $(a, b) \in [d, 1] \times [0, c]$ , one has*

$$\|c - a\| + \|d - b\| - \|a - b\| \leq c - d.$$

*Proof:* Denote by  $\Psi(a, b) = \|c - a\| + \|d - b\| - \|a - b\|$ . The proof is performed using a case by case study, summarized in the following tabular. Note that only 4 cases are used in the proof of Lemma 4.2, but the other cases will be used later. In this tabular, we denote by  $a \top b = \max(a, b)$  and  $a \perp b = \min(a, b)$ .

	$a \in [0, d]$	$a \in [d, c]$	$a \in [c, 1]$
$b \in [0, d]$	$\Psi(a, b) = c + d - 2(a \top b)$	$\Psi(a, b) = c + d - 2a$	$\Psi(a, b) = d - c$
$b \in [d, c]$	$\Psi(a, b) = c - d$	$\Psi(a, b) = c - d - 2(a - b)^+$	$\Psi(a, b) = 2b - (d + c)$
$b \in [c, 1]$	$\Psi(a, b) = c - d$	$\Psi(a, b) = c - d$	$\Psi(a, b) = 2(a \perp b) - (c + d)$

(21)

$\square$

**Lemma 4.3** *For all  $t \geq 0$ , one has*

$$|\mathcal{TV}(u_h(\cdot, t)) - TV(u_h(\cdot, t))| \leq 1.$$

*Proof:* Let  $t \in [t^n, t^{n+1})$ . Assume that  $(u_L^n, u_R^n) \in \mathcal{G}_2^n$ , then  $\mathcal{TV}(u_h(\cdot, t)) = TV(u_h(\cdot, t))$ . We now focus on the case where  $(u_L^n, u_R^n) = (A^n, B^n)$ . As seen in Proposition 3.1, this implies that  $u_{-1/2}^n \geq B^n$  and  $u_{1/2}^n \leq A^n$ . Since

$$\mathcal{TV}(u_h(\cdot, t)) - TV(u_h(\cdot, t)) = |u_{-1/2}^n - A^n| + |u_{1/2}^n - B^n| - |u_{-1/2}^n - u_{1/2}^n|,$$

then using the lemma 4.2 with  $a = u_{-1/2}^n$ ,  $b = u_{1/2}^n$ ,  $c = A^n$ ,  $d = B^n$  provides the result.  $\square$

Denote by  $\Lambda \subset \mathbb{N}$  the set of the times steps where the constraint is saturated, i.e.

$$p \in \Lambda \iff G(u_{-1/2}^p, u_{1/2}^p) \geq F^p,$$

by

$$\underline{\Lambda} = \{p \in \mathbb{N} \mid p \notin \Lambda, p+1 \in \Lambda\}, \quad \overline{\Lambda} = \{p \in \Lambda \mid p+1 \notin \Lambda\}, \quad \overset{\circ}{\Lambda} = \Lambda \setminus \overline{\Lambda},$$

and by

$$\Upsilon = \mathbb{N} \setminus (\underline{\Lambda} \cup \Lambda) = \{p \in \mathbb{N} \mid p \notin \Lambda, (p+1) \notin \Lambda\},$$

so that we have

$$\mathbb{N} = \Upsilon \cup \overset{\circ}{\Lambda} \cup \underline{\Lambda} \cup \overline{\Lambda}.$$

**Lemma 4.4** *Let  $p \in \Upsilon$ , then, under the CFL condition (13),*

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)).$$

*Proof:* Since  $p \notin \Lambda$ , then  $u_h(\cdot, t^{p+1})$  is the solution computed by the classical Godunov scheme without constraint. Hence, it follows from classical computations (see e.g. [15] or Lemma 5.7 in [14]) that

$$\sum_{i \in \mathbb{Z}} |u_{i+1/2}^{p+1} - u_{i-1/2}^{p+1}| \leq \sum_{i \in \mathbb{Z}} |u_{i+1/2}^p - u_{i-1/2}^p|. \quad (22)$$

Now, since  $p \in \Upsilon$  then, thanks to Proposition 3.1, for  $q \in \{p, p+1\}$ , either  $u_L^q = u_R^q = u_{-1/2}^q$  or  $u_L^q = u_R^q = u_{1/2}^q$ . As a consequence,

$$\mathcal{TV}(u_h(\cdot, t^q)) = \sum_{i \in \mathbb{Z}} |u_{i+1/2}^q - u_{i-1/2}^q|. \quad (23)$$

Lemma 4.4 is so a direct consequence of (22) and (23).  $\square$

**Lemma 4.5** *Let  $p \in \underline{\Lambda}$ , then, under the CFL condition (13),*

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)) + (A^{p+1} - B^{p+1}).$$

*Proof:* Since  $p \notin \Lambda$ , then, as previously,

$$\sum_{i \in \mathbb{Z}} |u_{i+1/2}^{p+1} - u_{i-1/2}^{p+1}| \leq \sum_{i \in \mathbb{Z}} |u_{i+1/2}^p - u_{i-1/2}^p| = \mathcal{TV}(u_h(\cdot, t^p)).$$

Since  $(u_L^{p+1}, u_R^{p+1}) = (A^{p+1}, B^{p+1})$ , then

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)) + \mathcal{R}^{p+1}, \quad (24)$$

where

$$\mathcal{R}^{p+1} = |u_{-1/2}^{p+1} - A^{p+1}| + |u_{1/2}^{p+1} - B^{p+1}| - |u_{1/2}^{p+1} - u_{-1/2}^{p+1}|. \quad (25)$$

Since  $p+1 \in \Lambda$ , then one has, thanks to Proposition 3.1,

$$u_{-1/2}^{p+1} \geq B^{p+1} \quad \text{and} \quad u_{1/2}^{p+1} \leq A^{p+1}.$$

Hence we can apply Lemma 4.2 to claim that

$$\mathcal{R}^{p+1} \leq A^{p+1} - B^{p+1}.$$

$\square$

We investigate now the cases where the constraint at the time step  $t^p$  is saturated, i.e. if  $p \in \overset{\circ}{\Lambda} \cup \bar{\Lambda}$ . In these cases,  $(u_L^p, u_R^p) = (A^p, B^p)$ , and adapting once again the computations of [14, Lemma 5.7], we get that, under the CFL condition (13),

$$\begin{aligned} \sum_{i>0} \left| u_{i+1/2}^{p+1} - u_{i-1/2}^{p+1} \right| + |u_{1/2}^{p+1} - B^p| &\leq \sum_{i>0} \left| u_{i+1/2}^p - u_{i-1/2}^p \right| + |u_{1/2}^p - B^p|. \\ \sum_{i<0} \left| u_{i+1/2}^{p+1} - u_{i-1/2}^{p+1} \right| + |u_{-1/2}^{p+1} - A^p| &\leq \sum_{i<0} \left| u_{i+1/2}^p - u_{i-1/2}^p \right| + |u_{-1/2}^p - A^p|. \end{aligned}$$

As a direct consequence of the inequalities stated above, one has

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)) + \mathcal{R}^{p+1}, \quad (26)$$

where

$$\mathcal{R}^{p+1} = |u_{-1/2}^{p+1} - u_L^{p+1}| - |u_{-1/2}^{p+1} - A^p| + |u_{1/2}^{p+1} - u_R^{p+1}| - |u_{1/2}^{p+1} - B^p|. \quad (27)$$

**Lemma 4.6** *Let  $p \in \overset{\circ}{\Lambda}$ , then, under the CFL condition (13),*

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)) + |(A^{p+1} - B^{p+1}) - (A^p - B^p)|.$$

*Proof:* Since  $p+1 \in \Lambda$ , one has  $(u_L^{p+1}, u_R^{p+1}) = (A^{p+1}, B^{p+1})$ . Replacing in (27) leads to, thanks to the triangle inequality,

$$\mathcal{R}^{p+1} \leq |A^{p+1} - A^p| + |B^{p+1} - B^p|$$

Since

$$A^{p+1} \geq A^p \Leftrightarrow B^{p+1} \leq B^p,$$

one obtains that

$$\mathcal{R}^{p+1} \leq |A^{p+1} - A^p| + |B^{p+1} - B^p| = |(A^{p+1} - B^{p+1}) - (A^p - B^p)|. \quad (28)$$

We conclude by using (28) in (26).  $\square$

**Lemma 4.7** *Let  $p \in \bar{\Lambda}$ , then, under the CFL condition (13),*

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) \leq \mathcal{TV}(u_h(\cdot, t^p)) + (B^{p+1} - A^{p+1}) + |A^{p+1} - A^p| + |B^{p+1} - B^p|.$$

*Proof:* Since, thanks to Proposition 3.1,  $u_L^{p+1} = u_R^{p+1} \in \{u_{-1/2}^{p+1}, u_{1/2}^{p+1}\}$ , then the expression (27) turns to

$$\mathcal{R}^{p+1} \leq \mathcal{R}_1^{p+1} + |A^{p+1} - A^p| + |B^{p+1} - B^p|,$$

with

$$\mathcal{R}_1^{p+1} = |u_{1/2}^{p+1} - u_{-1/2}^{p+1}| - |u_{-1/2}^{p+1} - A^{p+1}| - |u_{1/2}^{p+1} - B^{p+1}|$$

Since  $p+1 \notin \Lambda$ , we known from the case by case study carried out in Proposition 3.1 that  $u_{-1/2}^{p+1} \leq B^{p+1}$  or  $u_{1/2}^{p+1} \geq A^{p+1}$ . We deduce from the first column and the last line of (21) that  $\mathcal{R}_1^{p+1} \leq B^{p+1} - A^{p+1}$ .  $\square$

**Lemma 4.8** *Assume that  $u_0 \in BV(\mathbb{R})$ , then, under the CFL condition (13) and Assumption 1, there exists  $C$  depending only on  $A, B$  and  $T$  (but neither on  $h$  nor on  $k$ ) such that*

$$\mathcal{TV}(u_h(\cdot, T)) \leq \mathcal{TV}(u_0) + C. \quad (29)$$

*Proof:* Let  $n \in \mathbb{N}$  be such that  $T \in [t^n, t^{n+1})$ , then

$$\mathcal{TV}(u_h(\cdot, T)) = \mathcal{TV}(u_h(\cdot, 0)) + \sum_{p=0}^{n-1} (\mathcal{TV}(u_h(\cdot, t^{p+1})) - \mathcal{TV}(u_h(\cdot, t^p))).$$

From Lemmata 4.4, 4.5, 4.6 and 4.7, we deduce that

$$\mathcal{TV}(u_h(\cdot, t^{p+1})) - \mathcal{TV}(u_h(\cdot, t^p)) \leq \begin{cases} 0, & \text{if } p \in \Upsilon, \\ A^{p+1} - B^{p+1}, & \text{if } p \in \underline{\Delta}, \\ |(A^{p+1} - B^{p+1}) - (A^p - B^p)|, & \text{if } p \in \overset{\circ}{\Delta}, \\ B^{p+1} - A^{p+1} + |(A^{p+1} - B^{p+1}) - (A^p - B^p)| & \text{if } p \in \overline{\Delta}. \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{TV}(u_h(\cdot, T)) &\leq \mathcal{TV}(u_h(\cdot, 0)) + \sum_{\substack{p \leq n-1 \\ p \in \underline{\Delta}}} |(A^{p+1} - B^{p+1}) - (A^p - B^p)| \\ &\quad + \sum_{\substack{p \leq n-1 \\ p \in \underline{\Delta}}} (A^{p+1} - B^{p+1}) + \sum_{\substack{p \leq n-1 \\ p \in \overline{\Delta}}} (B^{p+1} - A^{p+1}). \end{aligned} \quad (30)$$

Since for all  $p, q \in \underline{\Delta}$  with  $p < q$ , there exists  $r \in \overline{\Delta}$  such that  $p < r < q$ , and since  $|A^{p+1} - B^{p+1}| \leq 1$ , it follows that

$$\sum_{\substack{p \leq n-1 \\ p \in \underline{\Delta}}} (A^{p+1} - B^{p+1}) + \sum_{\substack{p \leq n-1 \\ p \in \overline{\Delta}}} (B^{p+1} - A^{p+1}) \leq 1 + \sum_{\substack{p \leq n-1 \\ p \in \underline{\Delta}}} |(A^{p+1} - B^{p+1}) - (A^p - B^p)|. \quad (31)$$

Thus taking (31) into account in (30) yields

$$\begin{aligned} \mathcal{TV}(u_h(\cdot, T)) &\leq \mathcal{TV}(u_h(\cdot, 0)) + 1 + 2 \sum_{\substack{p \leq n-1 \\ p \in \underline{\Delta}}} |(A^{p+1} - B^{p+1}) - (A^p - B^p)| \\ &\leq \mathcal{TV}(u_h(\cdot, 0)) + 1 + 2TV_{[0, T]}(A_h - B_h). \end{aligned}$$

We conclude by using Lemmas 4.1 and 4.3.  $\square$

**Proposition 4.9** *Let  $u_0 \in BV(\mathbb{R})$ , then, under the CFL condition (13) and Assumption 1, there exists  $C$  depending only on  $u_0, T, A, B$  such that, for all  $\xi > 0$ ,*

$$\int_0^T \int_{\mathbb{R}} |u_h(x + \xi, t) - u_h(x, t)| dx dt \leq C\xi.$$

*Proof:* It follows from Lemma 4.8 that the function  $u_h(\cdot, t)$  has a bounded variation for all  $t \in [0, T]$ , thus there exists  $C$  depending only on  $u_0, A, B, T$  such that

$$\int_{\mathbb{R}} |u_h(x + \xi, t) - u_h(x, t)| dx \leq C\xi.$$

We conclude by integrating w.r.t. to  $t \in [0, T]$ .  $\square$

### 4.3 Time $BV$ estimate on the approximate solution

**Lemma 4.10** *Let  $u_0 \in BV(\mathbb{R})$ , then, one has*

$$\sum_{i \in \mathbb{Z}} |u_{i+1/2}^{n+1} - u_{i+1/2}^n| \leq \frac{2L_f k}{h} \mathcal{TV}(u_h(\cdot, t^n)). \quad (32)$$

*Proof:* The scheme (10) can be rewritten

$$\begin{aligned} u_{i+1/2}^{n+1} - u_{i+1/2}^n &= \frac{k}{h} \left( G(u_{i-1/2}^n, u_{i+1/2}^n) - f(u_{i+1/2}^n) - \left( G(u_{i+1/2}^n, u_{i+3/2}^n) - f(u_{i+1/2}^n) \right) \right) \text{ if } i \notin \{-1, 0\}, \\ u_{-1/2}^{n+1} - u_{-1/2}^n &= \frac{k}{h} \left( G(u_{-3/2}^n, u_{-1/2}^n) - f(u_{-1/2}^n) - \left( G(u_{-1/2}^n, u_L^n) - f(u_{-1/2}^n) \right) \right), \\ u_{1/2}^{n+1} - u_{1/2}^n &= \frac{k}{h} \left( G(u_R^n, u_{1/2}^n) - f(u_{1/2}^n) - \left( G(u_{1/2}^n, u_{3/2}^n) - f(u_{1/2}^n) \right) \right). \end{aligned}$$

Using the fact that  $G$  is  $L_f$ -Lipschitz continuous w.r.t. each of its variables, we obtain that

$$|u_{i+1/2}^{n+1} - u_{i+1/2}^n| \leq \frac{L_f k}{h} \left( |u_{i-1/2}^n - u_{i+1/2}^n| + |u_{i+1/2}^n - u_{i+3/2}^n| \right) \quad \text{if } i \notin \{-1, 0\}, \quad (33)$$

$$|u_{-1/2}^{n+1} - u_{-1/2}^n| \leq \frac{L_f k}{h} \left( |u_L^n - u_{-1/2}^n| + |u_{-1/2}^n - u_{-3/2}^n| \right), \quad (34)$$

$$|u_{1/2}^{n+1} - u_{1/2}^n| \leq \frac{L_f k}{h} \left( |u_{1/2}^n - u_R^n| + |u_{3/2}^n - u_{1/2}^n| \right). \quad (35)$$

Summing (33) for  $i \in \mathbb{Z} \setminus \{-1, 0\}$  with (34) and (35) yields (32).  $\square$

**Proposition 4.11** *Let  $u_0 \in BV(\mathbb{R})$ , then, under the CFL condition (13), (14) and Assumption 1, for all  $T > 0$ , there exists  $C$  depending only on  $A, B, T, u_0$  and  $L_f$  such that, for all  $\tau \in (0, T)$ ,*

$$\int_0^{T-\tau} \int_{\mathbb{R}} |u_h(x, t + \tau) - u_h(x, t)| dx dt \leq C\tau. \quad (36)$$

*Proof:* One has (with a slight abuse of notation, since  $\partial_t u_h$  is a bounded Radon measure on  $\mathbb{R} \times [0, T]$  which is not absolutely continuous w.r.t. to the Lebesgue measure)

$$\int_{\mathbb{R}} \int_0^T |\partial_t u_h(x, t)| dx dt = \sum_{i \in \mathbb{Z}} \sum_{n=0}^{\lfloor T/k \rfloor} |u_{i+1/2}^{n+1} - u_{i+1/2}^n| h.$$

Then it follows from Lemmata 4.10 and 4.8 that

$$\int_{\mathbb{R}} \int_0^T |\partial_t u_h(x, t)| dx dt \leq C \sum_{n=0}^{\lfloor T/k \rfloor} k.$$

Using (14), we obtain that

$$\int_{\mathbb{R}} \int_0^T |\partial_t u_h(x, t)| dx dt \leq C. \quad (37)$$

The inequality (36) is a classical consequence of the previous estimate (see e.g. [4]).  $\square$

#### 4.4 $BV$ estimates on the exact solution

Letting now  $h$  tend to 0, since we know, thanks to Theorem 2, that  $u_h$  tends to the unique solution  $u$  (at least under the more restrictive CFL condition (15)) we obtain the following regularity result on the exact solution  $u$ .

**Proposition 4.12** *Let  $u$  be the exact solution to the problem corresponding to  $u_0 \in BV(\mathbb{R})$ . Then, under Assumption 1, then, for all  $T > 0$ ,  $u \in BV(\mathbb{R} \times [0, T])$ .*

**Remark 4.1** *Because of the finite speed propagation property, the solution  $u$  to the problem depends, on the set  $\omega_R$  defined in (16), only on the restriction of the initial data  $u_0$  to  $[-R, R]$ . So, if  $u_0 \in BV_{loc}(\mathbb{R})$ , extending  $u_0$  by a constant outside of  $[-R, R]$  will not affect the solution  $u$  on  $\omega_R$ . Thus the Proposition 4.12 can be generalized in the following way. If  $u_0 \in BV_{loc}(\mathbb{R})$ , then, under Assumption 1,  $u$  belongs to  $BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$ .*

## 5 Entropy formulations for the discrete solution

### 5.1 Discrete entropy inequalities

Using the discrete traces  $u_L^n, u_R^n$  introduced in Section 3, the scheme can be rewritten under the form

$$H \left( u_{i+1/2}^{n+1}, u_{i+1/2}^n, u_{i-1/2}^n, u_{i+3/2}^n \right) = 0, \quad \forall i \notin \{-1, 0\},$$

$$H \left( u_{-1/2}^{n+1}, u_{-1/2}^n, u_{-3/2}^n, u_L^n \right) = 0,$$

$$H\left(u_{1/2}^{n+1}, u_{1/2}^n, u_R^n, u_{3/2}^n\right) = 0,$$

where, under the CFL condition (13), the function  $H$  is non-decreasing w.r.t. its first argument, and non-increasing w.r.t. its three last arguments. As a consequence, if  $(a, b, c, d) \in [0, 1]^4$  satisfies

$$H(a, b, c, d) = 0,$$

then, thanks to the fact that, for all  $\kappa \in [0, 1]$ ,

$$H(\kappa, \kappa, \kappa, \kappa) = 0,$$

then it follows from classical computations (see e.g. [14]) that

$$H(a \top \kappa, b \top \kappa, c \top \kappa, d \top \kappa) - H(a \perp \kappa, b \perp \kappa, c \perp \kappa, d \perp \kappa) \leq 0, \quad (38)$$

where  $a \top \kappa = \max(a, \kappa)$  and  $a \perp \kappa = \min(a, \kappa)$ . In the sequel, we denote by

$$\Phi_\kappa(a, b) = G(a \top \kappa, b \top \kappa) - G(a \perp \kappa, b \perp \kappa).$$

Note that for all  $a \in [0, 1]$ , for all  $\kappa \in [0, 1]$ ,  $\Phi_\kappa(a, a) = \Phi_\kappa(a)$ , where the notation  $\Phi_\kappa(a)$  has been introduced in (5). The following proposition follows from (38).

**Proposition 5.1** *For all  $\kappa \in [0, 1]$ , one has*

$$\frac{|u_{i+1/2}^{n+1} - \kappa| - |u_{i+1/2}^n - \kappa|}{k} h + \Phi_\kappa(u_{i+1/2}^n, u_{i+3/2}^n) - \Phi_\kappa(u_{i-1/2}^n, u_{i+1/2}^n) \leq 0, \quad \forall i \in \mathbb{Z} \setminus \{-1, 0\}, \quad (39)$$

$$\frac{|u_{-1/2}^{n+1} - \kappa| - |u_{-1/2}^n - \kappa|}{k} h + \Phi_\kappa(u_{-1/2}^n, u_L^n) - \Phi_\kappa(u_{-3/2}^n, u_{-1/2}^n) \leq 0, \quad (40)$$

$$\frac{|u_{1/2}^{n+1} - \kappa| - |u_{1/2}^n - \kappa|}{k} h + \Phi_\kappa(u_{1/2}^n, u_{3/2}^n) - \Phi_\kappa(u_R^n, u_{1/2}^n) \leq 0. \quad (41)$$

**Lemma 5.2** *For all  $\kappa \in [0, 1]$ ,*

$$\Phi_\kappa(u_{-1/2}^n, u_L^n) - \Phi_\kappa(u_L^n) \geq 0, \quad (42)$$

$$\Phi_\kappa(u_R^n) - \Phi_\kappa(u_R^n, u_{1/2}^n) \geq 0. \quad (43)$$

*Proof:* We only prove (42), since the proof of (43) is similar. Here again, for readers convenience, we drop the index  $n$ . We denote by  $I(a, b)$  the interval  $[a, b]$  if  $a \leq b$  and  $[b, a]$  otherwise.

Firstly, if  $\kappa \notin I(u_{-1/2}, u_L)$ , then, using that  $G(u_{-1/2}, u_L) = f(u_L)$ , one has

$$\Phi_\kappa(u_{-1/2}, u_L) = \Phi_\kappa(u_L).$$

Consider now the case where  $\kappa \in I(u_{-1/2}, u_L)$ . Since  $G(u_{-1/2}, u_L) = f(u_L)$ , the function  $a \mapsto G(a, u_L)$  is constant on  $I(u_{-1/2}, u_L)$ . Assume that  $u_{-1/2} \geq u_L$ , then

$$\Phi_\kappa(u_{-1/2}, u_L) = G(u_{-1/2}, \kappa) - G(\kappa, u_L) = G(u_{-1/2}, \kappa) - f(u_L).$$

Since  $G$  is non-increasing w.r.t. its second argument,  $G(u_{-1/2}, \kappa) \geq f(\kappa)$ , hence one has

$$\Phi_\kappa(u_{-1/2}, u_L) \geq f(\kappa) - f(u_L) = \Phi_\kappa(u_L).$$

Similarly, if  $u_{-1/2} \leq u_L$ , one obtains  $\Phi_\kappa(u_{-1/2}, u_L) \geq f(u_L) - f(\kappa) = \Phi_\kappa(u_L)$ .  $\square$

We now state the straightforward corollary, obtained by subtracting (42) to (40) and (43) to (41).

**Corollary 5.3** *For all  $\kappa \in [0, 1]$ ,*

$$\frac{|u_{-1/2}^{n+1} - \kappa| - |u_{-1/2}^n - \kappa|}{k} h + \Phi_\kappa(u_L^n) - \Phi_\kappa(u_{-3/2}^n, u_{-1/2}^n) \leq 0, \quad (44)$$

$$\frac{|u_{1/2}^{n+1} - \kappa| - |u_{1/2}^n - \kappa|}{k} h + \Phi_\kappa(u_{1/2}^n, u_{3/2}^n) - \Phi_\kappa(u_R^n) \leq 0. \quad (45)$$

## 5.2 Continuous entropy inequalities

For a  $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$  function  $\varphi$ , we denote by

$$\|\nabla\varphi\| = \|\partial_t\varphi\|_\infty + \|\partial_x\varphi\|_\infty.$$

Recall that when  $\varphi$  is compactly supported, i.e. if  $\varphi \in \mathcal{C}_c^1((-R, R) \times [0, T])$ , then there exists  $C$  depending only on  $R, T$  such that

$$\|\varphi\|_\infty \leq C\|\nabla\varphi\|.$$

**Lemma 5.4** *Let  $T > 0$ , and let  $\varphi \in \mathcal{D}^+((-R, R) \times [0, T])$ , then there exists  $C$  depending only on  $u_0, f, A, B, R$  and  $T$  such that, for all  $\kappa \in [0, 1]$ ,*

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u_h - \kappa| \partial_t \varphi dx dt + \int_{\mathbb{R}_-} |u_0 - \kappa| \varphi(\cdot, 0) dx \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_\kappa(u_h) \partial_x \varphi dx dt - \int_{\mathbb{R}_+} \Phi_\kappa(u_{L,h}) \varphi(0, \cdot) dt \geq -C\|\nabla\varphi\|h. \end{aligned} \quad (46)$$

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |u_h - \kappa| \partial_t \varphi dx dt + \int_{\mathbb{R}_+} |u_0 - \kappa| \varphi(\cdot, 0) dx \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_\kappa(u_h) \partial_x \varphi dx dt + \int_{\mathbb{R}_+} \Phi_\kappa(u_{R,h}) \varphi(0, \cdot) dt \geq -C\|\nabla\varphi\|h. \end{aligned} \quad (47)$$

*Proof:* We only prove (46) since the proof of (47) is similar. Let  $\varphi \in \mathcal{D}^+((-R, R) \times [0, T])$ , we denote by

$$\varphi_i^n = \varphi(x_i, t^n), \quad \varphi_{i+1/2}^n = \varphi(x_{i+1/2}, t^n), \quad \forall i \in \mathbb{Z}, \forall n \in \mathbb{N}.$$

Multiplying equations (39),(44), respectively by  $k\varphi_{i+1/2}^{n+1}$ ,  $k\varphi_{-1/2}^{n+1}$ , and summing on  $i < -1$  yields, after reorganization of the sums,

$$T_1 + T_2 + T_3 + T_4 \geq 0, \quad (48)$$

with

$$\begin{aligned} T_1 &= \sum_{n \in \mathbb{N}} \sum_{i < 0} |u_{i+1/2}^n - \kappa| (\varphi_{i+1/2}^{n+1} - \varphi_{i+1/2}^n) h + \sum_{i < 0} |u_{i+1/2}^0 - \kappa| \varphi_{i+1/2}^0 h \\ T_2 &= \sum_{n \in \mathbb{N}} k \sum_{i < 0} \Phi_\kappa(u_{i-1/2}^n, u_{i+1/2}^n) (\varphi_{i+1/2}^{n+1} - \varphi_{i-1/2}^{n+1}) h \\ T_3 &= - \sum_{n \in \mathbb{N}} k \Phi_\kappa(u_L^n) \varphi_0^{n+1} \\ T_4 &= - \sum_{n \in \mathbb{N}} k \Phi_\kappa(u_L^n) (\varphi_{-1/2}^{n+1} - \varphi_0^{n+1}). \end{aligned}$$

Firstly, it is easy to check that

$$|T_4| \leq C\|\nabla\varphi\|h, \quad (49)$$

and that

$$\left| T_3 + \int_{\mathbb{R}_+} \Phi_\kappa(u_{L,h}) \varphi(0, \cdot) dt \right| \leq C\|\nabla\varphi\|h, \quad (50)$$

It follows from Propositions 4.9 and 4.11 (we use here classical computations that we can deduce for example from [10]) and the CFL condition (13) that

$$\left| T_1 - \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u_h - \kappa| \partial_t \varphi dx dt - \int_{\mathbb{R}_-} |u_0 - \kappa| \varphi(\cdot, 0) dx \right| \leq Ch\|\nabla\varphi\|, \quad (51)$$

$$\left| T_2 - \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_\kappa(u_h) \partial_x \varphi dx dt \right| \leq Ch\|\nabla\varphi\|. \quad (52)$$

Then (46) follows from (48)–(52).  $\square$

As a direct consequence of Lemma 5.4, following the idea of R. EYMARD *et al.* [13], exploited by F. BOUCHUT and B. PERTHAME [6], we can state the following proposition.

**Proposition 5.5** *There exist positive local Radon measures  $\mu_{L,h}, \mu_{R,h}$  belonging to  $(C_c(\mathbb{R} \times \mathbb{R}_+))'$  such that there exists  $C$  depending only on  $R, L_f$*

$$\mu_{L,h}(\omega_R) \leq Ch, \quad \mu_{R,h}(\omega_R) \leq Ch, \quad (53)$$

and such that, for all  $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$ , one has

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{-\infty}^0 |u_h - \kappa| \partial_t \varphi dx dt + \int_{-\infty}^0 |u_0 - \kappa| \varphi(\cdot, 0) dx \\ & + \int_{\mathbb{R}_+} \int_{-\infty}^0 \Phi_\kappa(u_h) \partial_x \varphi dx dt - \int_{\mathbb{R}_+} \Phi_\kappa(u_{L,h}) \varphi(0, \cdot) dt \geq - \langle \mu_{L,h}, |\partial_t \varphi| + |\partial_x \varphi| \rangle, \end{aligned} \quad (54)$$

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_0^{+\infty} |u_h - \kappa| \partial_t \varphi dx dt + \int_0^{+\infty} |u_0 - \kappa| \varphi(\cdot, 0) dx \\ & + \int_{\mathbb{R}_+} \int_0^{+\infty} \Phi_\kappa(u_h) \partial_x \varphi dx dt + \int_{\mathbb{R}_+} \Phi_\kappa(u_{R,h}) \varphi(0, \cdot) dt \geq - \langle \mu_{R,h}, |\partial_t \varphi| + |\partial_x \varphi| \rangle. \end{aligned} \quad (55)$$

In section 6, we will use the doubling variable technique introduced by S. N. KRŪZKOV [17] and adapted to this frame by N. N. KUZNETSOV [18]. For this reason, we will assume that the exact solution  $u$  depends on the variable  $(y, s)$  instead of  $(x, t)$ . Since  $u$  admits strong traces  $u_L, u_R \in L^\infty(\mathbb{R}_+; [0, 1])$  on  $\{y = 0\} \times \mathbb{R}_+$  (either thanks to Proposition 4.12 or to [22]), then  $u$  satisfies the following entropy inequalities :  $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$ ,

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u - \kappa| \partial_s \psi dy ds + \int_{\mathbb{R}_-} |u_0 - \kappa| \psi(\cdot, 0) dy \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_\kappa(u) \partial_y \psi dy ds - \int_{\mathbb{R}_+} \Phi_\kappa(u_L) \psi(0, \cdot) ds \geq 0, \end{aligned} \quad (56)$$

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |u - \kappa| \partial_s \psi dy ds + \int_{\mathbb{R}_+} |u_0 - \kappa| \psi(\cdot, 0) dy \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_\kappa(u) \partial_y \psi dy ds + \int_{\mathbb{R}_+} \Phi_\kappa(u_R) \psi(0, \cdot) ds \geq 0. \end{aligned} \quad (57)$$

## 6 Proof of Theorem 3

The proof of the error estimate is based on the doubling variable technique, introduced by KRŪZKOV [17] for proving the uniqueness of the entropy solution to a multidimensional scalar conservation law. It has been later adapted by KUZNETSOV [18] to derive error estimates on the solutions provided by monotone finite difference schemes. Then we first need to introduce approximation of the units.

### 6.1 Approximation of the unit

Because of the presence of the interface  $\{x = 0\}$ , we need to introduce a family of non-even smoothing kernels  $(\rho_\epsilon)_{\epsilon > 0}$ . It is built as follows. Let  $\rho \in \mathcal{D}^+(\mathbb{R})$  such that  $\text{supp}(\rho) \subset [0, 1]$ ,  $\int_{\mathbb{R}} \rho(a) da = 1$  and such that  $(x - 1/2)\rho'(x) \leq 0$ . Let  $\epsilon \in (0, 1]$ , we denote by  $\rho_\epsilon(x) = \frac{1}{\epsilon} \rho(\frac{x}{\epsilon})$ , so that  $\text{supp}(\rho_\epsilon) \subset [0, \epsilon]$ , and

$$\int_{\mathbb{R}} |\rho'_\epsilon(a)| da = \frac{2}{\epsilon} \rho(1/2). \quad (58)$$

### 6.2 An estimate on the exact solution near the interface

It follows from Proposition 4.12 that for all  $T > 0$ , the exact solution  $u$  to the problem belongs to  $BV(\mathbb{R} \times (0, T))$ . As a consequence, for all  $\bar{x} \in \mathbb{R}$  and for almost all  $t \in (0, T)$ ,  $u$  admits strong traces (from the left and from the right) on  $\{\bar{x}\} \times (0, T)$ . We choose to define

$$\gamma u(\bar{x}, t) = \begin{cases} \text{esslim}_{\mu \rightarrow 0^+} u(\bar{x} + \mu, t) & \text{if } \bar{x} < 0, \\ \text{esslim}_{\mu \rightarrow 0^+} u(\bar{x} - \mu, t) & \text{if } \bar{x} > 0, \end{cases}$$

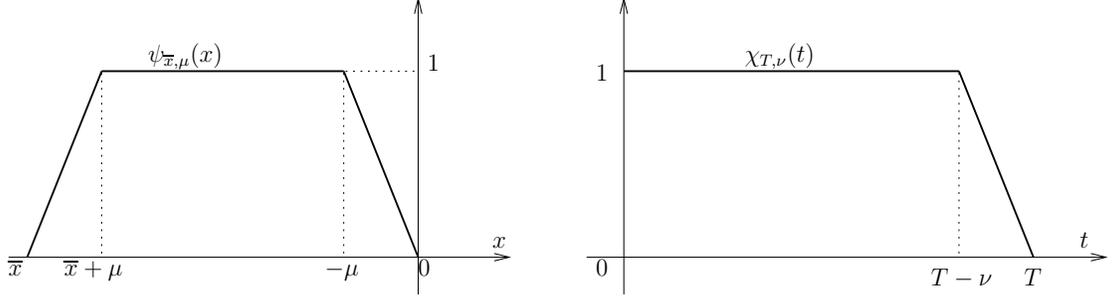


Figure 2: The functions  $\psi_{\bar{x}, \mu}$  and  $\chi_{T, \nu}$

i.e.

$$\lim_{\mu \rightarrow 0^+} \frac{1}{\mu} \int_0^T \int_{\bar{x}}^{\bar{x}+\mu} |u(x, t) - \gamma u(\bar{x}, t)| dx dt = 0 \quad \text{if } \bar{x} < 0, \quad (59)$$

$$\lim_{\mu \rightarrow 0^+} \frac{1}{\mu} \int_0^T \int_{\bar{x}-\mu}^{\bar{x}} |u(x, t) - \gamma u(\bar{x}, t)| dx dt = 0 \quad \text{if } \bar{x} > 0. \quad (60)$$

Recall that the traces on  $\{x = 0\}$  are defined from both side as  $u_L$  and  $u_R$ .

**Lemma 6.1** For all  $\kappa \in [0, 1]$ , for all  $\theta \in \mathcal{D}^+(\mathbb{R}_+)$ , then

$$\begin{aligned} \int_0^T \theta(t) (\Phi_\kappa(u_L(t)) - \Phi_\kappa(\gamma u(\bar{x}, t))) dt &\leq |\bar{x}|T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty) \quad \text{if } \bar{x} < 0, \\ \int_0^T \theta(t) (\Phi_\kappa(u_R(t)) - \Phi_\kappa(\gamma u(\bar{x}, t))) dt &\geq -|\bar{x}|T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty) \quad \text{if } \bar{x} > 0, \end{aligned}$$

*Proof:* We perform the proof for  $\bar{x} < 0$ , the case  $\bar{x} > 0$  being similar. For  $\mu \in (0, -\bar{x})$ , we denote by

$$\psi_{\bar{x}, \mu} = \left( 1, \frac{x^-}{\epsilon}, \frac{(x - \bar{x})^+}{\epsilon} \right).$$

For  $T > 0$  and  $\nu \in (0, T)$ , we denote by

$$\chi_{T, \nu}(t) = \min \left( 1, \frac{(T - t)^+}{\nu} \right).$$

It follows for classical density results that we can consider the function

$$\psi : (x, t) \mapsto \theta(t) \psi_{\bar{x}, \mu}(x) \chi_{T, \nu}(t)$$

as test function in (9). This yields

$$\begin{aligned} &\frac{1}{\mu} \int_0^T \theta(t) \chi_{T, \nu}(t) \left( \int_{-\mu}^0 \Phi_\kappa(u)(x, t) dx - \int_{\bar{x}}^{\bar{x}+\mu} \Phi_\kappa(u)(x, t) dx \right) dt \\ &\leq \int_0^T \int_{\bar{x}}^0 |u(x, t) - \kappa| \chi_{T, \nu}(t) \partial_t \theta(t) \psi_{\bar{x}, \mu}(x) dx dt \\ &\quad - \frac{1}{\nu} \int_{T-\nu}^T \int_{\bar{x}}^0 |u(x, t) - \kappa| \theta(t) \psi_{\bar{x}, \mu}(x) dx dt + \int_{\bar{x}}^0 |u_0(x) - \kappa| \theta(0) \psi_{\bar{x}, \mu}(x) dx. \end{aligned}$$

Since  $u$  belongs to  $\mathcal{C}(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$  (see [9]), one can let tend  $\nu$  to 0. This leads to

$$\begin{aligned} &\frac{1}{\mu} \int_0^T \theta(t) \left( \int_{-\mu}^0 \Phi_\kappa(u)(x, t) dx - \int_{\bar{x}}^{\bar{x}+\mu} \Phi_\kappa(u)(x, t) dx \right) dt \\ &\leq \int_0^T \int_{\bar{x}}^0 |u(x, t) - \kappa| \partial_t \theta(t) \psi_{\bar{x}, \mu}(x) dx dt \\ &\quad + \int_{\bar{x}}^0 (|u_0(x) - \kappa| \theta(0) - |u(x, T) - \kappa| \theta(T)) \psi_{\bar{x}, \mu}(x) dx. \end{aligned} \quad (61)$$

Now, letting  $\mu$  tend to 0 in (61) provides, thanks to the definition (59) of  $\gamma u(\bar{x}, t)$  that

$$\begin{aligned} & \int_0^T \theta(t) (\Phi_\kappa(u_L(t)) - \Phi_\kappa(\gamma u(\bar{x}, t))) dt \\ & \leq \int_0^T \int_{\bar{x}}^0 |u(x, t) - \kappa| \partial_t \theta(t) dx dt + \int_{\bar{x}}^0 (|u_0(x) - \kappa| \theta(0) - |u(x, T) - \kappa| \theta(T)) dx. \end{aligned} \quad (62)$$

We conclude the proof by noticing that since  $0 \leq \kappa, u_0(x), u(x, t) \leq 1$ , and since  $\theta \geq 0$ , one has

$$\begin{aligned} & \int_0^T \int_{\bar{x}}^0 |u(x, t) - \kappa| \partial_t \theta(t) dx dt \leq |\bar{x}| T \|\partial_t \theta\|_\infty, \\ & \int_{\bar{x}}^0 (|u_0(x) - \kappa| \theta(0) - |u(x, T) - \kappa| \theta(T)) dx \leq |\bar{x}| T \|\theta\|_\infty. \end{aligned}$$

□

Now, we give an integrated version of the previous lemma.

**Proposition 6.2** *Let  $u$  be the unique solution to the problem (1), then for all  $\kappa \in [0, 1]$ , for all  $\theta \in \mathcal{D}^+(\mathbb{R}_+)$ ,*

$$\int_0^T \int_{\mathbb{R}_-} (\Phi_\kappa(u_L(t)) - \Phi_\kappa(u(x, t))) \theta(t) \rho_\epsilon(-x) dx dt \leq \epsilon T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty), \quad (63)$$

$$\int_0^T \int_{\mathbb{R}_+} (\Phi_\kappa(u_R(t)) - \Phi_\kappa(u(x, t))) \theta(t) \rho_\epsilon(x) dx dt \geq -\epsilon T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty). \quad (64)$$

*Proof:* Since the proof of (63) and (64) are similar, we only explicit the proof of (63). Since almost every point  $(x, t) \in \mathbb{R} \times (0, T)$  is a Lebesgue point, one has

$$u(x, t) = \gamma u(x, t) \text{ a.e. in } \mathbb{R} \times (0, T).$$

Hence, since  $x \mapsto \rho_\epsilon(-x)$  is compactly supported in  $(-\epsilon, 0)$ , it follows from Lemma 6.1 that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}_-} (\Phi_\kappa(u_L(t)) - \Phi_\kappa(u(x, t))) \theta(t) \rho_\epsilon(-x) dx dt \\ & = \int_{-\epsilon}^0 \left( \int_0^T (\Phi_\kappa(u_L(t)) - \Phi_\kappa(\gamma u(x, t))) \theta(t) dt \right) \rho_\epsilon(-x) dx \\ & \leq T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty) \epsilon \int_{-\epsilon}^0 \rho_\epsilon(-x) dx = T (\|\theta\|_\infty + \|\partial_t \theta\|_\infty) \epsilon. \end{aligned}$$

□

### 6.3 The case where $F_h \equiv F$

In this section, we first assume that for almost all  $t \in [t^n, t^{n+1})$ ,  $F(t) = F^n$ . In this case, note that  $(u_{L,h}(t), u_{R,h}(t)) \in \mathcal{G}(t)$ .

Let  $\varphi \in \mathcal{D}^+((-R, R) \times [0, T])$ , then we define the functions  $\xi_L, \xi_R$  by

$$\xi_L(x, t, y, s) = \varphi(x, t) \rho_\epsilon(x - y) \rho_\epsilon(s - t), \quad \xi_R(x, t, y, s) = \varphi(x, t) \rho_\epsilon(y - x) \rho_\epsilon(s - t),$$

for some  $\epsilon > 0$  that will be fixed later, and where  $\rho_\epsilon$  is the approximation of the unit introduced in Section 6.1. The functions  $\xi_L$  and  $\xi_R$  are built so that

$$\xi_L(x, t, y, 0) = \xi_R(x, t, y, 0) = 0, \quad \forall (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+, \quad (65)$$

$$\xi_L(x, t, 0, s) = 0, \quad \forall (x, t, s) \in \mathbb{R}_- \times (\mathbb{R}_+)^2, \quad \xi_R(x, t, 0, s) = 0, \quad \forall (x, t, s) \in (\mathbb{R}_+)^3. \quad (66)$$

Let us choose  $\kappa = u(y, s)$  in (54) and integrate on  $\mathbb{R}_- \times \mathbb{R}_+$  w.r.t.  $(y, s)$ , and  $\kappa = u_h(x, t)$  in (56) and integrate on  $\mathbb{R}_- \times \mathbb{R}_+$  w.r.t.  $(x, t)$ , and then sum both contributions. This provides

$$D_1^L + D_2^L + D_3^L + D_4^L \geq D_5^L, \quad (67)$$

where

$$\begin{aligned}
D_1^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u_h(x, t) - u(y, s)| \partial_t \varphi(x, t) \rho_\epsilon(x - y) \rho_\epsilon(s - t) dx dt, \\
D_2^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} |u_0(x) - u(y, s)| \varphi(x, 0) \rho_\epsilon(x - y) \rho_\epsilon(s) dx dy ds, \\
D_3^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u(y, s)}(u_h(x, t)) \partial_x \varphi(x, t) \rho_\epsilon(x - y) \rho_\epsilon(s - t) dx dt dy ds, \\
D_4^L &= - \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \Phi_{u(y, s)}(u_{L, h}(t)) \varphi(0, t) \rho_\epsilon(-y) \rho_\epsilon(s - t) dt dy ds, \\
D_5^L &= - \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \langle \mu_{L, h}, (|\partial_t \varphi| + |\partial_x \varphi|) \rho_\epsilon(\cdot - y) \rho_\epsilon(s - \cdot) \rangle dy ds \\
&\quad - \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \langle \mu_{L, h}, \varphi (|\rho'_\epsilon(\cdot, -y)| \rho_\epsilon(s - \cdot) + \rho_\epsilon(\cdot - y) |\rho'_\epsilon(s - \cdot)|) \rangle dy ds
\end{aligned}$$

Among the above terms, only  $D_4^L$  is original in the sense that its treatment has not already been performed in the already mentioned works [18, 15, 14]. Let us first recall the classical results concerning the other terms.

It follows from Fubini-Tonelli theorem and from estimate (53) that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \langle \mu_{L, h}, (|\partial_t \varphi| + |\partial_x \varphi|) \rho_\epsilon(\cdot - y) \rho_\epsilon(s - \cdot) \rangle dy ds \leq Ch \|\nabla \varphi\|.$$

On the other hand, thanks to (58), one has

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \langle \mu_{L, h}, \varphi (|\rho'_\epsilon(\cdot, -y)| \rho_\epsilon(s - \cdot) + \rho_\epsilon(\cdot - y) |\rho'_\epsilon(s - \cdot)|) \rangle dy ds \leq C \frac{h}{\epsilon} \|\varphi\|_\infty,$$

hence

$$D_5^L \geq -C \left( h \|\nabla \varphi\| + \frac{h}{\epsilon} \|\varphi\|_\infty \right). \tag{68}$$

Let us now consider the term  $D_1$ . Obviously, one has

$$D_1^L \leq D_{1,1}^L + D_{1,2}^L, \tag{69}$$

where

$$\begin{aligned}
D_{1,1}^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u_h(x, t) - u(x, t)| \partial_t \varphi(x, t) dx dt, \\
D_{1,2}^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u(x, t) - u(y, s)| |\partial_t \varphi(x, t)| \rho_\epsilon(x - y) \rho_\epsilon(s - t) dx dt dy ds.
\end{aligned}$$

Using the fact that  $u$  belongs to  $BV(\text{supp} \varphi)$  (cf. Proposition 4.12), one thus obtains that

$$D_{1,2}^L \leq C \epsilon \|\partial_t \varphi\|_\infty, \tag{70}$$

where  $C$  only depends on  $\text{supp} \varphi$ ,  $u_0$ ,  $f$ ,  $A$ ,  $B$ .

In order to estimate  $D_2^L$ , we mimic the method proposed in [13, 10, 20], i.e. we choose

$$(y, s) \mapsto \psi(x, t, y, s) = \varphi(x, 0) \rho_\epsilon(x - y) \int_s^\infty \rho_\epsilon(\tau) d\tau$$

as test function in (56) for  $\kappa = u_0(x)$  and integrate with respect to  $x \in \mathbb{R}_-$ . This provides that

$$D_2^L \leq D_{2,1}^L + D_{2,2}^L + D_{2,3}^L, \tag{71}$$

where

$$\begin{aligned}
D_{2,1}^L &= \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} |u_0(y) - u_0(x)| \varphi(x, 0) \rho_\epsilon(x - y) dx dy, \\
D_{2,2}^L &= - \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u_0(y)}(u(y, s)) \varphi(x, 0) \rho'_\epsilon(x - y) \int_s^\infty \rho_\epsilon(\tau) d\tau dy ds dx, \\
D_{2,3}^L &= - \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} (\Phi_{u_0(x)}(u(y, s)) - \Phi_{u_0(y)}(u(y, s))) \varphi(x, 0) \rho'_\epsilon(x - y) \int_s^\infty \rho_\epsilon(\tau) d\tau dy ds dx.
\end{aligned}$$

Since  $u_0 \in BV(\mathbb{R}_-)$ , one has

$$D_{2,1}^L \leq C\epsilon \|\varphi\|_\infty. \quad (72)$$

Since  $b \mapsto \Phi_b(a)$  is  $L_f$ -Lipschitz continuous, since  $\text{supp}(s \mapsto \int_s^\infty \rho_\epsilon(\tau) d\tau) \subset [0, \epsilon]$  and  $0 \leq \int_s^\infty \rho_\epsilon(\tau) d\tau \leq 1$ , one has

$$|D_{2,3}^L| \leq L_f \epsilon \|\varphi\|_\infty \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} |u_0(x) - u_0(y)| |\rho'_\epsilon(x-y)| dx dy.$$

Now,  $a \mapsto \frac{\epsilon |\rho'_\epsilon(a)|}{2\rho(1/2)}$  is an approximation of the unit, therefore

$$|D_{2,3}^L| \leq C\epsilon. \quad (73)$$

Integrating  $D_{2,2}^L$  by parts with respect to the variable  $x$  provides

$$\begin{aligned} D_{2,2}^L &= \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u_0(y)}(u(y, s)) \partial_x \varphi(x, 0) \rho_\epsilon(x-y) \int_s^\infty \rho_\epsilon(\tau) d\tau dy ds dx \\ &\quad + \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u_0(y)}(u(y, s)) \varphi(0, 0) \rho_\epsilon(-y) \int_s^\infty \rho_\epsilon(\tau) d\tau dy ds. \end{aligned}$$

Using again that  $\text{supp}(s \mapsto \int_s^\infty \rho_\epsilon(\tau) d\tau) \subset [0, \epsilon]$  and  $0 \leq \int_s^\infty \rho_\epsilon(\tau) d\tau \leq 1$ , one obtains that

$$|D_{2,2}^L| \leq C\epsilon (\|\varphi\|_\infty + \|\nabla \varphi\|). \quad (74)$$

Concerning  $D_3^L$ , one has

$$D_3^L \leq D_{3,1}^L + D_{3,2}^L, \quad (75)$$

with, thanks to the fact that  $a \mapsto \text{sign}(a-b)(f(a) - f(b))$  is  $L_f$ -Lipschitz continuous,

$$\begin{aligned} D_{3,1}^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u(x,t)}(u_h(x, t)) \partial_x \varphi(x, t) dx dt, \\ D_{3,2}^L &= L_f \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u(x, t) - u(y, s)| |\partial_x \varphi(x, t)| \rho_\epsilon(x-y) \rho_\epsilon(s-t) dx dt dy ds. \end{aligned}$$

Therefore, we obtain that

$$D_{3,2}^L \leq C\epsilon \|\partial_x \varphi\|_\infty. \quad (76)$$

We can now focus on the term  $D_4$ . Here again, we can write

$$D_4^L = D_{4,1}^L + D_{4,2}^L, \quad (77)$$

where,

$$\begin{aligned} D_{4,1}^L &= - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_{u_L(s)}(u_{L,h}(t)) \varphi(0, t) \rho_\epsilon(s-t) dt ds, \\ D_{4,2}^L &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} (\Phi_{u_{L,h}(t)}(u_L(s)) - \Phi_{u_{L,h}(t)}(u(y, s))) \varphi(0, t) \rho_\epsilon(-y) dt dy ds. \end{aligned}$$

It follows from Proposition 6.2 that

$$D_{4,2}^L \leq C\epsilon \|\nabla \varphi\|. \quad (78)$$

Thus it follows from (67)–(78) that

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}_-} |u_h(x, t) - u(x, t)| |\partial_t \varphi(x, t)| dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u(x,t)}(u_h(x, t)) \partial_x \varphi(x, t) dx dt \\ &\quad - \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_{u_L(s)}(u_{L,h}(t)) \varphi(0, t) \rho_\epsilon(s-t) dt ds \geq -C \|\nabla \varphi\| \left( h + \epsilon + \frac{h}{\epsilon} \right). \end{aligned} \quad (79)$$

Similar calculations carried out for  $(x, t, y, s) \in (\mathbb{R}_+)^4$  with the test function  $\xi_R$  yield

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |u_h(x, t) - u(x, t)| \partial_t \varphi(x, t) dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_{u(x, t)}(u_h(x, t)) \partial_x \varphi(x, t) dx dt \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Phi_{u_R(s)}(u_{R, h}(t)) \varphi(0, t) \rho_\epsilon(s - t) dt ds \geq -C \|\nabla \varphi\| \left( h + \epsilon + \frac{h}{\epsilon} \right). \end{aligned} \quad (80)$$

Adding (79) and (80) provides

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}} |u_h(x, t) - u(x, t)| \partial_t \varphi(x, t) dx dt \\ & + \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \Phi_{u(x, t)}(u_h(x, t)) \partial_x \varphi(x, t) dx dt + R_{\epsilon, h}(\varphi) \geq -C \|\nabla \varphi\| \left( h + \epsilon + \frac{h}{\epsilon} \right), \end{aligned} \quad (81)$$

where

$$R_{\epsilon, h}(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\Phi_{u_R(s)}(u_{R, h}(t)) - \Phi_{u_L(s)}(u_{L, h}(t))] \varphi(0, t) \rho_\epsilon(s - t) dt ds.$$

**Lemma 6.3** *Let  $\mathcal{G}$  be a  $L^1$ -dissipative germ corresponding to the function  $f$ , then, for all  $(c_L, c_R) \in \mathcal{G}$ , for all  $(\kappa_L, \kappa_R) \in [0, 1]^2$ ,*

$$\Phi_{\kappa_R}(c_R) - \Phi_{\kappa_L}(c_L) \leq L_f \text{dist}_1((\kappa_L, \kappa_R), \mathcal{G}),$$

where

$$\text{dist}_1((\kappa_L, \kappa_R), \mathcal{G}) = \min_{(a_L, a_R) \in \mathcal{G}} (|\kappa_L - a_L| + |\kappa_R - a_R|).$$

*Proof:* Let  $(a_L, a_R) \in \mathcal{G}$ , then, thanks to the dissipative character of  $\mathcal{G}$ , one has

$$\Phi_{a_R}(c_R) - \Phi_{a_L}(c_L) \leq 0.$$

Now, since  $\kappa \mapsto \Phi_\kappa(s)$  is  $L_f$ -Lipschitz continuous for all  $s \in [0, 1]$ , we obtain that

$$\Phi_{\kappa_R}(c_R) - \Phi_{\kappa_L}(c_L) \leq L_f (|\kappa_L - a_L| + |\kappa_R - a_R|).$$

Since  $\mathcal{G}$  is closed in  $[0, 1]^2$ , the above relation thus still holds for the minimum w.r.t.  $(a_L, a_R) \in \mathcal{G}$ .  $\square$

**Lemma 6.4** *There exists  $C$  depending only on  $f, T, A, B$  such that*

$$R_{\epsilon, h}(\psi) \leq C \epsilon \|\psi\|_\infty, \quad \forall \psi \in \mathcal{C}_c([0, T]; \mathbb{R}).$$

*Proof:* Using the fact that  $\Phi_\kappa(u) = \Phi_u(\kappa)$ , it follows from Lemma 6.3 that

$$\Phi_{u_R(s)}(u_{R, h}(t)) - \Phi_{u_L(s)}(u_{L, h}(t)) \leq L_f \max \left\{ \text{dist}_1((u_L(s), u_R(s)), \mathcal{G}(t)), \text{dist}_1((u_{L, h}(t), u_{R, h}(t)), \mathcal{G}(s)) \right\}.$$

Now, it appears clearly that if  $(u_L(s), u_R(s)) \in \mathcal{G}(t)$  or  $(u_{L, h}(t), u_{R, h}(t)) \in \mathcal{G}(s)$ , then

$$\Phi_{u_R(s)}(u_{R, h}(t)) - \Phi_{u_L(s)}(u_{L, h}(t)) \leq 0.$$

Assume now that  $(u_L(s), u_R(s)) \notin \mathcal{G}(t)$  and  $(u_{L, h}(t), u_{R, h}(t)) \notin \mathcal{G}(s)$ . This implies that either

$$F(t) > F(s) \quad \text{and} \quad (u_L(s), u_R(s)) = (A(s), B(s)), \quad (82)$$

or

$$F(t) < F(s) \quad \text{and} \quad (u_{L, h}(t), u_{R, h}(t)) = (A(t), B(t)). \quad (83)$$

In the first case (82), one has

$$\text{dist}_1((u_L(s), u_R(s)); \mathcal{G}(t)) \leq |A(t) - A(s)| + |B(t) - B(s)|,$$

while in the second case (83), one has

$$\text{dist}_1((u_{L, h}(t), u_{R, h}(t)); \mathcal{G}(s)) \leq |A(t) - A(s)| + |B(t) - B(s)|.$$

Hence,

$$\Phi_{u_{R(s)}}(u_{R,h}(t)) - \Phi_{u_{L(s)}}(u_{L,h}(t)) \leq L_f (|A(t) - A(s)| + |B(t) - B(s)|). \quad (84)$$

Now, for  $\psi \in \mathcal{C}_c([0, T]; \mathbb{R})$ , recalling that  $\text{supp}(\rho_\epsilon) \subset [0, \epsilon]$ , one has

$$R_{\epsilon,h}(\psi) \leq L_f \|\psi\|_\infty \left( \sup_{\tau \in [0, \epsilon]} \int_0^T |A(t+\tau) - A(t)| dt + \sup_{\tau \in [0, \epsilon]} \int_0^T |B(t+\tau) - B(t)| dt \right),$$

thus we obtain

$$R_{\epsilon,h}(\psi) \leq L_f TV_{[0, T+1]}(A - B) \epsilon \|\psi\|_\infty.$$

□

Using Lemma 6.4 in (81) provides that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u_h(x, t) - u(x, t)| \partial_t \varphi(x, t) dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \Phi_{u(x,t)}(u_h(x, t)) \partial_x \varphi dx dt \leq C \|\nabla \varphi\| \left( h + \epsilon + \frac{h}{\epsilon} \right). \quad (85)$$

In order to conclude the proof of Theorem 3, it only remains to choose a convenient  $\varphi$ , that is

$$\varphi(x, t) = \begin{cases} \zeta(|x| - L_f t) \frac{T-t}{T} & \text{if } (x, t) \in \mathbb{R} \times [0, T], \\ 0 & \text{if } t \geq T. \end{cases}$$

where

$$\zeta(r) = \max(0, \min(1, R + 1 - r)), \quad \forall r \in \mathbb{R}^+.$$

## 6.4 The general case

Denote by  $\tilde{u}$  the unique solution to the problem corresponding to the constraint  $F_h$ . Then it has been proven previously that

$$\iint_{\omega_R} |u_h - \tilde{u}| dx dt \leq Ch^{1/2}.$$

In order to achieve the proof of Theorem 3, it only remains to show that

$$\iint_{\omega_R} |u - \tilde{u}| dx dt \leq Ch^{1/2}.$$

In fact, one has even something better, thanks to the following Proposition, proved in Appendix of [3].

**Proposition 6.5 ([3])** *Let  $F, \check{F} \in L^\infty(\mathbb{R}_+; [0, f(\bar{\pi})])$ , and let  $u, \check{u}$  be the solutions corresponding respectively to the constraint  $F, \check{F}$  and to a similar initial data  $u_0$ . Then,*

$$\int_0^T \int_{\mathbb{R}} |u - \check{u}| dx dt \leq 2 \int_0^T |F - \check{F}| dt.$$

Since  $A$  is supposed to belong to  $BV(0, T)$ , then  $F = f(A)$  also belongs to  $BV(0, T)$ . As a consequence, there exists  $C$  depending only on  $A, f, T$  such that

$$\|F - F_h\|_{L^1(0, T)} \leq Ch.$$

We deduce from the above estimate and from Proposition 6.5 the following corollary that achieves the proof of Theorem 3.

**Corollary 6.6** *Under Assumption 1, there exists  $C$  depending only on  $A, f, T$  such that*

$$\iint_{\omega_R} |u - \tilde{u}| dx dt \leq Ch.$$

## 7 Numerical illustration

We now present some numerical simulations in order to illustrate the error estimate (17). Two conservation laws are investigated: the first one is based on the flux (called the *hat flux* in the following)

$$f(u) = 1/2 - |u - 1/2|$$

which has the particularity of having linear two parts and the second one is based on the flux (called the *GNL flux* — as genuinely non linear — in the following)

$$f(u) = u(1 - u)$$

which is strictly concave. While the present work is devoted to the analysis of the Godunov scheme, we also present the results obtained with the Rusanov scheme:

$$G(a, b) = \frac{f(a) + f(b)}{2} - \frac{\max(|f'(a)|, |f'(b)|)}{2}(b - a)$$

and the constraint is still handled using the trick (12). The initial data for the test case is

$$u(x, 0) = \begin{cases} 0.4 & \text{if } -1/2 \leq x < 0, \\ 0.5 & \text{if } 0 \leq x \leq 1/2, \end{cases}$$

and the final time is 0.3. For each flux, the constraint is set to  $F = 0.2$  and is activated (see Figure 3, left). For the hat flux, the solution is composed of a left-going shock wave, a nonclassical stationary shock and a right-going linear wave. For the GNL flux, the solution is composed of a left-going shock wave, a nonclassical stationary shock and a right-going shock wave. The rates of convergence are displayed in Figure 3, right.

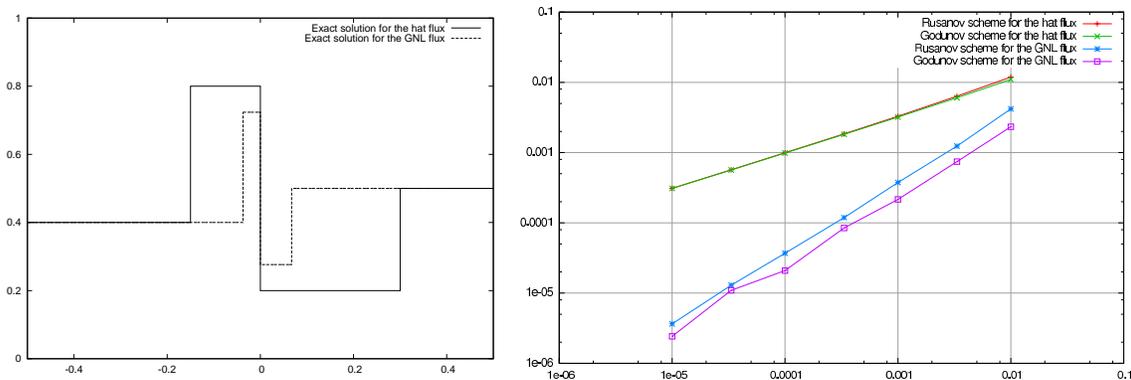


Figure 3: Left: Exact solutions for each flux ( $u$  vs  $x$ ). Right: Rates of convergence for each flux and each numerical scheme ( $L^1$  error vs  $h$  in Log-scale)

They are the same for both numerical schemes, which let us think that our result should be extended for any monotone numerical scheme. For the hat flux, the measured rate is  $1/2$  (and therefore it attests the optimality of our result) while the measured rate is  $1$  for the GNL flux. Note that in the latter case, it means that the constraint does not alter the classical rate of convergence.

## 8 Concluding remarks

### 8.1 *A posteriori* error estimate

As noticed by D. KRÖNER and M. OHLBERGER [16], the doubling variable approach used for obtaining error estimates provides *a posteriori* estimators, i.e. that for all compact subset  $K$  of  $\mathbb{R} \times \mathbb{R}_+$ , there exists  $\eta_K$  depending only  $f, K, u_0, A, B$  (but not on the exact solution  $u$ ) such that

$$\iint_K |u_h(x, t) - u(x, t)| dx dt \leq \eta_K(u_h).$$

Since the right-hand side in the above estimate is fully computable, this permits the localization of the error, and an adaptive mesh refinement strategy. We refer to [16] for more details on both the derivation of the *a posteriori* estimator and the mesh refinement algorithm.

## 8.2 Optimality of the result

The order  $h^{1/2}$  is optimal in the sense that it can be recovered in some particular cases. Indeed, choosing  $f(u) = 1/2 - |u - 1/2|$ ,  $F \equiv 1/2$  (this means that  $A \equiv B \equiv 1/2$ , so that the constraint is always inactive), and  $u_0$  in  $BV(\mathbb{R})$  such that  $0 \leq u_0 \leq 1/2$ . Then the problem turns to be the standard linear equation  $\partial_t u + \partial_x u = 0$ , and the Godunov scheme becomes the upwind scheme. It is well known that in this case, the error behaves as  $h^{1/2}$ , as illustrated in Figure 3. In the case where  $f$  is uniformly concave, the numerical experiments provide an error of order  $h$ .

## 8.3 The case of discontinuous flux function

Consider the case of a scalar conservation law with discontinuous flux function, i.e.

$$\partial_t u + \partial_x f(x, u) = 0,$$

where  $f(x, u) = f_L(u)$  if  $x < 0$  and  $f(x, u) = f_R(u)$  if  $x > 0$ , with  $f_L, f_R$  bell-shaped reaching their maximum respectively in  $\bar{u}_L, \bar{u}_R$ . As pointed out by ADIMURTHI and VEERAPPA GOWDA [1], an infinite number of  $L^1$ -contractive semi-groups can be built for such an equation, and a criterion has to be taken into account in order to select one. We refer to the recent contributions of R. BÜRGER *et al.* [7], B. ANDREIANOV *et al.* [2] and references therein for an overview of this topic, and in particular to the resolution of the Riemann problem arising at the interface, thanks to which we can define the Godunov scheme, and its discrete solution  $u_h$ . In the case where  $f_L \neq f_R$ , no  $BV$  estimate is available on  $u$  (neither on  $u_h$ ), but we can prove that the Temple function

$$(x, t) \mapsto \Phi_{\bar{u}}(u(x, t), x) := \begin{cases} \text{sign}(u(x, t) - \bar{u}_L)(f_L(u(x, t)) - f_L(\bar{u}_L)) & \text{if } x < 0, \\ \text{sign}(u(x, t) - \bar{u}_R)(f_R(u(x, t)) - f_R(\bar{u}_R)) & \text{if } x > 0 \end{cases}$$

belong to  $BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$  (see e.g. [24, 5, 8]). By the use of numerical diffusion introduced by the scheme (see [11, 25, 13, 10, 20]), it is still possible to derive an error estimate. Indeed, all the tools introduced in the paper, excepted in Section 4, can be adapted to the case of discontinuous flux functions. Nevertheless, the theoretical convergence speed will depend on the continuity modulus of the function  $(\Phi_{\bar{u}})^{-1}$ , and will be furthermore degraded by the fact that no strong  $BV$ -estimate is available on the exact solution itself.

## References

- [1] Adimurthi and G. D. Veerappa Gowda. Conservation law with discontinuous flux. *J. Math. Kyoto Univ.*, 43(1):27–70, 2003.
- [2] B. Andreianov, K. Karlsen, and N. Risebro. A theory of  $L^1$ -dissipative solvers for scalar conservation laws with discontinuous flux. *Arch. Ration. Mech. Anal.*, pages 1–60, 2011. 10.1007/s00205-010-0389-4.
- [3] B. Andreianov, P. Goatin, and N. Seguin. Finite volume schemes for locally constrained conservation laws. *Numer. Math.*, 115(4):609–645, 2010. With supplementary material available online.
- [4] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 6 of *MPS/SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006. Applications to PDEs and optimization.
- [5] F. Bachmann. Analysis of a scalar conservation law with a flux function with discontinuous coefficients. *Adv. Differential Equations*, 9(11-12):1317–1338, 2004.
- [6] F. Bouchut and B. Perthame. Kruřkov’s estimates for scalar conservation laws revisited. *Trans. Amer. Math. Soc.*, 350(7):2847–2870, 1998.
- [7] R. Bürger, K. H. Karlsen and J. D. Towers. An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections. *SIAM J. Numer. Anal.*, 47(3):1684–1712, 2009.
- [8] C. Cancès. Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space. I. Convergence to the optimal entropy solution. *SIAM J. Math. Anal.*, 42(2):946–971, 2010.

- [9] C. Cancès and T. Gallouët. On the time continuity of entropy solutions. *J. Evol. Equ.*, 11(1):43–55, 2011.
- [10] C. Chainais-Hillairet. Finite volume schemes for a nonlinear hyperbolic equation. Convergence towards the entropy solution and error estimate. *M2AN Math. Model. Numer. Anal.*, 33(1):129–156, 1999.
- [11] B. Cockburn, F. Coquel, and P. LeFloch. An error estimate for finite volume methods for multidimensional conservation laws. *Math. Comp.*, 63(207):77–103, 1994.
- [12] R. M. Colombo and P. Goatin. A well posed conservation law with a variable unilateral constraint. *J. Differential Equations*, 234(2):654–675, 2007.
- [13] R. Eymard, T. Gallouët, M. Ghilani, and R. Herbin. Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes. *IMA J. Numer. Anal.*, 18(4):563–594, 1998.
- [14] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. Ciarlet, P. G. (ed.) et al., in Handbook of numerical analysis. North-Holland, Amsterdam, pp. 713–1020, 2000.
- [15] E. Godlewski and P.-A. Raviart. *Hyperbolic systems of conservation laws*, volume 3/4 of *Mathématiques & Applications (Paris) [Mathematics and Applications]*. Ellipses, Paris, 1991.
- [16] D. Kröner and M. Ohlberger. A posteriori error estimates for upwind finite volume schemes for nonlinear conservation laws in multidimensions. *Math. Comp.*, 69(229):25–39, 2000.
- [17] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
- [18] N. N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. *USSR Comput. Math. and Math. Phys.*, 16:105–119, 1976.
- [19] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955.
- [20] M. Ohlberger and J. Vovelle. Error estimate for the approximation of nonlinear conservation laws on bounded domains by the finite volume method. *Math. Comp.*, 75(253):113–150 (electronic), 2006.
- [21] O. A. Oleĭnik. Discontinuous solutions of non-linear differential equations. *Amer. Math. Soc. Transl. (2)*, 26:95–172, 1963.
- [22] E. Yu. Panov. Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyperbolic Differ. Equ.*, 4(4):729–770, 2007.
- [23] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.
- [24] N. Seguin and J. Vovelle. Analysis and approximation of a scalar conservation law with a flux function with discontinuous coefficients. *Math. Models Methods Appl. Sci.*, 13(2):221–257, 2003.
- [25] J.-P. Vila. Convergence and error estimates in finite volume schemes for general multidimensional scalar conservation laws. I. Explicit monotone schemes. *RAIRO Modél. Math. Anal. Numér.*, 28(3):267–295, 1994.
- [26] A. I. Vol’pert. Spaces BV and quasilinear equations. *Mat. Sb. (N.S.)*, 73 (115):255–302, 1967.