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# SHARP GRADIENT BOUNDS FOR SOLUTIONS OF DEGENERATE SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

DAN CRISAN AND FRANÇOIS DELARUE

ABSTRACT. The paper is a continuation of the Kusuoka-Stroock programme of establishing smoothness properties of solutions of (possibly) degenerate partial differential equations by using probabilistic methods. We analyze here a class of semi-linear parabolic partial differential equations for which the linear part is a second order differential operator of the form  $V_0 + \sum_{i=1}^N V_i^2$ , where  $V_0, \dots, V_N$  are first order differential operators that do not necessarily satisfy the Hörmander condition. Instead we assume that  $V_0, \dots, V_N$  satisfy a weaker condition, the so-called UFG condition (see [17]). Specifically, we prove that the bounds of the higher order-derivatives of the solution along the vector fields coincide with those obtained in the Hörmander case, but that they may be affected by the non-linearity and thus may differ from the linear case. The methodology is also applied to partial differential equations with nonlinear terms with quadratic growth with respect to the first-order derivatives.

## 1. INTRODUCTION

In a series of papers [15, 16, 17, 18], Kusuoka and Stroock have analyzed the smoothness properties of solutions of linear parabolic partial differential equations of the form

$$(1) \quad \partial_t u(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 u(t, x) + V_0 u(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

with initial condition  $u(0, x) = h(x)$ ,  $x \in \mathbb{R}^d$ . The condition (called the UFG condition) imposed on the vector fields  $\{V_i, i = 0, \dots, N\}$  under which they prove their results is weaker than the Hörmander condition. This condition states that the  $C_b^\infty(\mathbb{R}^d)$ -module  $\mathcal{W}$  generated by the vector fields  $\{V_i, i = 1, \dots, N\}$  within the Lie algebra generated by  $\{V_i, i = 1, \dots, N\}$  is finite dimensional. In particular, the condition does not require that the vector space  $\{W(x) | W \in \mathcal{W}\}$  is homeomorphic to  $\mathbb{R}^d$  for any  $x \in \mathbb{R}^d$ . Hence, in this sense, the UFG condition is weaker than the Hörmander condition. It is important to emphasize that, under the UFG condition, the dimension of the space  $\{W(x) | W \in \mathcal{W}\}$  is not required to be constant over  $\mathbb{R}^d$ . Such generality makes any Frobenius type approach to prove smoothness of the solution very difficult. Indeed the authors are not aware of any alternative proof of the smoothness results of the solution of (1) (under the UFG condition) other than that given by Kusuoka and Stroock.

Kusuoka and Stroock use a probabilistic approach to deduce their results. To be more precise, they use the Feynman-Kac representation of the solution of the PDE in terms of the semigroup associated to a diffusion process. Let  $X = \{X_t^x, (t, x) \in [0, \infty) \times \mathbb{R}^d\}$  be the (time homogeneous) stochastic

flow

$$(2) \quad dX_t^x = x + \int_0^t V_0(X_s^x) ds + \sum_{i=1}^N \int_0^t V_i(X_s^x) \circ dB_s^i, \quad t \geq 0,$$

where the vector fields  $(V_i)_{0 \leq i \leq N}$  are smooth and bounded and the stochastic integrals in (2) are of Stratonovich type. The corresponding diffusion semigroup is then given by

$$[P_t g](x) = \mathbb{E}[g(X_t^x)], \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

for any given bounded measurable function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then the following representation holds true:

$$u(t, x) = P_t h(x), \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Kusuoka and Stroock prove that, under the UFG condition,  $P_t h$  is differentiable in the direction of any vector field  $W$  belonging to  $\mathcal{W}$ . Moreover they deduce sharp gradient bounds of the form:

$$(3) \quad \|W_1 \dots W_k P_t f\|_p \leq C^{p,k} t^{-l} \|f\|_p, \quad p \in [1, \infty],$$

where  $l$  is a constant that depends explicitly on the vector fields  $W_i$   $i = 1, \dots, k$ . Their results raise a number of fundamental questions related to the PDE (1). For example, the differentiability of  $P_t h$  in the  $V_0$  direction is not recovered. This is one of the fundamental differences between the UFG case and the Hörmander case where  $P_t h$  is shown to be differentiable in any direction, including  $V_0$ . So whilst in the Hörmander case, it is straightforward to show that  $P_t h$  is indeed the (unique) classical solution of (1), the situation is more delicate in the absence of the Hörmander condition. As explained in [7], it turns out that  $P_t h$  remains differentiable in the direction  $\mathcal{V}_0 = \partial_t - V_0$  when viewed as a function  $(t, x) \rightarrow P_t h(x)$  over the product space  $(0, \infty) \times \mathbb{R}^d$ . This together with the continuity at the origin implies that  $P_t h$  is the unique (classical) solution of the equation

$$(4) \quad \mathcal{V}_0 u(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 u(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d.$$

The introduction of a new class of numerical methods for approximating the law of solutions of SDE (and, implicitly, the solution of PDEs as computed by means of the Feynman-Kac formula) has brought a renewed interest in the work of Kusuoka and Stroock. Their fundamental results form the theoretical basis of a recently developed class of high accuracy numerical methods. In the last ten years, Kusuoka, Lyons, Ninomiya and Victoir [14, 19, 20, 21, 22] developed several numerical algorithms based on Chen's iterated integrals expansion (see [8] for a unified approach for the analysis of these methods). These new algorithms generate an approximation of the solution of the SDE in the form of the empirical distribution of a cloud of particles with deterministic trajectories. The particles evolve only in directions belonging to  $\mathcal{W}$ . This ensures that the particles remain within the support of the limiting diffusion, leading to more stable schemes. The global error of numerical schemes depends intrinsically on the smoothness of  $P_t h$  *but only in directions belonging to  $\mathcal{W}$* . As a result they work under the (weaker) UFG condition rather than the ellipticity/Hörmander condition. By contrast, the classical Euler based numerical method (combined with a Monte-Carlo procedure) sends the component particles in any direction, hence they require the Hörmander condition.

In recent work [5, 6] the applicability of these scheme has been extended to semilinear PDEs. One of the major hurdles in obtaining convergence results for these scheme has been the absence of

smoothness results of the type (3), again under the UFG condition. The authors are not aware of the existence of such bounds proved under the Hörmander condition either. In the following we will consider semilinear PDEs of the form:

$$(5) \quad \partial_t u(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 u(t, x) + V_0 u(t, x) + f(t, x, u(t, x), Vu(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d$$

with initial condition  $u(0, x) = h(x)$ ,  $x \in \mathbb{R}^d$ . In (5) we used the notation  $Vu(t, x)$  to denote the row vector  $(V_1 u(t, x), \dots, V_N u(t, x))$ . As we shall see,  $u(t, x)$  is differentiable in any direction  $W \in \mathcal{W}$  just as in the linear case. If, for example, the vectors  $V_i$   $i = 1, \dots, n$  satisfy the ellipticity condition, then  $u(t, x)$  is differentiable in any direction and the analysis cover semilinear PDEs written in the ‘standard’ format

$$\partial_t u(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 u(t, x) + V_0 u(t, x) + f(t, x, u(t, x), \nabla u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

where  $\nabla u$  is the usual gradient of  $u$ , i.e., the row vector of partial derivatives  $(\partial_1 u, \dots, \partial_N u)$ .

Following the tradition of Kusuoka and Stroock, we analyze the smoothness of the solution of the semilinear PDE using probabilistic methods. The basis of the analysis is the corresponding Feynman-Kac representation for the solution of (5). This representation was introduced by Pardoux and Peng in [24, 25] and involves the solution of a backward stochastic differential equation (see Section 2.1 below).

**1.1. The UFG condition.** Let  $(V_i)_{0 \leq i \leq N}$  be  $N+1$  vector fields,  $V_0$  belonging to  $\mathcal{C}_b^K(\mathbb{R}^d, \mathbb{R}^d)$  and  $V_i$ ,  $1 \leq i \leq N$ , to  $\mathcal{C}_b^{K+1}(\mathbb{R}^d, \mathbb{R}^d)$ ,  $K \geq 0$ ,  $\mathcal{C}_b^n(\mathbb{R}^d, \mathbb{R}^d)$  standing for the set of bounded and continuous functions that are  $n$ -times differentiable, with bounded and continuous partial derivatives up to order  $n$ . We will make use of the standard notation introduced in [18], (see also [7] and [8])

$$V_{[i]} = V_i, \quad V_{[\alpha \star i]} = [V_{[\alpha]}, V_i], \quad i \in \{0, \dots, N\},$$

where  $[\cdot, \cdot]$  stands for the Lie bracket of two vector fields, that is  $[V, W] = V \cdot \nabla W - W \cdot \nabla V$  and  $\alpha \star i$  stands for the multi-index  $(\alpha_1, \dots, \alpha_n, i)$  when  $\alpha$  is given by  $(\alpha_1, \dots, \alpha_n)$  with  $\alpha_i \in \{0, \dots, N\}$ ,  $i = 1, \dots, n$ . The following ‘lengths’ of a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  will be used:

$$|\alpha| = |(\alpha_1, \dots, \alpha_n)| = n, \quad \|\alpha\| = \|(\alpha_1, \dots, \alpha_n)\| = n + \#\{i : \alpha_i = 0\}.$$

The set of all multi-indices is denoted by  $\mathcal{A}$ , the set of all multi-indices  $\alpha$  different from (0) is denoted by  $\mathcal{A}_0$  and the set of non-empty multi-indices  $\alpha$  for which  $\|\alpha\| \leq m$  is denoted by  $\mathcal{A}_0(m)$ . For  $n$  multi-indices  $\alpha_1, \dots, \alpha_n$ ,  $n \geq 1$ , we often denote the  $n$ -tuple  $(\alpha_1, \dots, \alpha_n)$  by  $\alpha$  and then set  $\|\alpha\| = \|\alpha_1\| + \dots + \|\alpha_n\|$ .

**Definition 1.** Let  $m \in \mathbb{N}^*$  be a positive integer and assume that  $K \geq m + 3$ . (See footnote<sup>1</sup>.) The vector fields  $\{V_i, 0 \leq i \leq N\}$  satisfy the UFG condition of order  $m$  if for any  $\alpha \in \mathcal{A}_0(m+1)$  there exists  $\varphi_{\alpha, \beta} \in \mathcal{C}_b^{K+1-|\alpha|}(\mathbb{R}^d)$ , with  $\beta \in \mathcal{A}_0(m)$  such that

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}_0(m)} \varphi_{\alpha, \beta}(x) V_{[\beta]}(x), \quad x \in \mathbb{R}^d.$$

<sup>1</sup>In comparison with [7], the additional 3 follows from the nonlinear structure of the equation, see Theorem 1 below.

The following example illustrates the difference between the UFG and the Hörmander condition ( see [17]):

**Example 1.** Assume  $N = 1$  and  $d = 2$ . Let  $V_0$  and  $V_1$  be given by

$$V_0(x_1, x_2) = \sin x_1 \frac{\partial}{\partial x_1} \quad V_1(x_1, x_2) = \sin x_1 \frac{\partial}{\partial x_2}$$

The  $\{V_0, V_1\}$  satisfy the UFG condition of order  $m = 4$ , but not the Hörmander condition.

The vector fields  $\{V_i, 0 \leq i \leq N\}$  satisfy the *uniform* Hörmander condition if there exists  $m > 0$  such that

$$\inf_{\{x, \xi \in \mathbb{R}^d \mid \|\xi\|=1\}} \sum_{\beta \in \mathcal{A}_0(m)} (V_{[\beta]}(x), \xi)^2 > 0.$$

Obviously, if the vector fields  $\{V_i, 0 \leq i \leq N\}$  satisfy the uniform Hörmander condition then they satisfy the UFG condition. In particular if the vector fields  $\{V_i, 1 \leq i \leq N\}$  satisfy the strict ellipticity condition then they satisfy the UFG condition.

Definition 1 is a (slight) generalization of the corresponding one given in [18]. In [18], both the vector fields  $\{V_i, 0 \leq i \leq N\}$  and the coefficients  $\varphi_{\alpha, \beta}$  are assumed to be smooth (infinitely differentiable). If the smoothness assumption is imposed then  $V_{[\alpha]}$  is well defined for any  $\alpha \in \mathcal{A}$  and one can interpret the UFG condition in the following manner. Let  $\mathcal{W}$  be  $C_b^\infty(\mathbb{R}^d)$ -module  $\mathcal{W}$  generated by the vector fields  $\{V_i, i = 1, \dots, N\}$  within the Lie algebra generated by  $\{V_i, i = 1, \dots, N\}$ . Then  $\mathcal{W}$  is finitely generated as a vector space and  $\{V_{[\alpha]}, \alpha \in \mathcal{A}_0(m)\}$  is a finite set of generators for  $\mathcal{W}$ . In addition, the functions  $\varphi_{\alpha, \beta}$  appearing in the decomposition of any vector field  $V \in \mathcal{W}$  as a linear combination of the elements of the set  $\{V_{[\alpha]}, \alpha \in \mathcal{A}_0(m)\}$  are assumed to be smooth and uniformly bounded over  $\mathbb{R}^d$ . These are salient properties that are essential to make the proof of Kusuoka and Stroock work and justify the use of the acronym UFG - uniformly finitely generated - for the assumed property.

As shown in [7] the smoothness assumption on the vector fields  $\{V_i, 0 \leq i \leq N\}$  and the coefficients  $\varphi_{\alpha, \beta}$  is not necessary. The level of differentiability is dictated by the order of the UFG condition assumed. In other words, the vector fields have to be sufficiently many times differentiable for the repeated brackets to make sense up to the required level. Of course in this case we can no longer talk about the  $C_b^\infty(\mathbb{R}^d)$ -module  $\mathcal{W}$  or about the Lie algebra generated by  $\{V_i, i = 0, \dots, N\}$  as not all the Lie brackets will make sense (due to the reduced differentiability). In this case we will denote by  $\mathcal{W}$  the space of vector fields  $V$  for which there exist  $\varphi_{\alpha, \beta} \in C_b^{K+1-|\alpha|}(\mathbb{R}^d)$ , with  $\beta \in \mathcal{A}_0(m)$  such that

$$V(x) = \sum_{\beta \in \mathcal{A}_0(m)} \varphi_{\alpha, \beta}(x) V_{[\beta]}(x), \quad x \in \mathbb{R}^d.$$

Definition 1 then states that  $\{V_{[\alpha]}, \alpha \in \mathcal{A}_0(m+1)\} \subset \mathcal{W}$ . This extension allows us to identify the *minimal level of differentiability* that we need to impose on the coefficients of the PDE so as to deduce the desired gradient bounds.

**1.2. The Main Results.** Under the UFG condition (see [7] and [18]) the solution of the linear equation (1) is differentiable in any direction  $V \in \mathcal{W}$ . Moreover, if  $h$  is a bounded continuous function, the following gradient bound holds true:

$$(6) \quad |V_{[\alpha_1]} \cdots V_{[\alpha_n]} u(t, x)| \leq C \|h\|_\infty t^{-\|\alpha\|/2}$$

where  $C$  is a constant independent of  $h$  and  $(t, x)$  and  $\|\alpha\| = \|\alpha_1\| + \dots + \|\alpha_n\|$ . If  $h$  is a bounded Lipschitz continuous function with Lipschitz constant

$$\|h\|_{\text{Lip}} = \sup_{\{x, y \in \mathbb{R}^d, x \neq y\}} \frac{|h(x) - h(y)|}{|x - y|}$$

then there exists a constant  $C$  independent of  $h$  such that for all  $(t, x)$

$$(7) \quad |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C \|h\|_{\text{Lip}} t^{(1-\|\alpha\|)/2}.$$

In the current paper we investigate the counterpart of these results for the solution of the semilinear PDE (5). The results are summarized in the following:

**Theorem 1.** *Assume that the vector fields  $\{V_i, 0 \leq i \leq N\}$  satisfy the UFG condition of order  $m$ . Then, if  $h$  is bounded and continuous or Lipschitz continuous and if  $f$  satisfies additional conditions that are specified below, the semilinear PDE (5) is uniquely solvable in a suitable space of classical solutions and the solution is differentiable in any direction  $V \in \mathcal{W}$ . Moreover, if  $h$  is a bounded Lipschitz continuous function then there exists a constant  $C$ , such that for all  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$(8) \quad |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C t^{(1-\|\alpha\|)/2}, \quad n \leq K - m - 1,$$

with  $\alpha = (\alpha_1, \dots, \alpha_n) \in [\mathcal{A}_0(m)]^n$ . (See footnote<sup>2</sup>) If  $h$  is a bounded continuous function, but not necessarily Lipschitz, then there exists a constant  $C$ , such that for all  $(t, x) \in [0, T) \times \mathbb{R}^d$  such that

$$(9) \quad |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C t^{-\|\alpha\|/2}$$

if  $n \leq 2 \wedge (K - m - 1)$ . However, if  $n \geq 3$ , then there exists a constant  $C(\delta)$  such that

$$(10) \quad |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C(\delta) t^{-(\|\alpha\| + (n-2))/2}, \quad n \leq K - m - 1,$$

If  $h$  is bounded and measurable only, the semilinear PDE (5) is uniquely solvable as well, but in a suitable space of generalized solutions. The solution admits generalized derivatives in any direction  $V \in \mathcal{W}$  and satisfies (9) and (10) almost everywhere. (And footnote<sup>2</sup> applies as well.)

The details of the assumptions imposed on the function  $f$  are given in Sections 3 and 4 below. We make explicit the dependence of the constants appearing in equations (8), (9) and (10) on the initial condition  $h$  in Theorems 3 and 4. Theorems 3 and 4 also contain certain (nonlinear) Feynman-Kac representations for the derivatives  $V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)$ . Similar bounds and representations are also valid for  $V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)$ ,  $i = 1, \dots, N$ . These too are important for the analysis of numerical algorithms for the approximation of the solution of (5) representations for Theorems 3 and 4.

Let us comment on the bounds contained in (8), (9) and (10). Despite the introduction of the nonlinear term in (5), the solution of the semilinear PDE will have the same small time asymptotics as the solution of the linear PDE (1) when the initial condition  $h$  is a Lipschitz continuous function. The same applies for the case when  $h$  is a bounded (continuous) function as long as we differentiate no more than three times. For derivatives of order 3 or more the asymptotics deteriorates. In Section 5 we study specific examples where deterioration occurs.

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<sup>2</sup>The reader now understands why  $K$  is asked to be greater than  $m + 3$ : (8) holds at least for  $n = 1, 2$ , so that the partial derivatives in space in (5) make sense.

**1.3. Structure of the article.** The article is structured as follows: In section 2 we collect a number of preliminary results required for the proof of the main theorems. The Feynman-Kac formula for the solution of the equation (5) is presented that relates the solution of the PDE to the solution of a backward stochastic differential equation. We also give the rigorous definitions of a solution of (5). In Sections 3 and 4 we analyze the smoothness of the solution of (5) in the case when  $h$  is a bounded Lipschitz continuous function and, respectively, when  $h$  is a bounded (continuous) function. In Section 5 we study an example that shows that we cannot expect the same decay for the case when  $h$  is bounded, but not necessarily Lipschitz continuous, as in the linear case. Finally in Section 6 we relax the Lipschitz condition imposed on the function  $f$  appearing in (5) and treat the case when  $f$  has quadratic growth. This is an important case with applications in optimisation problems appearing in mathematical finance (see, e.g., [11, 26] and the references therein).

## 2. PRELIMINARY RESULTS

**2.1. The Feynman-Kac representation.** Let  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space endowed with an  $(\mathcal{F}_t)_{t \geq 0}$ -adapted Brownian motion  $(B_t)_{t \geq 0}$ . On  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  we consider the triplet  $(X, Y, Z) = \{(X_t, Y_t, Z_t), t \in [0, T]\}$  of  $\mathcal{F}_t$ -adapted stochastic processes satisfying the following system of equations

$$(11) \quad \begin{cases} dX_t &= V_0(X_t)dt + \sum_{j=1}^N V_j(X_t) \circ dW_t^j \\ -dY_t &= f(T-t, X_t, Y_t, Z_t)dt - Z_t dW_t \end{cases}.$$

The system (11) is called a forward-backward stochastic differential equation (FBSDE). The process  $X$ , called the forward component of the FBSDE, is a  $d$ -dimensional diffusion satisfying a stochastic differential equation with  $V_i : \mathbb{R}^d \rightarrow \mathbb{R}^d, i = 0, 1, \dots, d$ . The notation “ $\circ$ ” indicates that the stochastic term in the equation satisfied by  $X$  is a Stratonovitch integral. The process  $Y$ , called the backward component of the SDE is a one-dimensional stochastic process with final condition  $Y_T = h(X_T)$ , where  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  is a measurable function of polynomial growth. The function  $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  referred to as “the driver”, is assumed to be of polynomial growth in  $x$ , of linear growth in  $(y, z)$ , being bounded in time  $t$  and Lipschitz continuous<sup>3</sup> in  $y$  and  $z$ , uniformly in time  $t$  and space  $x$ .

The existence and uniqueness question for the system (11) was first addressed by Pardoux and Peng in [24, 25] and, since then, a large number of papers have been dedicated to the study of FBSDEs. Pardoux and Peng proved that the stochastic flow  $(X^{t,x}, Y^{t,x}, Z^{t,x}), t \in [0, T], x \in \mathbb{R}^d$  associated to the system (11), in other words, the solution of the system

$$(12) \quad \begin{cases} dX_s^{t,x} &= V_0(X_s^{t,x})ds + \sum_{j=1}^d V_j(X_s^{t,x}) \circ dW_s^j, \\ -dY_s^{t,x} &= f(T-s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x} dW_s, \quad s \in [t, T] \\ X_t^{t,x} &= x, \quad Y_T^{t,x} = h(X_T^{t,x}) \end{cases}$$

provides a non-linear Feynman-Kac representation for the solution of the semilinear PDE (5). More precisely they showed that when the functions  $f$  and  $h$  are continuous, then the function

$$(13) \quad u(T-t, x) = Y_t^{t,x},$$

<sup>3</sup>This assumption will be relaxed in Section 6.



is a continuous solution of (5) *in viscosity sense*. We remark that the triplet  $(X^{t,x}, Y^{t,x}, Z^{t,x})$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$  which solves the system (12) is adapted to the (augmented) filtration generated by the increments  $(B_s - B_t)_{t \leq s \leq T}$  so that  $Y_t^{t,x}$  has a deterministic value (up to a zero-measure event).

**2.2. Shift Operator.** When  $u$  is continuous on  $[0, T] \times \mathbb{R}^d$ , the relationship between the deterministic mapping  $u$  and the pair  $(Y, Z)$  extends as  $Y_t = u(T - t, X_t)$ ,  $t \in [0, T]$ . Given  $X_t = x$ , for some  $t \in [0, T]$ , this relationship reads:  $Y_s^{t,x} = u(T - s, X_s^{t,x})$ ,  $s \in [t, T]$ , so that

$$(14) \quad u(T - t, x) = \mathbb{E} \left[ h(X_T^{t,x}) + \int_t^T f(T - s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds \right].$$

Eq. (14) is the keystone for the probabilistic analysis of the regularity of  $u$ . Since  $X$  is a homogeneous diffusion process, we emphasize that  $(X_s^{t,x})_{t \leq s \leq T}$ ,  $t \in [0, T]$ , may be understood as a shifted version of  $(X_{s-t}^{0,x})_{0 \leq s-t \leq T-t}$ . Specifically, we can choose the canonical Wiener space for  $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and thus introduce the shift operator  $(\theta_t : \omega \mapsto \theta_t(\omega) = \omega(t + \cdot) - \omega(t))_{t \geq 0}$ . Then,  $(X_s^{t,x})_{t \leq s \leq T}$  reads as  $(X_{s-t}^{0,x} \circ \theta_t)_{0 \leq s-t \leq T-t}$ , or simply as  $(X_{s-t}^x \circ \theta_t)_{0 \leq s-t \leq T-t}$ , with the convention  $X^x = X^{0,x}$ .

As basic application, we remind the reader of following definition (see [7] and [18]). In the following let  $E$  be a separable Hilbert space and let  $\mathbb{D}^{n,\infty}(E)$  be the space of  $E$ -valued functionals admitting Malliavin derivatives up to order  $n$ , see the Appendix for details.

**Definition 2** (Kusuoka-Stroock functions). *Let  $E$  be a separable Hilbert space and let  $r \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . We denote by  $\mathcal{K}_r^T(E, n)$  the set of functions:  $g : (0, T] \times \mathbb{R}^d \rightarrow \mathbb{D}^{n,\infty}(E)$  satisfying the following:*

- (1)  $g(t, \cdot)$  is  $n$ -times continuously differentiable and  $\frac{\partial^\alpha g}{\partial x^\alpha}(\cdot, \cdot)$  is continuous in  $(t, x) \in (0, 1] \times \mathbb{R}^d$  a.s. for any  $\alpha \in \mathcal{A}$  satisfying  $|\alpha| \leq n$ .
- (2) For all  $k \in \mathbb{N}$ ,  $p \in [1, \infty)$ , and  $k \leq n - |\alpha|$

$$\sup_{t \in (0, T], x \in \mathbb{R}^d} t^{-r/2} \left\| \frac{\partial^\alpha g}{\partial x^\alpha} \right\|_{\mathbb{D}^{k,p}(E)} < \infty$$

Define  $\mathcal{K}_r^T(n) := \mathcal{K}_r^T(\mathbb{R}, n)$ .

The functions belonging to the set  $\mathcal{K}_r^T(E, n)$  satisfy the following properties which form the basis of our analysis (see [7] for details)

**Lemma 1** (Properties of Kusuoka-Stroock functions). *The following hold*

- (1) *The function  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto X_t^x$  belongs to  $\mathcal{K}_0^T(K)$ , for any  $T > 0$ .*
- (2) *Suppose  $g \in \mathcal{K}_r^T(E, n)$ , where  $r \geq 0$ . Then, for  $i = 1, \dots, d$ ,*

$$\int_0^\cdot g(s, x) dB_s^i \in \mathcal{K}_{r+1}^T(E, n) \quad \text{and} \quad \int_0^\cdot g(s, x) ds \in \mathcal{K}_{r+2}^T(E, n).$$

- (3) *If  $g_i \in \mathcal{K}_{r_i}^T(n_i)$  for  $i = 1, \dots, N$ , then*

$$\prod_{i=1}^N g_i \in \mathcal{K}_{r_1 + \dots + r_N}^T(\min_i n_i) \quad \text{and} \quad \sum_{i=1}^N g_i \in \mathcal{K}_{\min_i r_i}^T(\min_i n_i).$$

We then claim as a consequence of Lemmas 17 and 22 in [7] (see also page 265 in [18]):



**Lemma 2.** Define  $J_{t,x} = [\partial(X_t^x)_i / \partial x_j]_{1 \leq i, j \leq d}$ ,  $t \geq 0$ . Then, there exists a family of random functions  $(b_{\alpha,\beta} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R})_{\alpha,\beta \in \mathcal{A}_1^0(m)}$ ,  $b_{\alpha,\beta} \in \cap_{T>0} \mathcal{K}_{(\|\beta\| - \|\alpha\|)^+}^T(K-m)$ , such that for any  $x \in \mathbb{R}^d$  and  $\alpha \in \mathcal{A}_1^0(m)$ .

$$(15) \quad \theta_t^* [J_{s-t,x}] V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}_1^0(m)} \theta_t^* [b_{\alpha,\beta}] (s-t, x) V_{[\beta]}(X_s^{t,x}),$$

where  $\theta_t^* [J_{s-t,x}] = J_{s-t,x} \circ \theta_t$  and  $\theta_t^* [b_{\alpha,\beta}] (s-t, x) = [b_{\alpha,\beta} \circ \theta_t] (s-t, x)$ .

As we will see below, Lemma 2 is a key ingredient of the analysis.

**2.3. The Space of Classical Solutions for the PDE (5).** For an open ball  $\mathbb{B} \subset \mathbb{R}^d$  and for a function  $\varphi$  in  $\mathcal{C}_b^\infty(\mathbb{B})$ , that is a bounded (real-valued) function  $\varphi$  with bounded derivatives of any order on  $\mathbb{B}$ , we set  $\|\varphi\|_{\mathbb{B},\infty}^{V,1} = \|\varphi\|_{\mathbb{B},\infty} + \sum_{\alpha \in \mathcal{A}_1^0(m)} \|V_{[\alpha]}\varphi\|_{\mathbb{B},\infty}$  and then define  $\mathcal{D}_V^{1,\infty}(\mathbb{B})$  as the closure of  $\mathcal{C}_b^\infty(\mathbb{B})$  in  $\mathcal{C}_b(\mathbb{B})$  w.r.t.  $\|\cdot\|_{\mathbb{B},\infty}^{V,1}$ . (See Footnote<sup>4</sup> for the closability argument.)

More generally, for  $1 \leq k \leq K-m+2$ , we can define by induction

$$\|\varphi\|_{\mathbb{B},\infty}^{V,k} = \|\varphi\|_{\mathbb{B},\infty}^{V,k-1} + \sum_{\alpha_1, \dots, \alpha_k \in \mathcal{A}_1^0(m)} \|V_{[\alpha_1]} \dots V_{[\alpha_k]}\varphi\|_{\mathbb{B},\infty}, \quad \varphi \in \mathcal{C}_b^\infty(\mathbb{B}).$$

We emphasize that  $V_{[\alpha_1]} \dots V_{[\alpha_k]}\varphi$  makes sense for any smooth function because of the bound  $k \leq K-m+2$ : each  $V_{[\alpha_i]}$  is at least  $K-m+1$  times continuously differentiable, so that the last vector field  $V_{[\alpha_k]}$  in  $V_{[\alpha_1]} \dots V_{[\alpha_k]}$  can be differentiated  $K-m+1$  times.

We then define  $\mathcal{D}_V^{k,\infty}(\mathbb{B})$  as the closure of  $\mathcal{C}_b^\infty(\mathbb{B})$  in  $\mathcal{C}_b(\mathbb{B})$  w.r.t.  $\|\cdot\|_{\mathbb{B},\infty}^{V,k}$ . (The closability argument is the same as above.) In particular, we can define  $\mathcal{D}_\infty^{k,\infty}(\mathbb{R}^d)$  as

$$\mathcal{D}_V^{k,\infty}(\mathbb{R}^d) = \bigcap_{r \geq 1} \mathcal{D}_V^{k,\infty}(\mathbb{B}(0, r)), \quad 1 \leq k \leq K-m+2,$$

where  $\mathbb{B}(0, r)$  stands for the  $d$ -dimensional ball of center 0 and radius  $r$ . If finite, we set  $\|\varphi\|_\infty^{V,k} = \sup_{r \geq 1} \|\varphi\|_{\mathbb{B}(0,r),\infty}^{V,k}$ . For  $v \in \mathcal{D}_V^{k,\infty}(\mathbb{R}^d)$ ,  $1 \leq k \leq K-m+2$ ,  $V_{[\alpha_1]} \dots V_{[\alpha_k]}v$  is understood as the derivative of  $v$  in the directions  $V_{[\alpha_1]} \dots V_{[\alpha_k]}$ , with  $\alpha_1, \dots, \alpha_k \in \mathcal{A}_0(m)$ .

Similarly, for  $\varphi \in \mathcal{C}_b^\infty(\mathbb{B})$  and  $1 \leq k \leq K-m+1$ , we set

$$\|\varphi\|_{\mathbb{B},\infty}^{V,k+1/2} = \|\varphi\|_{\mathbb{B},\infty}^{V,k} + \sum_{i=1}^N \sum_{\alpha_1, \dots, \alpha_k \in \mathcal{A}_1^0(m)} \|V_{[\alpha_1]} \dots V_{[\alpha_k]} V_i \varphi\|_{\mathbb{B},\infty}, \quad k \geq 0.$$

(Above,  $\|\cdot\|_{\mathbb{B},\infty}^{V,0} = \|\cdot\|_{\mathbb{B},\infty}$ .) We then define  $\mathcal{D}_V^{k+1/2,\infty}(\mathbb{B})$  as the closure of  $\mathcal{C}_b^\infty(\mathbb{B})$  in  $\mathcal{C}_b(\mathbb{B})$  w.r.t.  $\|\cdot\|_{\mathbb{B},\infty}^{V,k+1/2}$  and we set

$$\mathcal{D}_V^{k+1/2,\infty}(\mathbb{R}^d) = \bigcap_{n \geq 1} \mathcal{D}_V^{k+1/2,\infty}(\mathbb{B}(0, n)).$$

<sup>4</sup> We emphasize that the closure is well-defined: if  $(\varphi_n, (V_{[\alpha]} \varphi_n)_{\alpha \in \mathcal{A}_1^0(m)})_{n \geq 1}$  tends to  $(0, (G_\alpha)_{\alpha \in \mathcal{A}_1^0(m)})$  uniformly on  $\mathbb{B}$  as  $n$  tends to  $+\infty$ , then for any test function  $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$  with compact support included in  $\mathbb{B}$ ,  $\int_{\mathbb{R}^d} G_\alpha^i(x) \psi(x) dx = \lim_{n \rightarrow +\infty} - \int_{\mathbb{R}^d} \varphi_n(x) \partial_{x_i} (V_{[\alpha]}^i \psi)(x) dx = 0$ ,  $i = 1, \dots, N$ , so that  $G_\alpha$  is zero.

If finite, we set  $\|\varphi\|_\infty^{V,k+1/2} = \sup_{n \geq 1} \|\varphi\|_{\mathbb{B}(0,n),\infty}^{V,k+1/2}$ .

A typical example of function in  $\mathcal{D}_V^{n,\infty}(\mathbb{R}^d)$ ,  $1 \leq n \leq K - m$ , is  $x \in \mathbb{R}^d \mapsto (P_t \varphi)(x)$ , for  $t > 0$  and  $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$ . For this we need to recall the following integration by parts formula (see Corollaries 26 and 30 in [7])

**Theorem 2.** *Let  $(V_i)_{0 \leq i \leq N}$  satisfy the assumptions in Definition 1. Then, for any  $T > 0$ ,  $n \leq K - m$  and  $\alpha_1, \dots, \alpha_n \in \mathcal{A}(m)$ , there exists  $\Phi_{\alpha_1, \dots, \alpha_n} \in \mathcal{K}_0^T(K - m - n)$  such that*

$$(16) \quad V_{[\alpha_1]} \dots V_{[\alpha_n]}(P_t h)(x) = t^{-\|\alpha\|/2} \mathbb{E} [\Phi_{\alpha_1, \dots, \alpha_n}(t, x) h(X_t^x)],$$

for any  $h \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ ,  $t \in (0, T]$ ,  $x \in \mathbb{R}^d$ , with  $\alpha = (\alpha_1, \dots, \alpha_n)$ . In particular, the following gradient bound holds true:

$$(17) \quad \|V_{[\alpha_1]} \dots V_{[\alpha_n]} P_t h\|_\infty \leq C \|h\|_\infty t^{-\|\alpha\|/2},$$

where  $C = \sup_{0 < t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} [|\Phi_{\alpha_1, \dots, \alpha_n}(t, x)|] < \infty$ . In addition, for any  $n \leq K - m$  and  $\alpha_1, \dots, \alpha_n \in \mathcal{A}(m)$  there exist  $\Phi_{\alpha_1, \dots, \alpha_n}^i \in \mathcal{K}_0^T(K - m - n + 1)$ ,  $i = 1, \dots, d$  such that

$$(18) \quad V_{[\alpha_1]} \dots V_{[\alpha_n]}(P_t h)(x) = t^{-(\|\alpha_1\| + \dots + \|\alpha_{n-1}\|)/2} \sum_{i=1}^d \mathbb{E} [\Phi_{\alpha_1, \dots, \alpha_n}^i(t, x) \partial_i h(X_t^x)],$$

for any  $h \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ ,  $t \in (0, T]$ ,  $x \in \mathbb{R}^d$ . Hence, in particular, the following gradient bound holds true:

$$(19) \quad \|V_{[\alpha_1]} \dots V_{[\alpha_n]} P_t h\|_\infty \leq C T^{(m-1)/2} \|\nabla h\|_\infty t^{(1-\|\alpha\|)/2},^5$$

where  $C = \max_{i=1, \dots, d} \sup_{0 < t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} [|\Phi_{\alpha_1, \dots, \alpha_n}^i(t, x)|] < \infty$ .

To prove that the mapping  $x \in \mathbb{R}^d \mapsto (P_t \varphi)(x)$ , for  $t \in (0, T]$  and  $\varphi \in \mathcal{C}_b^0(\mathbb{R}^d)$ , is in  $\mathcal{D}_{V,\infty}^n(\mathbb{R}^d)$ ,  $1 \leq n \leq K - m$ , it is sufficient to consider a sequence  $(\varphi_\ell)_{\ell \geq 1}$  of functions in  $\mathcal{C}_b^\infty(\mathbb{R}^d)$  converging towards  $\varphi$  uniformly on compact subsets of  $\mathbb{R}^d$  as  $\ell$  tends to  $+\infty$ . Then, from the above theorem, we have that

$$(20) \quad [V_{[\alpha_1]} \dots V_{[\alpha_n]} P_t \varphi_\ell](x) = t^{-\|\alpha\|/2} \mathbb{E} [\varphi_\ell(X_t^x) \psi(t, x)],$$

with  $\psi \in \mathcal{K}_0^T(K - m - n)$  is independent of  $\ell$ . Clearly, on every compact subsets of  $\mathbb{R}^d$ , the right-hand side in (20) converges towards the continuous function

$$x \in \mathbb{R}^d \mapsto t^{-\|\alpha\|/2} \mathbb{E} [\varphi(X_t^x) \psi(t, x)].$$

Therefore, the sequence  $(V_{[\alpha_1]} \dots V_{[\alpha_n]} P_t \varphi_\ell)_{\ell \geq 1}$  is Cauchy in any  $\mathcal{C}^0(\bar{\mathbb{B}}(0, r))$ ,  $r > 0$ , so that  $P_t \varphi$  belongs to  $\mathcal{D}_V^{n,\infty}(\mathbb{R}^d)$  for  $1 \leq n \leq K - m$  and (20) holds for  $\varphi$  as well.

<sup>5</sup>To be exact one has  $\|V_{[\alpha_1]} \dots V_{[\alpha_n]} P_t h\|_\infty \leq C \|\nabla h\|_\infty t^{-(\|\alpha_1\| + \dots + \|\alpha_{n-1}\|)/2}$  and inequality (17) follows as  $t^{-(\|\alpha_1\| + \dots + \|\alpha_{n-1}\|)} \leq T^{m-1} t^{1-\|\alpha\|}$  (recall that  $t \leq T$ ).

To define the notion of a classical solution to (4), we must define the operator  $\mathcal{V}_0$  first. Again, we proceed by a closure argument. For any  $r \geq 1$  and any time-space function  $\varphi \in \mathcal{C}_b^\infty([1/r, r] \times \mathbb{B}(0, r))$  with bounded derivatives of any order, we set

$$\|\varphi\|_{[1/r, r] \times \mathbb{B}(0, r), \infty}^{\mathcal{V}_0, 1} = \|\varphi\|_{[1/r, r] \times \mathbb{B}(0, r), \infty} + \|\partial_t \varphi - V_0 \varphi\|_{[1/r, r] \times \mathbb{B}(0, r), \infty}.$$

We then define  $\mathcal{D}_{\mathcal{V}_0}^{1, \infty}([1/r, r] \times \mathbb{B}(0, r))$  as the closure of  $\mathcal{C}_b^\infty([1/r, r] \times \mathbb{B}(0, r))$  w.r.t.  $\|\cdot\|_{[1/r, r] \times \mathbb{B}(0, r), \infty}^{\mathcal{V}_0, 1}$  and then define  $\mathcal{D}_{\mathcal{V}_0}^{1, \infty}((0, +\infty) \times \mathbb{R}^d)$  as the intersection of the spaces  $\mathcal{D}_{\mathcal{V}_0}^{1, \infty}([1/r, r] \times \mathbb{B}(0, r))$  over  $r \geq 1$ . (As above, the closability property is easily checked.)

We are now in position to define a classical solution to the PDE:

**Definition 3.** We call a function  $v = \{v(t, x), (t, x) \in [0, +\infty) \times \mathbb{R}^d\}$  a classical solution of the PDE (5) if the followings are satisfied

- (1)  $v$  belongs to  $\mathcal{D}_{\mathcal{V}_0}^{1, \infty}((0, +\infty) \times \mathbb{R}^d)$  and, for any  $t > 0$ ,  $v(t, \cdot)$  belongs to  $\mathcal{D}_V^{2, \infty}(\mathbb{R}^d)$  such that, for any  $\alpha_1, \alpha_2 \in \mathcal{A}_0(m)$ , the function

$$(t, x) \in (0, +\infty) \times \mathbb{R}^d \mapsto (V_{[\alpha_1]}v(t, x), V_{[\alpha_1]}V_{[\alpha_2]}v(t, x))$$

is continuous,

- (2) for any  $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ , it holds

$$\mathcal{V}_0 v(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 v(t, x) + f(t, x, v(t, x), Vv(t, x)),$$

- (3) the boundary condition  $\lim_{(t, y) \rightarrow (0, x)} v(t, y) = h(x)$  holds as well for any  $x \in \mathbb{R}^d$ .

As announced, here is the connection between the PDE and the BSDE:

**Proposition 1.** Under the standing assumption, if  $h$  is a bounded continuous function or a Lipschitz function, the function  $u$  given by (13) for a given  $T > 0$  is a classical solution to the PDE (5) on  $(0, T] \times \mathbb{R}^d$ .

Moreover, any other classical solution  $v$  of the semilinear PDE (5) that has polynomial growth, in other words for which there exist  $C, r \geq 0$  such that

$$(21) \quad \forall x \in \mathbb{R}^d, \quad |v(t, x)| \leq C(1 + |x|^r),$$

matches  $u$ .

The proof is postponed to Section 7. It mainly relies on the following version of Itô's formula (whose proof is also postponed to Section 7):

**Proposition 2.** Let  $v$  satisfy Point (1) in Definition 3 and be at most of polynomial growth as in (21). Then, for any  $T > 0$  and  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} v(T - s, X_s^{t, x}) &= v(T, x) + \int_t^s \left[ -\mathcal{V}_0 v(T - r, X_r^{t, x}) + \frac{1}{2} \sum_{i=1}^N V_i^2 v(T - r, X_r^{t, x}) \right] ds \\ &\quad + \sum_{i=1}^N \int_t^s V_i v(T - r, X_r^{t, x}) dB_r^i, \quad t \leq s < T. \end{aligned}$$

**2.4. The Space of Generalized Solutions to the PDE (5).** As we defined  $\mathcal{D}_V^{k,\infty}(\mathbb{B})$  as the closure of  $\mathcal{C}_b^\infty(\mathbb{B})$  in  $C_b(\mathbb{B})$  w.r.t.  $\|\cdot\|_{\mathbb{B},\infty}^{V,k}$  for a given ball  $\mathbb{B}$ , we can define  $\mathcal{D}_V^{k,p}(\mathbb{B})$ , for a given real  $p \geq 1$  and for  $1 \leq k \leq K - m + 2$ , as the closure of  $\mathcal{C}_b^\infty(\mathbb{B})$  in  $L^p(\mathbb{B})$  w.r.t.  $\|\cdot\|_{\mathbb{B},p}^{V,k}$ , where

$$\begin{aligned}\|\varphi\|_{\mathbb{B},p}^{V,1} &= \|\varphi\|_{\mathbb{B},p} + \sum_{\alpha \in \mathcal{A}_1^0(m)} \|V_{[\alpha]}\varphi\|_{\mathbb{B},p} \\ \|\varphi\|_{\mathbb{B},p}^{V,k} &= \|\varphi\|_{\mathbb{B},p}^{V,k-1} + \sum_{\alpha_1, \dots, \alpha_k \in \mathcal{A}_1^0(m)} \|V_{[\alpha_1]} \dots V_{[\alpha_k]}\varphi\|_{\mathbb{B},p}, \quad \varphi \in L^p(\mathbb{B}),\end{aligned}$$

the notation  $\|\cdot\|_{\mathbb{B},p}$  here standing for the  $L^p$  norm over  $\mathbb{B}$ . The closability argument is the same as in Footnote (4). Then, we can define  $\mathcal{D}_V^{k,p}(\mathbb{R}^d)$  as the intersection of all the  $\mathcal{D}_V^{k,p}(\mathbb{B}(0,r))$ ,  $r > 0$ . Similarly, we can define  $\mathcal{D}_V^{k+1/2,p}(\mathbb{B})$  and  $\mathcal{D}_V^{k+1/2,p}(\mathbb{R}^d)$  for  $1 \leq k \leq K - m + 1$ ,  $\mathcal{D}_V^{1,p}([\varepsilon, T] \times \mathbb{B})$  for  $0 < \varepsilon < T$ , and  $\mathcal{D}_V^{1,p}((0, +\infty) \times \mathbb{R}^d)$ .

A typical example of function in  $\mathcal{D}_V^{k,n}(\mathbb{R}^d)$ ,  $1 \leq n \leq K - m$ , is  $x \in \mathbb{R}^d \mapsto (P_t \varphi)(x)$ , for  $t > 0$  and  $\varphi \in L_{\text{loc}}^p(\mathbb{R}^d)$ ,  $\varphi$  being at most of polynomial growth at the infinity. The proof is almost the same as in the case when  $p = +\infty$ . The point is to consider an approximating sequence  $(\varphi_\ell)_{\ell \geq 1}$ , converging towards  $\varphi$  in  $L_{\text{loc}}^p(\mathbb{R}^d)$  (that is in any  $L^p(\mathbb{B}(0,R))$ ,  $R > 0$ ) and then to prove the right-hand side in (20) is Cauchy in  $L_{\text{loc}}^p(\mathbb{R}^d)$ . To prove it, we claim that for any  $R > 0$  and  $\ell, k \geq 0$ ,

$$(22) \quad \int_{|x| < R} |\mathbb{E}[(\varphi_{\ell+k} - \varphi_\ell)(X_t^x) \psi(t, x)]|^p dx \leq C \int_{|x| < R} \mathbb{E}[|\varphi_{\ell+k} - \varphi_\ell|^p(X_t^x)] dx,$$

with

$$C = \sup_{x \in \mathbb{R}^d} \mathbb{E}[\psi(t, x)^{p'}]^{p/p'} < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Now, the result follows from

**Lemma 3.** *Let  $\theta_1$  and  $\theta_2$  be two functions belonging to  $L_{\text{loc}}^p(\mathbb{R}^d)$ ,  $p \geq 1$ , and at most of polynomial growth of exponent  $r \geq 0$  (that is  $|\theta_i(x)| \leq C(1 + |x|^r)$ ,  $i = 1, 2$ , for some constant  $C \geq 0$ ), then, for any  $A, R > 0$ ,*

$$\int_{|x| < R} \mathbb{E}[|\theta_1 - \theta_2|^p(X_t^x)] dx \leq C' \int_{|y| < A} |\theta_1 - \theta_2|^p(y) dy + C' A^{-1/2} (1 + R^{r+1/2}),$$

the constant  $C'$  being independent of  $A$  and  $R$  and depending on  $\theta_1$  and  $\theta_2$  through  $C$  and  $r$  only.

The proof of Lemma 3 is postponed to the Appendix. Choosing  $\theta_1 = \varphi_{\ell+k}$  and  $\theta_2 = \varphi_\ell$  therein, we deduce that the right-hand side in (20) is indeed Cauchy in  $L_{\text{loc}}^p(\mathbb{R}^d)$ . (Clearly, we can assume the  $(\varphi_\ell)_{\ell \geq 1}$  to be of polynomial growth, uniformly in  $\ell$ .)

We are now in position to define the notion of generalized solution to the PDE (5). On the model of Definition 3, we set

**Definition 4.** *We call a function  $v = \{v(t, x), (t, x) \in [0, +\infty) \times \mathbb{R}^d\}$  a generalized solution of the PDE (5) if the followings are satisfied*

- (1)  $v$  belongs to  $\cap_{p \geq 1} \mathcal{D}_{V_0}^{1,p}((0, +\infty) \times \mathbb{R}^d)$  and, for any  $t > 0$ ,  $v(t, \cdot)$  belongs to  $\cap_{p \geq 1} \mathcal{D}_V^{2,p}(\mathbb{R}^d)$  such that, for any  $\alpha_1, \alpha_2 \in \mathcal{A}_1^0(m)$ , the function

$$(t, x) \in (0, +\infty) \times \mathbb{R}^d \mapsto (V_{[\alpha_1]}v(t, x), V_{[\alpha_1]}V_{[\alpha_2]}v(t, x))$$

is measurable and in any  $L_{\text{loc}}^p((0, +\infty) \times \mathbb{R}^d)$ ,  $p \geq 1$ ,

- (2) for almost every  $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ , it holds

$$\mathcal{V}_0 v(t, x) = \frac{1}{2} \sum_{i=1}^N V_i^2 v(t, x) + f(t, x, v(t, x), Vv(t, x)),$$

- (3) on any compact set,  $v(t, \cdot) \rightarrow h$  in Lebesgue-measure as  $t \searrow 0$ , that is, for any ball  $\mathbb{B} \subset \mathbb{R}^d$ , for any  $\varepsilon > 0$ ,

$$\lim_{t \searrow 0} |\{x \in \mathbb{B} : |v(t, x) - h(x)| \geq \varepsilon\}| = 0,$$

where  $|A|$  denotes the Lebesgue volume of  $A$  for a Borel subset  $A \subset \mathbb{R}^d$ .

In Section 7, we will show

**Proposition 3.** *Under the standing assumption, if  $h$  is bounded (and measurable), the function  $u$  given by (13) for a given  $T > 0$  is a generalized solution to the PDE (5) on  $(0, T] \times \mathbb{R}^d$ . Moreover, any other generalized solution  $v$  of the semilinear PDE (5) that has polynomial growth matches  $u$  almost everywhere.*

Again, the proof is based on a suitable version of Itô's formula. Because of the  $L^p$  setting, it cannot be true for any given starting point. We will prove in Section 7:

**Proposition 4.** *Let  $v$  satisfy Point (1) in Definition 4 and be at most of polynomial growth as in (21). Then, for any  $T > 0$  and any bounded  $\mathcal{F}_t$ -measurable and  $\mathbb{R}^d$ -valued random vector  $\xi$ ,  $0 \leq t < T$ , with an absolutely continuous distribution w.r.t. the Lebesgue measure on  $\mathbb{R}^d$ , Itô's formula holds on the same model as in Proposition 2, but replacing  $X_s^{t,x}$  by  $X_s^{t,\xi}$  therein.*

*In particular, the process  $(v(T - s, X_s^{t,\xi}))_{t \leq s \leq T}$  admits a continuous version.*

We emphasize that, in Itô's formula, all the terms are uniquely defined even if the derivatives of  $v$  are defined up to sets of zero Lebesgue measure. This a consequence of Lemma 3.

**2.5. Generalized Gronwall Lemma.** In the following we will make use of the following:

**Lemma 4.** *Consider two bounded measurable functions  $g_1, g_2 : [0, T] \rightarrow \mathbb{R}_+$  such that*

$$(23) \quad g_1(t) \leq C_1 + C_2 \int_t^T \frac{g_2(s)}{\sqrt{s-t}} ds,$$

for some constants  $C_1, C_2 \geq 0$ . Then there exist  $\lambda, \mu > 0$ , depending on  $C_2$  and  $T$  only, such that

$$(24) \quad \int_0^T g_1(t) \exp(\lambda t) dt \leq \mu C_1 + \frac{1}{2} \int_0^T g_2(t) \exp(\lambda t) dt.$$

In particular, if  $g_1 = g_2$ , then  $g_1$  is bounded by  $\mu' C_1$ , for a constant  $\mu'$  depending on  $C_2$  and  $T$  only.

**Proof.** Integrating (23) w.r.t  $\exp(\lambda t)$ , we obtain

$$\begin{aligned} \int_0^T g_1(t) \exp(\lambda t) dt &\leq C_1 \int_0^T \exp(\lambda t) dt + C_2 \int_0^T \left[ g_2(s) \int_0^s \frac{\exp(\lambda t)}{(s-t)^{1/2}} dt \right] ds \\ &= C_1 \int_0^T \exp(\lambda t) dt + C_2 \int_0^T \left[ g_2(s) \exp(\lambda s) \int_0^s \frac{\exp(\lambda(t-s))}{(s-t)^{1/2}} dt \right] ds \\ &= C_1 \int_0^T \exp(\lambda t) dt + C_2 \int_0^T \left[ g_2(s) \exp(\lambda s) \int_0^s \frac{\exp(-\lambda t)}{t^{1/2}} dt \right] ds \\ &\leq C_1 \int_0^T \exp(\lambda t) dt + C_2 \int_0^T \frac{\exp(-\lambda t)}{t^{1/2}} dt \int_0^T g_2(s) \exp(\lambda s) ds. \end{aligned}$$

Choosing  $\lambda$  large enough, this proves (24).

Assume now that  $g_1 = g_2$ . Then,  $\int_0^T \exp(\lambda t) g_1(t) dt \leq 2\mu C_1$ , so that  $\int_0^T g_1(t) dt \leq 2\mu C_1$ . Therefore, for  $\varepsilon > 0$ , (23) yields

$$\begin{aligned} g_1(t) &\leq C_1 + C_2 \int_t^{(t+\varepsilon)\wedge T} \frac{g_1(s)}{(s-t)^{1/2}} ds + C_2 \varepsilon^{-1/2} \int_{(t+\varepsilon)\wedge T}^T g_1(s) ds \\ &\leq C_1 + 2\mu C_1 C_2 \varepsilon^{-1/2} + C_2 \varepsilon^{1/2} \sup_{t \leq s \leq (t+\varepsilon)\wedge T} [g_1(s)]. \end{aligned}$$

Finally,

$$\sup_{0 \leq t \leq T} [g_1(t)] \leq C_1 + 2\mu C_1 C_2 \varepsilon^{-1/2} + C_2 \varepsilon^{1/2} \sup_{0 \leq s \leq T} [g_1(s)].$$

Choosing  $\varepsilon$  small enough, we complete the proof.  $\square$

### 3. LIPSCHITZ BOUNDARY CONDITION

**3.1. Setting and Main Result.** In the whole section, we assume that the boundary condition is at least Lipschitz continuous. We also assume that  $|f(t, x, y, z)| \leq \Lambda(1 + |x| + |y| + |z|)$ ,  $x, z \in \mathbb{R}^d$ ,  $y \in \mathbb{R}$ , and that  $f(t, \cdot)$  is  $K - m - 1$ -times continuously differentiable, the derivatives up to any order  $1 \leq n \leq K - m - 1$  being bounded by some constant  $\Lambda_n \geq 0$ . To simplify things, we will assume that  $\Lambda_n \geq \Lambda$ .

The objective is then to prove

**Theorem 3.** *Let  $(V_i)_{0 \leq i \leq N}$  be  $N + 1$  vector fields satisfying Definition 1. Then, for any  $t \in [0, T)$ ,  $u(t, \cdot)$  belongs to  $\mathcal{D}_V^{K-m-1/2, \infty}(\mathbb{R}^d)$  and is Lipschitz continuous;  $u(t, \cdot)$  is continuously differentiable if  $h$  is continuously differentiable, i.e.  $\nabla_x u(t, \cdot)$  exists as a continuous function.*

*Moreover, for any  $n \leq K - m - 1$  and  $\alpha_1, \dots, \alpha_n \in \mathcal{A}(m)$ , there exists a constant  $C_n(p)$ , depending on  $\Lambda_n, n, p, T$  and the vector fields  $V_0, \dots, V_N$  only, such that for all  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$|V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C_n(p) (T-t)^{(1-\|\alpha\|)/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{np}]^{1/p}],$$

$$|V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)| \leq C_n(p) (T-t)^{-\|\alpha\|/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{np}]^{1/p}], \quad 1 \leq i \leq N,$$

where  $\alpha$  stands for  $(\alpha_1, \dots, \alpha_n)$  and  $\|\alpha\|$  for  $\|\alpha_1\| + \dots + \|\alpha_n\|$ . In the casewhen  $\nabla h$  does not exist at point  $X_{T-x}^x$ ,  $|\nabla h(X_{T-t}^x)|$  is understood as  $\limsup_{\varepsilon \rightarrow 0, \varepsilon \neq 0} |\varepsilon|^{-1} |h(X_{T-t}^x + \varepsilon) - h(X_{T-t}^x)|$ .

Moreover, the derivative pair process

$$(Y_t^\alpha = (V_{[\alpha_1]} \dots V_{[\alpha_n]} u)(t, X_t^x), Z_t^\alpha = ((V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u)(t, X_t^x))_{1 \leq i \leq N})_{0 \leq t < T}$$

satisfies a generalized BSDE of the form

$$(25) \quad \begin{aligned} Y_t^\alpha &= (s-t)^{[1-\|\alpha\|]/2} \sum_{\beta \in \mathcal{A}_1^0(m)} \mathbb{E}[(V_{[\beta]} u)(s, X_s^x) \theta_t^*[\phi_\beta](s-t, X_t^x) | \mathcal{F}_t] \\ &\quad + \mathbb{E} \left[ \int_t^s (F_1(r, x) + F_2(r, x) Y_r^\alpha + F_3(r, x) Z_r^\alpha) | \mathcal{F}_t \right], \\ Z_t^\alpha &= (s-t)^{-\|\alpha\|/2} \sum_{\beta \in \mathcal{A}_1^0(m)} \mathbb{E}[(V_{[\beta]} u)(s, X_s^x) \theta_t^*[\psi_\beta](s-t, X_t^x) | \mathcal{F}_t] \\ &\quad + \mathbb{E} \left[ \int_t^s (r-t)^{-1/2} (G_1(t, r, x) + G_2(t, r, x) Y_r^\alpha + G_3(t, r, x) Z_r^\alpha) | \mathcal{F}_t \right], \end{aligned}$$

with  $t < s < T$ , where  $(\phi_\beta)_{\beta \in \mathcal{A}_1^0(m)}$  and  $(\psi_\beta)_{\beta \in \mathcal{A}_1^0(m)}$  stand for Kusuoka functions in  $\mathcal{K}_0^T(K - m - n - 1)$ , and  $F_1(r, x)$ ,  $G_1(t, r, x)$ ,  $F_2(r, x)$ ,  $G_2(t, r, x)$ ,  $F_3(r, x)$  and  $G_3(t, r, x)$  are random functionals, in any  $L^p(\Omega)$ ,  $p > 1$ , uniformly in  $x \in \mathbb{R}^d$  and in  $0 \leq t < r < s$ ,  $s$  in a compact subset of  $[0, T)$ . Moreover,  $F_2(r, x)$  and  $F_3(r, x)$  are bounded, uniformly in  $x \in \mathbb{R}^d$  and in  $0 \leq r < s$ ,  $s$  in a compact subset of  $[0, T)$ , and, for any  $p > 1$ ,  $\mathbb{E}[|G_2(t, r, x)|^p | \mathcal{F}_r]$  and  $\mathbb{E}[|G_3(t, r, x)|^p | \mathcal{F}_r]$  are bounded, uniformly in  $x \in \mathbb{R}^d$  and in  $0 \leq t < r < s$ ,  $s$  in a compact subset of  $[0, T)$ .

Equation (25) provides the stochastic dynamics of the derivative processes when the forward equation is initialized at  $x$  at time 0. It must be seen as a *non-linear integration by parts*, that is the equivalent to the integration by parts formula exhibited in the linear case. It must be also compared with the pathwise differentiation result in [25]. The difference between (25) and the result in [25] lies in the lack of well-defined boundary condition in (25): it would be the higher-order derivatives of  $h$  if they were well-defined. Here they don't exist as  $h$  is assumed to be Lipschitz only. As a consequence, the derivative processes are only defined up to any time  $s$  strictly less than maturity time and the boundary like type condition is expressed as a conditional expectation: the first-order term therein is bounded in  $t$  and  $s$  so that the leading coefficient  $(s-t)^{[1-\|\alpha\|]/2}$  stands for the typical order of the boundary condition in the neighborhood of  $T$ . Obviously, it is also well-understood that the coefficients  $F_1, F_2, F_3, G_1, G_2$  and  $G_3$  depend on the spatial derivatives of  $u$  of orders less than or equal to  $n-1$ : the explicit dependence being expressed in terms of additional Kusuoka functions. The actual dependence is detailed the proof of Proposition 5 below. Obviously, equation (25) also holds when the forward process starts from  $x$  at time  $t \in [0, T)$ .

A straightforward application of Lemma 4 shows that  $(Y^\alpha, Z^\alpha)$  is the unique solution to (25) with continuous paths such that  $\mathbb{E}[\sup_{0 \leq t \leq s} |Y_t^\alpha|^p + \sup_{0 \leq t \leq s} |Z_t^\alpha|^p] < +\infty$  for any  $s \in [0, T)$  and for any  $p > 1$ . This is done via a standard fixed point argument similar to that used in the classical proof of the unique solvability of BSDEs driven by  $Z$ -independent drivers. We also emphasize that, following Lemma 4, one can establish a stability property for equation (25), that is the stability of the solution under perturbation of the coefficients: again, the proof is similar to the one in the standard BSDE setting and we omit it.



The strategy of the proof of Theorem 3 consists in mollifying the coefficients of the equation first and then in letting the mollifying parameter vanish. Below, we will assume first that the coefficients  $f$  and  $h$  are smooth in  $x, y$  and  $z$ .

**3.2. One-Step Differentiation.** The following one-step differentiation lemma permits to switch from one derivative to another:

**Lemma 5.** *Let  $F$  be a differentiable function from  $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^N$  into  $\mathbb{R}$  and  $\varphi$  be in  $\mathcal{D}_V^{3/2, \infty}(\mathbb{R}^d)$ . Then, setting  $\Theta(X_s^x) = (X_s^x, \varphi(X_s^x), ((V_i \varphi)(X_s^x))_{1 \leq i \leq N})$ ,  $0 \leq s \leq T$ , the mapping  $x \mapsto F(\Theta(X_s^x))$  is in  $\mathcal{D}_V^1(\mathbb{R}^d)$  and, for any  $\alpha \in \mathcal{A}_1^0(m)$ ,*

$$\begin{aligned} V_{[\alpha]}[F(\Theta(X_s^x))] &= \sum_{\beta \in \mathcal{A}_1^0(m)} \left\{ b_{\alpha, \beta}(s, x) [V_{[\beta]}(X_s^x) \cdot \nabla_x F(\Theta(X_s^x)) \right. \\ &\quad \left. + \nabla_y F(\Theta(X_s^x))(V_{[\beta]} \varphi)(X_s^x) + \sum_{\ell=1 \dots N} \nabla_{z_\ell} F(\Theta(X_s^x))(V_{[\beta]} V_\ell \varphi)(X_s^x) \right\}. \end{aligned}$$

(Here,  $V_{[\alpha]}$  is understood as  $V_{[\alpha]}(x)$ .)

**Proof.** When  $\varphi$  is a smooth function, we can write

$$\begin{aligned} V_{[\alpha]}(x)[F(\Theta(X_s^x))] &= \sum_{i=1}^d V_{[\alpha]}^i(x) \sum_{j=1}^d \frac{\partial F \circ \Theta}{\partial x_j}(X_s^x) \frac{\partial (X_s^x)^j}{\partial x_i} \\ &= \sum_{i=1}^d \sum_{j=1}^d (J_{s,x})_{j,i} V_{[\alpha]}^i(x) \frac{\partial F \circ \Theta}{\partial x_j}(X_s^x). \end{aligned}$$

Applying Lemma 2 with  $t = 0$ , the result easily follows (when  $\varphi$  is smooth). By a closure argument, the result is still valid when  $\varphi$  is in  $\mathcal{D}_V^{3/2, \infty}(\mathbb{R}^d)$ .  $\square$

As a Corollary, we deduce

**Corollary 1.** *Let  $F$  be a  $(K - m - 1)$ -time differentiable function from  $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$  into  $\mathbb{R}$  and  $\varphi$  be in  $\mathcal{D}_V^{n+1/2}(\mathbb{R}^d)$ ,  $n \leq K - m - 1$ . For any  $k \in \{0, \dots, n\}$ , let  $\mathcal{U}_k(\varphi)$  be the set of  $k$ -tuples of functions of the form  $(v_1, \dots, v_k)$ , with  $v_i$  being equal either to  $\varphi$  or  $V_\ell \varphi$ ,  $1 \leq \ell \leq N$ . (When  $k = 0$ , we set  $\mathcal{U}_k(\varphi) = \emptyset$ .) Finally, let  $\mathcal{I}_k(n)$  be the set of non-decreasing sequences of (possibly zero) integers  $i_1, \dots, i_k$  such that  $i_1 + \dots + i_k \leq n$ .*

*Then, for any  $n$ -tuple of indices  $\alpha = (\alpha_1, \dots, \alpha_n) \in [\mathcal{A}_1^0(m)]^n$  and for any  $0 \leq k \leq n$ , we can find a subset  $\mathbf{V}_k \subset \mathcal{U}_k(\varphi)$ , a subset  $\mathbf{I}_k \subset \mathcal{I}_k(n)$  and, for any  $\mathbf{v} = (v_1, \dots, v_k) \in \mathcal{U}_k(\varphi)$  and  $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}_k(n)$ , we can find  $k$  subsets  $(\mathbf{A}_j^{\mathbf{i}, \mathbf{v}} \subset [\mathcal{A}_1^0(m)]^j)_{1 \leq j \leq k}$ , such that*

$$\begin{aligned} &V_{[\alpha_n]} \dots V_{[\alpha_1]} [F(\Theta(X_s^x))] \\ &= \sum_{k=0}^n \sum_{\mathbf{i} \in \mathbf{I}_k, \mathbf{v} \in \mathbf{V}_k} \sum_{\beta = (\beta_{i_\ell, j})_{\substack{1 \leq j \leq k \\ 1 \leq \ell \leq i_\ell}} \in \mathbf{A}^{\mathbf{i}, \mathbf{v}}} \left[ \prod_{j=1}^k (V_{[\beta_{i_\ell, j}]} \dots V_{[\beta_{i_\ell, j}]} v_j)(X_s^x) \phi_{\mathbf{i}, \mathbf{v}, \beta}(s, x) \psi_{\mathbf{i}, \mathbf{v}, \beta}(\Theta(X_s^x)) \right], \end{aligned}$$

where  $\mathbf{A}^{i,v} = \prod_{j=1}^k \mathbf{A}_j^{i,v}$ ,  $\phi_{i,v,\beta} \in \mathcal{K}_{(\|\beta\| - \|\alpha\|)^+}^T(K - m - n)$ , with  $\|\beta\| = \sum_{j=1}^k \sum_{i=1}^{i_j} \|\beta_{i,j}\|$  and  $\|\alpha\| = \sum_{i=1}^n \|\alpha_i\|$ , and  $\psi_{i,v,\beta}$  is  $(K - m - n - 1)$ -times differentiable with bounded derivatives.

**Proof.** We proceed by induction. Assume that the result holds true for a given  $n \geq 1$ . Then, for a given  $\alpha_{n+1} \in \mathcal{A}_1^0(m)$ , we are to consider for any  $(k, i, v, \beta)$  as above

$$V_{[\alpha_{n+1}]} \mathcal{E} \quad \text{with} \quad \mathcal{E} = \prod_{j=1}^k (V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j)(X_s^x) \phi_{i,v,\beta}(s, x) \psi_{i,v,\beta}(\Theta(X_s^x)).$$

Clearly, the term obtained by letting  $V_{[\alpha_{n+1}]}$  act on  $\phi_{i,v,\beta}$  gives a new Kusuoka function belonging to  $\mathcal{K}_{(\|\beta\| - \|\alpha\|)^+}^T(K - m - (n + 1))$ , which is included in  $\mathcal{K}_{(\|\beta\| - \|\alpha\| - \|\alpha_{n+1}\|)^+}^T(K - m - (n + 1))$ . To differentiate  $\psi_{i,v,\beta}(\Theta(X_s^x))$ , we apply Lemma 5. There are two cases: (i) the first term in Lemma 5 does not add a new term of the form  $V_{[\beta]} v$ ; (ii) the two last terms in Lemma 5 add new terms of the form  $V_{[\beta]} v$ . It is clear that (i) keeps the general form of the formula. We explain now what happens for (ii). Following Lemma 5, the functions  $\psi_{i,v,\beta}$  is modified in such a way that, for any  $\beta_{1,k+1} \in \mathcal{A}_1^0(m)$ , the term  $\mathcal{E}$  at rank  $n$  is multiplied by  $V_{[\beta_{1,k+1}]} v_{k+1}$  for  $v_{k+1}$  being either  $\varphi$  or one of the  $(V_\ell \varphi)_{1 \leq \ell \leq n}$  and that the sum is then performed over all the  $\beta_{1,k+1} \in \mathcal{A}_1^0(m)$ . It means that  $k$  is increased into  $k + 1$  and that  $\phi_{i,v,\beta}$  is changed into  $\phi_{i,v,\beta} b_{\alpha_{n+1}, \beta_{1,k+1}}$ . Now,  $b_{\alpha_{n+1}, \beta_{1,k+1}}$  is in  $\mathcal{K}_{(\|\beta_{1,k+1}\| - \|\alpha_{n+1}\|)^+}^T(K - m)$ . In particular, we can say that  $\phi_{i,v,\beta} b_{\alpha_{n+1}, \beta_{1,k+1}}$  belongs to  $\mathcal{K}_{(\|\beta\| - \|\alpha\|)^+ + (\|\beta_{1,k+1}\| - \|\alpha_{n+1}\|)^+}^T(K - m - n)$ . Since the positive part is sub-additive, that is  $(x + y)^+ \leq x^+ + y^+$ , we deduce that  $\phi_{i,v,\beta} b_{\alpha_{n+1}, \beta_{1,k+1}}$  belongs to  $\mathcal{K}_{(\|\beta\| + \|\beta_{1,k+1}\| - \|\alpha\| - \|\alpha_{n+1}\|)^+}^T(K - m - n)$ .

It now remains to say what happens when differentiating each of the terms  $(V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j)(X_s^x)$ . Again, we are using Lemma 2 with  $t = 0$  and  $\alpha = \alpha_{n+1}$ , i.e.

$$J_{s,x} V_{[\alpha_{n+1}]}(x) = \sum_{\beta \in \mathcal{A}_1^0(m)} b_{\alpha_{n+1}, \beta}(s, x) V_{[\beta]}(X_s^x).$$

The result is that we are increasing the length  $i_j$  for some  $1 \leq j \leq k$  from  $i_j$  to  $i_j + 1$ , all the other lengths being preserved, and that the Kusuoka function  $\phi_{i,v,\beta}$  is changed into  $\phi_{i,v,\beta} b_{\alpha_{n+1}, \beta}$  for any  $\beta \in \mathcal{A}_1^0(m)$ , which as we already argued belongs to  $\mathcal{K}_{(\|\beta\| + \|\beta_{1,k+1}\| - \|\alpha\| - \|\alpha_{n+1}\|)^+}^T(K - m - n)$ .

**3.3. Estimates of the Derivatives of  $u$  in the Mollified Setting.** We now go back to Eq. (12). As announced, we assume that all the coefficients of the backward equation (12) are  $\mathcal{C}^\infty$  functions with respect to the variables  $x, y$  and  $z$ . For any  $1 \leq n \leq K - m - 1$ , the common bound for the Lipschitz constant of  $h$  and for the derivatives of the coefficients up to the order  $n$  is denoted by  $\Lambda_n$ .

The bounds in Theorem 3 are proven by induction. For every  $n \geq 1$ , we denote by  $\mathcal{P}_n$  the following property: for any  $p > 1$ , there exists a constant  $C_n(p)$ , depending on  $\Lambda_n, n, p, T$  and the vector fields only, such that, for any  $(\alpha_1, \dots, \alpha_n) \in (\mathcal{A}_1^0(m))^n$  and any  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,

$$\begin{aligned} & |V_{[\alpha_1]} \cdots V_{[\alpha_n]} u(t, x)| \\ & \leq C_n(p) (T - t)^{(1 - \|\alpha\|)/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{np}]^{1/p}], \\ (\mathcal{P}_n) \quad & |V_{[\alpha_1]} \cdots V_{[\alpha_n]} V_i u(t, x)| \\ & \leq C_n(p) (T - t)^{-\|\alpha\|/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{np}]^{1/p}], \quad i \in \{1, \dots, N\}, \end{aligned}$$

with  $\|\alpha\| = \sum_{i=1}^n \|\alpha_i\|$ . The induction property relies on the following

**Proposition 5.** *For any  $p > 1$  and  $1 \leq n \leq K - m - 1$ , there exists a constant  $C_n(p)$ , depending on  $\Lambda_n$ ,  $n$ ,  $p$ ,  $T$  and the vector fields only, such that, for any  $(\alpha_1, \dots, \alpha_n) \in (\mathcal{A}_1^0(m))^n$  and any  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$\begin{aligned} |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| &\leq C_n(p) \left[ 1 + (T-t)^{(1-\|\alpha\|)/2} \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right] \\ &\quad + \int_t^{(T+t)/2} \sum_{k=1}^n \sum_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} (s-t)^{(\|\boldsymbol{\beta}\| - \|\alpha\|)^+/2} \prod_{j=1}^k \mathbb{E} \left[ |(V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j,j}]} v_j)(s, X_{s-t}^x)|^{np/i_j} \right]^{i_j/(np)} ds, \\ |V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)| &\leq C_n(p) \left[ 1 + (T-t)^{-\|\alpha\|/2} \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right] \\ &\quad + \int_t^{(T+t)/2} \sum_{k=1}^n \sum_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \left[ (s-t)^{[(\|\boldsymbol{\beta}\| - \|\alpha\|)^+ - 1]/2} \right. \\ &\quad \quad \left. \times \prod_{j=1}^k \mathbb{E} \left[ |(V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j,j}]} v_j)(s, X_{s-t}^x)|^{np/i_j} \right]^{i_j/(np)} \right] ds. \end{aligned}$$

Above, both sums run over the indices  $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}_k(n)$ ,  $\mathbf{v} = (v_1, \dots, v_k) \in \mathcal{U}_k(u(s, \cdot))$  and  $\boldsymbol{\beta} = ((\beta_{1,j}, \dots, \beta_{i_j,j}) \in [\mathcal{A}_0(m)]^j)_{1 \leq j \leq k}$ .

We first show how Proposition 5 implies the induction property.

We first prove  $\mathcal{P}_1$ . For a given  $p > 1$ , we set for any  $\beta_1 \in \mathcal{A}_1^0(m)$

$$\mathcal{Q}_{\beta_1}^1(t, s, x) = \mathbb{E} \left[ |(V_{[\beta_1]} u)(s, X_{s-t}^x)|^p \right]^{1/p} + (T-s)^{1/2} \sum_{j=1}^N \mathbb{E} \left[ |(V_{[\beta_1]} V_j u)(s, X_{s-t}^x)|^p \right]^{1/p}.$$

Choose  $n = 1$  in Proposition 5 and  $\alpha_1 \in \mathcal{A}_1^0(m)$ . Since  $s-t \leq T-s$  for any  $s \in [t, (T+t)/2]$ , we get

$$\begin{aligned} |V_{[\alpha_1]} u(t, x)| &\leq C_1(p) \left[ 1 + (T-t)^{(1-\|\alpha_1\|)/2} \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right. \\ &\quad \left. + \int_t^{(T+t)/2} \sum_{\beta_1 \in \mathcal{A}_1^0(m)} (s-t)^{[(\|\beta_1\| - \|\alpha_1\|)^+ - 1]/2} \mathcal{Q}_{\beta_1}^1(t, s, x) ds \right], \end{aligned}$$

together with

$$\begin{aligned} (T-t)^{1/2} |V_{[\alpha_1]} V_i u(t, x)| &\leq C_1(p) \left[ 1 + (T-t)^{(1-\|\alpha_1\|)/2} \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right. \\ &\quad \left. + \int_t^{(T+t)/2} \sum_{\beta_1 \in \mathcal{A}_1^0(m)} (s-t)^{[(\|\beta_1\| - \|\alpha_1\|)^+ - 1]/2} \mathcal{Q}_{\beta_1}^1(t, s, x) ds \right], \end{aligned}$$

$i$  running from 1 to  $N$ . Again, for  $s \in [t, (t+T)/2]$ ,  $T-s \geq (T-t)/2$ , so that the above inequalities can be written into a single one:

$$\begin{aligned} & (T-t)^{(\|\alpha_1\|-1)/2} \left[ |V_{[\alpha_1]}u(t, x)| + (T-t)^{1/2} \sum_{i=1}^N |V_{[\alpha_1]}V_i u(t, x)| \right] \\ & \leq C_1(p) \left[ 1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right. \\ & \quad \left. + \sum_{\beta_1 \in \mathcal{A}_1^0(m)} \left[ \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta_1\|-1)/2} \mathcal{Q}_{\beta_1}^1(t, s, x) ds \right] \right], \end{aligned}$$

the constant  $C_1(p)$  possibly varying from line to line here and after.

Choosing  $x$  of the form  $X_{t-r}^x$ , with  $0 \leq r \leq t$ , taking the  $L^p$  moment, applying Minkowski's integr inequality, and then making the sum over  $\alpha_1 \in \mathcal{A}_1^0(m)$  and  $i \in \{1, \dots, N\}$ , we eventually obtain

$$\begin{aligned} & \sum_{\alpha_1 \in \mathcal{A}_1^0(m)} (T-t)^{(\|\alpha_1\|-1)/2} \left[ \mathbb{E}[|(V_{[\alpha_1]}u)(t, X_{t-r}^x)|^p]^{1/p} \right. \\ & \quad \left. + \sum_{i=1}^N (T-t)^{1/2} \mathbb{E}[|(V_{[\alpha_1]}V_i u)(t, X_{t-r}^x)|^p]^{1/p} \right] \\ & \leq C_1(p) \left[ 1 + \mathbb{E}[|\nabla h(X_{T-r}^x)|^p]^{1/p} \right. \\ & \quad \left. + \sum_{\beta_1 \in \mathcal{A}_1^0(m)} \left[ \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta_1\|-1)/2} \mathcal{Q}_{\beta_1}^1(r, s, x) ds \right] \right]. \end{aligned}$$

We emphasize that the left-hand side is nothing but  $\sum_{\alpha_1 \in \mathcal{A}_1^0(m)} (T-t)^{(\|\alpha_1\|-1)/2} \mathcal{Q}_{\beta_1}^1(r, t, x)$ . By Lemma 4, we complete the proof of  $\mathcal{P}_1$ .

We turn to the proof of the induction property. Assume that  $\mathcal{P}_k$  holds for every  $1 \leq k \leq n$ , for some rank  $n \geq 1$ . We then apply Proposition 5, but we take care to make there the difference between the following cases: ( $i_k = n$ ) and ( $i_k < n$ ). When  $i_k = n$ , the sum over  $\beta$  actually reduces to a sum over  $\beta = (\beta_1, \dots, \beta_n) \in [\mathcal{A}_1^0(m)]^n$  and the product of the  $V$ 's reduces to a single term of the form  $V_{[\beta_1]} \dots V_{[\beta_n]}v$ ,  $v$  running over the set  $\{u(s, \cdot), V_\ell u(s, \cdot), 1 \leq \ell \leq N\}$ . In this case, we do not use the induction property. When  $i_k < n$ , all the possible  $i_j$ 's,  $1 \leq j \leq k$ , are also (strictly) less than  $n$ . That is, the terms of the form  $V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j,j}]}v_j$  fulfill the induction property, i.e., for any  $1 \leq j \leq k$ ,

$$\begin{aligned} & |(V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j,j}]}v_j)(s, X_{s-t}^x)| \\ & \leq C_n(p) (T-s)^{\delta/2 - (\sum_{\ell=1}^{i_j} \|\beta_{\ell,j}\|)/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{i_j p}]^{1/p}], \end{aligned}$$

with  $\delta$  being equal to 1 when  $v_j(s, \cdot)$  matches  $u(s, \cdot)$  and being equal to 0 when  $v_j(s, \cdot)$  matches some  $V_i u(s, \cdot)$ ,  $1 \leq i \leq N$ . Clearly, the worst case is

$$|(V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j,j}]}v_j)(s, X_{s-t}^x)| \leq C_n(p) (T-s)^{-(\sum_{\ell=1}^{i_j} \|\beta_{\ell,j}\|)/2} [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{i_j p}]^{1/p}].$$

We then obtain

$$\begin{aligned} & \prod_{j=1}^k |(V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j)(s, X_{s-t}^x)| \\ & \leq C_n(p) (T-s)^{-\sum_{j=1}^k \sum_{\ell=1}^{i_j} \|\beta_{\ell,j}\|/2} \prod_{j=1}^k [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^{i_j p}]]^{1/p} \\ & \leq C_n(p) (T-s)^{-\|\beta\|/2} \prod_{j=1}^k \mathbb{E}[(1 + |\nabla h(X_{T-t}^x)|)^{i_j p}]^{1/p}, \end{aligned}$$

where  $\beta$  stands for the  $k$ -tuple of multi-indices  $((\beta_{\ell,j})_{1 \leq \ell \leq i_j})_{1 \leq j \leq k}$ .

Plugging these bounds into Proposition 5, we obtain (up to a modification of the constant  $C_n$ )

$$\begin{aligned} (26) \quad & |V_{[\alpha_1]} \cdots V_{[\alpha_n]} u(t, x)| \leq C_n(p) \left[ 1 + (T-t)^{(1-\|\alpha\|)/2} \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p} \right] \\ & + C_n(p) \sum_{\beta=(\beta_1, \dots, \beta_k) \in [\mathcal{A}_1^0(m)]^k, k < n} \int_t^{(T+t)/2} (T-s)^{-\|\beta\|/2} (s-t)^{(\|\beta\|-\|\alpha\|)^+/2} \mathcal{R}^k(s, x) ds \\ & + C_n(p) \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_1^0(m)} \int_t^{(T+t)/2} (s-t)^{[(\|\beta\|-\|\alpha\|)^+ - 1]/2} \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(t, s, x) ds \\ & = T_1(t, x) + T_2(t, x) + T_3(t, x), \end{aligned}$$

with

$$\begin{aligned} \mathcal{R}^k(s, x) &= \sum_{i_1 + \dots + i_k \leq n} \prod_{j=1}^k \mathbb{E}[(1 + |\nabla h(X_{T-t}^x)|)^{i_j p}]^{1/p} \\ &\leq \sum_{i_1 + \dots + i_k \leq n} \prod_{j=1}^k \mathbb{E}[(1 + |\nabla h(X_{T-t}^x)|)^{np}]^{i_j/(np)} \leq C_n(p) \mathbb{E}[(1 + |\nabla h(X_{T-t}^x)|)^{np}]^{1/p} \\ \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(t, s, x) &= \mathbb{E}[|(V_{[\beta_1]} \cdots V_{[\beta_n]} u)(s, X_{s-t}^x)|^p]^{1/p} \\ &\quad + (T-s)^{1/2} \sum_{i=1}^N \mathbb{E}[|(V_{[\beta_1]} \cdots V_{[\beta_n]} V_i u)(s, X_{s-t}^x)|^p]^{1/p}. \end{aligned}$$

Up to a modification of  $C_n(p)$  from line to line, we obtain by replacing  $x$  with  $X_{t-r}^x$ ,  $0 \leq r \leq t$ , and by taking the  $L^p$  moment

$$\begin{aligned} (27) \quad & (T-t)^{(\|\alpha\|-1)/2} \mathbb{E}[|T_2(t, X_{t-r}^x)|^p]^{1/p} \\ & \leq C_n(p) \mathbb{E}[(1 + |\nabla h(X_{T-r}^x)|)^{np}]^{1/p} \int_t^{(T+t)/2} (s-t)^{-1/2} ds \\ & \leq C_n(p) (T-t)^{1/2} \mathbb{E}[(1 + |\nabla h(X_{T-r}^x)|)^{np}]^{1/p}. \end{aligned}$$

Similarly, by Minkowski's integral inequality,

$$(28) \quad \begin{aligned} & (T-t)^{(\|\alpha\|-1)/2} \mathbb{E} \left[ |T_3(t, X_{t-r}^x)|^p \right]^{1/p} \\ & \leq C_n(p) \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_1^0(m)} \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta\|-1)/2} \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(r, s, x) ds. \end{aligned}$$

By (26), (27) and (28), we deduce

$$(29) \quad \begin{aligned} & (T-t)^{(\|\alpha\|-1)/2} \mathbb{E} \left[ |(V_{[\alpha_1]} \dots V_{[\alpha_n]} u)(t, X_{t-r}^x)|^p \right]^{1/p} \\ & \leq C_n(p) \mathbb{E} \left[ (1 + |\nabla h(X_{T-r}^x)|)^{np} \right]^{1/p} \\ & \quad + C_n(p) \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_1^0(m)} \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta\|-1)/2} \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(r, s, x) ds. \end{aligned}$$

By a similar argument,

$$(30) \quad \begin{aligned} & (T-t)^{\|\alpha\|} \sum_{i=1}^N \mathbb{E} \left[ |(V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u)(t, X_{t-r}^x)|^p \right]^{1/p} \\ & \leq C_n(p) \mathbb{E} \left[ (1 + |\nabla h(X_{T-r}^x)|)^{np} \right]^{1/p} \\ & \quad + C_n(p) \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_1^0(m)} \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta\|-1)/2} \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(r, s, x) ds. \end{aligned}$$

Summing (29) and (30) over  $(\alpha_1, \dots, \alpha_n) \in [\mathcal{A}_1^0(m)]^n$ , we obtain

$$(31) \quad \begin{aligned} & \sum_{\alpha_1, \dots, \alpha_n \in \mathcal{A}_1^0(m)} (T-t)^{(\|\alpha\|-1)/2} \mathcal{Q}_{\alpha_1, \dots, \alpha_n}^n(r, t, x) \\ & \leq C_n(p) \mathbb{E} \left[ (1 + |\nabla h(X_{T-r}^x)|)^{np} \right]^{1/p} \\ & \quad + C_n(p) \sum_{\beta_1, \dots, \beta_n \in \mathcal{A}_1^0(m)} \int_t^{(T+t)/2} (s-t)^{-1/2} (T-s)^{(\|\beta\|-1)/2} \mathcal{Q}_{\beta_1, \dots, \beta_n}^n(r, s, x) ds. \end{aligned}$$

By Lemma 4, we complete the proof.  $\square$

It now remains to prove Proposition 5.

**Proof (Proposition 5).** The first point is to prove that, for any  $t \in [0, T]$ ,  $u(t, \cdot)$  is  $(K - m - 1)$ -times continuously differentiable w.r.t.  $x$  in all the directions of the space. Again, keep in mind that the coefficients of the backward component are here assumed to be smooth as well with bounded derivatives, by a mollifying argument. (See Subsection 3.3.)

**Lemma 6.** *In the mollified setting, for any  $t \in [0, T]$ , the mappings  $u(t, \cdot)$  and  $(V_i u(t, \cdot))_{1 \leq i \leq N}$  are  $(K - m - 1)$ -times continuously differentiable in all the directions of the space and, for any  $1 \leq n \leq K - m - 1$ ,  $(\nabla_x^n u(t, \cdot))_{0 \leq t \leq T}$  and  $((\nabla_x^n V_i u(t, \cdot))_{0 \leq t \leq T})_{1 \leq i \leq N}$  are uniformly bounded in  $t$ .*

The proof of Lemma 6 is rather technical. For this reason, we feel more convenient to postpone it to the end of the section.

We recall from [25] that, in the mollified setting, the function  $u$  is continuously differentiable with respect to the variable  $x$  and that

$$(32) \quad \sup_{0 \leq t \leq T} \|\nabla_x u(t, \cdot)\|_\infty \leq C(\Lambda_1, T),$$

where  $C(\Lambda_1, T)$  depends on  $\Lambda_1, T$  and the bounds of the derivatives of the vector fields  $V_0, \dots, V_N$  only. Basically, by Proposition 3.2 in Briand et al. [2], we can even say that for any  $p > 1$

$$(33) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |\nabla_x u(t, x)| \leq C(\Lambda_1, p, T) [1 + \mathbb{E}[|\nabla h(X_{T-t}^x)|^p]^{1/p}],$$

for some constant  $C(\Lambda_1, p, T)$  depending on  $\Lambda_1, p, T$  and the bounds of the derivatives of the vector fields  $V_0, \dots, V_1$  only. Below, we aim at estimating the higher order-derivatives of  $u$  along the vector fields. We write

$$\begin{aligned} \forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad u(t, x) &= P_{(T-t)/2} \left[ u\left(\frac{T+t}{2}, \cdot\right) \right](x) \\ &\quad + \int_t^{(T+t)/2} P_{s-t} [f(s, \cdot, u(s, \cdot), (V_i D_x u(s, \cdot))_{1 \leq i \leq N})](x) ds. \end{aligned}$$

For  $n$  given multi-indices  $\alpha_1, \dots, \alpha_n$  in  $\mathcal{A}_1^0(m)$ ,

$$\begin{aligned} V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x) &= V_{[\alpha_1]} \dots V_{[\alpha_n]} P_{(T-t)/2} \left[ u\left(\frac{T+t}{2}, \cdot\right) \right](x) \\ &\quad + \int_t^{(T+t)/2} V_{[\alpha_1]} \dots V_{[\alpha_n]} P_{s-t} [f(s, \cdot, u(s, \cdot), (V_i D_x u(s, \cdot))_{1 \leq i \leq N})](x) ds \\ &= T_1(t, x) + T_2(t, x). \end{aligned}$$

By Corollaries 24 and 27 in [7], we can find a family of Kusuoka function  $(\phi_{\alpha_1, \dots, \alpha_n}^j)_{1 \leq j \leq N}$  in  $\mathcal{K}_0^T(K - m - n + 1)$  such that

$$\begin{aligned} (34) \quad T_1(t, x) &= V_{[\alpha_1]} \dots V_{[\alpha_{n-1}]} \mathbb{E} [V_{[\alpha_n]} (u(\frac{T+t}{2}, X_{(T-t)/2}^x))] \\ &= \sum_{j=1}^d V_{[\alpha_1]} \dots V_{[\alpha_{n-1}]} \mathbb{E} [(J_{(T-t)/2, x} V_{[\alpha_n]}(x))_j \frac{\partial u}{\partial x_j} (\frac{T+t}{2}, X_{(T-t)/2}^x)] \\ &= (T-t)^{-(1/2) \sum_{i=1}^{n-1} \|\alpha_i\|} \sum_{j=1}^d \mathbb{E} [\phi_{\alpha_1, \dots, \alpha_n}^j (\frac{T-t}{2}, x) \frac{\partial u}{\partial x_j} (\frac{T+t}{2}, X_{(T-t)/2}^x)]. \end{aligned}$$

Therefore, for any  $p > 1$ , we can find a constant  $C_n(p)$ , depending on  $T$  and the bounds for the higher-order derivatives of the vector fields only and possibly varying from line to line, such that

$$\begin{aligned} (35) \quad |T_1(t, x)| &\leq C_n(p) (T-t)^{1/2 - (1/2) \|\alpha\|} \mathbb{E} [|\nabla_x u(\frac{T+t}{2}, X_{(T-t)/2}^x)|^p]^{1/p} \\ &\leq C_n(p) (T-t)^{1/2 - (1/2) \|\alpha\|} \mathbb{E} [1 + |\nabla h(X_{T-t}^x)|^p]^{1/p}, \end{aligned}$$



the last line following from (33). We emphasize that the exponent in  $(T - t)$  is  $1/2 - \|\alpha\|$ , where  $\|\alpha\| = |\alpha_1| + \dots + |\alpha_n|$ . Compared with (34), the additional  $1/2$  follows from the term  $|\alpha_n|$ , which is not taken into account in (34). We here see that the smoothing decay of the boundary condition behaves as in the linear case exactly, as well-guessed. The big deal is now to handle the nonlinear term. To do so, we follow the strategy developed in the non-degenerate case.

By Corollary 1 with  $(\varphi, \theta)$  therein possibly depending on  $s$ , that is with  $\varphi$  of the form  $u(s, \cdot)$  and  $\Theta(X_{s-t}^x)$  of the form  $\Theta(s, X_{s-t}^x) = \Theta(s, X_{s-t}^x, u(s, X_{s-t}^x), (V_i u(s, X_{s-t}^x))_{1 \leq i \leq N})$ , we write

$$\begin{aligned} T_2(t, x) &= \sum_{k, \mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \int_t^{(t+T)/2} \mathbb{E} \left[ \prod_{j=1}^k (V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j, j}]} v_j)(s, X_{s-t}^x) \phi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}(s-t, x) \psi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}(\Theta(s, X_{s-t}^x)) \right] ds, \end{aligned}$$

where the shorten notation  $(k, \mathbf{i}, \mathbf{v}, \boldsymbol{\beta})$  is as in Corollary 1: it stands for  $k \in \{0, \dots, n\}$ ,  $\mathbf{i} \in \mathbf{I}_k$ ,  $\mathbf{v} \in \mathbf{V}_k$  and  $\boldsymbol{\beta} = (\beta_{1,\ell}, \dots, \beta_{i_\ell, \ell})_{1 \leq \ell \leq k} \in \mathbf{A}^{\mathbf{i}, \mathbf{v}}$ . Keeping in mind that  $\phi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \in \mathcal{K}_{(\|\boldsymbol{\beta}\| - \|\alpha\|)^+ (K - m - n)}$  and that  $\psi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}$  is bounded, we deduce that, for any  $p > 1$ ,

$$\begin{aligned} |T_2(t, x)| &\leq C_n(p) \sum_{k, \mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \int_t^{(T+t)/2} (s-t)^{(\|\boldsymbol{\beta}\| - \|\alpha\|)^+ / 2} \mathbb{E} \left[ \prod_{j=1}^k |V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j, j}]} v_j(s, X_{s-t}^x)|^p \right]^{1/p} ds \\ &\leq C_n(p) \sum_{k, \mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \int_t^{(T+t)/2} (s-t)^{(\|\boldsymbol{\beta}\| - \|\alpha\|)^+ / 2} \prod_{j=1}^k \mathbb{E} \left[ |V_{[\beta_{1,j}]} \dots V_{[\beta_{i_j, j}]} v_j(s, X_{s-t}^x)|^{np/i_j} \right]^{i_j/(np)} ds, \end{aligned}$$

the constant  $C_n(p)$  possibly depending on  $\Lambda_n$  as well. Similarly, we can compute, for any index  $i \in \{1, \dots, N\}$ ,  $V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)$  may be expressed as

$$\begin{aligned} &V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x) \\ &= V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i P_{(T-t)/2} \left[ u\left(\frac{T+t}{2}, \cdot\right) \right](x) \\ (36) \quad &+ \int_t^{(T+t)/2} V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i P_{s-t} \left[ f(s, \cdot, u(s, \cdot), (V_{i'} D_x u(s, \cdot))_{1 \leq i' \leq N}) \right](x) ds \\ &= S_1(t, x) + S_2(t, x). \end{aligned}$$

Following the proof of (35), we obtain

$$\begin{aligned} |S_1(t, x)| &\leq C_n(p) (T-t)^{-\|\alpha\|/2} \mathbb{E} \left[ \left| \nabla_x u\left(\frac{T+t}{2}, X_{(T-t)/2}^x \right) \right|^p \right]^{1/p} \\ &\leq C_n(p) (T-t)^{-\|\alpha\|/2} \left[ 1 + \mathbb{E} \left[ \left| \nabla h(X_{(T-t)}^x) \right|^p \right]^{1/p} \right]. \end{aligned}$$

We now turn to  $S_2$ . By Integration by Parts (see Corollary 24 in [7]), we emphasize that

$$\begin{aligned} V_i P_{s-t} \left[ f(s, \cdot, u(s, \cdot), (V_{i'} D_x u(s, \cdot))_{1 \leq i' \leq N}) \right](x) &= V_i \mathbb{E} \left[ f(\Theta(s, X_{s-t}^x)) \right] \\ &= (s-t)^{-1/2} \mathbb{E} \left[ f(\Theta(s, X_{s-t}^x)) \phi_i^0(s-t, x) \right], \end{aligned}$$

for some Kusuoka function  $\phi_i^0 \in \mathcal{K}_0^T(K - m - 1)$ .

Therefore,

$$\begin{aligned} & V_{[\alpha_1]} \cdots V_{[\alpha_n]} V_i P_{s-t} [f(s, \cdot, u(s, \cdot), (V_{i'} D_x u(s, \cdot))_{1 \leq i' \leq N})](x) \\ &= (s-t)^{-(1/2)} V_{[\alpha_1]} \cdots V_{[\alpha_n]} \mathbb{E} [F(\Theta(s, X_{s-t}^x)) \phi_i^0(s-t, x)](x). \end{aligned}$$

Differentiating the product, we obtain

$$\begin{aligned} & V_{[\alpha_1]} \cdots V_{[\alpha_n]} V_i P_{s-t} [f(s, \cdot, u(s, \cdot), (V_{i'} D_x u(s, \cdot))_{1 \leq i' \leq N})](x) \\ &= (s-t)^{-(1/2)} \sum_{k=0}^n \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n} \mathbb{E} [V_{[\alpha_{\ell_1}]} \cdots V_{[\alpha_{\ell_k}]} \{F(\Theta(s, X_{s-t}^x))\} \phi_i^{\ell_1, \dots, \ell_k}(s-t, x)], \end{aligned}$$

for new Kusuoka functions  $\phi_i^{\ell_1, \dots, \ell_k} \in \mathcal{K}_0^T(K-m-n-1)$ .

We now apply Corollary 1 again. For any  $1 \leq \ell_1 < \cdots < \ell_k \leq n$ , we can write

$$\begin{aligned} & V_{[\alpha_{\ell_1}]} \cdots V_{[\alpha_{\ell_k}]} \{F(\Theta(s, X_{s-t}^x))\} \\ &= \sum_{k'=0}^k \sum_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} \left[ \prod_{j=1}^{k'} (V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j(s, X_{s-t}^x)) \phi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}^{\ell_1, \dots, \ell_k}(s-t, x) \psi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}^{\ell_1, \dots, \ell_k}(\Theta(s, X_{s-t}^x)) \right], \end{aligned}$$

where the shorten notation  $(\mathbf{i}, \mathbf{v}, \boldsymbol{\beta})$  stands for  $\mathbf{i} \in \mathbf{I}_{k'} \subset \mathcal{I}_{k'}(k)$ ,  $\mathbf{v} \in \mathbf{V}_{k'} \subset \mathcal{U}_{k'}(u(s, \cdot))$  and  $\boldsymbol{\beta} = (\beta_{1,j}, \dots, \beta_{i_j,j})_{1 \leq j \leq k'} \in \mathbf{A}^{\mathbf{i}, \mathbf{v}} \subset [\mathcal{A}_0(m)]^k$ ,  $\phi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}^{\ell_1, \dots, \ell_k}$  stands for a Kusuoka function belonging to  $\mathcal{K}_{(\|\boldsymbol{\beta}\| - \sum_{p=1}^k \|\alpha_{\ell_p}\|)^+}^T(K-m-k)$  and  $\psi_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}}^{\ell_1, \dots, \ell_k}$  stands for a bounded function.

Therefore, denoting by  $\ell$  the increasing sequence  $1 \leq \ell_1 < \cdots < \ell_k \leq n$ ,

$$\begin{aligned} & V_{[\alpha_1]} \cdots V_{[\alpha_n]} V_i P_{s-t} [f(s, \cdot, u(s, \cdot), (V_{i'} D_x u(s, \cdot))_{1 \leq i' \leq N})](x) \\ &\leq C_n(p) (s-t)^{-1/2} \\ &\quad \times \sum_{k=0}^n \sum_{\boldsymbol{\ell}} \sum_{k'=0}^k \sum_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} (s-t)^{-(\|\boldsymbol{\beta}\| - \|\boldsymbol{\alpha}\|)^+/2} \prod_{j=1}^{k'} \mathbb{E} \left[ |V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j(s, X_{s-t}^x)|^{np/i_j} \right]^{i_j/(np)} \\ &\leq C_n(p) \sum_{k=0}^n \sum_{\mathbf{i}, \mathbf{v}, \boldsymbol{\beta}} (s-t)^{-(1/2) - (\|\boldsymbol{\beta}\| - \|\boldsymbol{\alpha}\|)^+/2} \prod_{j=1}^k \mathbb{E} \left[ |V_{[\beta_{1,j}]} \cdots V_{[\beta_{i_j,j}]} v_j(s, X_{s-t}^x)|^{np/i_j} \right]^{i_j/(np)}, \end{aligned}$$

where the shorten notation in the last line above stands for  $\mathbf{i} \in \mathcal{I}_k(n)$ ,  $\mathbf{v} \in \mathcal{U}_k(u(s, \cdot))$  and  $\boldsymbol{\beta} = (\beta_{1,j}, \dots, \beta_{i_j,j})_{1 \leq j \leq k} \in [\mathcal{A}_0(m)]^n$ . We emphasize that the case  $k=0$  is the constant case: the product is understood as a 1. In the right-hand sides of the two estimates in the statement of Proposition 5, the sum over  $k$  starts from  $k=1$ : the case when  $k=0$  is hidden in the additional 1 in the boundary term. □

### 3.4. Proof of Lemma 6. We start with

*Case  $n=1$ .* The first-order continuous differentiability of  $u(t, \cdot)$  is a straightforward consequence of Pardoux and Peng [25]. Moreover, for any initial condition  $(t, x) \in [0, T] \times \mathbb{R}^d$ , the solution

$(\nabla_x Y_s^{t,x}, \nabla_x Z_s^{t,x})_{t \leq s \leq T}$  to the derivative BSDE

$$(37) \quad \begin{aligned} \nabla_x Y_s^{t,x} &= \nabla h(X_T^{t,x}) \nabla_x X_T^{t,x} \\ &+ \int_s^T [\nabla_x f(\Theta_r^{t,x}) \nabla_x X_r^{t,x} + \nabla_y f(\Theta_r^{t,x}) \nabla_x Y_r^{t,x} + \nabla_z f(\Theta_r^{t,x}) \nabla_x Z_r^{t,x}] dr \\ &- \int_s^T \nabla_x Z_r^{t,x} dB_r, \end{aligned}$$

with  $\Theta_r^{t,x} = (r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})$ , satisfies

$$(38) \quad \begin{aligned} \mathbb{P} - \text{a.s.}, \quad \nabla_x Y_s^{t,x} &= \lim_{h \rightarrow 0} \frac{Y_s^{t,x+h} - Y_s^{t,x}}{h}, \quad t \leq s \leq T, \quad x \in \mathbb{R}^d, \\ \nabla_x Y_s^{t,x} &= \lim_{h \rightarrow 0} \nabla_x Y_s^{t,x+h}, \quad t \leq s \leq T, \quad x \in \mathbb{R}^d, \end{aligned}$$

and

$$(39) \quad \begin{aligned} \lim_{h \rightarrow 0} \mathbb{E} \left[ \int_t^T \left| \frac{Z_s^{t,x+h} - Z_s^{t,x}}{h} - \nabla_x Z_s^{t,x} \right|^2 ds \right] &= 0, \\ \lim_{h \rightarrow 0} \mathbb{E} \left[ \int_t^T \left| \nabla_x Z_s^{t,x+h} - \nabla_x Z_s^{t,x} \right|^2 ds \right] &= 0. \end{aligned}$$

Clearly, (37) yields  $\sup_{0 \leq t \leq T} \|\nabla_x u(t, \cdot)\|_\infty < +\infty$ , since  $\nabla_x f$ ,  $\nabla_y f$  and  $\nabla_z f$  are bounded.

We now go back to the backward formulation of  $u(t, \cdot)$ :

$$u(t, x) = \mathbb{E}[h(X_T^{t,x})] + \int_t^T \mathbb{E}[f(s, X_s^{t,x}, u(s, X_s^{t,x}), (V_i u(s, X_s^{t,x}))_{1 \leq i \leq N})] ds.$$

By the example in Subsection 2.3 and by Lebesgue dominated theorem, we know that the right-hand side is in  $\mathcal{D}_V^{1/2}(\mathbb{R}^d)$  and that for any  $1 \leq i \leq N$ ,

$$\begin{aligned} V_i u(t, x) &= \mathbb{E}[\nabla h(X_T^{t,x}) V_i X_T^{t,x}] \\ &+ \int_t^T (s-t)^{-1/2} \mathbb{E}[f(s, X_s^{t,x}, u(s, X_s^{t,x}), (V_i u(s, X_s^{t,x}))_{1 \leq i \leq N}) \theta_t^*(\psi)(t-s, x)] ds. \end{aligned}$$

Above,  $\psi$  stands for a Kusuoka function in  $\mathcal{K}_0^T(K-m-1)$  and  $\theta_t^*(\psi)$  indicates that the randomness is evaluated after shifting. (See Subsection 2.2.) Clearly, we can rewrite the above expression as

$$(40) \quad \begin{aligned} V_i u(t, x) &= \mathbb{E}[\nabla h(X_T^{t,x}) V_i X_T^{t,x}] \\ &+ \int_t^T (s-t)^{-1/2} \mathbb{E}[f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \theta_t^*(\psi)(t-s, x)] ds, \end{aligned}$$

$V_i X_T^{t,x}$  being understood as  $V_i(x) \cdot \nabla_x X_T^{t,x}$ . At this stage of the proof, we would like to apply (38) and (39) to differentiate the right-hand side under the integral. The point is that  $(\nabla_x Z_s^{t,x})_{t \leq s \leq T}$  is in  $L^2([t, T] \times \Omega)$  only so that the convergence of the integral of  $(s-t)^{-1/2} |\nabla_x Z_s^{t,x}|$  is not guaranteed.

The proof is then completed by

**Lemma 7.** Consider two random jointly measurable functions, denoted by  $(\Psi_1(t, x))_{0 \leq t \leq T, x \in \mathbb{R}^d}$  and  $(\Psi_2(t, s, x))_{0 \leq t \leq s \leq T, x \in \mathbb{R}^d}$ , with values in  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$  respectively such that, a.s.,  $\Psi_1(t, \cdot)$  is continuously differentiable and  $\Psi_2(t, s, \cdot)$  is continuously differentiable for any  $0 \leq t \leq s \leq T$ . Assume in addition that  $(\nabla_x \Psi_1(t, x))_{0 \leq t \leq T, x \in \mathbb{R}^d}$  and  $(\nabla_x \Psi_2(t, s, x))_{0 \leq t \leq s \leq T, x \in \mathbb{R}^d}$  are in  $L^p(\Omega)$ , uniformly in  $x$  for any  $p \geq 1$ . Consider a random function  $F : (\omega, t, s, \xi, \zeta) \in \Omega \times [0, T]^2 \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_1} \mapsto F(\omega, t, s, \xi, \zeta) \in \mathbb{R}^{d_1}$ , continuously differentiable in  $(\xi, \zeta)$  with derivatives in  $\cap_{p \geq 1} L^p(\Omega)$ , uniformly in  $(\xi, \zeta)$ , and assume that  $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d_1}$  satisfies

$$(41) \quad v(t, x) = \mathbb{E} \left[ \Psi_1(t, x) + \int_t^T (s-t)^{-1/2} F(t, s, \Psi_2(t, s, x), v(s, X_s^{t,x})) ds \right],$$

then  $v(t, \cdot)$  is Lipschitz continuous, uniformly in  $t$ .

Before we prove Lemma 7, we explain how it applies to the proof of Lemma 6. We choose  $\xi = (x, y)$ ,  $\zeta = z$ , and  $F(t, s, \xi, \zeta) = f(s, x, y, z) \theta_t^*(\psi)(t-s, x)$ ,  $\Psi_2(t, s, x) = (X_s^{t,x}, Y_s^{t,x})$  and  $\Psi_1(t, x) = \nabla h(X_T^{t,x}) V_i X_T^{t,x}$ , and obviously,  $v(t, x) = (V_i u(t, x))_{1 \leq i \leq N}$ . We then deduce that  $(V_i u(t, \cdot))_{1 \leq i \leq N}$  is Lipschitz continuous, uniformly in  $t$ .

In particular, for any  $0 \leq t \leq s \leq T$ , the mapping  $x \mapsto Z_s^{t,x} = (V_i u(s, X_s^{t,x}))$  is locally Lipschitz continuous, i.e. for any  $x \in \mathbb{R}^d$ ,

$$\sup_{y, y' \in \mathbb{B}(x, 1)} |Z_s^{t,y'} - Z_s^{t,y}| \leq \vartheta(x) |y' - y|,$$

where  $\vartheta$  is a random variable in any  $L^p$ , uniformly in  $x$  and  $s$ . In particular, by (39),  $\nabla_x Z_s^{t,x}$  is any  $L^p(\Omega)$ , uniformly in  $s$  and  $x$ .

We now go back to (40). By (39), we know that the term inside the integral is continuously differentiable for any  $s > t$ . Since  $\nabla_x Z_s^{t,x}$  is any  $L^p(\Omega)$ , uniformly in  $s$  and  $x$ , we deduce that  $(V_i u(t, \cdot))_{1 \leq i \leq N}$  is continuously differentiable as well and that  $(D_x V_i u(t, \cdot))_{1 \leq i \leq N}$  is bounded uniformly in  $t$ .

*Induction.* We now assume that, for a given  $1 \leq n \leq K - m - 2$ ,  $u(t, \cdot)$  and  $(V_i u(t, \cdot))_{1 \leq i \leq N}$  are  $n$ -times continuously differentiable in all the directions of the space, with bounded derivatives, uniformly in  $t$ .

By Lemma 1, we can differentiate  $n$  times the pair  $(Y_s^{t,x}, Z_s^{t,x})_{t \leq s \leq T}$ . The dynamics of the derivative process  $(\nabla_x^n Y_s^{t,x}, \nabla_x^n Z_s^{t,x})_{t \leq s \leq T}$  may be summarized as follows:

$$(42) \quad \begin{aligned} \nabla_x^n Y_s^{t,x} &= H^n(t, x) + \int_s^T [F^n(t, s, x) + \nabla_y f(\Theta_r^{t,x}) \nabla_x^n Y_r^{t,x} + \nabla_z f(\Theta_r^{t,x}) \nabla_x^n Z_r^{t,x}] dr \\ &\quad - \int_s^T \nabla_x^n Z_r^{t,x} dB_r, \end{aligned}$$

where  $H^n(t, x)$  is an  $\mathcal{F}_T$ -measurable r.v., bounded in any  $L^p(\Omega)$ ,  $p \geq 1$ , uniformly in  $(t, x)$ , and  $(F^n(t, s, x))_{t \leq s \leq T}$  is a progressively-measurable process (w.r.t.  $s$ ), bounded in any  $L^p(\Omega)$ ,  $p \geq 1$ , uniformly in  $(t, x)$ . Obviously,  $H^n(t, x)$  is given by the differentiation of the boundary condition and  $F^n(t, s, x)$  by the differentiation of the driver of the BSDE:  $F^n(t, s, x)$  contains all the derivatives of  $X$  up to order  $n$  and all the derivatives of  $(Y, Z)$  up to order  $n - 1$ . In particular,  $F^n(t, s, x)$  is

a.s. continuously differentiable w.r.t.  $x$ , with bounded derivatives in any  $L^p(\Omega)$ ,  $p \geq 1$ , uniformly in  $(t, x)$  (by the induction assumption).

We then recover the framework of Eq. (37). By the same strategy, we deduce that the pair  $(\nabla_x^{n+1}Y_s^{t,x}, \nabla_x^{n+1}Z_s^{t,x})_{t \leq s \leq T}$  exists as in (38) and (39). Clearly,  $\nabla_x^{n+1}Y_t^{t,x}$  is continuous and bounded, uniformly in  $(t, x)$ , i.e.  $\bar{\nabla}_x^{n+1}u(t, \cdot)$  is well-defined, continuous and bounded, uniformly in  $(t, x)$ .

To obtain the continuous differentiability of  $(\nabla_x^n V_i u(t, \cdot))_{1 \leq i \leq N}$ , we follow the strategy developed in (40). Differentiating  $n$  times, we obtain a generic equation of the form

$$\begin{aligned} \nabla_x^n V_i u(t, x) &= \mathbb{E}[H^{n+1/2}(t, x)] \\ &+ \int_t^T \frac{\mathbb{E}[(F^n(t, s, x) + \nabla_y f(\Theta_r^{t,x}) \nabla_x^n Y_r^{t,x} + \nabla_z f(\Theta_r^{t,x}) \nabla_x^n Z_r^{t,x}) \theta_t^*(\psi)(t-s, x)]}{(s-t)^{1/2}} dr, \end{aligned}$$

for some  $\psi \in \mathcal{K}_0^T(K-m-1)$ . Above,  $F^n$  is the same as in (42) and  $H^{n+1/2}(t, x)$  is a.s. continuously differentiable, with derivatives in any  $L^p(\Omega)$ , for any  $p \geq 1$ . (Basically,  $H^{n+1/2}(t, x)$  is obtained by differentiating  $(n+1)$ -times the boundary condition. Since  $n+2 \leq K$  and  $(t, x) \mapsto X_{T-t}^x$  is in  $\mathcal{K}_T^0(K)$ ,  $H^{n+1/2}$  is continuously differentiable w.r.t.  $x$ . A similar argument holds for  $F_n$ .) We then recover the framework of Lemma 7 exactly. The end of the proof then follows the case  $n = 1$ .  $\square$

**Proof (Lemma 7).** The point now is to prove Lemma 7. We introduce the following mapping

$$\begin{aligned} \Phi : (v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d_1}) \\ \mapsto \left( w : (t, x) \in [0, T] \times \mathbb{R}^d \mapsto \mathbb{E} \left[ \Psi_1(t, x) + \int_t^T \frac{F(t, s, \Psi_2(t, s, x), v(s, X_s^{t,x}))}{(s-t)^{1/2}} ds \right] \right). \end{aligned}$$

Clearly, the right-hand side is well-defined if  $v$  is assumed to be bounded. Moreover, it is plain to prove that there exists a constant  $C$  such that, for any  $t \in [0, T]$ ,

$$(43) \quad \|w_1(t, \cdot) - w_2(t, \cdot)\|_\infty \leq C \int_t^T \frac{\|v_1(s, \cdot) - v_2(s, \cdot)\|_\infty}{(t-s)^{1/2}} ds,$$

with  $w_1 = \Phi(v_1)$  and  $w_2 = \Phi(v_2)$ ,  $v_1$  and  $v_2$  being bounded.

By Lemma 4, we can find  $\lambda > 0$  such that

$$\int_0^T [\exp(\lambda t) \|w_1(t, \cdot) - w_2(t, \cdot)\|_\infty] dt \leq \frac{1}{2} \int_0^T [\exp(\lambda s) \|v_1(s, \cdot) - v_2(s, \cdot)\|_\infty] ds$$

Clearly, the mapping  $\Phi$  is a contraction on the space of bounded functions from  $[0, T] \times \mathbb{R}^d$  to  $\mathbb{R}^{d_1}$  endowed with the semi-norm  $v \mapsto \int_0^T \exp(\lambda t) \|v(t, \cdot)\|_\infty dt$ . In particular, if  $v$  satisfies (41) and  $\tilde{v}$  is a fixed point of  $\Phi$ , then, for a.e.  $t \in [0, T]$ ,  $\tilde{v}(t, \cdot) = v(t, \cdot)$ . By (43), this proves that  $\tilde{v}(t, \cdot) = v(t, \cdot)$  for any  $t \in [0, T]$ . Similarly, if we construct by induction a sequence of the form  $(v_{n+1} = \Phi(v_n))_{n \geq 0}$ ,  $v_0 = 0$ , we get

$$\lim_{n \rightarrow +\infty} \int_0^T [\exp(\lambda t) \|v_n(t, \cdot) - v(t, \cdot)\|_\infty] dt = 0.$$

In particular, up to a subsequence,  $\|v_n(t, \cdot) - v(t, \cdot)\|_\infty \rightarrow 0$  for a.e.  $t \in [0, T]$ . By (43) again, the convergence holds for any  $t \in [0, T]$ . Therefore, if we prove that the  $((v_n(t, \cdot))_{t \in [0, T]})_{n \geq 1}$  are Lipschitz continuous, uniformly in  $t$  and in  $n$ ,  $v(t, \cdot)$  is Lipschitz continuous as well, uniformly in  $t \in [0, T]$ .

By induction, it is clear that all the  $v_n(t, \cdot)$  are continuously differentiable and that

$$(44) \quad \|\nabla_x v_{n+1}(t, \cdot)\|_\infty \leq C + C \int_t^T \frac{\|\nabla_x v_n(s, \cdot)\|_\infty}{(t-s)^{1/2}} ds.$$

(The value of  $C$  may vary from line to line.) We use Lemma 4 again. For a possibly new value of  $\lambda$ ,

$$\int_0^T \exp(\lambda t) \|\nabla_x v_{n+1}(t, \cdot)\|_\infty \leq C + \frac{1}{2} \int_0^T \exp(\lambda t) \|\nabla_x v_n(t, \cdot)\|_\infty.$$

Iterating the bound, we obtain (for a possibly new value of  $C$ )

$$\int_0^T \exp(\lambda t) \|\nabla_x v_n(t, \cdot)\|_\infty \leq C.$$

In particular, for any  $\varepsilon > 0$ , (44) yields

$$\begin{aligned} \|\nabla_x v_{n+1}(t, \cdot)\|_\infty &\leq C + C \int_t^T \frac{\|\nabla_x v_n(s, \cdot)\|_\infty}{(t-s)^{1/2}} ds \\ &\leq C + C \int_t^{(t+\varepsilon) \wedge T} \frac{\|\nabla_x v_n(s, \cdot)\|_\infty}{(t-s)^{1/2}} ds + C\varepsilon^{-1/2} \int_0^T \|\nabla_x v_n(s, \cdot)\|_\infty ds \\ &\leq C'(\varepsilon) + C\varepsilon^{1/2} \sup_{0 \leq s \leq T} [\|\nabla_x v_n(s, \cdot)\|_\infty], \end{aligned}$$

for some new  $C'(\varepsilon)$ , independent of  $n$  and possibly depending on  $\varepsilon$ . For  $\varepsilon$  small enough, we deduce that

$$\|\nabla_x v_{n+1}(t, \cdot)\|_\infty \leq C'(\varepsilon) + \frac{1}{2} \sup_{0 \leq s \leq T} [\|\nabla_x v_n(s, \cdot)\|_\infty].$$

Iterating, we obtain that  $\sup_{n \geq 1} \sup_{0 \leq t \leq T} \|\nabla_x v_n(t, \cdot)\|_\infty < +\infty$ . □

**3.5. Getting out from the Mollified Setting.** To complete the proof of Theorem 3, we must get out from the mollified setting.

In the mollified setting, we clearly obtain (25) as a consequence of Corollary 1, so that Theorem 3 holds in this case. The point to let the mollifying parameter vanish is to prove some uniform continuity property for the derivatives in the mollified setting. Specifically, if we denote by  $(h^M, f^M)_{M \geq 1}$  a sequence of mollified coefficients converging towards  $(h, f)$  uniformly on compact sets and if we denote by  $(u^M)_{M \geq 1}$  the associated family of solutions, the point is to prove that, for any  $\alpha_1, \dots, \alpha_n$  in  $\mathcal{A}_1^0(m)$ ,  $n \leq Q$ , and  $t \in [0, T)$ , the families  $(V_{[\alpha_1]} \dots V_{[\alpha_n]} u^M(t, \cdot))_{t \geq 0}$  and  $((V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u^M(t, \cdot))_{1 \leq i \leq N})_{t \geq 0}$  are uniformly continuous in compact subsets of  $\mathbb{R}^d$ . This would prove that  $u(t, \cdot)$  is in  $\mathcal{D}_V^{Q+1/2}(\mathbb{R}^p)$  and satisfies the announced bound. Passing to the limit in (25) along the mollifying sequence, (25) would hold for  $u(t, \cdot)$  as well<sup>6</sup>

The proof of uniform continuity may be seen as a straightforward consequence of (25) taking into account the explicit form of the coefficients given by Corollary 1 and using a stability argument based on Lemma 4.

<sup>6</sup>The argument is a bit short since  $F_1, F_2, F_3, G_1, G_2$  and  $G_3$  in (46) depends on the derivatives of lower orders. We let the reader check that an induction argument would apply quite easily.

4. BOUNDARY CONDITION IN  $L^\infty$ 

In this section we dispense with the Lipschitz condition and assume that the boundary condition  $h$  is continuous and bounded. As already stated (see Corollary 2) when  $f = 0$ , it is indeed known that, for any  $p > 1$ ,  $n \geq 1$ , and  $(\alpha_1, \dots, \alpha_n) \in [\mathcal{A}_1^0(m)]^n$  and any  $(t, x) \in [0, T] \times \mathbb{R}^d$ ,

$$(45) \quad |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C_n(p) (T-t)^{-\|\alpha\|/2} \mathbb{E}[|h(X_{T-t}^x)|^p]^{1/p}.$$

for some constant  $C_n(p)$ , independent of  $h$ . The main result of this section is

**Theorem 4.** *Let  $(V_i)_{0 \leq i \leq N}$  be  $N+1$  vector fields belonging to  $\mathcal{C}^k(\mathbb{R}^d, \mathbb{R}^d)$ , the set of all bounded continuous function  $k$ -times differentiable and partial derivatives bounded and continuous, that satisfy the UFG condition of order  $m$ , where  $k > m$ . For any  $t \in [0, T]$ ,  $u(t, \cdot)$  is continuous and belongs to  $\mathcal{D}_V^{k-m+1/2}(\mathbb{R}^d)$ . Moreover, for any  $p > 1$  and  $1 \leq n \leq k - m$ , there exists a constant  $C_n(\delta, p)$ , depending on  $\Lambda_n$ ,  $n$ ,  $p$ ,  $T$  and the vector fields  $V_0, \dots, V_N$  only, such that for all  $\alpha_1, \dots, \alpha_n \in \mathcal{A}_1^0$  and all  $(t, x) \in [0, T] \times \mathbb{R}^d$ , *Did I get the exponents right below ?**

$$|V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| \leq C_n(\delta, p) (T-t)^{-(n-2)^+/2 - \|\alpha\|/2} [1 + \mathbb{E}[|h(X_{T-t}^x)|^{np}]^{1/p}],$$

$$|V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)| \leq C_n(\delta, p) (T-t)^{-(n-1)^+/2 - \|\alpha\|/2} [1 + \mathbb{E}[|h(X_{T-t}^x)|^{np}]^{1/p}],$$

with  $1 \leq i \leq N$ . Moreover, the derivative pair process

$$(Y_t^\alpha = (V_{[\alpha_1]} \dots V_{[\alpha_n]} u)(t, X_t^x), Z_t^\alpha = ((V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u)(t, X_t^x))_{1 \leq i \leq N})_{0 \leq t < T}$$

satisfies the generalized BSDE

$$(46) \quad \begin{aligned} Y_t^\alpha &= (s-t)^{-\|\alpha\|/2} \mathbb{E}[u(s, X_s^x) \theta_t^*[\phi](s-t, X_t^x) | \mathcal{F}_t] \\ &\quad + \mathbb{E}\left[\int_t^s (F_1(r, x) + F_2(r, x) Y_r^\alpha + F_3(r, x) Z_r^\alpha) | \mathcal{F}_t\right], \\ Z_t^\alpha &= (s-t)^{-(1+\|\alpha\|)/2} \mathbb{E}[u(r, X_r^x) \theta_t^*[\psi](s-t, X_t^x) | \mathcal{F}_t] \\ &\quad + \mathbb{E}\left[\int_t^s (r-t)^{-1/2} (G_1(t, r, x) + G_2(t, r, x) Y_r^\alpha + G_3(t, r, x) Z_r^\alpha) | \mathcal{F}_t\right], \end{aligned}$$

with  $t < s < T$ , the coefficients satisfying the same properties as in Theorem 3.

## 5. COUNTER-EXAMPLES

We here provide two counter-examples:

- (1) The first example is driven by the one-dimensional Laplace operator and by a discontinuous boundary condition. The operator being uniformly elliptic, Theorem 4 says that the decay of the derivatives of order less than 3 is the same as in the linear case and that the decay of the derivatives of order 4 is almost the same as in the linear case, up to a small correction of the exponent. On the opposite, Theorem 4 suggests that the decay of the derivatives of order greater than 5 might be worse. For a specific choice of the boundary condition and of the nonlinear term, we show that the decay of the derivatives of order greater than 5 is indeed worse than the corresponding decay in the linear setting. This thus sounds as a confirmation of Theorem 4: order 5 appears as a threshold above which the decay of the derivatives deteriorates because of the nonlinearity.



- (2) In the second example, we investigate a nonlinear equation driven by a weak Hörmander operator of dimension 2, close to the hypoelliptic Kolmogorov operator. Basically, the operator is driven by two vector fields  $V_0$  and  $V_1$  such that  $\{V_1(x), [V_1, V_0](x)\}$  spans the whole space at any  $x \in \mathbb{R}^2$ . Theorem 4 says that the decay of the derivatives of order less than 2 is the same as in the linear case but suggests that a threshold might exist at order 3. For a suitable boundary condition and a suitable nonlinear term, we show that the decay of the derivatives of order 3 is indeed worse than in the linear case. In other words, the concomitancy of the nonlinearity and of the degeneracy here modifies the threshold above of which the decay of the derivatives deteriorates.

In both cases, we show that the right exponent for the decay of the derivatives exactly fits the exponent suggested by Theorem 4, up to the additional correction  $\delta$  therein. This may be seen as a justification of the title of the paper: “sharp estimates”.

**5.1. Counter-Example in the Linear Setting.** In the whole subsection, we assume that  $d = N = 1$  and we choose a smooth function  $f$  from  $\mathbb{R}$  to  $[-1, 1]$ . By Theorem 4, we know that the solution  $u$  to the nonlinear equation

$$(47) \quad \partial_t u(t, x) = \frac{1}{2} \partial_{x,x}^2 u(t, x) + f(\partial_x u(t, x)), \quad t \in (0, 1], \quad x \in \mathbb{R},$$

with  $u(0, x) = \mathbf{1}_{\{x > 0\}}$  as boundary condition satisfies

$$|\partial_{x,\dots,x}^n u(t, x)| \leq C_n t^{-n/2}, \quad t \in (0, 1], \quad x \in \mathbb{R}, \quad n = 1, 2, 3,$$

where  $C_n$  is some nonnegative constant. Moreover, for any  $\delta > 0$  and any  $n \geq 4$ , there exists a constant  $C_n(\delta)$  such that

$$|\partial_{x,\dots,x}^n u(t, x)| \leq C_n(\delta) t^{2-n-\delta}, \quad t \in (0, 1], \quad x \in \mathbb{R}.$$

**5.1.1. Diffusive Scaling.** Having in mind to take advantage of the diffusive scaling, we then set, for any integer  $p \in \mathbb{N}^*$ ,

$$u_p(t, x) = u(p^{-2}t, p^{-1}x),$$

so that, for any  $t \in (0, 1)$ ,  $x \in \mathbb{R}$ ,

$$(48) \quad \begin{aligned} |\partial_{x,\dots,x}^n u_p(t, x)| &\leq C_n t^{-n/2}, \quad n = 1, 2, 3, \\ |\partial_{x,\dots,x}^n u_p(t, x)| &\leq C_n(\delta) p^{2\delta+n-4} t^{2-n-\delta}, \quad \delta > 0, \quad n \geq 4. \end{aligned}$$

and

$$(49) \quad \partial_t u_p(t, x) = \frac{1}{2} \partial_{x,x}^2 u_p(t, x) + p^{-2} f(p \partial_x u_p(t, x)), \quad t \in (0, 1), \quad x \in \mathbb{R}.$$

In particular, the functions  $(\partial_t u_p)_{p \geq 1}$  are uniformly bounded in compact subsets of  $(0, 1] \times \mathbb{R}$ , so that the functions  $(u_p)_{p \geq 1}$  are uniformly convergent on compact subsets of  $(0, 1] \times \mathbb{R}$  towards the solution of the linear equation

$$\partial_t u_0(t, x) = \frac{1}{2} \partial_{x,x}^2 u_0(t, x), \quad t \in (0, 1], \quad x \in \mathbb{R},$$

with  $u(0, x) = \mathbf{1}_{\{x>0\}}$  as boundary condition. That is,

$$u_0(t, x) = (2\pi t)^{-1/2} \int_{-x}^{+\infty} \exp[-y^2/(2t)] dy.$$

We first specify the rate of convergence:

**Lemma 8.** *For any  $(t, x) \in (0, 1] \times \mathbb{R}$  and any  $p \geq 1$ ,*

$$|u_p(t, x) - u_0(t, x)| \leq p^{-2}, \quad |\partial_x u_p(t, x) - \partial_x u_0(t, x)| \leq Cp^{-2},$$

for some universal constant  $C \geq 0$ .

**Proof.** It is clear that

$$u_p(t, x) = u_0(t, x) + p^{-2} \int_0^t \int_{\mathbb{R}} f(p\partial_x u_p(t-s, y)) g(s, x-y) ds dy,$$

where  $g$  is the standard Gaussian kernel, hence the first inequality. To get the second inequality, we differentiate the above formula to obtain

$$|\partial_x u_p(t, x) - \partial_x u_0(t, x)| \leq p^{-2} \int_0^t s^{-1} \int_{\mathbb{R}} |f|(p\partial_x u_p(t-s, y)) |x-y| g(s, x-y) ds dy.$$

This completes the proof. □

The rate of convergence of the second-order derivative is a bit different:

**Lemma 9.** *There exists a constant  $C \geq 0$ , such that for any  $(t, x) \in (0, 1] \times \mathbb{R}$ ,*

$$|\partial_{x,x}^2(u_p - u_0)(t, x)| \leq Cp^{-1}t^{-1/2}.$$

**Proof.** We write

$$\begin{aligned} (u_p - u_0)(t, x) &= \int_{\mathbb{R}} (u_p - u_0)(t/2, x-y) g(t/2, y) dy \\ &\quad + p^{-2} \int_0^{t/2} \int_{\mathbb{R}} f(p\partial_x u_p(t-s, y)) g(s, x-y) ds dy, \end{aligned}$$

so that (differentiating once, making a change of variable and differentiating once again)

$$\begin{aligned} \partial_{x,x}^2(u_p - u_0)(t, x) &= -2t^{-1} \int_{\mathbb{R}} \partial_x(u_p - u_0)(t/2, y) (x-y) g(t/2, x-y) dy \\ &\quad - p^{-1} \int_0^{t/2} s^{-1} \int_{\mathbb{R}} f(p\partial_x u_p(t-s, y)) \partial_{x,x}^2 u_p(t-s, y) (x-y) g(s, x-y) ds dy. \end{aligned}$$

Therefore, by (48) and by Lemma 8, we can find a constant  $C$ , such that

$$|\partial_{x,x}^2(u_p - u_0)(t, x)| \leq Ct^{-1/2}p^{-2} + Cp^{-1}t^{-1/2}.$$

This completes the proof. □

5.1.2. *Sharpness of the Bounds of the Derivatives.* We are now ready to complete the analysis of the first counter-example. By differentiating PDE (47)  $n$  times and by applying the chain rule formula (or so-called Faà di Bruno's formula),

$$\begin{aligned} & \partial_t \partial_{x, \dots, x} u_p^n(t, x) \\ &= \frac{1}{2} \partial_t \partial_{x, \dots, x} u_p^{n+2}(t, x) \\ &+ p^{-2} \sum \beta_{n, m_1, \dots, m_n} p^{m_1 + \dots + m_n} f^{(m_1 + \dots + m_n)}(p \partial_x u_p(t, x)) \prod_{j=1}^n (\partial_{x, \dots, x}^{j+1} u_p(t, x))^{m_j}, \end{aligned}$$

for some weights  $(\beta_{n, m_1, \dots, m_n})_{n, m_1, \dots, m_n}$ , the sum running over  $n$ -tuples  $(m_j)_{1 \leq j \leq n}$  such that  $m_1 + 2m_2 + \dots + nm_n = n$ .

By Itô's formula, we deduce for a given stopping time  $\tau$  less than some prescribed real  $\theta < 1/2$ ,

$$\begin{aligned} & \partial_{x, \dots, x} u_p^n(1, -1) \\ &= \mathbb{E}[\partial_{x, \dots, x} u_p^n(1 - \tau, -1 + W_\tau)] \\ &+ p^{-2} \sum \beta_{n, m_1, \dots, m_n} p^{m_1 + \dots + m_n} \mathbb{E} \int_0^\tau [f^{(m_1 + \dots + m_n)}(p \partial_x u_p(1 - s, -1 + W_s)) \\ &\quad \times \prod_{j=1}^n (\partial_{x, \dots, x}^{j+1} u_p(1 - s, -1 + W_s))^{m_j}] ds \\ &= \mathbb{E}[\partial_{x, \dots, x} u_p^n(1 - \tau, -1 + W_\tau)] + \sum \beta_{n, m_1, \dots, m_n} p^{m_1 + \dots + m_n - 2} T_{n, m_1, \dots, m_n}^{(p)}, \end{aligned} \tag{50}$$

where  $(W_t)_{t \geq 0}$  stands for a one-dimensional Brownian motion.

Below, we choose  $\tau$  as the first exit time  $\tau = \inf\{t \geq 0 : |W_t| \geq \theta p^{-1}\} \wedge (\theta^2 p^{-2})$ , so that  $\tau$  has the same law as  $\theta^2 p^{-2}(\rho \wedge 1)$ , where  $\rho$  stands for the first exit time of a Brownian motion from  $(-1, 1)$ . We deduce that  $\theta^2 p^{-2} \mathbb{P}\{\rho \geq 1\} \leq \mathbb{E}(\tau) \leq \theta^2 p^{-2} \mathbb{E}(\rho)$ .

By (48), for every  $\delta > 0$ , we can find a constant  $C_\delta$  such that

$$\begin{aligned} & p^{m_1 + \dots + m_n - 2} |T_{n, m_1, \dots, m_n}^{(p)}| \leq C_\delta \theta^2 p^{\delta - 4} p^{\sum_{j=1}^n m_j} \prod_{j=1}^n p^{(j-3)^+ m_j} \\ & \leq C_\delta \theta^2 p^{\delta - 4} p^{\sum_{j=1}^n m_j} p^{\sum_{j=1}^n (j-3)m_j + \sum_{j=1}^2 (3-j)m_j} \\ & = C_\delta \theta p^{n + \delta - 4} p^{-2 \sum_{j=3}^n m_j - m_2}. \end{aligned} \tag{51}$$

(Keep in mind that  $\sum_{j=1}^n j m_j = n$ .) Therefore, when  $m_1 < n$  (i.e.  $m_i \geq 1$  for some  $i \in \{2, \dots, n\}$ ),

$$\limsup_{p \rightarrow +\infty} p^{4-n} p^{m_1 + \dots + m_n} |T_{n, m_1, \dots, m_n}^{(p)}| = 0.$$

Now, when  $m_1 = n$ ,

$$p^{n-2} T_{n, n, 0, \dots, 0}^{(p)} = p^{n-2} \mathbb{E} \int_0^\tau f^{(n)}(p \partial_x u_p(1 - s, -1 + W_s)) (\partial_{x, x}^2 u_p(1 - s, -1 + W_s))^n ds.$$

By Lemmas 8 and 9 and by Taylor's formula, we can find a constant  $C \geq 1$  such that

$$\begin{aligned} & p^{n-2} T_{n,n,0,\dots,0}^{(p)} \\ &= p^{n-2} \mathbb{E} \int_0^\tau f^{(n)}(p\partial_x u_0(1-s, -1+W_s)) (\partial_{x,x}^2 u_0(1-s, -1+W_s))^n ds + O_p(p^{n-3}) \mathbb{E}(\tau) \\ &\geq p^{n-2} \mathbb{E}(\tau) \inf_{|x| \leq C\theta} [f^{(n)}(p\partial_x u_0(1, -1) + x)] \inf_{|x| \leq C\theta} [\partial_{x,x}^2 u_0(1, -1) + x]^n + O_p(p^{n-3}) \mathbb{E}(\tau) \\ &\geq C\theta^2 p^{n-4} \inf_{|x| \leq C\theta} [f^{(n)}(p\partial_x u_0(1, -1) + x)] \inf_{|x| \leq C\theta} [\partial_{x,x}^2 u_0(1, -1) + x]^n + O_p(p^{n-5}), \end{aligned}$$

where  $O_p(\cdot)$  stands for the Landau notation (as  $p$  tends to  $+\infty$ ).

We now compute

$$\begin{aligned} \partial_x u_0(t, x) &= (2\pi t)^{-1/2} \exp[-x^2/(2t)], \\ \partial_{x,x}^2 u_0(t, x) &= -(2\pi)^{-1/2} t^{-3/2} x \exp[-x^2/(2t)], \end{aligned}$$

so that

$$\partial_x u_0(1, -1) = c_1 > 0, \quad \partial_{x,x}^2 u_0(1, -1) = c_2 > 0.$$

Choose now  $f(z) = \cos[(2\pi/c_1)z - n(\pi/2)]$ . Then,  $f^{(n)}(z) = (2\pi/c_1)^n \cos[(2\pi/c_1)z]$ , so that

$$f^{(n)}(p\partial_x u_0(1, -1) + x) = (2\pi/c_1)^n \cos[(2\pi/c_1)x] \geq (2\pi/c_1)^n / 2,$$

for  $(2\pi/c_1)|x| \leq \pi/4$ .

Therefore, for  $\theta$  small enough,

$$p^{n-2} T_{n,n,0,\dots,0}^{(p)} \geq c_3 p^{n-4} + O_p(p^{n-5}),$$

with  $c_3 > 0$ . Therefore,

$$(52) \quad \liminf_{p \rightarrow +\infty} [p^{4-n} (p^{n-2} T_{n,n,0,\dots,0}^{(p)})] > 0.$$

5.1.3. *Conclusion.* Assume now that, for some  $\delta > 0$  and  $n \geq 5$ , the bound

$$|\partial_{x,\dots,x}^n u(t, x)| \leq C_n t^{-n+2+\delta}, \quad t \in (0, 1], \quad x \in \mathbb{R}.$$

By scaling,

$$|\partial_{x,\dots,x}^n u_p(t, x)| \leq C_n p^{n-4-2\delta} t^{-n+2+\delta}, \quad t \in (0, 1], \quad x \in \mathbb{R}.$$

Plugging the above inequality in (50) and multiplying (50) by  $p^{4-n}$ , we understood from (5.1.2) that all the terms but  $p^{4-n} (p^{n-2} T_{n,n,0,\dots,0}^{(p)})$  vanish as  $p$  tends to  $+\infty$ . By (52), there is a contradiction.  $\square$

5.2. **Counter-Example in the Degenerate Setting.** Consider now the following family of PDEs:

$$(53) \quad \partial_t u_p(t, x, y) = \frac{1}{2} \partial_{x,x}^2 u_p(t, x, y) + \varphi(x) \partial_y u_p(t, x, y) + f(\partial_x u_p(t, x, y)), \quad t > 0, \quad (x, y) \in \mathbb{R}^2,$$

with  $u_p(0, x, y) = \text{sign}(x)\text{sign}(y) + \lambda \text{sign}(x + 1/p)$  as boundary condition, the function  $f$  being bounded and the parameter  $\lambda$  being real. Both  $f$  and  $\lambda$  will be chosen later on.

In Eq. (53) above,  $\varphi$  stands for a bounded smooth function with bounded derivatives of any order such that  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ . In particular, Eq. (53) is degenerate but satisfies weak Hörmander condition since  $[\partial_x, \varphi(x)\partial_y]_{|x=0} = \varphi'(0)\partial_y = \partial_y$ . It may be seen as a nonlinear generalization of

the so-called Kolmogorov hypoelliptic example: in the earlier paper [13], Kolmogorov noticed that the operator driving the nonlinear equation above admitted a smooth density of Gaussian type when  $\varphi(x) = x$ , despite the degeneracy of the diffusion matrix. (Below, the operator  $(1/2)\partial_{x,x}^2 + x\partial_y$  will be referred to as Kolmogorov operator.)

5.2.1. *Gaussian Fundamental Solution when  $\varphi(x) = x$ .* In what follows, we will choose  $\varphi(x)$  very close to  $x$  in the neighborhood of zero so that the derivatives of the solution  $u$  to (53) be close to the derivatives of the solution to (53) but driven by the Kolmogorov operator. (Obviously, we cannot choose  $\varphi(x) = x$ ,  $x \in \mathbb{R}$ , since it is not bounded.)

Kolmogorov operator is of great interest since its fundamental solution is explicitly known. It is given by the Gaussian density associated with the covariance matrix of the two-dimensional Gaussian process

$$G_t = \left( W_t, \int_0^t W_s ds \right)_{t \geq 0},$$

$(W_t)_{t \geq 0}$  here standing for a one-dimensional Brownian motion.

The covariance matrix of  $G_t$ , at a given time  $t > 0$ , reads

$$K_t = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}.$$

Therefore, the kernel of Eq. (53) when  $\varphi(x) = x$ , may be expressed as

$$(54) \quad \begin{aligned} g(t, x, y) &= \frac{3^{1/2}}{\pi t^2} \exp\left(-\frac{|K_t^{-1/2}(x, y)^*|^2}{2}\right) \\ &= \frac{3^{1/2}}{\pi t^2} \exp\left(-2\frac{x^2}{t} - 6\frac{y^2}{t^3} + 6\frac{xy}{t^2}\right). \end{aligned}$$

That is,  $u_p$  has the form

$$\begin{aligned} u_p(t, x, y) &= \int_{\mathbb{R}^2} u_p(0, x', y') g(t, x - x', y + tx - y') dx' dy' \\ &\quad + \int_{\mathbb{R}^2} \int_0^t f(\partial_x u_p(t - s, x', y')) g(s, x - x', y + sx - y') dx' dy', \quad t > 0, x, y \in \mathbb{R}^2, \end{aligned}$$

when  $\varphi(x) = x$ .

We observe that the covariance matrix has two scales:  $1/2$  stands for the exponent of the fluctuations of the coordinate  $x$  and  $3/2$  for the exponent of the fluctuations of the coordinate  $y$ ;  $1/2$  may also be understood as the half-length of the vector field  $V_1(x) = 1$  and  $3/2$  as the half-length of the vector field  $[V_1, V_0]$ , with  $V_0 = x\partial_y$ .

5.2.2. *Rescaling Argument.* Following the previous subsection, we consider a rescaled version of  $u_p$  according to the scaling exponents  $(1/2, 3/2)$ . We set:

$$\hat{u}_p(t, x, y) = u_p(p^{-2}t, p^{-1}x, p^{-3}y), \quad t > 0, x, y \in \mathbb{R},$$

for any  $p \geq 1$ .

By Theorem 4, we have

**Lemma 10.** *There exists a constant  $C$ , independent of  $p$ , such that*

$$\begin{aligned} |\partial_x \hat{u}_p(t, x, y)| &\leq Ct^{-1/2}, \quad |\partial_y \hat{u}_p(t, x, y)| \leq Ct^{-3/2}, \quad |\partial_{x,x}^2 \hat{u}_p(t, x, y)| \leq Ct^{-1}, \\ |\partial_{x,y}^2 \hat{u}_p(t, x, y)| &\leq Ct^{-2}, \quad |\partial_{x,x,y}^3 \hat{u}_p(t, x, y)| \leq Ct^{-5/2}, \quad |\partial_{x,y,y}^3 \hat{u}_p(t, x, y)| \leq Ct^{-7/2}, \end{aligned}$$

$x, y \in \mathbb{R}$  and  $t > 0$ . Moreover, for any  $\delta > 0$  and any  $n \geq 3$ , there exists a constant  $C_n(\delta)$ , independent of  $p$ , such that

$$\begin{aligned} |\partial_{y,\dots,y}^n \hat{u}_p(t, x, y)| &\leq C_n(\delta) p^{n-8/3+2\delta} t^{-2n+4/3-\delta}, \\ |\partial_{x,y,\dots,y}^{n+1} \hat{u}_p(t, x, y)| &\leq C_n(\delta) p^{n-7/3+2\delta} t^{-2n+2/3-\delta}, \\ |\partial_{x,x,y,\dots,y}^{n+2} \hat{u}_p(t, x, y)| &\leq C_n(\delta) p^{n-2+2\delta} t^{-2n-\delta}, \end{aligned}$$

$x, y \in \mathbb{R}$  and  $t > 0$ . The last inequality above is also true when  $n = 2$ .

We now investigate the limit behaviour of  $\hat{u}_p$ , as  $p$  tends to  $+\infty$ . The equation for  $\hat{u}_p$  has the form

$$\partial_t \hat{u}_p(t, x, y) = \frac{1}{2} \partial_{x,x}^2 \hat{u}_p(t, x, y) + p\varphi(x/p) \partial_y \hat{u}_p(t, x, y) + p^{-2} f(p\partial_x \hat{u}_p(t, x, y)), \quad t > 0, \quad x, y \in \mathbb{R},$$

with  $\hat{u}_p(0, x, y) = \text{sign}(x)\text{sign}(y) + \lambda\text{sign}(x+1)$  as boundary condition. Below, we set  $\hat{u}(0, x, y) = \text{sign}(x)\text{sign}(y) + \lambda\text{sign}(x+1)$ . (That is, we get rid of the index  $p$  in  $\hat{u}_p(0, \cdot, \cdot)$  since it is independent of  $p$ .) Since  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ , the limit is expected to be  $\hat{u}_0$ , solution to the PDE

$$(55) \quad \partial_t \hat{u}_0(t, x, y) = \frac{1}{2} \partial_{x,x}^2 \hat{u}_0(t, x, y) + x\partial_y \hat{u}_0(t, x, y), \quad t > 0, \quad x, y \in \mathbb{R},$$

with  $\hat{u}_0(0, x, y) = \hat{u}(0, \cdot, \cdot)$  as boundary condition. It is plain to see that Eq. (55) is well-posed and that the solution  $\hat{u}_0$  writes

$$\hat{u}_0(t, x, y) = \int_{\mathbb{R}^2} \hat{u}(0, x', y') g(t, x - x', y + tx - y') dx' dy',$$

with  $g$  as in (54).

As a corollary, we deduce

**Lemma 11.** *Choose  $\varphi(x) = x(1 - \exp(-x^2))$ ,  $x \in \mathbb{R}$ . Then, for any  $T > 0$ , we can find a constant  $C$  such that*

$$|\hat{u}_p(t, x, y) - \hat{u}_0(t, x, y)| \leq Cp^{-2}t^{-1/2}(1 + |x|^3), \quad t \in (0, T], \quad x, y \in \mathbb{R}.$$

**Proof.** We write  $\hat{u}_p$  as the solution of the PDE

$$\begin{aligned} \partial_t \hat{u}_p(t, x, y) &= \frac{1}{2} \partial_{x,x}^2 \hat{u}_p(t, x, y) + x\partial_x \hat{u}_p(t, x, y) \\ &\quad + (p\varphi(x/p) - x)\partial_y \hat{u}_p(t, x, y) + p^{-2} f(\partial_x \hat{u}_p(t, x, y)), \quad t \in (0, T], \quad x, y \in \mathbb{R}, \end{aligned}$$

so that

$$\begin{aligned}
(56) \quad \hat{u}_p(t, x, y) &= \hat{u}_0(t, x, y) \\
&+ \int_0^t \int_{\mathbb{R}^2} (p\varphi(x'/p) - x') \partial_y \hat{u}_p(t-s, x', y') g(s, x-x', y+sx-y') dx' dy' ds \\
&+ p^{-2} \int_0^t \int_{\mathbb{R}^2} f(p\partial_x \hat{u}(t-s, x', y')) g(s, x-x', y+sx-y') dx' dy' ds \\
&= \hat{u}_0(t, x, y) + R_p^{(1)}(t, x, y) + R_p^{(2)}(t, x, y).
\end{aligned}$$

By boundedness of  $f$ , we can find a constant  $C$ , independent of  $p$  and  $T$ , such that  $|R^{(2)}(t, x, y)| \leq Cp^{-2}$ ,  $t \in (0, T]$ ,  $x, y \in \mathbb{R}$ . (The value of  $C$  may vary below.)

Turn now to  $R^{(1)}(t, x, y)$ . By integration by parts,

$$\begin{aligned}
R_p^{(1)}(t, x, y) &= \int_0^t \left\{ \left( \int_{\mathbb{R}^2} (p\varphi(x'/p) - x') \partial_y \hat{u}_p(t-s, x', y') g(s, x-x', y+sx-y') dx' dy' \right)^{1/2} \right. \\
&\quad \times \left. \left( \int_{\mathbb{R}^2} (p\varphi(x'/p) - x') \hat{u}_p(t-s, x', y') \partial_y g(s, x-x', y+sx-y') dx' dy' \right)^{1/2} \right\} ds \\
&= \int_0^t \left\{ \left( \int_{\mathbb{R}^2} (R_p^{(1,1)}(t-s, x', y') g(s, x-x', y+sx-y') dx' dy' \right)^{1/2} \right. \\
&\quad \times \left. \left( \int_{\mathbb{R}^2} (R_p^{(1,2)}(t-s, s, x', y') g(s, x-x', y+sx-y') dx' dy' \right)^{1/2} \right\} ds.
\end{aligned}$$

By the choice we made for  $\varphi$ ,  $|p\varphi(x/p) - x| \leq |x|[1 - \exp(-x^2/p^2)] \leq Cp^{-2}|x|^3$ ,  $x \in \mathbb{R}$ . By Lemma 10, we deduce that  $|R_p^{(1,1)}(t-s, x', y')| \leq C(t-s)^{-3/2}p^{-2}|x'|^3$ ,  $0 \leq s < t \leq T$ ,  $x', y' \in \mathbb{R}$ , for some possibly new value of  $C$ . Similarly, by (54),  $|R_p^{(1,2)}(t-s, x', y')| \leq Cs^{-3/2}p^{-2}|x'|^3(s^{-1/2}|x'-x| + s^{-3/2}|y+sx-y'|)$ ,  $0 \leq s < t \leq T$ ,  $x', y' \in \mathbb{R}$ . Performing a change of variable in the integrals above, we obtain

$$|R_p^{(1)}(t, x, y)| \leq Cp^{-2}(1 + |x|^3) \int_0^t s^{-3/4}(t-s)^{-3/4} ds \leq Cp^{-2}(1 + |x|^3)t^{-1/2}.$$

This completes the proof. □

As a Corollary, we deduce

**Lemma 12.** *Choose  $\varphi(x) = x(1 - \exp(-x^2))$ ,  $x \in \mathbb{R}$ . Then, for any  $T > 0$ , we can find a constant  $C$  such that*

$$\begin{aligned}
|\partial_x \hat{u}_p(t, x, y) - \partial_x \hat{u}_0(t, x, y)| &\leq Cp^{-2}t^{-1}(1 + |x|^3), \\
|\partial_{x,y} \hat{u}_p(t, x, y) - \partial_{x,y} \hat{u}_0(t, x, y)| &\leq Cp^{-1}t^{-5/2}(1 + |x|^3),
\end{aligned}$$

$t > 0$ ,  $x, y \in \mathbb{R}$ .

**Proof.** We consider a variation of (56).

$$\begin{aligned}
& \hat{u}_p(t, x, y) \\
&= \hat{u}_0(t, x, y) + \int_{\mathbb{R}^2} [\hat{u}_p(t/2, x', y') - \hat{u}_0(t/2, x', y')] g(t/2, x - x', y + (t/2)x - y') dx' dy' \\
(57) \quad &+ \int_0^{t/2} \int_{\mathbb{R}^2} (p\varphi(x'/p) - x') \partial_y \hat{u}_p(t - s, x', y') g(s, x - x', y + sx - y') dx' dy' ds \\
&+ p^{-2} \int_0^{t/2} \int_{\mathbb{R}^2} f(p\partial_x \hat{u}(t - s, x', y')) g(s, x - x', y + sx - y') dx' dy' ds \\
&= \hat{u}_0(t, x, y) + S_p^{(1)}(t, x, y) + S_p^{(2)}(t, x, y) + S_p^{(3)}(t, x, y).
\end{aligned}$$

*Convergence of  $\partial_x \hat{u}_p$ .* We start with  $\partial_x S_p^{(1)}$ .

$$\begin{aligned}
& \partial_x S_p^{(1)}(t, x, y) \\
&= \int_{\mathbb{R}^2} [\hat{u}_p(t/2, x', y') - \hat{u}_0(t/2, x', y')] \partial_x [g(t/2, x - x', y + (t/2)x - y')] dx' dy',
\end{aligned}$$

so that

$$\begin{aligned}
(58) \quad & |\partial_x S_p^{(1)}(t, x, y)| \leq Cp^{-2}t^{-1/2} \int_{\mathbb{R}^2} \{(1 + |x'|^3)(t^{-1}|x - x'| + t^{-2}|y + sx - y'|) \\
&\quad \times g(s, x - x', y + sx - y')\} dx' dy' \\
&\leq Cp^{-2}t^{-1}(1 + |x|^3).
\end{aligned}$$

By a similar argument and by Lemma 10,

$$\begin{aligned}
(59) \quad & |\partial_x S_p^{(2)}(t, x, y)| \leq Cp^{-2}t^{-3/2} \int_0^{t/2} \int_{\mathbb{R}} |x'|^3 (s^{-1}|x - x'| + s^{-2}|y + sx - y'|) \\
&\quad \times g(s, x - x', y + sx - y') dx' dy' ds \\
&\leq Cp^{-2}t^{-1}(1 + |x|^3).
\end{aligned}$$

The same method applies to  $S_p^{(3)}(t, x, y)$ . It is plain to check that

$$(60) \quad |\partial_x S_p^{(3)}(t, x, y)| \leq Cp^{-2}.$$

By (58), (59) and (60), we complete the proof of the convergence of  $\partial_x \hat{u}_p$ .

*Convergence of  $\partial_{x,y}^2 \hat{u}_p$ .* We start with  $\partial_{x,y}^2 S_p^{(1)}$ . Following (58),

$$\begin{aligned}
(61) \quad & |\partial_{x,y} S_p^{(1)}(t, x, y)| \\
&\leq Cp^{-2}t^{-1/2} \int_{\mathbb{R}^2} \{(1 + |x'|^3) [t^{-1}(t^{-1}|x - x'| + t^{-2}|y + sx - y'|)^2 + t^{-2}] \\
&\quad \times g(s, x - x', y + sx - y')\} dx' dy' \\
&\leq Cp^{-2}t^{-5/2}(1 + |x|^3).
\end{aligned}$$



To deal with  $\partial_{x,y}S_p^{(2)}(t, x, y)$ , we perform a change of variable:

$$\begin{aligned} & \partial_{x,y}S_p^{(2)}(t, x, y) \\ &= \int_0^{t/2} \int_{\mathbb{R}^2} (p\varphi(x'/p) - x') \partial_{y,y}^2 \hat{u}_p(t-s, x', y+sx-y') \partial_x [g(s, x-x', y')] dx' dy' ds, \end{aligned}$$

so that, by Lemma 10,

$$\begin{aligned} & |\partial_{x,y}S_p^{(2)}(t, x, y)| \\ (62) \quad & \leq Ct^{-3}p^{-2} \int_0^{t/2} \int_{\mathbb{R}^2} |x'|^3 (s^{-1}|x-x'| + s^{-2}|y'|) g(s, x-x', y') dx' dy' ds \\ & \leq Ct^{-5/2}p^{-2}(1+|x|^3). \end{aligned}$$

By a similar argument,

$$\begin{aligned} \partial_{x,y}S_p^{(3)}(t, x, y) &= p^{-1} \int_{\mathbb{R}^2} \int_0^{t/2} \{f'(p\partial_x \hat{u}_p(t-s, x', y+sx-y')) \\ & \quad \times \partial_{x,y}^2 \hat{u}_p(t-s, x', y+sx-y') \partial_x [g(s, x-x', y')]\} dx' dy', \end{aligned}$$

so that, by Lemma 10,

$$\begin{aligned} (63) \quad |\partial_{x,y}S_p^{(3)}(t, x, y)| &\leq Cp^{-1}t^{-2} \int_0^{t/2} \int_{\mathbb{R}^2} (s^{-1}|x-x'| + s^{-2}|y'|) g(s, x-x', y') dx' dy' ds \\ &\leq Cp^{-1}t^{-3/2}. \end{aligned}$$

By (61), (62) and (63), the proof is over.  $\square$

**5.2.3. Criticality at Order 3.** We now investigate  $\partial_{y,y,y}^3 \hat{u}_p$ . Specifically, we assume that it satisfies the decay  $|\partial_{y,y,y}^3 \hat{u}_p(t, x, y)| \leq C(\delta)t^{-9/2-1/3+\delta}$  for some  $\delta > 0$ . We will establish below a contradiction showing that the order 3 in  $y$  is critical.

In what follows, we denote by  $(X_t^{1,p}, X_t^{2,p})_{t \geq 0}$  the two-dimensional process associated with the operator  $(1/2)\partial_{x,x}^2 + p\varphi(x/p)\partial_y$ . Differentiating three times Eq. (53) w.r.t.  $y$ , we apply Itô's formula to  $(\partial_{y,y,y}^3 \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}))_{0 \leq s < t}$ ,  $t > 0$  being given. For a stopping time  $\tau$  less than  $\theta$ , for  $\theta$  small (in particular,  $\theta < t/2$ ),

$$\begin{aligned} (64) \quad \partial_{y,y,y}^3 \hat{u}_p(t, x, y) &= \mathbb{E}[\partial_{y,y,y}^3 \hat{u}_p(t-\tau, X_\tau^{1,p}, X_\tau^{2,p})] \\ & \quad + p\mathbb{E} \int_0^\tau f^{(3)}(p\partial_x \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p})) (\partial_{x,y}^2 \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}))^3 ds \\ & \quad + 3\mathbb{E} \int_0^\tau f^{(2)}(p\partial_x \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p})) \partial_{x,y}^2 \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}) \\ & \quad \quad \quad \times \partial_{x,y,y} \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}) ds \\ & \quad + p^{-1}\mathbb{E} \int_0^\tau f' (p\partial_x \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p})) \partial_{x,y,y,y} \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}) ds \\ &= T_p^{(1)}(t, x, y) + T_p^{(2)}(t, x, y) + T_p^{(3)}(t, x, y) + T_p^{(4)}(t, x, y). \end{aligned}$$

By Lemma 10, for any  $\delta > 0$ ,  $p^{-2/3-\delta}\partial_{x,y,y}^4\hat{u}_p$  is bounded on every compact subset of  $(0, +\infty)\times\mathbb{R}^2$ , uniformly in  $p$ . Similarly,  $\partial_{x,y,y}^3\hat{u}_p$  is bounded on every compact subset of  $(0, +\infty)\times\mathbb{R}^2$ , uniformly in  $p$ . When  $\tau$  is the first exit time of a compact subset of  $(0, +\infty)\times\mathbb{R}^2$ ,  $T_p^{(2)}(t, x, y)$  and  $T_p^{(4)}(t, x, y)$  are bounded, uniformly in  $p$ .

By Lemma 12, the asymptotic behavior of  $T_p^{(2)}(t, x, y)$  is given by

$$(65) \quad T_p^{(2)}(t, x, y) = p\mathbb{E} \int_0^\tau f^{(3)}(p\partial_x\hat{u}_0(t-s, X_s, Y_s))(\partial_{x,y}^2\hat{u}_0(t-s, X_s, Y_s))^3 ds + O_p(1),$$

where  $O_p(1)$  stands for the Landau symbol and denotes a bounded sequence in  $p$ .

Assume now that we can find  $t > 0$  such that  $\partial_y\hat{u}_0(t, 0, 0) = \partial_{x,x}^2\hat{u}_0(t, 0, 0) = \partial_{x,x,x}^3\hat{u}_0(t, 0, 0) = 0$ . Choose then  $X_0 = Y_0 = 0$  and  $\tau$  as the first exit time  $\tau = \inf\{t \geq 0 : |X_t| \geq \theta p^{-1/3}, |Y_t| \geq \theta^3 p^{-1}\} \wedge \theta^2 p^{-2/3}$ . Differentiating PDE (55) w.r.t.  $x$ , we also have  $\partial_{t,x}^2\hat{u}_0(t, 0, 0) = 0$ . Performing a Taylor expansion in (65), we obtain

$$(66) \quad \begin{aligned} & T_p^{(2)}(t, x, y) \\ &= p\mathbb{E} \int_0^\tau f^{(3)}(p\partial_x\hat{u}_0(t, 0, 0) + \theta O_p(1))(\partial_{x,y}^2\hat{u}_0(t, 0, 0) + \theta O_p(p^{-1/3}))^3 ds + O_p(1). \end{aligned}$$

In particular, there exists a constant  $\gamma \geq 0$ , such that, for any power  $\delta > 0$ ,

$$(67) \quad \begin{aligned} \liminf_{p \rightarrow +\infty} p^{-\delta} T_p^{(2)}(t, x, y) &\geq \liminf_{p \rightarrow +\infty} \{p^{1-\delta} \mathbb{E}[\tau] \inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x\hat{u}_0(t, 0, 0) + x)] \\ &\quad \times \inf_{|x| \leq \gamma\theta} [(\partial_{x,y}^2\hat{u}_0(t, 0, 0) + x)^3]\}. \end{aligned}$$

Come now back to (64). We claim that the bound  $|\partial_{y,y,y}^3\hat{u}_p(t, x, y)| \leq Cp^\eta t^{-9/2-\eta/2}$ ,  $t > 0$ ,  $x, y \in \mathbb{R}$ , cannot be true if the infimum limit below is infinite:

$$(68) \quad \liminf_{p \rightarrow +\infty} \{p^{1-\eta} \mathbb{E}[\tau] \inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x\hat{u}_0(t, 0, 0) + x)] \inf_{|x| \leq \gamma\theta} [(\partial_{x,y}^2\hat{u}_0(t, 0, 0) + x)^3]\} = +\infty.$$

Indeed, by (67), (68) implies  $\liminf_{p \rightarrow +\infty} p^{-\eta} T_p^{(2)}(t, x, y) = +\infty$ . Multiplying (64) by  $p^{-\eta}$ , we then obtain a contradiction.

In particular, the bound  $|\partial_{y,y,y}^3 u_p(t, x, y)| \leq Ct^{-9/2-\eta/2}$ ,  $t > 0$ ,  $x, y \in \mathbb{R}$ , cannot be true if (68) holds true. Indeed, if  $|\partial_{y,y,y}^3 u_p(t, x, y)| \leq Ct^{-9/2-\eta/2}$ , then

$$|\partial_{y,y,y}^3 \hat{u}_p(t, x, y)| = p^{-9} |\partial_{y,y,y}^3 u_p(p^{-2}t, p^{-1}x, p^{-3}y)| \leq Cp^\eta t^{-9/2-\eta/2}, \quad t > 0, \quad x, y.$$

5.2.4. *Lower Bound for  $\mathbb{E}[\tau]$ .* It now remains to bound  $\mathbb{E}(\tau)$  from below. Define  $\tau' = \inf\{t \geq 0 : |X_t^{1,p}| \geq \theta p^{-1/3}\}$ . Since

$$|X_t^{2,p}| = \left| p \int_0^t \varphi(X_s^{1,p}/p) ds \right| \leq \int_0^t |X_s^{1,p}| ds, \quad t \geq 0,$$

we obtain that

$$|X_t^{2,p}| \leq \theta t p^{-1/3}, \quad t \leq \tau'.$$

In particular,

$$|X_t^{2,p}| \leq \theta^3 p^{-1}, \quad t \leq \tau' \wedge \theta^2 p^{-2/3}.$$

Therefore,

$$\mathbb{E}[\tau] \geq \theta^2 \mathbb{P}\{\tau' \geq \theta^2 p^{-2/3}\} p^{-2/3}.$$

Since  $\tau' \sim \theta^2 p^{-2/3} \rho$ , where  $\rho$  is the first exit time of a Brownian motion from  $(-1, 1)$ , we deduce that

$$(69) \quad \mathbb{E}[\tau] \geq \theta^2 \mathbb{P}\{\rho \geq 1\} p^{-2/3}.$$

Therefore, (68) holds for  $\eta < 1/3$ , provided

$$(70) \quad \liminf_{p \rightarrow +\infty} \left\{ \inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x \hat{u}_0(t, 0, 0) + x)] \inf_{|x| \leq \gamma\theta} [(\partial_{x,y}^2 \hat{u}_0(t, 0, 0) + x)^3] \right\} > 0.$$

That is, the bound  $|\partial_{y,y,y}^3 u_p(t, x, y)| \leq Ct^{-9/2-\eta/2}$ ,  $t > 0$ ,  $x, y \in \mathbb{R}$ , cannot be true for  $\eta < 1/3$ . This exactly fits the threshold in Theorem 4.

**5.2.5. Computation of the Derivatives.** It now remains that to find  $t > 0$  such that  $\partial_y \hat{u}_0(t, 0, 0) = \partial_{x,x}^2 \hat{u}_0(t, 0, 0) = \partial_{x,x,x}^3 \hat{u}_0(t, 0, 0) = 0$  and to check (70).

We first notice that  $\hat{u}_0$  can be splitted into terms  $\hat{u}_0 = \hat{u}_0^{(1)} + \lambda \hat{u}_0^{(2)}$ ,  $\hat{u}_0^{(1)}$  and  $\hat{u}_0^{(2)}$  both satisfying Eq. (55) but with different boundary conditions:

$$\hat{u}_0^{(1)}(0, x, y) = \text{sign}(x)\text{sign}(y), \quad \hat{u}_0^{(2)}(0, x) = \text{sign}(x + 1).$$

We emphasize that

$$\hat{u}_0^{(1)}(t, x, y) = \int_{\mathbb{R}^2} \hat{u}_0^{(1)}(0, x', y') g(t, x - x', y + tx - x') dx'$$

Since  $\hat{u}_0^{(1)}(0, -x', -y') = \hat{u}_0^{(1)}(0, x', y')$ , it is plain to see, by change of variable, that

$$\hat{u}_0^{(1)}(t, -x, -y) = \hat{u}_0^{(1)}(t, x, y), \quad t > 0, \quad x, y \in \mathbb{R}.$$

By differentiation, we deduce that  $\partial_y \hat{u}_0^{(1)}(t, 0, 0) = \partial_{x,x,x}^3 \hat{u}_0^{(1)}(t, 0, 0) = 0$ .

We now compute

$$\begin{aligned} \partial_x \hat{u}_0^{(1)}(t, x, y) &= 2 \int_{\mathbb{R}} \text{sign}(y + tx - y') g(t, x, y') dy' + 2t \int_{\mathbb{R}} \text{sign}(x - x') g(t, x', y + tx) dx' \\ \partial_{x,y}^2 \hat{u}_0^{(1)}(t, x, y) &= 4g(t, x, y + tx) + 2t \int_{\mathbb{R}} \text{sign}(x - x') \left(-12 \frac{y + tx}{t^3} + 6 \frac{x'}{t^2}\right) g(t, x', y + tx) dx'. \end{aligned}$$

In particular,

$$\partial_{x,y}^2 \hat{u}_0^{(1)}(t, 0, 0) = 4g(t, 0, 0) + 12t^{-1} \int_{\mathbb{R}} \text{sign}(-x') x' g(t, x', 0) dx' = c_1 t^{-2},$$

with  $c_1 > 0$ .

We now investigate  $\hat{u}_0^{(2)}(t, x)$ . It is given by

$$\hat{u}_0^{(2)}(t, x) = (2\pi)^{-1/2} \int_{\mathbb{R}} \text{sign}(x - t^{1/2} x' + 1) \exp\left(-\frac{(x')^2}{2}\right) dx'.$$

Therefore,

$$\begin{aligned}\partial_x \hat{u}_0^{(2)}(t, x) &= (2\pi)^{-1/2} t^{-1/2} \exp\left(-\frac{(x+1)^2}{2t}\right) \\ \partial_{x,x}^2 \hat{u}_0^{(2)}(t, x) &= -(2\pi)^{-1/2} t^{-3/2} (x+1) \exp\left(-\frac{(x+1)^2}{2t}\right) \\ \partial_{x,x,x}^3 \hat{u}_0^{(2)}(t, x) &= (2\pi)^{-1/2} (t^{-5/2} (x+1)^2 - t^{-3/2}) \exp\left(-\frac{(x+1)^2}{2t}\right).\end{aligned}$$

In particular,

$$\begin{aligned}\partial_{x,x}^2 \hat{u}_0^{(2)}(1, 0) &= -c_2 < 0 \\ \partial_{x,x,x}^3 \hat{u}_0^{(2)}(1, 0) &= 0.\end{aligned}$$

Finally,

$$\begin{aligned}\partial_{x,x}^2 \hat{u}_0(1, 0, 0) &= \partial_{x,x}^2 \hat{u}_0^{(1)}(1, 0, 0) + \lambda \partial_{x,x}^2 \hat{u}_0^{(2)}(1, 0) = \partial_{x,x}^2 \hat{u}_0^{(1)}(1, 0, 0) + \lambda c_2, \\ (71) \quad \partial_{x,x,x}^3 \hat{u}_0(1, 0) &= \partial_{x,x,x}^3 \hat{u}_0^{(1)}(1, 0, 0) + \lambda \partial_{x,x,x}^3 \hat{u}_0^{(2)}(1, 0) = 0. \\ \partial_{x,y}^2 \hat{u}_0(1, 0, 0) &= \partial_{x,y}^2 \hat{u}_0^{(1)}(1, 0, 0) = c_1 > 0.\end{aligned}$$

Choose now  $\lambda$  so that  $\partial_{x,x}^2 \hat{u}_0^{(1)}(1, 0, 0) + \lambda c_2 = 0$ . (This is possible since  $c_2 > 0$ .) For this choice, the required conditions  $\partial_y \hat{u}_0(1, 0, 0) = \partial_{x,x}^2 \hat{u}_0(1, 0, 0) = \partial_{x,x,x}^3 \hat{u}_0(1, 0, 0) = 0$  are satisfied.)

5.2.6. *Conclusion.* We now choose  $f$ :

$$(72) \quad \begin{aligned}f(z) &= -\sin(2\pi z / |\partial_x \hat{u}_0(1, 0, 0)|), \quad z \in \mathbb{R}, \quad \text{if } \partial_x \hat{u}_0(1, 0, 0) \neq 0, \\ f(z) &= -\sin(z), \quad z \in \mathbb{R}, \quad \text{if } \partial_x \hat{u}_0(1, 0, 0) = 0.\end{aligned}$$

In particular, there are two cases in (70). If  $\partial_x \hat{u}_0(1, 0, 0) \neq 0$ ,

$$\begin{aligned}\inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x \hat{u}_0(1, 0, 0) + x)] &\geq (2\pi / |\partial_x \hat{u}_0(1, 0, 0)|)^3 \inf_{|x| \leq \gamma\theta} [\cos(\pm 2\pi p + 2\pi x / \partial_x \hat{u}_0(1, 0, 0))] \\ &= (2\pi / |\partial_x \hat{u}_0(1, 0, 0)|)^3 \inf_{|x| \leq \gamma\theta} [\cos(2\pi x / |\partial_x \hat{u}_0(1, 0, 0)|)].\end{aligned}$$

Choosing  $\gamma\theta < |\partial_x \hat{u}_0(1, 0, 0)|/4$ , we then obtain

$$(73) \quad \inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x \hat{u}_0(1, 0, 0) + x)] \geq 2^{-1/2} (2\pi / |\partial_x \hat{u}_0(1, 0, 0)|)^3.$$

If  $\partial_x \hat{u}_0(1, 0, 0) = 0$ ,

$$\inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x \hat{u}_0(1, 0, 0) + x)] = \inf_{|x| \leq \gamma\theta} [\cos(x)].$$

Choosing  $\gamma\theta < \pi/4$ , we then obtain

$$(74) \quad \inf_{|x| \leq \gamma\theta} [f^{(3)}(p\partial_x \hat{u}_0(1, 0, 0) + x)] \geq 2^{-1/2}.$$

Examine now the second term in (68). For  $\gamma\theta < c_1/2$ ,

$$(75) \quad \inf_{|x| \leq \gamma\theta} [(\partial_{x,y}^2 \hat{u}_0(t, 0, 0) + x)^3] \geq (c_1/2)^3.$$

From (69), (73), (74) and (74), we deduce that (68) holds true with  $\eta < 1/3$ . This shows criticality at order 3.

5.2.7. *Generalization at any Order  $n \geq 3$ .* Following (67), we can generalize the result to any order  $n \geq 3$ . The point is to differentiate (53)  $n$  times w.r.t.  $y$  and to apply Itô's formula as in (64). We then obtain

$$\begin{aligned}
& \partial_{y,y,y}^3 \hat{u}_p(t, x, y) \\
&= \mathbb{E} [\partial_{y,y,y}^3 \hat{u}_p(t - \tau, X_\tau^{1,p}, X_\tau^{2,p})] \\
&+ p^{-2} \sum \beta_{n,m_1,\dots,m_n} p^{m_1+\dots+m_n} \mathbb{E} \int_0^\tau [f^{(m_1+\dots+m_n)}(p\partial_x \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p})) \\
&\quad \times \prod_{j=1}^n (\partial_{x,y,\dots,y}^{j+1} \hat{u}_p(t-s, X_s^{1,p}, X_s^{2,p}))^{m_j}] ds \\
&= \mathbb{E} [\partial_{y,y,y}^3 \hat{u}_p(t - \tau, X_\tau^{1,p}, X_\tau^{2,p})] + \sum \beta_{n,m_1,\dots,m_n} p^{m_1+\dots+m_n-2} T_{n,m_1,\dots,m_n}^{(p)}.
\end{aligned} \tag{76}$$

Following (51) and applying Lemma 10, for any  $\delta > 0$ , we can find a constant  $C_\delta > 0$  such that

$$\begin{aligned}
p^{m_1+\dots+m_n-2} |T_{n,m_1,\dots,m_n}^{(p)}| &\leq C_\delta \mathbb{E}(\tau) p^{\delta-2} p^{\sum_{j=1}^n m_j} \prod_{j=1}^n p^{(j-7/3)+m_j} \\
&\leq C_\delta \mathbb{E}(\tau) p^{\delta-2} p^{\sum_{j=1}^n m_j} p^{\sum_{j=1}^n (j-7/3)m_j + m_2/3 + 4m_1/3} \\
&= C_\delta \mathbb{E}(\tau) p^{n+\delta-2} p^{-(4/3)\sum_{j=3}^n m_j - m_2}.
\end{aligned}$$

Keeping in mind that  $\tau \leq p^{-2/3}$ , we deduce that

$$\lim_{p \rightarrow +\infty} p^{-n+8/3} p^{m_1+\dots+m_n-2} |T_{n,m_1,\dots,m_n}^{(p)}| = 0$$

when  $m_1 < n$ .

When  $m_1 = n$ , we can follow (66), (67) and (69). We deduce

$$\liminf_{p \rightarrow +\infty} p^{-n+8/3} T_{n,1,0,\dots,0}^{(p)} > 0,$$

provided

$$\liminf_{p \rightarrow +\infty} \inf_{|x| \leq \gamma\theta} [f^{(n)}(p\partial_x \hat{u}_0(t, 0, 0) + x)] \inf_{|x| \leq \gamma\theta} [(\partial_{x,x} \hat{u}_0(t, 0, 0) + x)^n] > 0. \tag{77}$$

Following (72), (77) holds true for

$$\begin{aligned}
f(z) &= \cos(2\pi z / |\partial_x \hat{u}_0(1, 0, 0)| - n(\pi/2)), \quad z \in \mathbb{R}, \quad \text{if } \partial_x \hat{u}_0(1, 0, 0) \neq 0, \\
f(z) &= \cos(z - n(\pi/2)), \quad z \in \mathbb{R}, \quad \text{if } \partial_x \hat{u}_0(1, 0, 0) = 0.
\end{aligned}$$

Going back to (76), we deduce that the bound

$$|\partial_{y,\dots,y}^n \hat{u}_p(t, x, y)| \leq C p^{n-8/3-\delta} t^{-n+4/3+\delta}, \quad t > 0, x, y, \in \mathbb{R},$$

cannot be true for some  $\delta > 0$ . By scaling, we deduce that the bound

$$|\partial_{y,\dots,y}^n u_p(t, x, y)| \leq C t^{-n+4/3+\delta}, \quad t > 0, x, y, \in \mathbb{R},$$

cannot be true. This shows sharpness of the bound in Theorem 4 in the current example.

□

5.2.8. *Crossed Derivatives.* We here show that the same method applies to crossed derivatives. To simplify, we investigate  $\partial_{x,y,y,y}^4 \hat{u}_p$  only.

The bound given by Lemma 10 reads

$$(78) \quad |\partial_{x,y,y,y}^4 \hat{u}_p(t, x, y)| \leq C_4(\delta) p^{2/3+2\delta} t^{-16/3-\delta}, \quad t > 0, x, y \in \mathbb{R}.$$

for any  $\delta > 0$ . Below, we show that it is sharp, up to the additional exponent  $\delta$ .

The strategy is the same as above and consists in differentiating (53) once w.r.t.  $x$  and three times w.r.t.  $y$ . Applying Itô's formula, we obtain ( $\tau$  standing for the same stopping time as above)

$$(79) \quad \begin{aligned} \partial_{x,y,y,y}^4 \hat{u}_p(t, x, y) &= \mathbb{E}[\partial_{x,y,y,y}^4 \hat{u}_p(t - \tau, X_\tau^{1,p}, X_\tau^{2,p})] \\ &\quad + p^2 \mathbb{E} \int_0^\tau [f^{(4)}(p \partial_x \hat{u}_p) \partial_{x,x}^2 \hat{u}_p (\partial_{x,y}^2 \hat{u}_p)^3](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + 3p \mathbb{E} \int_0^\tau [f^{(3)}(p \partial_x \hat{u}_p) \partial_{x,x,y}^3 \hat{u}_p (\partial_{x,y}^2 \hat{u}_p)^2](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + 3p \mathbb{E} \int_0^\tau [f^{(3)}(p \partial_x u_p) \partial_{x,x}^2 \hat{u}_p \partial_{x,y}^2 \hat{u}_p \partial_{x,y,y}^3 \hat{u}_p](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + 3 \mathbb{E} \int_0^\tau [f^{(2)}(p \partial_x u_p) \partial_{x,x,y}^3 \hat{u}_p \partial_{x,y,y}^3 \hat{u}_p](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + 3 \mathbb{E} \int_0^\tau [f^{(2)}(p \partial_x u_p) \partial_{x,y}^2 \hat{u}_p \partial_{x,x,y,y}^4 \hat{u}_p](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + \mathbb{E} \int_0^\tau [f^{(2)}(p \partial_x u_p) \partial_{x,x}^2 \hat{u}_p \partial_{x,y,y,y}^4 \hat{u}_p](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &\quad + p^{-1} \mathbb{E} \int_0^\tau [f'(p \partial_x u_p) \partial_{x,x,y,y,y}^5 \hat{u}_p](t - s, X_s^{1,p}, X_s^{2,p}) ds \\ &= \mathbb{E}[\partial_{x,y,y,y}^4 \hat{u}_p(t - \tau, X_\tau^{1,p}, X_\tau^{2,p})] + \sum_{j=1 \dots 7} T_p^{(j)}(t, x, y). \end{aligned}$$

Keep in mind that  $\mathbb{E}(\tau) \leq p^{-2/3} \wedge (t/2)$ . By Lemma 10, we obtain, for any  $\delta > 0$ ,

$$(80) \quad \begin{aligned} |T_p^{(2)}(t, x, y)|, |T_p^{(3)}(t, x, y)| &\leq Cp^{1/3}, \\ |T_p^{(4)}(t, x, y)| &\leq Cp^{-2/3}, \\ |T_p^{(5)}(t, x, y)|, |T_p^{(7)}(t, x, y)| &\leq C_\delta p^{-2/3+\delta}, \\ |T_p^{(6)}(t, x, y)| &\leq C_\delta p^\delta, \end{aligned}$$

the constants  $C$  and  $C_\delta$  possibly depending on  $t$ . (As above,  $t$  is taken as 1 below.)

We then focus on  $T_p^{(1)}(t, x, y)$ . Following (65),

$$T_p^{(1)}(t, x, y) = p^2 \mathbb{E} \int_0^\tau [f^{(4)}(p \partial_x \hat{u}_0) \partial_{x,x}^2 \hat{u}_0 (\partial_{x,y}^2 \hat{u}_0)^3](t - s, X_s^{1,p}, X_s^{2,p}) ds + O_p(p^{1/3}).$$

As above, we choose  $t = 1$  and  $x = y = 0$ . We then expand the right-hand side above by Taylor's formula. The terms  $(\partial_x \hat{u}_0(1-s, X_s^{1,p}, X_s^{2,p}))_{0 \leq s \leq \tau}$  and  $(\partial_{x,y}^2 \hat{u}_0(1-s, X_s^{1,p}, X_s^{2,p}))_{0 \leq s \leq \tau}$  is expanded

as in (65). The big deal is to expand  $(\partial_{x,x}^2 \hat{u}_0(1-s, X_s^{1,p}, X_s^{2,p}))_{0 \leq s \leq \tau}$ . By Itô's formula,

$$\begin{aligned} & \partial_{x,x}^2 \hat{u}_0(1-s, X_s^{1,p}, X_s^{2,p}) \\ &= \partial_{x,x}^2 \hat{u}_0(1, 0, 0) + \int_0^s [-\partial_{t,x,x}^3 \hat{u}_0 + (1/2)\partial_{x,x,x,x}^4 \hat{u}_0 + X_r^{1,p} \partial_{x,x,y}^2 \hat{u}_0](1-r, X_r^{1,p}, X_r^{2,p}) dr \\ & \quad + \int_0^s \partial_{x,x,x}^3 \hat{u}_0(1-s, X_r^{1,p}, X_r^{2,p}) dX_r^{1,p} \\ &= \partial_{x,x}^2 \hat{u}_0(1, 0, 0) + U_1^{(p)}(s) + U_2^{(p)}(s). \end{aligned}$$

Recall that  $\partial_{x,x}^2 \hat{u}_0(1, 0, 0) = \partial_{x,x,x}^3 \hat{u}_0(1, 0, 0) = 0$ . (See (71).) In particular,

$$\mathbb{E}|U_2^{(p)}(s \wedge \tau)| \leq \left( \mathbb{E} \int_0^\tau |\partial_{x,x,x}^3 \hat{u}_0(1-s, X_r^{1,p}, X_r^{2,p})|^2 ds \right)^{1/2} \leq C\theta^2 p^{-2/3}.$$

Compute now

$$\begin{aligned} & [-\partial_{t,x,x}^3 \hat{u}_0 + (1/2)\partial_{x,x,x,x}^4 \hat{u}_0 + X_r^{1,p} \partial_{x,x,y}^2 \hat{u}_0](1-r, X_r^{1,p}, X_r^{2,p}) \\ &= -\partial_{x,y}^2 \hat{u}_0(1-r, X_r^{1,p}, X_r^{2,p}) - [f^{(2)}(p\partial_x \hat{u}_0) \partial_{x,x}^2 \hat{u}_0](1-r, X_r^{1,p}, X_r^{2,p}) \\ & \quad - p^{-1} [f'(p\partial_x \hat{u}_0) \partial_{x,x,x}^3 \hat{u}_0](1-r, X_r^{1,p}, X_r^{2,p}) \\ &= -c_1 + \theta O_p(p^{-1/3}), \end{aligned}$$

so that

$$U_1^{(p)}(s) = -c_1 s + \theta^3 O_p(p^{-1}).$$

Finally,

$$\mathbb{E}[|\partial_{x,x}^2 \hat{u}_0(1-s, X_s^{1,p}, X_s^{2,p})|] \leq C\theta^2 p^{-2/3}.$$

We deduce

$$\begin{aligned} & T_p^{(1)}(t, x, y) \\ &= p^2 (\partial_{x,y}^2 \hat{u}_0(1, 0, 0))^3 \mathbb{E} \int_0^\tau [f^{(4)}(p\partial_x \hat{u}_0(1, 0, 0) + p\partial_{x,x,x,x}^4 \hat{u}_0(1, 0, 0)(X_s^{1,p})^3) \\ & \quad \times \partial_{x,x}^2 \hat{u}_0(t-s, X_s^{1,p}, X_s^{2,p})] ds + O_p(p^{1/3}) \\ &= p^2 (\partial_{x,y}^2 \hat{u}_0(1, 0, 0))^3 f^{(4)}(p\partial_x \hat{u}_0(1, 0, 0)) \mathbb{E} \int_0^\tau [\partial_{x,x}^2 \hat{u}_0(t-s, X_s^{1,p}, X_s^{2,p})] ds \\ & \quad + \theta^7 O_p(p^{2/3}) + O_p(p^{1/3}). \end{aligned}$$

In the end,

$$\begin{aligned} & T_p^{(1)}(t, x, y) \\ &= -c_4 \theta^4 p^{2/3} (\partial_{x,y}^2 \hat{u}_0(1, 0, 0))^3 f^{(4)}(p\partial_x \hat{u}_0(1, 0, 0)) + \theta^7 O_p(p^{2/3}) + O_p(p^{1/3}), \end{aligned}$$

where  $c_4 \neq 0$  is independent of  $\theta$  and  $p$ . As in (72), we can choose  $f$  so that  $f^{(4)}(p\partial_x \hat{u}_0(1, 0, 0)) \neq 0$ . For  $\theta$  small enough, we deduce that

$$\liminf_{p \rightarrow +\infty} [p^{-2/3} T_p^{(1)}(t, x, y)] > 0.$$

As above, this shows that the decay of  $|\partial_{x,y,y}\hat{u}_p(t,x,y)|$  cannot be of the form

$$|\partial_{x,y,y}\hat{u}_p(t,x,y)| \leq C_4(\delta)p^{2/3-2\delta}t^{-16/3+\delta}, \quad t > 0, x, y \in \mathbb{R}.$$

That is, (78) is sharp up to the additional  $\delta$  therein. □

## 6. THE QUADRATIC CASE

Semilinear PDEs with quadratic nonlinearities appear in solving certain optimization problems encountered in mathematical finance (see [11, 26]). Their corresponding BSDE (11) is said to be quadratic if

$$(81) \quad |f(t, x, y, z)| \leq \Lambda_1(|x| + |y| + |z|^2),$$

for some constant  $\Lambda_1$  (independent of  $t$ ). The exponent 2 is the critical one for the growth of the nonlinear term with respect to the spatial derivatives: it is known that existence and uniqueness may fail for higher exponents.

Below, we show how to estimate the first-order derivatives of  $u(t, \cdot)$  when  $h$  is bounded and  $f(t, \cdot)$  is continuously differentiable. In particular we will assume that there exists  $\Lambda_1 > 0$  such that

$$(82) \quad \|h\|_\infty < \Lambda_1, \quad |\nabla_x f(t, x, y, z)|, |\nabla_y f(t, x, y, z)|, |\nabla_z f(t, x, y, z)| \leq \Lambda_1(1 + |z|).$$

Quadratic equations are known to be well-posed provided the boundary condition  $h$  is bounded: we refer the reader to the original paper by Kobylanski [12]. Basically, the boundedness property ensures that the martingale driving the BSDE (11) is BMO and thus one can use Girsanov transformation to get rid of the quadratic part of the equation: we refer to Hu, Imkeller and Müller [11] for a review of this strategy. For this reason, the most natural approach is to estimate the first-order derivatives in  $L^\infty$  only. We remind the reader of the following (see e.g. Ankirchner et al. [1]):

**Proposition 6.** *Assume that (81) and (82) hold, then (11) is uniquely solvable for any starting point  $(t, x)$  of  $X$ . Moreover, the BMO-norm of the martingale part*

$$\left\| \int \langle Z, dW \rangle \right\|_{\text{BMO}} = \sup_{\text{Stopping Times } \tau \leq T} \mathbb{E} \left[ \int_\tau^T Z_s^2 ds | \mathcal{F}_\tau \right]^{1/2},$$

is finite and bounded by a constant  $C$ , depending on  $\Lambda_1$  and  $T$  only.

As announced, Girsanov assumption holds under BMO property:

**Proposition 7.** *For any progressively-measurable process  $(\lambda_t)_{0 \leq t \leq T}$  with values in  $\mathbb{R}^N$  such that*

$$\left( M_t = \int_0^t \langle \lambda_s, dW_s \rangle \right)_{0 \leq t \leq T}$$

has a finite BMO-norm, there exists an exponent  $q^* > 1$ , depending on the BMO-norm of  $(M_t)_{0 \leq t \leq T}$  only, such that the  $L^{q^*}(\mathbb{P})$ -norm of the exponential martingale of  $(M_t)_{0 \leq t \leq T}$  is finite and bounded by a constant, depending on the BMO-norm of  $(M_t)_{0 \leq t \leq T}$  only.

We have the following:



**Theorem 5.** *Let  $(V_i)_{0 \leq i \leq N}$  be  $N + 1$  vector fields belonging to  $C^k(\mathbb{R}^d, \mathbb{R}^d)$  that satisfy the UFG condition of order  $m$ , where  $k \geq m$ . Assume that (81) and (82) hold and that  $h$  is a Lipschitz continuous function. Then for any  $t \in [0, T)$ ,  $\nabla_x u(t, \cdot)$  exists as a continuous function and  $u(t, \cdot)$  belongs to  $\mathcal{D}_V^{m+1/2}(\mathbb{R}^d)$ . Moreover, for any  $n \leq k - m$  and  $\alpha_1, \dots, \alpha_n \in \mathcal{A}(m)$ , there exists a constant  $C_n(p, h)$ , depending on  $\Lambda_1, n, p$ , the Lipschitz constant of  $h$  and the vector fields  $V_0, \dots, V_N$  only, such that for all  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$\begin{aligned} |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| &\leq C_n(p, h)(T - t)^{(1 - \|\alpha\|)/2} \\ |V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)| &\leq C_n(p, h)(T - t)^{-\|\alpha\|/2}, \quad 1 \leq i \leq N. \end{aligned}$$

*Proof.* The proof is identical with the case when  $f$  is assumed to be Lipschitz. The reason is quite simple: in this case, the gradient is known to be bounded in any directions of the space in terms of the Lipschitz constant of  $h$ . This goes back to the work by Ankirchner et al. [1]. As consequence, quadratic growth does not affect the decay of the higher order derivatives, but only the dependence of the constant  $C_n(p, h)$  on the Lipschitz constant of  $h$ .  $\square$

The non-Lipschitz case is much more involved. Here we no longer have available the result of Ankirchner et al. [1] for the control of the first order derivatives. The first step is to obtain a bound for the first order derivatives. Once obtained, the analysis is handled as in the non-quadratic case.

**Lemma 13.** *Let  $(V_i)_{0 \leq i \leq N}$  be  $N + 1$  vector fields belonging to  $C^k(\mathbb{R}^d, \mathbb{R}^d)$  that satisfy the UFG condition of order  $m$ , where  $k \geq m$ . Assume that the boundary condition  $h$  is continuous that (81) and (82) hold. Then  $u(t, \cdot)$  belongs to  $\mathcal{D}_V^{3/2}(\mathbb{R}^d)$  and, for any  $t \in [0, T)$ , there exists a constant  $C$ , depending on  $\Lambda_1, n, T$  and the vector fields only, such that, for any  $1 \leq i \leq N$  and any  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$|V_i u(t, x)| \leq C(T - t)^{-1/2}.$$

Moreover, we can write  $(V_i u(t, X_t^x))_{1 \leq i \leq N}$  as

$$\begin{aligned} V_i u(t, X_t^x) &= (T - t)^{-1/2} \mathbb{E}[h(X_T^x) \theta_t^*[\phi^i](T - t, X_t^x) | \mathcal{F}_t] \\ &\quad + \mathbb{E}\left[\int_t^T (s - t)^{-1/2} f(s, X_s^x, Y_s^x, Z_s^x) \theta_s^*[\phi^i](s - t, X_t^x) ds | \mathcal{F}_t\right], \end{aligned}$$

where  $(\phi^i)_{1 \leq i \leq N}$  denotes Kusuoka functions in  $\mathcal{K}_0$ .

*Proof.* As above, we first mollify the coefficients to assume them infinitely differentiable and truncate them to make the derivatives of any order bounded. We need to prove that in the mollified setting the announced estimates in terms of the parameters  $\Lambda_1, n, p$  and  $T$  only.

By Kobylanski [12], we know that  $u$  is bounded in terms of  $\Lambda_1$  and  $T$  only. This point is crucial in what follows. Let  $(X_t, Y_t, Z_t)_{0 \leq t \leq T}$  be a generic solution of the equation (11). The initial condition of  $X$  will be specified later on. The basic argument then relies on a Girsanov transformation. Indeed, for a given  $\alpha \in \mathcal{A}_1^0(m)$ , we can always write (keep in mind that the coefficients are smooth)

$$\begin{aligned} d[V_{[\alpha]}(x) Y_t] &= -\langle \nabla_x(\Theta_t), V_{[\alpha]}(x) X_t \rangle dt - \langle \nabla_y f(\Theta_t), V_{[\alpha]}(x) Y_t \rangle dt - \langle \nabla_z f(\Theta_t), V_{[\alpha]}(x) Z_t \rangle dt \\ &\quad + \langle V_{[\alpha]}(x) Z_t, dB_t \rangle, \end{aligned}$$

where  $\Theta_t = (t, X_t, Y_t, Z_t)$ .

Owing to Proposition 7 (or taking advantage of the mollified setting), we know that the martingale process the Radon-Nykodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \langle \nabla_z f(\Theta_t), dB_t \rangle - \frac{1}{2} \int_0^T |\nabla_z f(\Theta_t)|^2 dt\right)$$

defines a new probability measure  $\mathbb{Q}$  under which the process

$$\bar{B}_t = B_t - \int_0^t \nabla_z f(\Theta_t) dt, \quad 0 \leq t \leq T,$$

is a Brownian motion.

In particular, under  $\mathbb{Q}$ , the process  $(V_{[\alpha]}(x)Y_t)_{0 \leq t \leq T}$  admits the following semi-martingale decomposition:

$$(83) \quad d[V_{[\alpha]}(x)Y_t] = -\langle \nabla_x f(\Theta_t), V_{[\alpha]}(x)X_t \rangle dt - \langle \nabla_y f(\Theta_t), V_{[\alpha]}(x)Y_t \rangle dt + \langle V_{[\alpha]}(x)Z_t, d\bar{B}_t \rangle.$$

Similarly, we can compute the Malliavin derivative of  $Y_t$ . (We refer to Pardoux and Peng [25] for a review of Malliavin calculus for BSDEs.) For any  $0 \leq s \leq t \leq T$ , we get, for any  $1 \leq j \leq N$ ,

$$\begin{aligned} d[D_s^j Y_t] &= -\langle \nabla_x f(\Theta_t), D_s^j X_t \rangle dt - \langle \nabla_y f(\Theta_t), D_s^j Y_t \rangle dt - \langle \nabla_z f(\Theta_t), D_s^j Z_t \rangle dt \\ &\quad + \langle D_s^j Z_t, dB_t \rangle \\ &= -\langle \nabla_x f(\Theta_t), D_s^j X_t \rangle dt - \langle \nabla_y f(\Theta_t), D_s^j Y_t \rangle dt + \langle D_s^j Z_t, d\bar{B}_t \rangle, \end{aligned}$$

with  $D_s^j Y_s = Z_s^j$  as initial condition, so that

$$(84) \quad D_s^j Y_t = Z_s^j + \int_s^t [\langle \nabla_x f(\Theta_r), D_s^j X_r \rangle + \langle \nabla_y f(\Theta_r), D_s^j Y_r \rangle] dr - \int_s^t \langle D_s^j Z_r, d\bar{B}_r \rangle.$$

Choosing  $(t, x)$  as initial condition for  $X$ , we deduce from (83) that

$$\begin{aligned} (85) \quad V_{[\alpha]}(x)u(t, x) &= \mathbb{E}^{\mathbb{Q}} \left\{ V_{[\alpha]}(x) \left[ u\left(\frac{T+t}{2}, X_{(T+t)/2}^{t,x}\right) \right] \right\} \\ &\quad + \mathbb{E}^{\mathbb{Q}} \int_t^{(T+t)/2} [\langle \nabla_x f(\Theta_s^{t,x}), V_{[\alpha]}(x)X_s^{t,x} \rangle + \langle \nabla_y f(\Theta_s^{t,x}), V_{[\alpha]}(x)Y_s^{t,x} \rangle] ds \\ &= S_1 + S_2. \end{aligned}$$

Surprisingly, the most difficult term to handle in (85) is the first one in the right-hand side.

The idea is to go back to the original measure  $\mathbb{P}$  and then to perform an integration by parts under  $\mathbb{P}$ . To do so, we apply Proposition 20 in [7],

$$\begin{aligned} &V_{[\alpha]}(x) \left[ u\left(\frac{T+t}{2}, X_{(T+t)/2}^{t,x}\right) \right] \\ &= \sum_{j=1}^N \sum_{\beta \in \mathcal{A}_1^0(m)} \left(\frac{T-t}{2}\right)^{-\frac{\|\alpha\|+\|\beta\|}{2}} \theta_t^* [M_{\alpha,\beta}^{-1}] \left(\frac{T-t}{2}, x\right) \int_t^{\frac{T+t}{2}} \theta_t^* [a_{j,\beta}](s-t, x) D_s^j Y_{(T+t)/2}^{t,x} ds, \end{aligned}$$

where  $a_{\alpha,\beta}$  stands for some Kusuoka function in  $\mathcal{K}_{(\|\beta\|-\|\alpha\|)^+}$  and  $M_{\alpha,\beta}$  for (see Proposition 20 in [7])  $M_{\alpha,\beta}(r, x) = r^{-(\|\alpha\|+\|\beta\|)/2} \int_0^r a_{i,\alpha}(s, x) a_{i,\beta}(s, x) ds, r > 0$ . It is the key point of the Malliavin calculus theory to prove that the matrix  $(M_{\alpha,\beta}(r, x))_{\alpha,\beta \in \mathcal{A}_1^0(m)}$  is invertible and that  $M_{\alpha,\beta}^{-1}$  is a Kusuoka function in  $\mathcal{K}_0$ .

We can plug the above expression into the right-hand side in (85) to get

$$(86) \quad \begin{aligned} & \left(\frac{T-t}{2}\right)^{\|\alpha\|/2} \mathbb{E}^{\mathbb{Q}} \left\{ V_{[\alpha]}(x) \left[ u\left(\frac{T+t}{2}, X_{(T+t)/2}^{t,x}\right) \right] \right\} \\ &= \sum_{j=1}^N \sum_{\beta \in \mathcal{A}_1^0(m)} \left(\frac{T-t}{2}\right)^{-\frac{\|\beta\|}{2}} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) \int_t^{\frac{T+t}{2}} \theta_t^* [a_{j,\beta}](s-t, x) D_s^j Y_{\frac{T+t}{2}}^{t,x} ds \right]. \end{aligned}$$

To simplify the integration by parts we will perform below, we write first

$$(87) \quad \begin{aligned} & \sum_{j=1}^N \int_t^{(T+t)/2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) \theta_t^* [a_{j,\beta}](s-t, x) D_s^j Y_{(T+t)/2}^{t,x} \right] ds \\ &= \sum_{j=1}^N \int_t^{(T+t)/2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \mathbb{E} \left\{ \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) | \mathcal{F}_s \right\} \theta_t^* [a_{j,\beta}](s-t, x) D_s^j Y_{(T+t)/2}^{t,x} \right] ds \\ &+ \sum_{j=1}^N \int_t^{(T+t)/2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \left\{ \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) - \mathbb{E} \left\{ \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) | \mathcal{F}_s \right\} \right\} \right. \\ &\quad \left. \times \theta_t^* [a_{j,\beta}](s-t, x) D_s^j Y_{(T+t)/2}^{t,x} \right] ds \\ &= T_1 + T_2. \end{aligned}$$

Now, by the conditioning performed in  $T_1$  and by (84), we observe that

$$(88) \quad \begin{aligned} T_1 &= \sum_{j=1}^N \int_t^{(T+t)/2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \mathbb{E} \left\{ \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) | \mathcal{F}_s \right\} \theta_t^* [a_{j,\beta}](s-t, x) (Z_s^{t,x})^j \right] ds \\ &+ \sum_{j=1}^N \int_t^{(T+t)/2} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \mathbb{E} \left\{ \theta_t^* [M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) | \mathcal{F}_s \right\} \theta_t^* [a_{j,\beta}](s-t, x) \right. \\ &\quad \left. \times \left( \int_s^{(T+t)/2} [\langle \nabla_x f(\Theta_r), D_s^j X_r^{t,x} \rangle + \langle \nabla_y f(\Theta_r), D_s^j Y_r^{t,x} \rangle] dr \right) \right] ds \\ &= T_{1,1} + T_{1,2}. \end{aligned}$$

We analyze first  $T_{1,1}$ . By the BMO condition, we know that the density  $d\mathbb{Q}/d\mathbb{P}$  belongs to some  $L^{q^*}(\mathbb{P})$  space, for  $q^* > 1$ , the  $L^{q^*}(\mathbb{P})$ -norm being bounded in terms of known parameters. Denoting

by  $p^*$  the adjoint exponent, we deduce that

$$\begin{aligned}
& |T_{1,1}| \\
& \leq C \mathbb{E} \left[ \left| \int_t^{(T+t)/2} \mathbb{E} \{ \theta_t^* [M_{\alpha,\beta}^{-1}] \left( \frac{T-t}{2}, x \right) | \mathcal{F}_s \} \langle \theta_t^* [a_{\cdot,\beta}] (s-t, x), Z_s^{t,x} \rangle ds \right|^{p^*} \right]^{1/p^*} \\
& \leq C (T-t)^{1/2} \mathbb{E} \left[ \left( \int_t^{(T+t)/2} \left[ \mathbb{E} \{ |\theta_t^* [M_{\alpha,\beta}^{-1}] \left( \frac{T-t}{2}, x \right) | \mathcal{F}_s \} |\theta_t^* [a_{\cdot,\beta}] (s-t, x)| |Z_s^{t,x}| \right]^2 \right)^{p^*/2} \right]^{1/p^*} \\
& \leq C (T-t)^{1/2} \mathbb{E} \left[ \sup_{t \leq s \leq T} \mathbb{E} \{ |\theta_t^* [M_{\alpha,\beta}^{-1}] \left( \frac{T-t}{2}, x \right)|^{p^*} | \mathcal{F}_s \} \right. \\
& \quad \times \sup_{1 \leq j \leq N} \sup_{t \leq s \leq T} [|\theta_t^* [a_{j,\beta}] (s-t, x)|^{p^*}] \left( \int_t^{(T+t)/2} |Z_s^{t,x}|^2 ds \right)^{p^*/2} \left. \right]^{1/p^*} \\
& \leq C (T-t)^{1/2} \mathbb{E} \left[ |\theta_t^* [M_{\alpha,\beta}^{-1}] \left( \frac{T-t}{2}, x \right)|^{4p^*} \right]^{1/(4p^*)} \sup_{1 \leq j \leq N} \mathbb{E} \left[ \sup_{t \leq s \leq T} [|\theta_t^* [a_{j,\beta}] (s-t, x)|^{4p^*}]^{1/(4p^*)} \right. \\
& \quad \times \mathbb{E} \left[ \left( \int_t^{(T+t)/2} |Z_s^{t,x}|^2 ds \right)^{p^*} \right]^{1/(2p^*)}.
\end{aligned}$$

By Briand and Confortola [4], the last term above can be bounded by known parameters. By Lemma 21 in [7], we deduce from a Cauchy-Schwarz argument that

$$(89) \quad |T_{1,1}| \leq C (T-t)^{\|\beta\|/2}.$$

We analyze next  $T_{1,2}$ . We obtain by a similar argument that

$$\begin{aligned}
|T_{1,2}| & \leq C (T-t)^{(1+\|\beta\|)/2} \mathbb{E} \left[ \left( \int_t^{(T+t)/2} (1 + |Z_s^{t,x}|^2) ds \right)^{q^*} \right]^{1/(2q^*)} \\
& \quad + C (T-t)^{1+\|\beta\|/2} \mathbb{E} \left[ \sup_{t \leq s \leq r \leq (T+t)/2} |D_s Y_r^{t,x}|^{2p^*} \left( \int_t^{(T+t)/2} (1 + |Z_s^{t,x}|^2) ds \right)^{p^*} \right]^{1/(2p^*)} \\
& \leq C (T-t)^{1+\|\beta\|/2} \left[ 1 + \mathbb{E} \left[ \sup_{t \leq s \leq r \leq (T+t)/2} |D_s Y_r^{t,x}|^{4p^*} \right]^{1/(4p^*)} \right].
\end{aligned}$$

We now use the connection between  $D_s Y_r$  and the derivatives of  $u(r, \cdot)$ . Indeed, from the proof of Proposition 20 in [7], we know that

$$D_s^j Y_r^{t,x} = \sum_{\gamma \in \mathcal{A}_1^0(m)} V_{[\gamma]}(u(r, X_r^{t,x})) \theta_t^* [a_{j,\gamma}](s-t, x)$$

Since  $T-r \geq s-t$ , we deduce

$$\begin{aligned}
(90) \quad |T_{1,2}| & \leq C (T-t)^{1+\|\beta\|/2} (s-t)^{(\|\gamma\|-1)/2} \sup_{x \in \mathbb{R}^d} [|V_{[\gamma]} u(r, x)|] \\
& \leq C (T-t)^{1/2+\|\beta\|/2} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]} u(r, x)|].
\end{aligned}$$

Finally, we handle  $T_2$ . The idea is to represent  $M_{\alpha,\beta}^{-1}((T-t)/2, x)$  as a stochastic integral through the Clark-Ocone formula:

$$\begin{aligned} \theta_t^*[M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) &= \mathbb{E}\left[\theta_t^*[M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) \middle| \mathcal{F}_s\right] \\ &\quad + \sum_{j=1}^N \int_s^{(T+t)/2} \mathbb{E}\left\{D_r^j\left[\theta_t^*[M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right)\right] \middle| \mathcal{F}_r\right\} dB_r^j. \end{aligned}$$

From Kusuoka [18], we know that the Malliavin derivative of  $D_s[\theta_t^*[M_{\alpha,\beta}^{-1}]]((T-t)/2, x)$  has finite  $L^p$  moments for any  $p \geq 1$ , the moments being bounded by known parameters. We then deduce that, for any  $p \geq 1$ , there exists a constant  $C_p$  such that

$$\mathbb{E}\left[|\theta_t^*[M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) - \mathbb{E}\{\theta_t^*[M_{\alpha,\beta}^{-1}]\left(\frac{T-t}{2}, x\right) \middle| \mathcal{F}_s\}|^p\right]^{1/p} \leq C_p(T-t)^{1/2}.$$

Using the same strategy as above, it is therefore quite simple to prove that

$$(91) \quad |T_2| \leq C(T-t)^{1/2+\|\beta\|/2} \sup_{t \leq r \leq T} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]}u(r, x)|].$$

By (86), (87), (88), (89), (90), (91), we deduce

$$(92) \quad \begin{aligned} |S_1| &= \left| \left(\frac{T-t}{2}\right)^{\|\alpha\|/2} \mathbb{E}^{\mathbb{Q}}\left\{V_{[\alpha]}(x)\left[u\left(\frac{T+t}{2}, X_{(T+t)/2}^{t,x}\right)\right]\right\} \right| \\ &\leq C + C(T-t)^{1/2} \sup_{t \leq r \leq T} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]}u(r, x)|]. \end{aligned}$$

We now handle  $S_2$  in (85). By (15) and by Cauchy-Schwarz inequality, we obtain

$$(93) \quad \begin{aligned} |S_2| &\leq C(T-t)^{1/2} \mathbb{E}\left[\int_t^T (1 + |Z_s^{t,x}|^2) ds\right]^{1/2} \\ &\quad \times (1 + (T-t)^{-\|\alpha\|/2} \sup_{t \leq r \leq T} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]}u(r, x)|]) \\ &\leq C(T-t)^{1/2} (1 + (T-t)^{-\|\alpha\|/2} \sup_{t \leq r \leq T} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]}u(r, x)|]). \end{aligned}$$

Eventually, from (85), (92) and (93), we obtain

$$\begin{aligned} &\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} [(T-t)^{\|\alpha\|/2} |V_{[\alpha]}u(t, x)|] \\ &\leq C + T^{1/2} \sum_{\gamma \in \mathcal{A}_1^0(m)} \sup_{0 \leq r \leq T} \sup_{x \in \mathbb{R}^d} [(T-r)^{\|\gamma\|/2} |V_{[\gamma]}u(r, x)|]. \end{aligned}$$

Clearly, this proves the result when  $T$  is small enough, i.e.  $T$  less than some  $\delta$ . The result easily follows for  $T$  of arbitrary length by choosing as initial condition  $u(t, \cdot)$  itself and then by applying the above inequality on  $[t - \delta, t]$ .  $\square$

The lemma gives us the gradient bounds for the higher order derivatives as in the case when  $f$  is Lipschitz.

**Theorem 6.** *Let  $(V_i)_{0 \leq i \leq N}$  be  $N + 1$  vector fields belonging to  $C^k(\mathbb{R}^d, \mathbb{R}^d)$  that satisfy the UFG condition of order  $m$ , where  $k \geq m$ . Assume that the boundary condition  $h$  is continuous that (81) and (82) hold. Then for any  $t \in [0, T)$ ,  $u(t, \cdot)$  is continuous and belongs to  $\mathcal{D}_V^{Q+1/2}(\mathbb{R}^d)$ . Moreover, for any  $\delta > 0$ ,  $p > 1$  and  $1 \leq n \leq k$ , there exists a constant  $C_n$  depending on  $\delta$ ,  $\Lambda_1$ ,  $n$ ,  $T$  and the vector fields  $V_0, \dots, V_N$  only, such that for all  $\alpha_1, \dots, \alpha_n \in \mathcal{A}_1^0$  and all  $(t, x) \in [0, T) \times \mathbb{R}^d$ ,*

$$\begin{aligned} |V_{[\alpha_1]} \dots V_{[\alpha_n]} u(t, x)| &\leq C_n (T - t)^{-(n-4)^+/2 - \|\alpha\|/2 - \delta \mathbf{1}_{\{n \geq 4\}}} \\ |V_{[\alpha_1]} \dots V_{[\alpha_n]} V_i u(t, x)| &\leq C_n (T - t)^{-(n-3)^+/2 - \|\alpha\|/2 - \delta \mathbf{1}_{\{n \geq 3\}}}. \end{aligned}$$

with  $1 \leq i \leq N$ .

## 7. CONNECTION WITH PDES

We here prove Propositions 1, 2, 3 and 4.

**7.1. Proof of Proposition 1.** We first assume that Proposition 2 holds true and then prove that the unique solvability of the PDE (5) holds as well.

**7.1.1. Solvability.** We know that  $u$  satisfies (1) in Definition 3. Taking the expectation in (12), we can even write  $\mathbb{E}[Y_t^{t,x}]$  as  $\mathbb{E}[h(X_T^{t,x})] + O((T - t)^{1/2})$ , the Landau notation  $O(\cdot)$  being uniform w.r.t.  $x$  on compact subsets, so that  $u$  is continuous up to the boundary. (That is (3) holds as well.)

To prove that it satisfies (2), we shall apply Itô's formula (i.e. Proposition 2.) By Markov property, it is indeed well known that  $Y_s^{t,x} = u(T - s, X_s^{t,x})$ ,  $t \leq s < T$ . The dynamics given by Itô's formula and by the BSDE (12) thus coincide, that is, at time  $t$  and at point  $x$ , the PDE (5) is satisfied.

**7.1.2. Uniqueness.** Uniqueness also follows from Proposition 2. Note first that the martingale term in Proposition 2 is local only. Anyhow, we can prove it to be a true martingale under the standing assumption (see Subsection 2.1). Indeed, by the PDE structure, for any starting point  $(t, x) \in [0, T) \times \mathbb{R}^d$ , the pair  $(v(T - s, X_s^{t,x}), Vv(T - s, X_s^{t,x}))_{t \leq s < T}$  satisfies the BSDE (12) on  $[t, T)$ . By standard Young's inequality, it is then possible to prove that

$$\mathbb{E} \int_t^T |Vv(T - s, X_s^{t,x})|^2 ds \leq C \sup_{t \leq s \leq T} \mathbb{E}[|v(T - s, X_s^{t,x})|^2],$$

for a constant  $C$  possibly depending on  $T$ . By the growth property of  $v$ , this proves that the martingale term is square integrable. Moreover, by the continuity of  $v$  up to the boundary, Eq. (12) is shown to hold up to time  $T$ . The initial condition of the diffusion being given, uniqueness of the classical solution easily follows by uniqueness of the solution to the BSDE (12).

**7.2. Proof of Proposition 2.** Clearly, Proposition 2 is true when  $v$  is smooth. When  $v$  is not smooth, the point is to approximate it by a sequence of smooth functions  $(v_p)_{p \geq 1}$  such that

$$(94) \quad \forall r \geq 1, \quad \lim_{p \rightarrow +\infty} \sup_{1/r \leq t \leq T} \|v_p(t, \cdot) - v(t, \cdot)\|_{\mathbb{B}(0,r), \infty}^{V,2} = 0, \quad \lim_{p \rightarrow +\infty} \|v_p - v\|_{[1/r, T] \times \mathbb{B}(0,r), \infty}^{V_0,1} = 0.$$

Indeed, introducing the stopping times  $(\tau_q = \inf\{s \geq t : |X_s^{t,x}| \geq q\})_{q \geq 1}$  ( $\inf \emptyset = +\infty$ ), we can apply Itô's formula to  $(u_p(T - s, X_s^x))_{0 \leq s \leq \tau_q \wedge (T - \varepsilon)}$ ,  $\varepsilon$  standing for a small positive real, and then let  $p$  tend to  $+\infty$ . Property (94) then implies Itô's formula for  $(u(T - s, X_s^x))_{0 \leq s \leq \tau_q \wedge (T - \varepsilon)}$  up time  $\tau_q \wedge (T - \varepsilon)$ . Letting  $q$  tend to  $+\infty$ , this completes the proof.

It thus remains to prove (94). It is a consequence of

**Lemma 14.** *For two smooth densities  $\rho_1$  and  $\rho_d$  over  $\mathbb{R}$  and  $\mathbb{R}^d$ , both with compact support, and for a solution  $v$  to the PDE as in Definition 3, define for all  $\varepsilon > 0$*

$$v^\varepsilon(t, x) = \int_{\mathbb{R}^{d+1}} v(t - \varepsilon s, x - \varepsilon y) \mathbf{1}_{\{t - \varepsilon s > 0\}} \rho_1(s) \rho_d(y) ds dy.$$

Then,

$$\forall r \geq 1, \quad \lim_{\varepsilon \rightarrow 0} \sup_{1/r \leq t \leq r} \|v^\varepsilon(t, \cdot) - v(t, \cdot)\|_{\mathbb{B}(0, r), \infty}^{V, 2} = 0, \quad \lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - v\|_{[1/r, r] \times \mathbb{B}(0, r), \infty}^{\mathcal{V}_0, 1} = 0.$$

The proof of Lemma 14 is not so straightforward: it is postponed to the end of the section.  $\square$

**7.3. Proof of Proposition 3.** We first assume that Proposition 4 holds true and then prove that the unique solvability of the PDE (5) holds as well.

**7.3.1. Solvability.** We know that  $u$  satisfies (1) in Definition 4. To prove that it satisfies (2), we shall apply Itô's formula (i.e. Proposition 4). To do so, we consider  $\xi$  as in Proposition 4. Basically, the first point is to prove that  $(Y_s^{t, \xi})_{t \leq s \leq T}$  writes  $(u(T - s, X_s^{t, \xi}))_{t \leq s \leq T}$ . When  $h$  is smooth, it holds true since  $((Y_s^{t, x})_{t \leq s \leq T})_{t \in [0, T], x \in \mathbb{R}^d}$  defines a continuous flow (w.r.t the initial condition  $x$ ): see Pardoux and Peng [25]. In the case when  $h$  is measurable only, things are less obvious since  $u$  might be discontinuous. Nevertheless, it can be proven that  $(Y_s^{t, \xi})_{t \leq s \leq T}$  and  $(u(T - s, X_s^{t, \xi}))_{t \leq s \leq T}$  coincide by approximating the terminal condition: we can approximate  $h$  by a sequence of uniformly bounded smooth functions  $(h_\ell)_{\ell \geq 1}$ , converging towards  $h$  almost everywhere (for the Lebesgue measure). Then, by standard stability results on BSDEs, it is known that

$$(95) \quad \mathbb{E} \left[ \sup_{t \leq s \leq T} |Y_s^{t, \xi} - u_\ell(T - s, X_s^{t, \xi})|^2 \right] \leq C \mathbb{E} [|h(X_T^{t, \xi}) - h_\ell(X_T^{t, \xi})|^2],$$

where  $u_\ell$  is associated with the boundary condition  $h_\ell$  by (13). Above, the right-hand side tends to 0 since the law of  $X_T^{t, \xi}$  is absolutely continuous w.r.t. the Lebesgue measure (apply Lemma 3).

Denoting by  $\mu$  the density of  $\xi$ , we obtain by the same argument:

$$(96) \quad \begin{aligned} & \sup_{t \leq s \leq T} \mathbb{E} [|u(T - s, X_s^{t, \xi}) - u_\ell(T - s, X_s^{t, \xi})|^2] \\ &= \sup_{t \leq s \leq T} \int_{\mathbb{R}^d} \mathbb{E} [|u(T - s, X_s^{t, x}) - u_\ell(T - s, X_s^{t, x})|^2] \mu(x) dx \\ &\leq C \int_{\mathbb{R}^d} \mathbb{E} [|h(X_T^{t, x}) - h_\ell(X_T^{t, x})|^2] \mu(x) dx. \end{aligned}$$

Again by Lemma 3, the above right-hand side converges to 0 as  $\ell$  tends to  $+\infty$ . Comparing (95) and (96), we understand that, for any  $s \in [t, T]$ ,  $Y_s^{t, \xi}$  and  $u(T - s, X_s^{t, \xi})$  coincide. Since both processes have a continuous version (see the statement of Proposition 4), this means that the processes  $(u(T - s, X_s^{t, \xi}))_{t \leq s \leq T}$  and  $(Y_s^{t, \xi})_{t \leq s \leq T}$  coincide.

Since processes are the same, we can compare the dynamics of  $(u(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  and  $(Y_s^{t,\xi})_{t \leq s \leq T}$ . We then deduce that, for any  $0 \leq t < T$ ,

$$\begin{aligned} & \mathbb{E} \int_t^s \left[ (\mathcal{V}_0 u - \frac{1}{2} \sum_{i=1}^N V_i^2 u)(T-r, X_r^{t,\xi}) \right. \\ & \quad \left. - f(T-r, X_r^{t,\xi}, u(T-r, X_r^{t,\xi}), Vu(T-r, X_r^{t,\xi})) \right] dr = 0, \quad t \leq s < T. \end{aligned}$$

Then, Point (2) in Definition 4 follows from

**Lemma 15.** *Let  $\psi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a function such that, for any  $t \in [0, T - \varepsilon]$ ,  $\varepsilon \in (0, T)$ ,  $|\psi(t, x)| \leq C_\varepsilon(1 + |x|^r)$ , and, for any  $t \in [0, T]$ ,*

$$\forall s \in [t, T), \quad \mathbb{E} \int_t^s \psi(r, X_r^{t,\xi}) dr = 0.$$

*Then,  $\psi$  is zero almost-everywhere for the Lebesgue measure.*

**Proof (Lemma 15).** By Lebesgue differentiation theorem, for any  $t \in [0, T)$ , there exists a Borel subset  $\mathcal{N}_t \subset [t, T]$ , of zero Lebesgue measure, such that for all  $s \in \mathcal{N}_t^c \cap [t, T)$ ,

$$\int_{\mathbb{R}^d} \psi(s, y) d\mathbb{P}_{X_s^{t,\xi}}(y) = 0.$$

Setting  $\mathcal{N} = \cup_{t \in \mathbb{Q} \cap [0, T)} \mathcal{N}_t$ , we deduce, that for all  $s \in \mathcal{N}^c \cap [0, T)$ , for all  $t \in [0, s) \cap \mathbb{Q}$ , the above equality holds true. In particular, we can let  $t$  tend to  $s$ . To do so, keep in mind that  $\mathbb{P}_{X_s^{t,\xi}}$  is absolutely continuous w.r.t. the Lebesgue measure with  $y \in \mathbb{R}^d \mapsto \mathbb{E}[\mu(\mathbb{X}_{s-t}^{-1}(y)) | J_{s-t,y}^{-1}]$  as density ( $\mathbb{X}^{-1}$  denoting the converse of the flow of  $X$  and  $J$  the associated Jacobian matrix): there is no need of continuity on  $\psi$  to pass to the limit in the above expression. We deduce that, for all  $s \in \mathcal{N}^c \cap [0, T)$ ,

$$\int_{\mathbb{R}^d} \psi(s, y) \mu(y) dy = 0.$$

Choosing  $\mu$  running over a countable total subset of densities with compact support, we deduce that  $\psi$  is zero almost-everywhere. □

It finally remains to check that  $u$  satisfies the boundary condition (3) in Definition 4. Since  $h$  is bounded, the solution  $u$  is bounded as well by the maximum principle (or equivalently by Gronwall's lemma). Taking the expectation in (12), we then write  $\mathbb{E}[Y_t^{t,x}]$  as  $\mathbb{E}[h(X_T^{t,x})] + O((T-t)^{1/2})$ , the Landau notation  $O(\cdot)$  being uniform w.r.t.  $x$  on compact subsets. Therefore, with  $\mu$  as above,

$$\lim_{t \rightarrow T} \int_{\mathbb{R}^d} |u(T-t, x) - \mathbb{E}[h(X_T^{t,x})]| \mu(x) dx = 0.$$

We deduce that

$$(97) \quad \lim_{t \rightarrow T} \int_{\mathbb{R}^d} |u(T-t, x) - h(x)| \rho(x) dx = 0,$$

provided

$$(98) \quad \lim_{t \rightarrow T} \int_{\mathbb{R}^d} |\mathbb{E}[h(X_T^{t,x})] - h(x)| \rho(x) dx = 0.$$



Convergence (98) holds true when  $h$  is continuous. When  $h$  is not continuous, we can approximate it by a smooth function in  $L^1_{\text{loc}}(\mathbb{R}^d)$  and then apply Lemma 3. This implies (3) in Definition 4.  $\square$

7.3.2. *Uniqueness.* The uniqueness property is checked in a similar way. Given a solution  $v$  to the PDE with polynomial growth, the point is to prove that  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  satisfies the BSDE (12) (for the same  $\xi$  as above). Basically, this follows from Itô's formula. As in the continuous case, the polynomial growth property together with the standing assumption on  $f$  imply the martingale part in the BSDE to be square integrable on  $[t, T]$ , that is

$$\mathbb{E} \int_t^T |Vv(T-s, X_s^{t,\xi})|^2 ds < +\infty.$$

As a consequence, the martingale part

$$\left( \sum_{i=1}^N \int_t^s V^i v(T-s, X_s^{t,\xi}) dB_s^i \right)_{t \leq s < T},$$

has an a.s. limit as  $s$  tends to  $T$  as the limit of an  $L^2$ -martingale. Similarly, by Cauchy criterion,

$$\left( \int_t^s f(T-r, X_r^{t,\xi}, v(T-r, X_r^{t,\xi}), Vv(T-r, X_r^{t,\xi})) dr \right)_{t \leq s < T}$$

has an a.s. limit as well. Therefore,  $(v(T-s, X_s^{t,\xi}))_{t \leq s < T}$  has also an a.s. limit as  $s$  tends to  $T$ . We can identify it as an  $L^1$  limit:

$$\mathbb{E}[|v(T-s, X_s^{t,\xi}) - h(X_T^{t,\xi})|] \leq \mathbb{E}[|v(T-s, X_s^{t,\xi}) - h(X_s^{t,\xi})|] + \mathbb{E}[|h(X_T^{t,\xi}) - h(X_s^{t,\xi})|].$$

By Lemma 3 and by (3) in Definition 4, the first term in the right-hand side tends to 0 as  $s$  tends to  $T$ . The second one also tends to 0 when  $h$  is continuous: approximating  $h$  in  $L^1_{\text{loc}}(\mathbb{R}^d)$  by a continuous function and applying Lemma 3 again, it tends to 0 as well when  $h$  is measurable only.

Finally,  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  satisfies (12) with  $h(X_T^{t,\xi})$  as boundary condition. By uniqueness of the solution to the BSDE, we deduce that  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  and  $(Y_s^{t,\xi})_{t \leq s \leq T}$  coincide, that is  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  and  $(u(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  coincide. Here, we emphasize that we cannot choose  $s = t$  directly since both  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  and  $(u(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  are seen as continuous versions of the original processes  $(v(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$  and  $(u(T-s, X_s^{t,\xi}))_{t \leq s \leq T}$ : they might differ from the original ones for some values of  $s$  because of the possible discontinuities of  $v$  and  $u$ . Anyhow, we can always claim that

$$\forall t \in [0, T), \forall t \leq s < T, \quad \mathbb{E} \int_t^s |v(T-r, X_r^{t,\xi}) - u(T-r, X_r^{t,\xi})| dr = 0.$$

By Lemma 15, we deduce that  $u$  and  $v$  match almost everywhere.  $\square$

**7.4. Proof of Proposition 4.** Again, the proof follows from a mollification argument. The whole point is to find a sequence  $(v_\ell)_{\ell \geq 1}$  of smooth functions such that, for all  $p \geq 1$ ,

$$(99) \quad \forall r \geq 1, \quad \lim_{\ell \rightarrow +\infty} \sup_{1/r \leq t \leq T} \|v_\ell(t, \cdot) - v(t, \cdot)\|_{\mathbb{B}(0,r),p}^{V,2} = 0, \quad \lim_{\ell \rightarrow +\infty} \|v_\ell - v\|_{[1/r,T] \times \mathbb{B}(0,r),p}^{\mathcal{V}_0,1} = 0.$$

Indeed, introducing the stopping times  $(\tau_q = \inf\{s \geq t : |X_s^{t,x}| \geq q\})_{q \geq 1}$  ( $\inf \emptyset = +\infty$ ), we can apply Itô's formula to  $(v_\ell(T-s, X_s^x))_{0 \leq s \leq \tau_q \wedge (T-\varepsilon)}$ , for some small positive real  $\varepsilon$ .

Therefore, for any  $\ell \geq 0$  and any  $t \leq s < T$ , we have

$$v_\ell(T-s, X_s^{t,\xi}) - v_\ell(T-t, \xi) = \mathcal{I}_\ell(s),$$

with

$$\mathcal{I}_\ell(s) = \int_t^s \left[ -\mathcal{V}_0 v_\ell(T-r, X_r^{t,\xi}) + \frac{1}{2} \sum_{i=1}^N V_i^2 v_\ell(T-r, X_r^{t,\xi}) \right] ds + \sum_{i=1}^N \int_t^s V_i v_\ell(T-r, X_r^{t,\xi}) dB_r^i.$$

Setting

$$\mathcal{I}(s) = \int_t^s \left[ -\mathcal{V}_0 v(T-r, X_r^{t,\xi}) + \frac{1}{2} \sum_{i=1}^N V_i^2 v(T-r, X_r^{t,\xi}) \right] ds + \sum_{i=1}^N \int_t^s V_i v(T-r, X_r^{t,\xi}) dB_r^i,$$

which makes sense by Lemma 3, we deduce from Lemma 3 that

$$\lim_{\ell \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \leq s \leq \tau_q \wedge (T-\varepsilon)} |\mathcal{I}(s) - \mathcal{I}_\ell(s)| \right] = 0.$$

Therefore,

$$\lim_{\ell \rightarrow +\infty} \sup_{k \geq 0} \mathbb{E} \left[ \sup_{t \leq s \leq \tau_q \wedge (T-\varepsilon)} |v_{\ell+k}(T-s, X_s^{t,\xi}) - v_\ell(T-s, X_s^{t,\xi})| \right] = 0.$$

We deduce that we can find a continuous adapted process  $(\Xi_s)_{t \leq s < T}$  such that

$$(100) \quad \lim_{\ell \rightarrow +\infty} \mathbb{E} \left[ \sup_{t \leq s \leq \tau_q \wedge (T-\varepsilon)} |\Xi_s - v_\ell(T-s, X_s^{t,\xi})| \right] = 0.$$

The point is now to identify  $(\Xi_s)_{t \leq s < T}$  as a version of  $(v(T-s, X_s^{t,\xi}))_{t \leq s < T}$ . By Lemma 3, we know that

$$(101) \quad \lim_{\ell \rightarrow +\infty} \mathbb{E} [ |v(T-s, X_s^{t,\xi}) - v_\ell(T-s, X_s^{t,\xi})| ] = 0.$$

By (100) and (101), we understand that, for any  $s \in [t, T)$ ,

$$\mathbb{P} \left\{ \Xi_s \neq v(T-s, X_s^{t,\xi}), \sup_{t \leq s \leq T} |X_s^{t,\xi}| \leq q \right\} = 0.$$

Letting  $q$  tend to  $+\infty$ , this completes the proof. □

Now, (99) follows from

**Lemma 16.** For two smooth densities  $\rho_1$  and  $\rho_d$  over  $\mathbb{R}$  and  $\mathbb{R}^d$ , both with compact support, and for a solution  $v$  to the PDE as in Definition 4, define for all  $\varepsilon > 0$

$$v^\varepsilon(t, x) = \int_{\mathbb{R}^{d+1}} v(t-\varepsilon s, x-\varepsilon y) \mathbf{1}_{\{t-\varepsilon s > 0\}} \rho_1(s) \rho_d(y) ds dy.$$

Then, for all  $p \geq 1$ ,

$$\forall r \geq 1, \quad \lim_{\varepsilon \rightarrow 0} \sup_{1/r \leq t \leq r} \|v^\varepsilon(t, \cdot) - v(t, \cdot)\|_{\mathbb{B}(0,r),p}^{V,2} = 0, \quad \lim_{\varepsilon \rightarrow 0} \|v^\varepsilon - v\|_{[1/r,r] \times \mathbb{B}(0,r),p}^{Y_0,1} = 0.$$

The proof is postponed to the next subsection.

**7.5. Convolution Arguments: Proof of Lemma 14.** We here prove Lemma 14. We start with:

**Lemma 17.** For a smooth density  $\rho_d$  over  $\mathbb{R}$  with compact support and a function  $\varphi \in \mathcal{D}_V^2(\mathbb{R}^d)$ , define for all  $\varepsilon > 0$

$$\varphi^\varepsilon(x) = \int_{\mathbb{R}^d} \varphi(x - \varepsilon y) \rho(y) dy, \quad x \in \mathbb{R}^d.$$

Then,

$$\forall r \geq 1, \quad \lim_{\varepsilon \rightarrow 0} \|\varphi^\varepsilon - \varphi\|_{\mathbb{B}(0,r),\infty}^{V,2} = 0.$$

**Proof.** Start with the case when  $\varphi$  is smooth. Then, for  $\alpha \in \mathcal{A}_1^0(m)$ ,

$$\begin{aligned} V_{[\alpha]} \varphi^\varepsilon(x) &= \int_{\mathbb{R}^d} \langle V_{[\alpha]}(x - \varepsilon y), \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &+ \int_{\mathbb{R}^d} \langle V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y), \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy. \end{aligned}$$

By integration by parts, we obtain

$$\begin{aligned} (102) \quad V_{[\alpha]} \varphi^\varepsilon(x) &= \int_{\mathbb{R}^d} V_{[\alpha]} \varphi(x - \varepsilon y) \rho(y) dy \\ &+ \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \operatorname{div}(V_{[\alpha]})(x - \varepsilon y) \rho(y) dy \\ &+ \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \left\langle \frac{V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)}{\varepsilon}, \nabla \rho(y) \right\rangle dy. \end{aligned}$$

In the general case when  $\varphi$  is not smooth, we can approximate the pair  $(\varphi, V_{[\alpha]} \varphi)$  by a sequence  $(\varphi_p, V_{[\alpha]} \varphi_p)_{p \geq 1}$ , uniformly on compact subsets. All the mappings  $(\varphi_p)_{p \geq 1}$  satisfy (102). Letting  $p$  tend to  $+\infty$ , we deduce that  $\varphi$  satisfies (102) as well.

Since  $(\varphi, V_{[\alpha]} \varphi)$  is continuous, we deduce that  $V_{[\alpha]} \varphi^\varepsilon$  converges towards  $V_{[\alpha]} \varphi$  uniformly on compact sets.

Turn now to the second-order derivatives. When  $\varphi$  is smooth, we obtain for a given  $\beta \in \mathcal{A}_1^0(m)$

$$\begin{aligned} V_{[\alpha]} V_{[\beta]} \varphi^\varepsilon(x) &= \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \cdot \nabla V_{[\beta]}](x), \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &+ \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \otimes V_{[\beta]}](x), \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &= T_1^\varepsilon + T_2^\varepsilon. \end{aligned}$$

Clearly,

$$\begin{aligned} T_1^\varepsilon &= \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \cdot \nabla V_{[\beta]}](x - \varepsilon y), \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &\quad + \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \cdot \nabla V_{[\beta]}](x) - [V_{[\alpha]} \cdot \nabla V_{[\beta]}](x - \varepsilon y), \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &= T_{1,1}^\varepsilon + T_{1,2}^\varepsilon, \end{aligned}$$

with

$$\begin{aligned} T_{1,2}^\varepsilon &= \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \left\langle \frac{[V_{[\alpha]} \cdot \nabla V_{[\beta]}](x) - [V_{[\alpha]} \cdot \nabla V_{[\beta]}](x - \varepsilon y)}{\varepsilon}, \nabla \rho(y) \right\rangle dy \\ &\quad + \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \operatorname{div}(V_{[\alpha]} \cdot \nabla V_{[\beta]})(x - \varepsilon y) \rho(y) dy, \end{aligned}$$

by integration by parts. Similarly, we can write

$$\begin{aligned} T_2^\varepsilon &= \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \otimes V_{[\beta]}](x - \varepsilon y), \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &\quad + \int_{\mathbb{R}^d} \langle [V_{[\alpha]} \otimes V_{[\beta]}](x) - [V_{[\alpha]} \otimes V_{[\beta]}](x - \varepsilon y), \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &= T_{2,1}^\varepsilon + T_{2,2}^\varepsilon. \end{aligned}$$

Now,

$$\begin{aligned} T_{2,2}^\varepsilon &= \int_{\mathbb{R}^d} \langle [V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)] \otimes [V_{[\beta]}(x) - V_{[\beta]}(x - \varepsilon y)], \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &\quad + \int_{\mathbb{R}^d} \langle V_{[\alpha]}(x - \varepsilon y) \otimes [V_{[\beta]}(x) - V_{[\beta]}(x - \varepsilon y)], \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &\quad + \int_{\mathbb{R}^d} \langle [V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)] \otimes V_{[\beta]}(x - \varepsilon y), \nabla^2 \varphi(x - \varepsilon y) \rangle \rho(y) dy \\ &= T_{2,2,1}^\varepsilon + T_{2,2,2}^\varepsilon + T_{2,2,3}^\varepsilon. \end{aligned}$$

By integration by parts,

$$\begin{aligned} &T_{2,2,1}^\varepsilon \\ &= \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \left\langle \frac{V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)}{\varepsilon} \otimes \frac{V_{[\beta]}(x) - V_{[\beta]}(x - \varepsilon y)}{\varepsilon}, \nabla^2 \rho(y) \right\rangle dy \\ &\quad + \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \operatorname{div}(V_{[\alpha]})(x - \varepsilon y) \left\langle \frac{V_{[\beta]}(x) - V_{[\beta]}(x - \varepsilon y)}{\varepsilon}, \nabla \rho(y) \right\rangle dy \\ &\quad + \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \operatorname{div}(V_{[\beta]})(x - \varepsilon y) \left\langle \frac{V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)}{\varepsilon}, \nabla \rho(y) \right\rangle dy \\ &\quad + \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \nabla^2 \cdot \frac{[V_{[\beta]}(x) - V_{[\beta]}(x - \varepsilon y)] \otimes [V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)]}{\varepsilon^2} \rho(y) dy. \end{aligned}$$

Similarly,

$$\begin{aligned} T_{2,2,3}^\varepsilon &= \int_{\mathbb{R}^d} \langle [V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)], \nabla [V_{[\beta]}\varphi(x - \varepsilon y)] \rangle \rho(y) dy \\ &\quad - \int_{\mathbb{R}^d} \langle [V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)], \nabla V_{[\beta]}(x - \varepsilon y) \nabla \varphi(x - \varepsilon y) \rangle \rho(y) dy. \end{aligned}$$

We deduce

$$\begin{aligned} T_{2,2,3}^\varepsilon &= \int_{\mathbb{R}^d} (V_{[\beta]}\varphi(x - \varepsilon y) - V_{[\beta]}\varphi(x)) \langle \frac{V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y)}{\varepsilon}, \nabla \rho(y) \rangle dy \\ &\quad + \int_{\mathbb{R}^d} (V_{[\beta]}\varphi(x - \varepsilon y) - V_{[\beta]}(x)) \operatorname{div}(V_{[\alpha]})(x - \varepsilon y) \rho(y) dy \\ &\quad - \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \langle \nabla V_{[\alpha]}(x - \varepsilon y), \nabla V_{[\beta]}(x - \varepsilon y) \rangle \rho(y) dy \\ &\quad - \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \langle \frac{V_{[\alpha]}(x - \varepsilon y) - V_{[\alpha]}(x)}{\varepsilon}, \nabla V_{[\beta]}(x - \varepsilon y) \nabla \rho(y) \rangle dy \\ &\quad - \int_{\mathbb{R}^d} (\varphi(x - \varepsilon y) - \varphi(x)) \langle V_{[\alpha]}(x) - V_{[\alpha]}(x - \varepsilon y), \nabla \operatorname{div} V_{[\beta]}(x - \varepsilon y) \rangle \rho(y) dy. \end{aligned}$$

Obviously, a similar expression holds for  $T_{2,2,2}^\varepsilon$ .

We now emphasize that

$$T_{1,1}^\varepsilon + T_{2,1}^\varepsilon = \int_{\mathbb{R}^d} (V_{[\alpha]}V_{[\beta]}\varphi)(x - \varepsilon y) \rho(y) dy.$$

Clearly, this term makes sense when  $\varphi$  is in  $\mathcal{D}_V^2(\mathbb{R}^d)$  only and then converges towards  $V_{[\alpha]}V_{[\beta]}\varphi(x)$ , uniformly on compact subsets, as  $\varepsilon$  tends to 0. We also notice that all the remaining terms  $T_{1,2}^\varepsilon$ ,  $T_{2,2,2}^\varepsilon$  and  $T_{2,2,3}^\varepsilon$  make sense when  $\varphi$  is in  $\mathcal{D}_V^2(\mathbb{R}^d)$  only and then converge towards 0, uniformly on compact set, as  $\varepsilon$  tends to 0. The proof is then completed as in the first-order case.  $\square$

Similarly, we claim

**Lemma 18.** *For two smooth densities  $\rho_1$  and  $\rho_d$  over  $\mathbb{R}$  and  $\mathbb{R}^d$ , both with compact support, and for a function  $\psi \in \mathcal{D}_{\mathcal{V}_0}^1(\mathbb{R}_+ \times \mathbb{R}^d)$ , define for all  $\varepsilon > 0$*

$$\psi^\varepsilon(t, x) = \int_{\mathbb{R}^{d+1}} \psi(t - \varepsilon s, x - \varepsilon y) \mathbf{1}_{\{t - \varepsilon s > 0\}} \rho_1(s) \rho_d(y) ds dy.$$

Then,

$$\forall r \geq 1, \quad \lim_{\varepsilon \rightarrow 0} \|\psi^\varepsilon - \psi\|_{[1/r, r] \times \mathbb{B}(0, r), \infty}^{\mathcal{V}_0, 1} = 0.$$

**Proof.** The proof is similar to the proof of Lemma 17. We first assume  $\psi$  to be smooth. Given  $t > 0$ , we can assume  $\varepsilon$  small enough to get rid of the indicator function in the definition of  $\psi^\varepsilon$ . Then,

$$\begin{aligned} \mathcal{V}_0 \psi^\varepsilon(t, x) &= \int_{\mathbb{R}^{d+1}} \partial_t \psi(t - \varepsilon s, x - \varepsilon y) \rho_1(s) \rho_d(y) ds dy \\ &\quad - \int_{\mathbb{R}^{d+1}} \langle V_0(x), \nabla \psi(t - \varepsilon s, x - \varepsilon y) \rangle \rho_1(s) \rho_d(y) ds dy, \end{aligned}$$

so that

$$\begin{aligned}
\mathcal{V}_0\psi^\varepsilon(t, x) &= \int_{\mathbb{R}^{d+1}} [\mathcal{V}_0\psi](t - \varepsilon s, x - \varepsilon y) \rho_1(s) \rho_d(y) ds dy \\
&\quad - \int_{\mathbb{R}^{d+1}} \langle V_0(x) - V_0(x - \varepsilon y), \nabla\psi(t - \varepsilon s, x - \varepsilon y) \rangle \rho_1(s) \rho_d(y) ds dy \\
&= \int_{\mathbb{R}^{d+1}} [\mathcal{V}_0\psi](t - \varepsilon s, x - \varepsilon y) \rho_1(s) \rho_d(y) ds dy \\
&\quad - \int_{\mathbb{R}^{d+1}} (\psi(t - \varepsilon s, x - \varepsilon y) - \psi(t, x)) \left\langle \frac{V_0(x) - V_0(x - \varepsilon y)}{\varepsilon}, \nabla\rho_d(y) \right\rangle \rho_1(s) ds dy \\
&\quad - \int_{\mathbb{R}^{d+1}} (\psi(t - \varepsilon s, x - \varepsilon y) - \psi(t, x)) \operatorname{div}[V_0](x - \varepsilon y) \rho_1(s) \rho_d(y) ds dy.
\end{aligned}$$

We then complete the proof as in Lemma 17.  $\square$

**End of the Proof of Lemma 14.** The second point is a straightforward consequence of Lemma 18. For the first one, we note that, for any  $\alpha, \beta \in \mathcal{A}_1^0(m)$ ,

$$V_{[\alpha]}V_{[\beta]}u^\varepsilon(t, x) = \int_{\mathbb{R}} \mathbf{1}_{\{t-\varepsilon s > 0\}} \rho_1(s) V_{[\alpha]}(x) \cdot \nabla \left[ V_{[\beta]}(x) \cdot \nabla \int_{\mathbb{R}^d} u(t - \varepsilon s, x - \varepsilon y) \rho_d(y) dy \right] ds.$$

Using the time-space continuity of the function  $(t, x) \mapsto V_{[\alpha]}V_{[\beta]}u(t, x)$  and following the proof of Lemma 17, we can prove that

$$V_{[\alpha]}(x) \cdot \nabla \left[ V_{[\beta]}(x) \cdot \nabla \int_{\mathbb{R}^d} u(t - \varepsilon s, x - \varepsilon y) \rho_d(y) dy \right] \rightarrow V_{[\alpha]}V_{[\beta]}u(t, x),$$

as  $\varepsilon$  tends to 0, uniformly on compact subsets.

The proof is easily completed.

**7.6. Proof of Lemma 16.** We here give a sktech of the proof only, since quite similar to the proof of Lemma 14. First, we let the reader check that Lemmas 17 and 18 can be adapted to the  $L_{\text{loc}}^p$  framework,  $p \geq 1$ . (Keep in mind that, for any  $L_{\text{loc}}^1(\mathbb{R}^d)$  function  $\varphi$ , the mapping  $x \in \mathbb{R}^d \mapsto \varphi(\cdot + \varepsilon y) - \varphi(\cdot)$  tends to zero in  $L_{\text{loc}}^1(\mathbb{R}^d)$  as  $\varepsilon$  tends to 0,  $y$  being given in  $\mathbb{R}^d$ : this follows from a standard approximation of  $\varphi$  in  $L_{\text{loc}}^1(\mathbb{R}^d)$  by a continuous function.)

By Point (1) in Definition 4, we know that the second order derivatives are time-space  $L_{\text{loc}}^p$ ,  $p \geq 1$ . Again, we can follow the end of the proof of Lemma 14.

## 8. APPENDIX

**8.1. Malliavin Differentiation.** Consider the *Cameron-Martin* space

$$H = \{h \in \Omega; h' \in L^2([0, \infty); \mathbb{R}^d)\} \in \Omega,$$

with the inner product

$$\langle h, g \rangle_H := \langle h', g' \rangle_{L^2([0, \infty); \mathbb{R}^d)} := \int_0^\infty h'(u) \cdot g'(u) du.$$

**Definition 5** (Malliavin Derivative). *Let  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $h'_1, \dots, h'_n \in L^2([0, \infty); \mathbb{R}^d)$  and  $F : \Omega \rightarrow \mathbb{R}$  be the random variable:*

$$(103) \quad F(\omega) = f \left( \int_0^\infty h'_1(t) dB_t(\omega), \dots, \int_0^\infty h'_n(t) dB_t(\omega) \right).$$

We call such random variables smooth and denote by  $\mathcal{S}$ .

The set of smooth random variables  $\mathcal{S}$  is dense in  $L^2(\Omega)$ . See for example Nualart [23]. Using the density property of  $\mathcal{S}$  one extends the definition of the Malliavin derivative to the set of all square integrable random variable for which there exist an approximating sequence of smooth random variables such that the corresponding Malliavin derivatives converge too. This approach will work provided the Malliavin derivatives of two convergent sequences of smooth random variables who converge to the same  $L^2(\Omega)$ -limit have the same  $L^2([0, \infty) \times \Omega)$ -limit. This amounts to showing the following:

**Corollary 2** (Closability of the Malliavin Derivative operator). *The Malliavin derivative, a linear unbounded operator  $D : \mathcal{S} \rightarrow L^2([0, \infty) \times \Omega; \mathbb{R}^d)$  is closable as an operator from  $L^2(\Omega; \mathbb{R}^d)$  into  $L^2([0, \infty) \times \Omega; \mathbb{R}^d)$ . In other words if  $\{F_n\} \subset \mathcal{S}$  is a sequence of smooth random variables such that:  $\|F_n\|_{L^2(\Omega)} \rightarrow 0$  and  $\|DF_n\|_{L^2([0, \infty) \times \Omega)}$  is convergent then it follows that*

$$\|DF_n\|_{L^2([0, \infty) \times \Omega)} \rightarrow 0.$$

Since smooth random variables are dense in  $L^p$  for  $p \geq 1$  the same results one has just obtained hold for any such  $p$ . For  $p \neq 2$  we use with the norm:

$$(104) \quad \|DF\|_{L^p(\Omega; H)}^p := \mathbb{E} [\|DF\|_H^p]$$

The closability property still holds i.e.  $D$  is closable from  $L^p(\Omega)$  to  $L^p(\Omega; H)$ .

Denote the domain of  $D$  by  $\mathbb{D}^{1,p}$ , meaning that  $\mathbb{D}^{1,p}$  is the closure of smooth random variables  $\mathcal{S}$  with respect to the norm:

$$\|F\|_{\mathbb{D}^{1,p}} = [\mathbb{E}(|F|^p) + \mathbb{E}(\|DF\|_H^p)]^{\frac{1}{p}}.$$

One may also define the iteration of the Malliavin derivative  $D$  in such a way that for smooth random variables, the iterated derivative  $D^k F$  is a random variable with values in  $H^{\otimes k}$ . Define

$$D^k F := \sum_{i_1, \dots, i_k=1}^n \partial_{i_1, \dots, i_k} f \left( \int_0^\infty h'_1(u) dB_u, \dots, \int_0^\infty h'_n(u) dB_u \right) h_{i_1} \otimes \dots \otimes h_{i_k},$$

where  $h_i(\cdot) := \int_0^\cdot h'_i(s) ds$ .

In an analogous way, one can close the operator  $D^k$  from  $L^p(\Omega)$  to  $L^p(\Omega; H^{\otimes k})$ . So, for any  $p \geq 1$  and natural  $k \geq 1$ , define  $\mathbb{D}^{k,p}$  to be the closure of  $\mathcal{S}$  with respect to the norm:

$$\|F\|_{\mathbb{D}^{k,p}} := \left[ \mathbb{E}(|F|^p) + \sum_{j=1}^k \mathbb{E}(\|D^j F\|_{H^{\otimes j}}^p) \right]^{\frac{1}{p}}.$$

Note that for  $p = 2$  the following isometry holds  $L^2(\Omega \times [0, \infty)^k; \mathbb{R}^d) \simeq L^2(\Omega; H^{\otimes k})$ . Hence one may identify  $D^k F$  as a process:  $D_{t_1, \dots, t_k}^k F$ .

A random variable is said to be *smooth in the Malliavin sense* if  $F \in \mathbb{D}^{k,p}$  for all  $p \geq 1$  and  $k \in \mathbb{N}$ .

Moreover, there is nothing which pins consideration to  $\mathbb{R}^d$ -valued random variables. Indeed, it one could consider more general Hilbert space-valued random variables, and the theory would extend in an appropriate way. To this end, denote  $\mathbb{D}^{k,p}(E)$  to be the appropriate space of  $E$ -valued random variables, where  $E$  is some separable Hilbert space. Also we define  $\mathbb{D}^{k,\infty}(E)$  to be the space

$$(105) \quad \mathbb{D}^{k,\infty}(E) = \bigcap_{p \geq 1} \mathbb{D}^{k,p}(E).$$

Similarly, let  $\mathcal{S}_t$  be the set of all  $\mathcal{F}_t$ -measurable random variables. Then for any  $p \geq 1$  and natural  $k \geq 1$ , one define  $\tilde{\mathbb{D}}^{k,p}$  to be the closure of  $\mathcal{S}$  with respect to the norm:

$$\|F\|_{\tilde{\mathbb{D}}^{k,p}} := \left[ \mathbb{E}(|F|^p) + \sum_{j=1}^k \mathbb{E} \left( \sup_{t_1, \dots, t_j \in [0,t]} \|D_{t_1, \dots, t_j}^j F\|^p \right) \right]^{\frac{1}{p}}.$$

and similarly we define  $\tilde{\mathbb{D}}^{k,\infty}(E)$  to be the space

$$(106) \quad \tilde{\mathbb{D}}^{k,\infty}(E) = \bigcap_{p \geq 1} \tilde{\mathbb{D}}^{k,p}(E).$$

**8.2. Proof of Lemma 3.** For a given  $A \gg R$ ,

$$(107) \quad \begin{aligned} & \int_{|x| < R} \mathbb{E}[|\theta_1 - \theta_2|^p(X_t^x)] dx \\ &= \int_{|x| < R} \mathbb{E}[|\theta_1 - \theta_2|^p(X_t^x); |X_t^x| \leq A] dx + \int_{|x| < R} \mathbb{E}[|\theta_1 - \theta_2|^p(X_t^x); |X_t^x| > A] dx \\ &= T_1(A, R) + T_2(A, R). \end{aligned}$$

By the change of variable formula

$$T_1(A, R) = \int_{|y| \leq A} |\varphi_{\ell+k} - \varphi_\ell|^p(y) \mathbb{E}[|J_{t,y}^{-1}| \mathbf{1}_{\{|X_t^{-1}(y)| \leq R\}}] dy \leq C \int_{|y| \leq A} |\varphi_{\ell+k} - \varphi_\ell|^p(y) dy,$$

where  $X_t^{-1}$  stands for the converse of the flow of  $X$  and  $J_{t,y}$  for the associated Jacobian matrix.

To handle  $T_2(A, R)$ , we make use of the polynomial growth assumption. By Cauchy-Schwarz inequality,

$$\begin{aligned} T_2(A, R) &\leq C' \int_{|x| < R} \mathbb{E}[1 + (X_t^x)^{rp}; |X_t^x| > A] dx \\ &\leq C' \int_{|x| < R} \mathbb{E}[1 + (X_t^x)^{2rp}]^{1/2} \mathbb{P}\{|X_t^x| > A\}^{1/2} dx \\ &\leq C'(1 + R^{rp}) \sup_{|x| < R} \mathbb{P}\{|X_t^x| > A\}^{1/2}, \end{aligned}$$

the constant  $C'$  here changing from line to line. Since  $\sup_{|x| < R} \mathbb{P}\{|X_t^x| > A\} \leq C'(1 + R)A^{-1}$ , this completes the proof.



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