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RESEARCH ARTICLE

Homogenization of multi-coated inclusion-reinforced linear elastic composites with eigenstrains: application to the thermo-elastic behavior

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A new micromechanical approach for arbitrary multi-coated ellipsoidal elastic inclusions with general eigenstrains is developed. We start from the integral equation of the linear elastic medium with eigenstrains adopting the Green's function technique and we apply a "$(n+1)$-phase" model with a self-consistent condition to determine the homogenized behavior of multi-coated inclusion-reinforced composites. The effective elastic moduli and eigenstrains are obtained as well as the residual stresses through the local stress concentration equations. The effective eigenstrains are determined either with the concentration tensors here obtained by the present model, or, more classically, with Levin's formula. In order to assess our micromechanical model, some applications to the isotropic thermo-elastic behavior of composites with and without interphase are given. In particular, "4-phase" and "3-phase" models are derived for isotropic homothetic spherical inclusion-reinforced materials, and, the results are successfully compared to exact analytical solutions regarding the effective elastic moduli and the effective thermal expansion.

Keywords: Interphase; Multi-coated inclusion; Eigenstrains; Thermo-elastic Behavior; Micro-mechanics

1. Introduction

Numerous composite materials contain an interphase between a matrix phase and reinforcements like fibers, inclusions etc. This interphase may significantly change the mechanical properties of composites. Classical linear elastic properties were extensively studied by various authors in solid mechanics and in civil engineering applications to account for the role of interphase on the homogenized behavior of composites [1–15]. Meanwhile, the influence of other materials properties in the interphase giving rise to the so called *eigenstrains* [16–20] are less explored. These eigenstrains (or stress-free strains) are inelastic strains which occur in many physical problems (e.g. thermal strain, lattice mismatch strain, static configurations of dislocations, chemical compositions, magnetostrictive strain, poroelasticity etc.). The purpose of the paper is twofold. The first issue is to extend a recently developed self-consistent model for linear elastic heterogeneous composites based on the multi-coated inclusion problem [21] to the case of same types of composites

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undergoing eigenstrains in addition to elastic strains. This means that the problem needs to be solved from the integral equation of the linear elastic microheterogeneous solid with eigenstrains. Thus, inclusions, interphases and matrix will have different elastic properties and different eigenstrains (i.e. piecewise uniform). The extensions of the technique based on the so called *interfacial operators* [22–24] to this multi-coated inclusion problem is here highlighted. The second objective is to apply the present model to the thermo-elastic behavior of multiple coated inclusion-reinforced composites. Actually, this problem can be solved analytically using exact results derived by Hervé and Zaoui [3] for the effective elastic properties who extended the "3-phase" model of Christensen and Lo [2]. Later, Hervé [25] derived the effective thermal expansion using the results of Hervé and Zaoui [3] and the Levin's formula [26, 27]. Mean field approximations based on the multi-inclusion model derived by Nemat-Nasser and Hori [28] were also applied to this particular case by Li [29]. Variational principles to derive sharp bounds on the effective thermal expansion coefficients of multiphase composites were given by Willis [30], Gibiansky and Torquato [31] and Stolz [32, 33]. In particular, the exact solutions for the "3-phase" model are retrieved [32, 33]. Results of the present model are given in the particular case of a spherical inclusion-reinforced material, i.e. containing an interphase, and compared to exact solutions [3, 25]. In contrast to [3, 25], the presented methodology can be applied to more complex anisotropic problems and to ellipsoidal inclusions of arbitrary shapes. The paper is organized as follows. In section 2, the micromechanical approach for arbitrary multi-coated ellipsoidal elastic inclusions with general eigenstrains is detailed. We start from the integral equation of the problem adopting the Green's function technique, and we use interfacial relations for perfect bonded interfaces with an averaging procedure, which gives solutions in the form of recurrence relations. Then, we apply a generalized self-consistent scheme or "(n+1)-phase" model [2, 3] to determine the homogenized behavior of multi-coated inclusion-reinforced composites. As a result, we obtain the effective elastic properties and the effective eigenstrains as well as the residual stresses due to eigenstrains. It is highlighted that the effective eigenstrains can be determined by two different ways: the one detailed in the present paper using concentration tensors, and, the one derived from Levin's formula [26, 27]. In section 3, we apply the present micromechanical approach ("4-phase" and "3-phase" versions) to layered homothetic spherical inclusion-reinforced composite materials with isotropic elastic properties and thermal properties in each phase. In particular, we compare the results of our "4-phase" and "3-phase" approaches respectively to the exact analytical results of Hervé [25] and Stolz [32, 33] regarding the effective thermal expansion. Section 4 is devoted to conclusions and perspectives of this work.

2. Micromechanical approach

2.1. *Field Equations and Integral Equation*

On the boundary ∂V of V, a prescribed displacement $\mathbf{u^d}$ (Dirichlet conditions) is considered:

$$\mathbf{u^d} = \mathbf{E} \cdot \mathbf{x} \text{ on } \partial V, \tag{1}$$

where $\mathbf{E}$ is a uniform imposed strain on ∂V.

The other field equations are constituted of:

- the stress equilibrium condition for the symmetric Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\mathbf{div}\ \boldsymbol{\sigma}(\mathbf{x}) = 0 \text{ in } V, \tag{2}$$

- the compatibility relation for total strain $\boldsymbol{\epsilon}$ where $\mathbf{u}$ is the displacement field:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \frac{1}{2}\left(\nabla\mathbf{u}(\mathbf{x}) + \nabla^t\mathbf{u}(\mathbf{x})\right), \tag{3}$$

- the total strain in the small perturbation hypothesis which writes as the sum of an elastic strain $\boldsymbol{\epsilon}^e$ and an *eigenstrain* $\boldsymbol{\epsilon}^\star$:

$$\boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}^e(\mathbf{x}) + \boldsymbol{\epsilon}^\star(\mathbf{x}), \tag{4}$$

- the constitutive equation for linear elasticity (Hooke's law) with the presence of eigenstrains:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\epsilon}^e(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : (\boldsymbol{\epsilon}(\mathbf{x}) - \boldsymbol{\epsilon}^\star(\mathbf{x})) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\epsilon}(\mathbf{x}) + \boldsymbol{\lambda}(\mathbf{x}), \tag{5}$$

where $\mathbf{C}(\mathbf{x})$ denotes the elastic moduli. In this problem, the unknown fields are the displacement $\mathbf{u}$, from which the total strain $\boldsymbol{\epsilon}$, and, the Cauchy stress $\boldsymbol{\sigma}$ are derived.

In the following, we consider $\boldsymbol{\lambda}(\mathbf{x}) = -\mathbf{C}(\mathbf{x}) : \boldsymbol{\epsilon}^\star(\mathbf{x})$ as the *eigenstress* associated to the *eigenstrain* $\boldsymbol{\epsilon}^\star(\mathbf{x})$. Then, first order spatial variations of elastic properties and eigenstresses are respectively denoted $\delta\mathbf{C}(\mathbf{x})$ and $\delta\boldsymbol{\lambda}(\mathbf{x})$ so that:

$$\begin{aligned} \mathbf{C}(\mathbf{x}) &= \mathbf{C}^0 + \delta\mathbf{C}(\mathbf{x}), \\ \boldsymbol{\lambda}(\mathbf{x}) &= \boldsymbol{\lambda}^0 + \delta\boldsymbol{\lambda}(\mathbf{x}), \end{aligned} \tag{6}$$

where $\mathbf{C}^0$ and $\boldsymbol{\lambda}^0 = -\mathbf{C}^0 : \boldsymbol{\epsilon}^{\star 0}$ denote respectively the homogeneous elastic moduli and eigenstresses of the infinite reference medium (0) described in Fig. 1.

By introducing these fluctuations in the set of field equations (eqs.(1) to (5)), we obtain the so-called integral equation [34–36] of the problem as follows:

$$\epsilon_{ij}(\mathbf{x}) = E_{ij} - \int_V \Gamma^0_{ijkl}(\mathbf{x}-\mathbf{x}')\left(\delta C_{klmn}(\mathbf{x}')\epsilon_{mn}(\mathbf{x}') + \delta\lambda_{kl}(\mathbf{x}')\right) dV', \tag{7}$$

where $\boldsymbol{\Gamma}^0(\mathbf{x}-\mathbf{x}')$ is the modified Green's tensor associated to $\mathbf{C}^0$. According to [37], this one classically writes:

$$\Gamma^0_{ijkl}(\mathbf{x}-\mathbf{x}') = -\frac{1}{2}\left(G^0_{ik,jl}(\mathbf{x}-\mathbf{x}') + G^0_{jk,il}(\mathbf{x}-\mathbf{x}')\right), \tag{8}$$

with $\mathbf{G}^0$ being the Green's function of the infinite homogeneous medium $\mathbf{C}^0$.

2.2. *Case of a n-phase multi-coated composite inclusion*

2.2.1. Concentration equations

We first apply the integral equation (eq.(7)) to the case of a n-phase multi-coated composite inclusion of volume V_I embedded in an infinite reference medium denoted

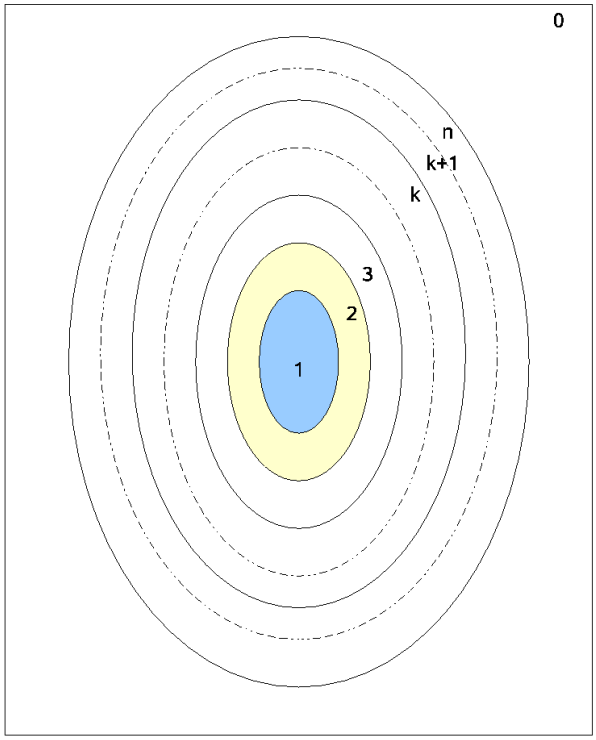

Figure 1. Multi-coated inclusion problem with eigenstrains considering n phases. The inclusion is embedded in an infinite reference medium (0). In this decription, Phase 2 is called the *interphase* when $n = 3$ for instance.

0 (see Fig.1), and, containing a set of phases k with $k \in [1, 2, .., n]$ characterized by their volume V_k and the characteristic function $\theta_k(\mathbf{x})$ defined by:

$$\theta_k(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in V_k \\ 0 \text{ if } \mathbf{x} \notin V_k, \end{cases} \tag{9}$$

Then, the first order variations of elastic moduli and eigenstresses follow a piecewise uniform decomposition in the form of:

$$\begin{aligned} \delta\mathbf{C}(\mathbf{x}) &= \sum_{k=1}^{n} \Delta\mathbf{C}^{(k/0)}\theta_k(\mathbf{x}) \text{ with } \Delta\mathbf{C}^{(k/0)} = \mathbf{C}^k - \mathbf{C}^0, \\ \delta\boldsymbol{\lambda}(\mathbf{x}) &= \sum_{k=1}^{n} \Delta\boldsymbol{\lambda}^{(k/0)}\theta_k(\mathbf{x}) \text{ with } \Delta\boldsymbol{\lambda}^{(k/0)} = \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^0. \end{aligned} \tag{10}$$

In the following and according to eq.(10), $\mathbf{C}^k$ and $\boldsymbol{\lambda}^k$ denote piecewise constant values per phase k respectively for $\mathbf{C}(\mathbf{x})$ and $\boldsymbol{\lambda}(\mathbf{x})$.

By construction, the volume V_I of the composite inhomogeneity is constituted of the first inhomogeneity (1) and $n-1$ other coatings. Thus:

$$V_I = \sum_{k=1}^{n} V_k, \tag{11}$$

and the volume fraction ϕ_k of phase k in V_I is defined by:

$$\phi_k = \frac{V_k}{V_I} \text{ for k} \in [1,2,..,n]. \tag{12}$$

The average strain $\overline{\boldsymbol{\epsilon}}^I$ over V_I is defined by:

$$\overline{\boldsymbol{\epsilon}}^I = \frac{1}{V_I}\int_{V_I} \boldsymbol{\epsilon}(\mathbf{x})dV, \tag{13}$$

After simple manipulations using eq.(7) and eq.(10), we obtain:

$$\overline{\boldsymbol{\epsilon}}^I = \mathbf{E} - \mathbf{T}^I(\mathbf{C}^0) : \sum_{k=1}^{n} \phi_k \left(\Delta\mathbf{C}^{(k/0)} : \overline{\boldsymbol{\epsilon}}^k + \Delta\boldsymbol{\lambda}^{(k/0)}\right), \tag{14}$$

where:

$$\mathbf{T}^I(\mathbf{C}^0) = \frac{1}{V_I}\int_{V_I}\int_{V_I} \boldsymbol{\Gamma}^0(\mathbf{x}-\mathbf{x}')dV'dV, \tag{15}$$

and,

$$\overline{\boldsymbol{\epsilon}}^k = \frac{1}{V_k}\int_{V_k} \boldsymbol{\epsilon}(\mathbf{x})dV. \tag{16}$$

From eq.(13) and eq.(16), we can also write:

$$\overline{\boldsymbol{\epsilon}}^I = \sum_{k=1}^{n} \phi_k \overline{\boldsymbol{\epsilon}}^k. \tag{17}$$

Thus, by comparing eq.(14) and eq.(17), we find out:

$$\mathbf{E} = \sum_{k=1}^{n} \phi_k \left[\overline{\boldsymbol{\epsilon}}^k + \mathbf{T}^I(\mathbf{C}^0) : \left(\Delta\mathbf{C}^{(k/0)} : \overline{\boldsymbol{\epsilon}}^k + \Delta\boldsymbol{\lambda}^{(k/0)}\right)\right]. \tag{18}$$

The concentrations tensors $\mathbf{A}^I$ and $\mathbf{b}^I$ for the composite formed by volume V_I, and the concentrations tensors $\boldsymbol{\alpha}^k$ and $\boldsymbol{\beta}^k$ for each phase k can be introduced [38–43] so that :

$$\begin{aligned} \overline{\boldsymbol{\epsilon}}^I &= \mathbf{A}^I : \mathbf{E} + \mathbf{b}^I, \\ \overline{\boldsymbol{\epsilon}}^k &= \boldsymbol{\alpha}^k : \overline{\boldsymbol{\epsilon}}^I + \boldsymbol{\beta}^k, \end{aligned} \tag{19}$$

with from eq.(17):

$$\begin{aligned} \sum_{k=1}^{n} \phi_k \boldsymbol{\alpha}^k &= \mathbf{I}, \\ \sum_{k=1}^{n} \phi_k \boldsymbol{\beta}^k &= \mathbf{0}, \end{aligned} \tag{20}$$

where $\mathbf{0}$ and $\mathbf{I}$ are respectively null and unit fourth order tensors. From eqs.(19), we can also write:

$$\overline{\boldsymbol{\epsilon}}^k = \mathbf{A}^k : \mathbf{E} + \mathbf{b}^k, \tag{21}$$

where:

$$\begin{aligned} \mathbf{A}^k &= \boldsymbol{\alpha}^k : \mathbf{A}^I, \\ \mathbf{b}^k &= \boldsymbol{\alpha}^k : \mathbf{b}^I + \boldsymbol{\beta}^k. \end{aligned} \tag{22}$$

Using eq. (18), it comes:

$$\begin{aligned} \mathbf{A}^I &= \left(\mathbf{I} + \mathbf{T}^I(\mathbf{C}^0) : \left(\sum_{k=1}^{n} \phi_k \Delta \mathbf{C}^{(k/0)} : \boldsymbol{\alpha}^k \right) \right)^{-1}, \\ \mathbf{b}^I &= -\mathbf{A}^I : \mathbf{T}^I(\mathbf{C}^0) : \left(\sum_{k=1}^{n} \phi_k \left[\Delta \mathbf{C}^{(k/0)} : \boldsymbol{\beta}^k + \Delta \boldsymbol{\lambda}^{(k/0)} \right] \right). \end{aligned} \tag{23}$$

2.2.2. Interfacial relations

We consider a perfect bonding at the interface $k/k+1$ (Fig. 1), denoted (k), for which essential properties at any point $\mathbf{x}$ of the interface are the continuity of displacement vector $[u]_i^{(k)}(\mathbf{x}) \equiv u_i^{k+1}(\mathbf{x}) - u_i^k(\mathbf{x}) = 0$ and the continuity of the interfacial traction vector $[\sigma]_{ij}^{(k)} N_j^{(k)} \equiv \sigma_{ij}^{k+1} N_j^{(k)} - \sigma_{ij}^k N_j^{(k)} = 0$, where $N_i^{(k)}$ is the outward unit normal on the interface (k). As detailed elsewhere [22, 24, 44], the strains and the stresses are discontinuous and their jumps can be related to the so-called interfacial operators introduced by Walpole [23] or Hill [24]. The strain jump is obtained from the compatibility relation at any point of the interface and taking the symmetric part of the displacement gradient jump:

$$[\epsilon]_{ij}^{(k)}(\mathbf{x}) \equiv \epsilon_{ij}^{k+1}(\mathbf{x}) - \epsilon_{ij}^k(\mathbf{x}) = \frac{1}{2}\left(\nu_i^{(k)} N_j^{(k)} + \nu_j^{(k)} N_i^{(k)} \right) \tag{24}$$

where $\nu_i^{(k)}$ is a vector describing the direction and the magnitude of the strain jump on the interface (k). Starting from eq.(5) in k and $k+1$ respectively:

$$\begin{aligned} \sigma_{ij}^{k+1} &= C_{ijkl}^{k+1} \epsilon_{kl}^{k+1} + \lambda_{ij}^{k+1}, \\ \sigma_{ij}^{k} &= C_{ijkl}^{k} \epsilon_{kl}^{k} + \lambda_{ij}^{k}, \end{aligned} \tag{25}$$

and using the continuity of the interfacial traction vector at the interface, we obtain:

$$C_{ijkl}^{k+1} [\epsilon]_{kl}^{(k)} N_j^{(k)} + [C]_{ijkl}^{(k)} \epsilon_{kl}^k N_j^{(k)} + [\lambda]_{ij}^{(k)} N_j^{(k)} = 0, \tag{26}$$

where $[C]_{ijkl}^{(k)} = C_{ijkl}^{k+1} - C_{ijkl}^k$ and $[\lambda]_{ij}^{(k)} = \lambda_{ij}^{k+1} - \lambda_{ij}^k$. In eq.(26), we replace $[\epsilon]_{kl}^{(k)}$ by its expression given by eq.(24). It gives:

$$h_{ik}^{k+1} \nu_k^{(k)} + [C]_{ijkl}^{(k)} \epsilon_{kl}^k N_j^{(k)} + [\lambda]_{ij}^{(k)} N_j^{(k)} = 0, \tag{27}$$

where h_{ik}^{k+1} is the Christoffel matrix associated with $k+1$ and defined by $h_{ik}^{k+1} = C_{ijkl}^{k+1} N_j^{(k)} N_l^{(k)}$. The Christophel matrix is found symmetric if the classic symmetry

for the elastic moduli are assumed (i.e. $C^{k+1}_{ijkl} = C^{k+1}_{jikl} = C^{k+1}_{ijlk}$). Following Walpole [23] and Hill [24], the interfacial operator P^{k+1}_{ijkl} only depends on the elastic moduli $\mathbf{C}^{k+1}$ of phase $k+1$ and on the unit normal $\mathbf{N}$ of the interface (k):

$$P^{k+1}_{ijkl} = \frac{1}{4}\left((h^{k+1})^{-1}_{ik}N_jN_l + (h^{k+1})^{-1}_{jk}N_iN_l + (h^{k+1})^{-1}_{il}N_jN_k + (h^{k+1})^{-1}_{jl}N_iN_k\right), \tag{28}$$

From eq.(27) (assuming the determinant of h^{k+1}_{ik} is non zero) and eq.(28), it comes directly :

$$\epsilon^{k+1}_{ij}(\mathbf{x}) = \left(I_{ijmn} + P^{k+1}_{ijkl}(C^{k}_{klmn} - C^{k+1}_{klmn})\right)\epsilon^{k}_{mn}(\mathbf{x}) + P^{k+1}_{ijkl}(\lambda^{k}_{kl} - \lambda^{k+1}_{kl}), \tag{29}$$

where $\boldsymbol{\lambda}^k$, $\mathbf{C}^k$, and, $\boldsymbol{\lambda}^{k+1}$, $\mathbf{C}^{k+1}$ are respectively uniform in phases k and $k+1$.

2.2.3. Approximation by an averaging procedure and solutions

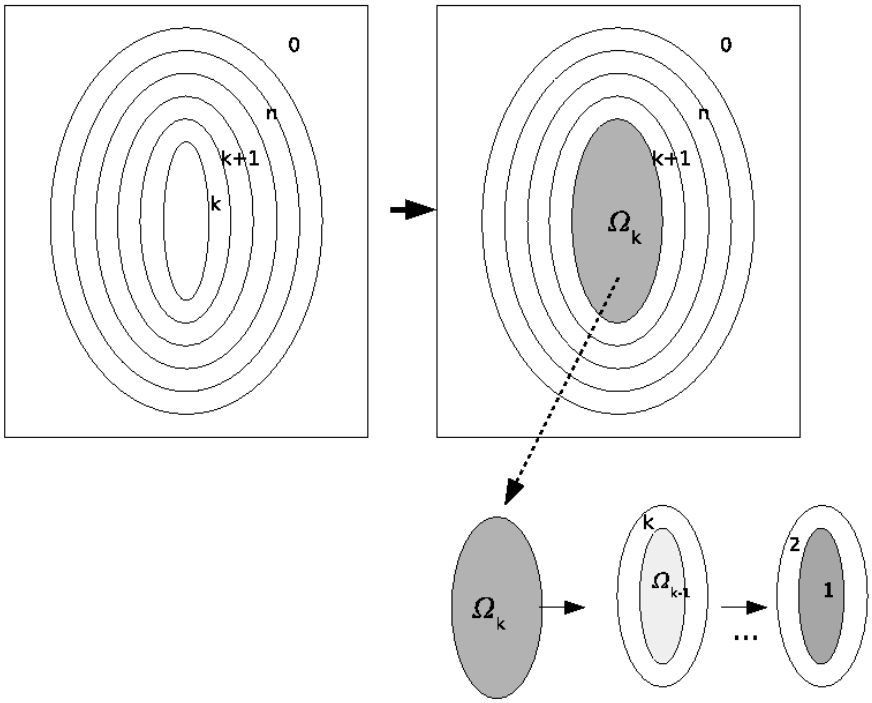

Figure 2. Multi-coated inclusion problem with eigenstrains considering n phases. Approximation using the average strain fields over $\Omega_k = V_1 \cup ... \cup V_k$, the volume of the composite formed by the phases 1 to k.

For each level (k) (Fig. 2), we denote by $\Omega_k = V_1 \cup ... \cup V_k$ the volume of the composite formed by the phases 1 to k. Then, in order to solve the problem, we adopt the following assumption which avoids complex full field calculations: $\boldsymbol{\epsilon}^k(\mathbf{x})$ is substituted by the averaged value of $\boldsymbol{\epsilon}(\mathbf{x})$ over Ω_k denoted $\overline{\boldsymbol{\epsilon}}^{\Omega_k}$. Thus, by performing the average strain over the coating of volume V_{k+1} denoted $\overline{\boldsymbol{\epsilon}}^{k+1}$, we obtain the following recurrence relation at each level (k) from eq.(29):

$$\overline{\epsilon_{ij}}^{k+1} = \left(I_{ijmn} + T^{k+1}_{ijkl}(\mathbf{C}^{k+1})(C^{\Omega_k}_{klmn} - C^{k+1}_{klmn})\right)\overline{\epsilon_{mn}}^{\Omega_k} + T^{k+1}_{ijkl}(\mathbf{C}^{k+1})(\lambda^{\Omega_k}_{kl} - \lambda^{k+1}_{kl}), \tag{30}$$

where:

$$\mathbf{T}^{k+1}(\mathbf{C}^{k+1}) = \frac{1}{V_{k+1}} \int_{V_{k+1}} \mathbf{P}^{k+1} dV, \tag{31}$$

and,

$$\overline{\boldsymbol{\epsilon}}^{\Omega_k} = \sum_{i=1}^{k} \frac{V_i}{\Omega_k} \overline{\boldsymbol{\epsilon}}^i = \frac{\sum_{i=1}^{k} \phi_i \overline{\boldsymbol{\epsilon}}^i}{\sum_{i=1}^{k} \phi_i}. \tag{32}$$

$\mathbf{C}^{\Omega_k}$ and $\boldsymbol{\lambda}^{\Omega_k}$ are respectively the elastic moduli and eigenstresses of the composite inclusion formed by volume Ω_k. It is noteworthy that $\mathbf{C}^{\Omega_k}$ and $\boldsymbol{\lambda}^{\Omega_k}$ will be naturally eliminated in the following equations by recurrence relations starting from the basic configuration described by n=2. Indeed, for $n = 2$, we have $\overline{\boldsymbol{\epsilon}}^{\Omega_1} = \overline{\boldsymbol{\epsilon}}^1$ from eq.(32), $\mathbf{C}^{\Omega_1} = \mathbf{C}^1$ and $\boldsymbol{\lambda}^{\Omega_1} = \boldsymbol{\lambda}^1$ because $\Omega_1 = V_1$ (Fig. 2). Thus, eq.(30) reduces to:

$$\overline{\epsilon_{ij}}^2 = \left(I_{ijmn} + T^2_{ijkl}(\mathbf{C}^2)(C^1_{klmn} - C^2_{klmn})\right) \overline{\epsilon_{mn}}^1 + T^2_{ijkl}(\mathbf{C}^2)(\lambda^1_{kl} - \lambda^2_{kl}). \tag{33}$$

Following [6], we can demonstrate in the general case of non homothetic ellipsoidal inclusions that the expression of $\mathbf{T}^{k+1}(\mathbf{C}^{k+1})$ takes the form of:

$$\mathbf{T}^{k+1}(\mathbf{C}^{k+1}) = \mathbf{T}^{\Omega_k}(\mathbf{C}^{k+1}) - \frac{\sum_{i=1}^{k} \phi_i}{\phi_{i+1}} \left(\mathbf{T}^{\Omega_{k+1}}(\mathbf{C}^{k+1}) - \mathbf{T}^{\Omega_k}(\mathbf{C}^{k+1})\right), \tag{34}$$

with:

$$\begin{aligned} \mathbf{T}^{\Omega_k}(\mathbf{C}^{k+1}) &= \frac{1}{\Omega_k} \int_{\Omega_k} \boldsymbol{\Gamma}(\mathbf{C}^{k+1}) dV, \\ \mathbf{T}^{\Omega_{k+1}}(\mathbf{C}^{k+1}) &= \frac{1}{\Omega_{k+1}} \int_{\Omega_{k+1}} \boldsymbol{\Gamma}(\mathbf{C}^{k+1}) dV. \end{aligned} \tag{35}$$

Furthermore, eq.(30) can be written in the following form:

$$\overline{\boldsymbol{\epsilon}}^{k+1} = \overline{\boldsymbol{\epsilon}}^{\Omega_k} - \mathbf{T}^{k+1}(\mathbf{C}^{k+1}) : \left(\Delta\mathbf{C}^{(k+1/\Omega_k)} : \overline{\boldsymbol{\epsilon}}^{\Omega_k} + \Delta\boldsymbol{\lambda}^{(k+1/\Omega_k)}\right), \tag{36}$$

where:

$$\begin{aligned} \Delta\mathbf{C}^{(k+1/\Omega_k)} &= \mathbf{C}^{k+1} - \mathbf{C}^{\Omega_k}, \\ \Delta\boldsymbol{\lambda}^{(k+1/\Omega_k)} &= \boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^{\Omega_k}. \end{aligned} \tag{37}$$

Using simple manipulations on the averaged values with the Hooke's law of each

phase from 1 to k, we find out:

$$\Delta\mathbf{C}^{(k+1/\Omega_k)} : \overline{\boldsymbol{\epsilon}}^{\Omega_k} + \Delta\boldsymbol{\lambda}^{(k+1/\Omega_k)} = \frac{\sum_{i=1}^{k} \phi_i \left(\Delta\mathbf{C}^{(k+1/i)} : \overline{\boldsymbol{\epsilon}}^{i} + \Delta\boldsymbol{\lambda}^{(k+1/i)} \right)}{\sum_{i=1}^{k} \phi_i}. \tag{38}$$

For $n = 2$, both sides of eq.(38) give $\Delta\mathbf{C}^{(2/1)} : \overline{\boldsymbol{\epsilon}}^{1} + \Delta\boldsymbol{\lambda}^{(2/1)}$ because $\overline{\boldsymbol{\epsilon}}^{\Omega_1} = \overline{\boldsymbol{\epsilon}}^{1}$, $\mathbf{C}^{\Omega_1} = \mathbf{C}^{1}$ and $\boldsymbol{\lambda}^{\Omega_1} = \boldsymbol{\lambda}^{1}$.

By using eqs.(32)(36)(38), the expressions of $\boldsymbol{\alpha}^{k+1}$ and $\boldsymbol{\beta}^{k+1}$ in eq.(19) are obtained:

$$\begin{aligned} \boldsymbol{\alpha}^{k+1} &= \frac{\sum_{i=1}^{k} \phi_i \mathbf{w}^{(k+1/i)} : \boldsymbol{\alpha}^{i}}{\sum_{i=1}^{k} \phi_i}, \\ \boldsymbol{\beta}^{k+1} &= \frac{\sum_{i=1}^{k} \phi_i \left(\mathbf{w}^{(k+1/i)} : \boldsymbol{\beta}^{i} - \mathbf{v}^{(k+1/i)} \right)}{\sum_{i=1}^{k} \phi_i}, \end{aligned} \tag{39}$$

where:

$$\begin{aligned} \mathbf{w}^{(k+1/i)} &= \mathbf{I} - \mathbf{T}^{k+1}(\mathbf{C}^{k+1}) : \Delta\mathbf{C}^{(k+1/i)}, \\ \mathbf{v}^{(k+1/i)} &= \mathbf{T}^{k+1}(\mathbf{C}^{k+1}) : \Delta\boldsymbol{\lambda}^{(k+1/i)}. \end{aligned} \tag{40}$$

Then, by recurrence, we can transform eqs.(39) into the following equations:

$$\begin{aligned} \boldsymbol{\alpha}^{k+1} &= \mathbf{X}^{k+1} : \boldsymbol{\alpha}^{1}, \\ \boldsymbol{\beta}^{k+1} &= \mathbf{X}^{k+1} : \boldsymbol{\beta}^{1} + \mathbf{Y}^{k+1}, \end{aligned} \tag{41}$$

with the recurrence relations for $\mathbf{X}^{k+1}$ and $\mathbf{Y}^{k+1}$:

$$\begin{aligned} \mathbf{X}^{k+1} &= \frac{\sum_{i=1}^{k} \phi_i \mathbf{w}^{(k+1/i)} : \mathbf{X}^{i}}{\sum_{i=1}^{k} \phi_i}, \\ \mathbf{Y}^{k+1} &= \frac{\sum_{i=1}^{k} \phi_i \left(\mathbf{w}^{(k+1/i)} : \mathbf{Y}^{i} - \mathbf{v}^{(k+1/i)} \right)}{\sum_{i=1}^{k} \phi_i}. \end{aligned} \tag{42}$$

Thus, it is sufficient to derive $\boldsymbol{\alpha}^{1}$ and $\boldsymbol{\beta}^{1}$ to completely solve the problem. This is

done by applying eqs.(20) such that:

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \left(\sum_{k=1}^{n} \phi_k \mathbf{X}^k \right)^{-1}, \\ \boldsymbol{\beta}^1 &= -\boldsymbol{\alpha}^1 : \sum_{k=1}^{n} \phi_k \mathbf{Y}^k. \end{aligned} \tag{43}$$

Let us note that the purely linear elastic solution for the n-phase composite inclusion described by Fig. 1 recently obtained by Lipinski et al. [21] is retrieved in eq.(43) by setting $\boldsymbol{\lambda}^k = \mathbf{0}$ and $\boldsymbol{\beta}^k = \mathbf{0}$ for all k in eq.(36) and eq.(38). This can be considered as a particular case of the general formalism developed in the present paper.

2.3. *"$(n+1)$-phase" self-consistent model*

In this section, we adopt the same methodology as the Generalized Self-Consistent Scheme (GSCS) introduced by Christensen and Lo [2] (the so-called "3-phase" model) for composite spheres (or cylinders), and, extended by Hervé and Zaoui [3, 45] to a "$(n+1)$-phase" model with a self-consistent condition. A Homogeneous Equivalent Medium (HEM) is introduced in Fig. 3 in addition to the n phases including the matrix phase denoted (0). In the following, the effective properties associated with the HEM are denoted "eff". We consider an imposed strain $\mathbf{E}$ (Fig. 3) such that $\mathbf{u^d} = \mathbf{E} \cdot \mathbf{x}$ on the boundary of the Representative Volume Element (RVE) of total volume V_T. As highlighted by Hervé and Zaoui [46] and by Zaoui [10], the Christensen and Lo's self-consistent energy condition [2] coincides with the following average strain condition:

$$\mathbf{E} = \overline{\boldsymbol{\epsilon}}^{V_T} = \frac{1}{V_T} \int_{V_T} \boldsymbol{\epsilon}(\mathbf{x}) dV = \sum_{q=0}^{n-1} f_q \boldsymbol{\epsilon}^q = f_0 \boldsymbol{\epsilon}^0 + \sum_{q=1}^{n-1} f_q \boldsymbol{\epsilon}^q \tag{44}$$

where $f_0 = \dfrac{V_0}{V_T}$ for the matrix phase (0), and, $f_q = \dfrac{V_q}{V_T}$ for the other phases (q) from 1 to $n-1$ such that:

$$\sum_{q=0}^{n-1} f_q = 1. \tag{45}$$

Thus, the homogenized behavior (or effective behavior associated with the HEM) writes:

$$\boldsymbol{\Sigma} = \mathbf{C}^{eff} : \mathbf{E} + \boldsymbol{\lambda}^{eff}, \tag{46}$$

where $\mathbf{C}^{eff}$ and $\boldsymbol{\lambda}^{eff}$ are unknown effective elastic moduli and eigenstresses. In eq.(46), $\boldsymbol{\Sigma}$ is the volume stress average in the RVE:

$$\boldsymbol{\Sigma} = f_0 \boldsymbol{\sigma}^0 + \sum_{q=1}^{n-1} f_q \boldsymbol{\sigma}^q. \tag{47}$$

The respective constitutive behaviors for the matrix (0) and the other phases (q) are:

$$\begin{aligned}\boldsymbol{\sigma}^0 &= \mathbf{C}^0 : \boldsymbol{\epsilon}^0 + \boldsymbol{\lambda}^0,\\ \boldsymbol{\sigma}^q &= \mathbf{C}^q : \boldsymbol{\epsilon}^q + \boldsymbol{\lambda}^q.\end{aligned} \tag{48}$$

The strain concentration equation for each phase (q) (including the matrix phase) writes:

$$\boldsymbol{\epsilon}^q = \mathbf{A}^q : \mathbf{E} + \mathbf{b}^q, \tag{49}$$

with:

$$\begin{aligned}\mathbf{A}^q &= \boldsymbol{\alpha}^q : \mathbf{A},\\ \mathbf{b}^q &= \boldsymbol{\alpha}^q : \mathbf{b} + \boldsymbol{\beta}^q,\end{aligned} \tag{50}$$

where the concentration tensors $\boldsymbol{\alpha}^q$, $\boldsymbol{\beta}^q$ were introduced in section 2.2 through eqs.(41) (42) (43), and, $\mathbf{A}$, $\mathbf{b}$ are adapted from eqs.(23) for the "$(n+1)$-phase" model:

$$\begin{aligned}\mathbf{A} &= \left(\mathbf{I} + \mathbf{T}^I(\mathbf{C}^{eff}) : \left(\sum_{q=0}^{n-1} f_q \Delta\mathbf{C}^{(q/eff)} : \boldsymbol{\alpha}^q\right)\right)^{-1},\\ \mathbf{b} &= -\mathbf{A} : \mathbf{T}^I(\mathbf{C}^{eff}) : \left(\sum_{q=0}^{n-1} f_q \left[\Delta\mathbf{C}^{(q/eff)} : \boldsymbol{\beta}^q + \Delta\boldsymbol{\lambda}^{(q/eff)}\right]\right).\end{aligned} \tag{51}$$

Applying the strain average condition (eq.(44)) yields:

$$\begin{aligned}\sum_{q=0}^{n-1} f_q \mathbf{A}^q &= \overline{\mathbf{A}}^{V_T} = \mathbf{I},\\ \sum_{q=0}^{n-1} f_q \mathbf{b}^q &= \overline{\mathbf{b}}^{V_T} = \mathbf{0}.\end{aligned} \tag{52}$$

The stress concentration equations yield:

$$\boldsymbol{\sigma}^q = \mathbf{B}^q : \boldsymbol{\Sigma} + \mathbf{d}^q, \tag{53}$$

where from eqs.(48) and (46):

$$\begin{aligned}\mathbf{B}^q &= \mathbf{C}^q : \mathbf{A}^q : (\mathbf{C}^{eff})^{-1},\\ \mathbf{d}^q &= -\mathbf{C}^q : \mathbf{A}^q : (\mathbf{C}^{eff})^{-1} : \boldsymbol{\lambda}^{eff} + \mathbf{C}^q : \mathbf{b}^q + \boldsymbol{\lambda}^q.\end{aligned} \tag{54}$$

$\mathbf{d}^q$ denote the residual stresses in each phase (q) (i.e., for $\boldsymbol{\Sigma} = \mathbf{0}$). From the stress average condition (eq.(47)), we have:

$$\begin{aligned} \sum_{q=0}^{n-1} f_q \mathbf{B}^q &= \overline{\mathbf{B}}^{V_T} = \mathbf{I}, \\ \sum_{q=0}^{n-1} f_q \mathbf{d}^q &= \overline{\mathbf{d}}^{V_T} = \mathbf{0}. \end{aligned} \tag{55}$$

The last equation ($\overline{\mathbf{d}}^{V_T} = \mathbf{0}$) means that the residual stresses are self-equilibrated over the RVE. Thus, using the previous concentration equations in eq.(47), we find out the expressions for $\mathbf{C}^{eff}$ and $\boldsymbol{\lambda}^{eff}$ as:

$$\begin{aligned} \mathbf{C}^{eff} &= \mathbf{C}^0 + \sum_{q=1}^{n-1} f_q \Delta\mathbf{C}^{(q/0)} : \mathbf{A}^q, \\ \boldsymbol{\lambda}^{eff} &= \boldsymbol{\lambda}^0 + \sum_{q=1}^{n-1} f_q \left(\Delta\mathbf{C}^{(q/0)} : \mathbf{b}^q + \Delta\boldsymbol{\lambda}^{(q/0)} \right), \end{aligned} \tag{56}$$

After a few manipulations and algebra using eq.(56), eq.(50) and eq.(51), we can also write $\mathbf{C}^{eff}$ and $\boldsymbol{\lambda}^{eff}$ as:

$$\begin{aligned} \mathbf{C}^{eff} &= \sum_{q=0}^{n-1} f_q \mathbf{C}^q : \boldsymbol{\alpha}^q : \mathbf{A}, \\ \boldsymbol{\lambda}^{eff} &= \sum_{q=0}^{n-1} f_q \left(\boldsymbol{\lambda}^q + \mathbf{C}^q : \boldsymbol{\beta}^q \right). \end{aligned} \tag{57}$$

It is worth noticing that another way to obtain the expression of $\boldsymbol{\lambda}^{eff}$ is to use directly the Levin's formula [26, 27] applied to the composite depicted in Fig. 3. From the knowledge of $\mathbf{A}^q$ (eq.(50)), we deduce from the Levin's formula:

$$\boldsymbol{\lambda}^{eff} = \sum_{q=0}^{n-1} f_q \mathbf{A}^{qT} : \boldsymbol{\lambda}^q, \tag{58}$$

where $\mathbf{A}^{qT}$ is the transpose of $\mathbf{A}^q$ such that $A^{qT}_{ijkl} = A^q_{klij}$.

To conclude this part, two significant results of the present model can be reached through the last equations: the local mechanical fields like the residual stresses (respectively residual strains), and, the homogenized elastic properties and the effective eigenstresses (respectively the effective eigenstrains). The Levin's formula can only achieve the latter because this does not need a relevant description of the microstructure.

3. Application to isotropic thermo-elastic materials with homothetic spherical inclusions

In the case of layered homothetic spherical inclusion-reinforced n-phase composite with isotropic properties in each phase, Hervé and Zaoui [3] and Hervé [25] re-

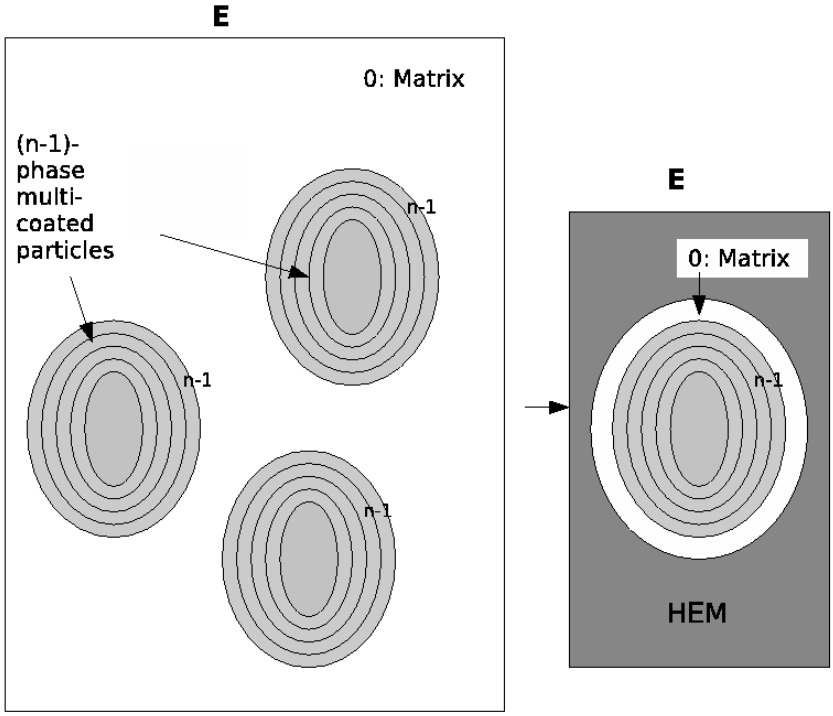

Figure 3. Schematic principle of the "$(n+1)$-phase" model (self-consistent condition).

spectively found the exact expressions for the effective elastic properties and the effective thermal expansion. In the following, we adapt the equations of our model to this peculiar case in order to assess the good quality of the estimation of effective properties resulting from the "$(n+1)$-phase" self-consistent model described previously. In particular, "4-phase" and "3-phase" models are respectively derived in the cases of 3-phase and 2-phase composites.

3.1. *"4-phase" model*

As a first application, let us apply the concept of interphase as described here in a "4-phase" model. Thus, we consider a heterogeneous elastic material with eigenstrains globally isotropic with isotropic phases and the peculiar case of a layered homothetic spherical inclusion-reinforced material is examined. Thus, the 3-phase composite material is formed by a matrix phase (denoted 0), inclusions (reinforcements, denoted 1) and interphases between inclusions and matrix (denoted 2). Thus, the RVE of this material is reported in Fig. 4. For the description of fourth order isotropic tensors, we use the orthogonal projection tensors [44, 47] denoted $\mathbf{J}$ and $\mathbf{K}$ such that the unit tensor $\mathbf{I}$ decomposes as:

$$\mathbf{I} = \mathbf{J} + \mathbf{K}, \tag{59}$$

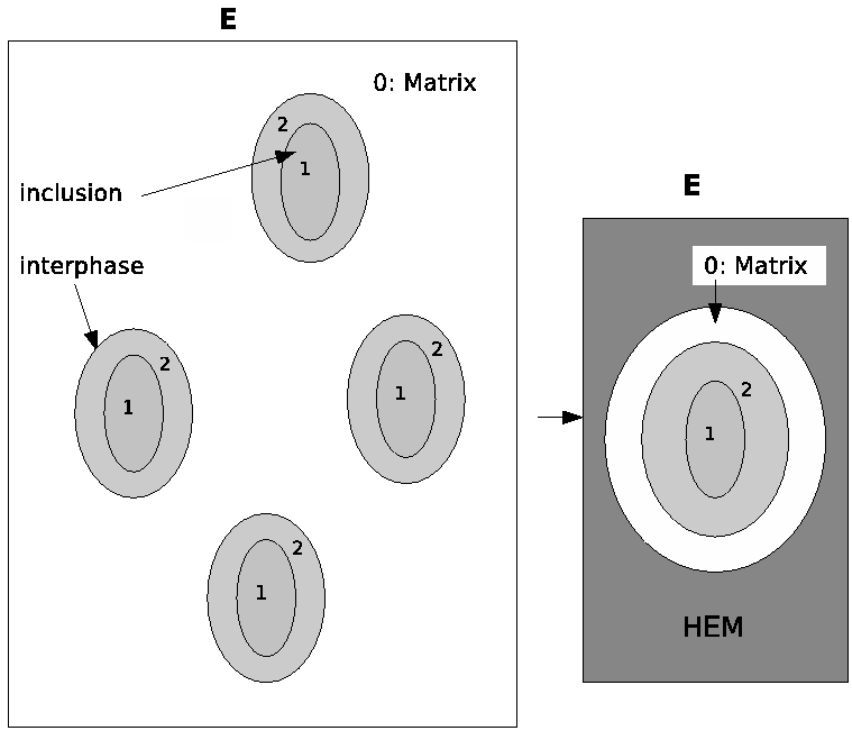

Figure 4. Schematic principle of the "4-phase" scheme (self-consistent condition). The inclusion phase (phase (1)) is coated by the interphase (phase (2)). Phases (1) and (2) are coated inclusions embedded in the matrix phase (0). In this decription, we estimate the effective behavior of the Homogenized Equivalent Medium (denoted HEM).

with:

$$
\begin{aligned}
I_{ijkl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \\
J_{ijkl} &= \frac{1}{3}\delta_{ij}\delta_{kl}, \\
K_{ijkl} &= \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right),
\end{aligned}
\tag{60}
$$

where δ_{ij} is the Kronecker operator. Furthermore, $\mathbf{J}$ and $\mathbf{K}$ have the following properties:

$$
\begin{aligned}
&\mathbf{J} : \mathbf{J} = \mathbf{J}, \\
&\mathbf{K} : \mathbf{K} = \mathbf{K}, \\
&\mathbf{J} : \mathbf{K} = \mathbf{K} : \mathbf{J} = \mathbf{0}, \\
&\mathbf{J} : \mathbf{I_2} = \mathbf{I_2} : \mathbf{J} = \mathbf{I_2}, \\
&\mathbf{K} : \mathbf{I_2} = \mathbf{I_2} : \mathbf{K} = \mathbf{0},
\end{aligned}
\tag{61}
$$

where $\mathbf{I_2}$ is the second order unit tensor, i.e. $(I_2)_{ij} = \delta_{ij}$.

For each phase $q = 0, 1, 2$ of the composite, the elastic moduli are supposed

isotropic:

$$\mathbf{C}^q = 3k_q\mathbf{J} + 2\mu_q\mathbf{K}, \tag{62}$$

where k_q and μ_q denote the bulk modulus and the shear modulus, respectively. The thermal stresses $\boldsymbol{\lambda}^q$ are also supposed isotropic:

$$\boldsymbol{\lambda}^q = -3k_q\alpha_q\mathbf{I_2}\Theta \tag{63}$$

where α_q is the thermal expansion of phase (q) and Θ is the temperature rise relative to a reference temperature which is arbitrarily chosen as zero.

In the isotropic case, the effective elastic moduli $\mathbf{C}^{eff}$ and the effective thermal stresses $\boldsymbol{\lambda}^{eff}$ are of the form:

$$\begin{aligned} \mathbf{C}^{eff} &= 3k_{eff}\mathbf{J} + 2\mu_{eff}\mathbf{K}, \\ \boldsymbol{\lambda}^{eff} &= -3k_{eff}\alpha_{eff}\mathbf{I_2}\Theta. \end{aligned} \tag{64}$$

In Eq.(64), k_{eff} and μ_{eff} are respectively the effective elastic bulk and shear moduli, and, α_{eff} is the effective thermal expansion coefficient.

In the particular case of concentric homothetic ellipsoidal inclusions, eq.(34) reduces to (see e.g. [6]):

$$\mathbf{T}^{q+1}(\mathbf{C}^{q+1}) = \mathbf{T}^{I}(\mathbf{C}^{q+1}) = \mathbf{T}^{\Omega_1}(\mathbf{C}^{q+1}) = ... = \mathbf{T}^{\Omega_q}(\mathbf{C}^{q+1}) = \mathbf{T}^{\Omega_{q+1}}(\mathbf{C}^{q+1}). \tag{65}$$

Furthermore, for spherical inclusions, $\mathbf{T}^{I}(\mathbf{C}^q)$ reads [6]:

$$\mathbf{T}^{I}(\mathbf{C}^q) = \frac{\mathbf{J}}{3k_q + 4\mu_q} + \frac{3(k_q + 2\mu_q)\mathbf{K}}{5\mu_q(3k_q + 4\mu_q)} \tag{66}$$

Let us apply the general equations obtained in the section for the homogenized behavior of the composite described in Fig. 4. In this case, by applying eq.(57), $\mathbf{C}^{eff}$ reads:

$$\mathbf{C}^{eff} = \left(f_0\mathbf{C}^0 : \boldsymbol{\alpha}^0 + f_1\mathbf{C}^1 : \boldsymbol{\alpha}^1 + f_2\mathbf{C}^2 : \boldsymbol{\alpha}^2\right) : \mathbf{A}, \tag{67}$$

and, $\boldsymbol{\lambda}^{eff}$ reads:

$$\boldsymbol{\lambda}^{eff} = f_0\left(\boldsymbol{\lambda}^0 + \mathbf{C}^0 : \boldsymbol{\beta}^0\right) + f_1\left(\boldsymbol{\lambda}^1 + \mathbf{C}^1 : \boldsymbol{\beta}^1\right) + f_2\left(\boldsymbol{\lambda}^2 + \mathbf{C}^2 : \boldsymbol{\beta}^2\right). \tag{68}$$

where the concentration tensors $\mathbf{A}$, $\boldsymbol{\alpha}^0$, $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\beta}^0$, $\boldsymbol{\beta}^1$, $\boldsymbol{\beta}^2$ can write in the isotropic symmetry using eqs.(39),(40),(41),(42),(43),(51):

$$\begin{aligned} \mathbf{A} &= M\mathbf{J} + D\mathbf{K}, \\ \boldsymbol{\alpha}^0 &= m^0\mathbf{J} + d^0\mathbf{K}, \\ \boldsymbol{\alpha}^1 &= m^1\mathbf{J} + d^1\mathbf{K}, \\ \boldsymbol{\alpha}^2 &= m^2\mathbf{J} + d^2\mathbf{K}, \\ \boldsymbol{\beta}^0 &= n^0\mathbf{I_2}\Theta, \\ \boldsymbol{\beta}^1 &= n^1\mathbf{I_2}\Theta, \\ \boldsymbol{\beta}^2 &= n^2\mathbf{I_2}\Theta. \end{aligned} \tag{69}$$

In the last equation, the expressions of M, D, m^0, d^0, m^1, d^1, m^2, d^2, n^0, n^1, n^2 are determined in the Appendix. Let us note that the expressions of n^0, n^1, n^2 represent new contributions due to thermal effects in comparison with the work of Lipinski et al. [21]. M and D depend on k_{eff} and μ_{eff}, then, from eq.(67), k_{eff} and μ_{eff} are the solutions of the following system of non linear equations:

$$\begin{aligned} k_{eff} - k_0 - f_1 (k_1 - k_0) m^1 M - f_2 (k_2 - k_0) m^2 M = 0, \\ \mu_{eff} - \mu_0 - f_1 (\mu_1 - \mu_0) d^1 D - f_2 (\mu_2 - \mu_0) d^2 D = 0. \end{aligned} \tag{70}$$

To solve this system of equations, a standard Levenberg-Marquardt procedure [48] is chosen with a starting guess at the solutions corresponding to the volume averages of elastic bulk and shear moduli over the RVE (i.e., a Voigt approximation).

To show the importance of the interphase elastic properties on the effective elastic moduli of the composite, we used same materials parameters as the ones used by Hervé and Zaoui [3] (their phase (3) corresponds to our phase (0)), i.e. identical Poisson's ratios for all the phases: $\nu_1 = \nu_2 = \nu_0 = 0.3$, $\mu_1/\mu_0 = 6$ and $f_1 + f_2 = 0.2$. The results concerning the normalized effective shear moduli μ_{eff}/μ_0 are reported in Fig. 5 and Fig. 6. These results are compared with the ones obtained by Hervé and Zaoui [3] from their equations numbered (47) and (51). Fig. 5 shows the evolution of μ_{eff}/μ_0 as a function of the interphase volume fraction f_2 for different mechanical contrasts between the interphase and the matrix phase characterized by the ratio $\beta = \mu_2/\mu_0$. When f_2 tends to 0.2, it is noteworthy that no influence of interphase occurs and the "3-phase" model's solution [2] is retrieved for all values of β, or, when $\beta = 6$ (i.e. $\mu_1 = \mu_2$ for all values of f_2). In Fig. 6, the interphase volume fraction f_2 is fixed to 0.02 and the evolution of μ_{eff}/μ_0 is plotted as a function of $\beta = \mu_2/\mu_0$. Two different regimes are observed: a first strong increase of the effective shear moduli when the values of β are lower than 3 followed by a saturation for the values of β larger than 5. As already noticed by Lipinski et al. [21] for the elastic regime only, the present approach give same results as the exact solution of Hervé and Zaoui [3].

Once the effective bulk modulus k_{eff} is obtained, the effective thermal expansion coefficient α_{eff} can be computed using eq.(64) and eq.(68) as:

$$\alpha_{eff} = \frac{1}{k_{eff}} \left(-f_1 n^1 k_1 - f_2 n^2 k_2 - f_0 n^0 k_0 + f_1 k_1 \alpha_1 + f_2 k_2 \alpha_2 + f_0 k_0 \alpha_0 \right). \tag{71}$$

The obtained effective thermal expansion coefficient α_{eff} is reported in Fig. 7 in the case of a composite material made of coated inclusions as described in Fig. 4. Here, the volume fraction of the interphase f_2 is fixed to 0.01 and the one of the inclusions f_1 is fixed to 0.1. The strong influence of both interphase properties (α_2 and k_2) on α_{eff}/α_1 is here demonstrated. The results are found to be coherent with Hervé's result [25] reported in Fig. 8 using the equation numbered (69) from her paper. Furthermore, she used the effective elastic properties obtained by [3] (see Fig. 5 and Fig. 6) and Levin's formula [26, 27] to derive α_{eff}.

In order to double check the results reported in Fig. 7, we also applied the Levin's formula [26, 27] according to eq.(58) applied to the composite described in Fig. 4. In this case (isotropic configuration), α_{eff} can be derived by the following formula:

$$\alpha_{eff} = \frac{M}{k_{eff}} \left(f_1 m^1 k_1 \alpha_1 + f_2 m^2 k_2 \alpha_2 + f_0 m^0 k_0 \alpha_0 \right), \tag{72}$$

where M, m^1, m^2, m^0 are given by eq.(69) and are detailed in the appendix. By

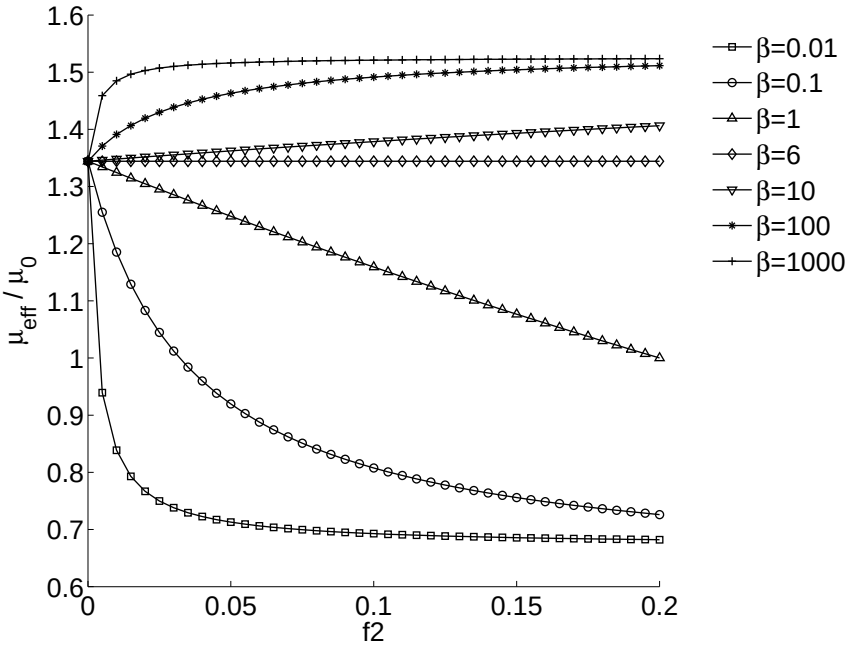

Figure 5. Estimation from the present model of normalized effective shear modulus μ_{eff}/μ_0 (*lines*) and comparison with the exact solutions given by Hervé and Zaoui [3] (*points*) for a composite material constituted of coated inclusions, versus the volume fraction f_2 of the interphase (phase 2) (phase 0 denotes the matrix and phase 1 denotes the inclusions). The results are plotted for different values of $\beta = \mu_2/\mu_0$ with $\nu_1 = \nu_2 = \nu_0 = 0.3$ (Poisson's ratios), $\mu_1/\mu_0 = 6$ and $f_1 + f_2 = 0.2$.

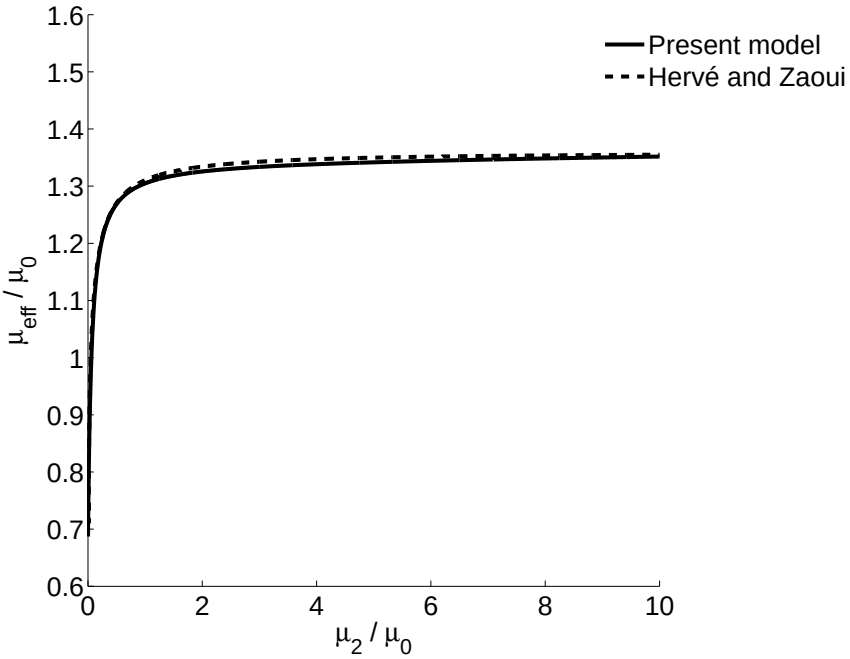

Figure 6. Estimation from the present model of normalized effective shear modulus μ_{eff}/μ_0 (*lines*) and comparison with the exact solutions given by Hervé and Zaoui [3](*dashed lines*) for a composite material constituted of coated inclusions, versus the normalized shear modulus μ_2/μ_0 of the interphase (phase 2) (phase 0 denotes the matrix and phase 1 denotes the inclusions). The results are plotted for $\nu_1 = \nu_2 = \nu_0 = 0.3$, $\mu_1/\mu_0 = 6$, $f_1 = 0.18$ and $f_2 = 0.02$.

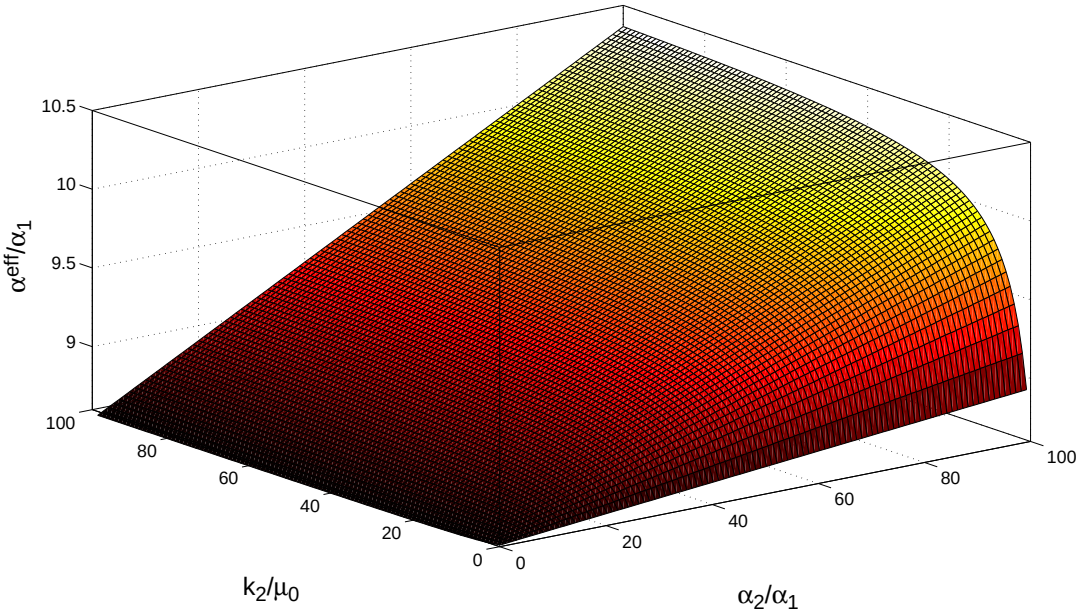

Figure 7. Normalized effective thermal expansion coefficient α_{eff}/α_1 of a composite material made of coated inclusions as described in Fig. 4: result from the present model (eq.(57)) with $f_1 = 0.1$, $f_2 = 0.01$. The elastic properties of the different phases are characterized by: $k_1/\mu_0 = 15$, $\mu_1/\mu_0 = 7$, $k_0/\mu_0 = 2$, $\mu_2/\mu_0 = 4$ and $\alpha_0/\alpha_1 = 10$.

applying this last formula, we find the same numerical results as previously (Fig. 9).

3.2. *"3-phase" model*

In the case of a "3-phase" model (Fig. 10), and, using the same conventions and notations as in section 3.1, k_{eff}, μ_{eff}, and, α_{eff} are obtained from eqs.(70) (71) or (72) by setting $f_2 = 0$ (no interphase). We set $f_1 = f$ and $f_0 = 1 - f$ for the 2-phase composite ((0) being the matrix phase and (1) the inclusions). Then, both eqs.(70) and (71) of the present modeling reduce to:

$$\begin{aligned} k_{eff} - k_0 - f\,(k_1 - k_0)\,m^1 M = 0, \\ \mu_{eff} - \mu_0 - f\,(\mu_1 - \mu_0)\,d^1 D = 0, \end{aligned} \tag{73}$$

with $f_2 = 0$, $f_1 = f$, and, $f_0 = 1 - f$ in m^1, M, d^1 and D given in the Appendix, and,

$$\alpha_{eff} = \frac{1}{k_{eff}} \left(-f n^1 k_1 - (1-f) n^0 k_0 + f k_1 \alpha_1 + (1-f) k_0 \alpha_0 \right), \tag{74}$$

with $f_2 = 0$, $f_1 = f$, and, $f_0 = 1 - f$ in n^1 and n^0 given in the Appendix. In the case of thermo-elastic heterogeneous materials, Stolz [32] found rigorous bounds for different morphological assemblages starting from the free energy and developing a variational procedure. In the case of the "3-phase" model [2], the effective bulk modulus is exactly determined (i.e., the bounds give the same value) and corresponds to the analytical solution given by the Composite Sphere Assemblage

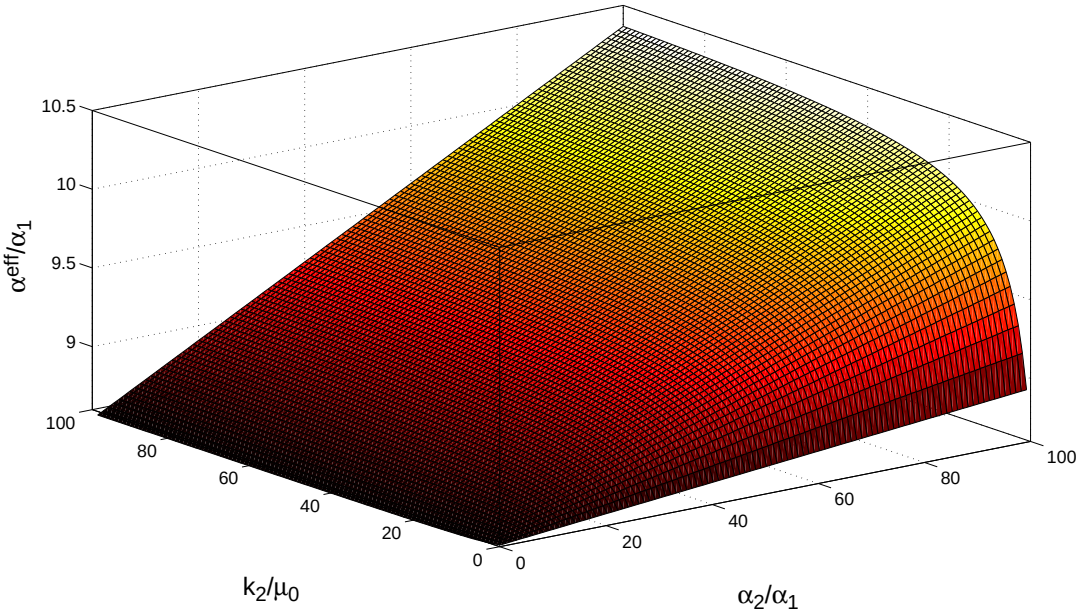

Figure 8. Normalized effective thermal expansion coefficient α_{eff}/α_1 of a composite material made of coated inclusions as described in Fig. 4: result from [25] using the effective elastic properties obtained by [3] and Levin's formula [26, 27] with $f_1 = 0.1$, $f_2 = 0.01$. The elastic properties of the different phases are characterized by: $k_1/\mu_0 = 15$, $\mu_1/\mu_0 = 7$, $k_0/\mu_0 = 2$, $\mu_2/\mu_0 = 4$ and $\alpha_0/\alpha_1 = 10$.

model [1]:

$$k_{eff} = k_0 + \frac{f\,(k_1 - k_0)\,(3k_0 + 4\mu_0)}{3k_0 + 4\mu_0 + 3\,(1-f)\,(k_1 - k_0)}, \tag{75}$$

and, α_{eff} reads using the Levin's formula:

$$\alpha_{eff} = \overline{\alpha}^{V_T} + \frac{1/k_{eff} - \overline{1/k}^{V_T}}{1/k_0 - 1/k_1}\,(\alpha_0 - \alpha_1)\,, \tag{76}$$

where:

$$\begin{aligned} \overline{\alpha}^{V_T} &= f\alpha_1 + (1-f)\alpha_0, \\ \overline{1/k}^{V_T} &= f/k_1 + (1-f)/k_0. \end{aligned} \tag{77}$$

According to Fig. 11, the results of the present "3-phase" model in terms of effective thermal expansion given by eqs.(73) and (74) are found in excellent agreement with the ones given by eqs.(75) and (76) [32] for 2-phase materials with different volume fractions f of inclusions and various ratios $\gamma = \mu_1/\mu_0$ (with $\nu_1 = \nu_0 = 0.3$ and $\alpha_0/\alpha_1 = 50$).

4. Concluding remarks

In this paper, we investigated the effective behavior of multi-coated inclusion-reinforced composites containing interphases with a linear elastic behavior with eigenstrains. These eigenstrains may be encountered in many physical situations of

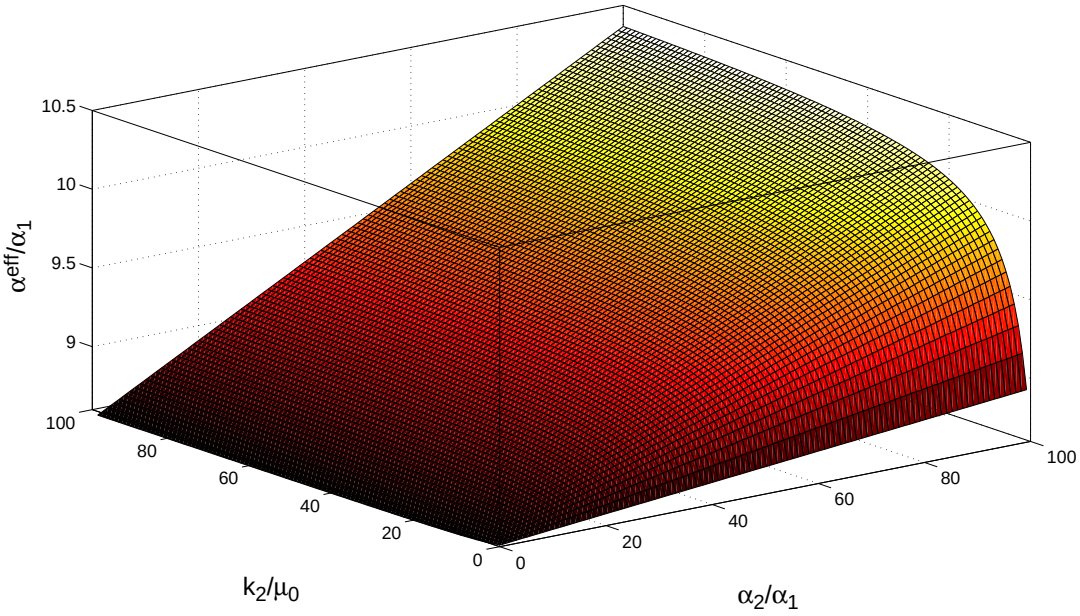

Figure 9. Normalized effective thermal expansion coefficient α_{eff}/α_1 of a composite material made of coated inclusions as described in Fig. 4: result from the present model using Levin's formula (eq.(58)) with $f_1 = 0.1$, $f_2 = 0.01$. The elastic properties of the different phases are characterized by: $k_1/\mu_0 = 15$, $\mu_1/\mu_0 = 7$, $k_0/\mu_0 = 2$, $\mu_2/\mu_0 = 4$ and $\alpha_0/\alpha_1 = 10$.

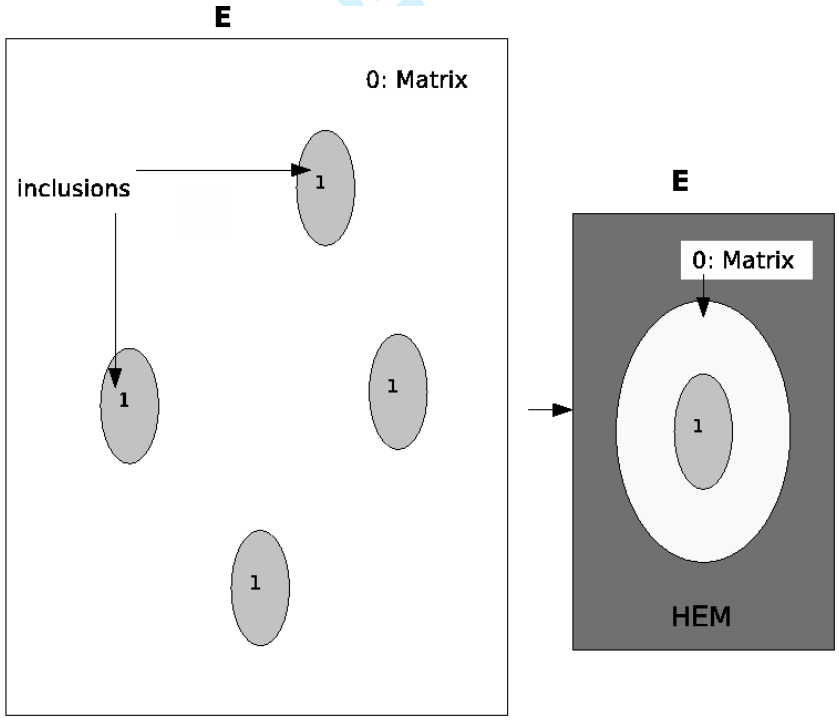

Figure 10. Schematic principle of the "3-phase" scheme (self-consistent condition). The inclusions (phase (1)) are embedded in the matrix phase (0). In this decription, we estimate the effective behavior of the Homogenized Equivalent Medium (denoted HEM).

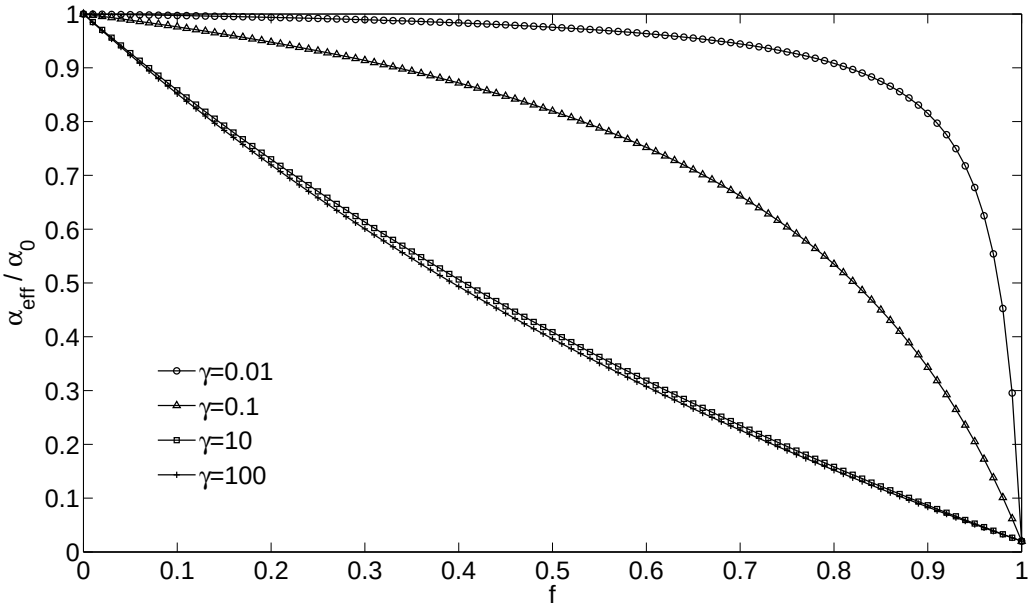

Figure 11. Estimation from the present model (eqs.(73) and (74)) (*lines*) of the normalized effective thermal expansion coefficient α_{eff}/α_0 of a composite material made of spherical inclusions (phase 1) in a matrix phase (phase 0) (see Fig. 10) as a function of the volume fraction f of inclusions, and, comparison with the exact solution given by the Levin's formula using the Composite Sphere Assemblage model for k_{eff} [1, 32] (eqs.(75) and (76)) (*points*). The results are plotted for different values of $\gamma = \mu_1/\mu_0$ with $\nu_1 = \nu_0 = 0.3$ (Poisson's ratios) and $\alpha_0/\alpha_1 = 50$.

importance in functional materials. By assuming perfectly bonded interfaces, the effective elastic properties and the effective eigenstrains of multi-coated inclusion-reinforced composites are retrieved with a "(n+1)-phase" self-consistent procedure. Even though the present micromechanical model is not exact due to averaging procedures introduced in section 2.2, this one can be applied to any anisotropic behaviors and reinforced composites with non-homothetic multi-coated ellipsoidal inclusions. Here, we illustrated the efficiency of the model to study the influences of the interphase thermal expansion and elastic bulk modulus on the effective thermal expansion with our "4-phase" model. Furthermore, local and overall thermo-elastic behaviors are assumed isotropic. In this case, the results of the present formulation match the exact results reported by Hervé [25] for the "4-phase" model, and, the ones reported by Stolz [32, 33] for the "3-phase" model. Such framework is scheduled to be extended to the case of composite elastic materials with eigenstrains and with imperfect interfaces [49, 50] and to functional nanocomposites [51, 52] involving ellipsoidal nano-inhomogeneities.

Appendix: Expressions of M, D, m^0, d^0, m^1, d^1, m^2, d^2, n^0, n^1, n^2 used in section 3

In this appendix, we give the details for the complete expressions of M, D, m^0, d^0, m^1, d^1, m^2, d^2, n^0, n^1, n^2 present in eqs.(69) to derive the effective elastic moduli k_{eff} and μ_{eff} (eqs.(70)) and the effective thermal expansion coefficient α_{eff} (eq.(71)).

By using simple algebra using eq.(51) and the properties of the orthogonal projection tensors [44, 47] denoted $\mathbf{J}$ and $\mathbf{K}$ introduced in section 3 (see also the developments in [21]), the following expressions of M and D (eq.(69)) are obtained:

$$
\begin{aligned}
M &= \left(1+\frac{3f_1(k_1-k_{eff})}{3k_{eff}+4\mu_{eff}}m^1+\frac{3f_2(k_2-k_{eff})}{3k_{eff}+4\mu_{eff}}m^2+\frac{3f_0(k_0-k_{eff})}{3k_{eff}+4\mu_{eff}}m^0\right)^{-1}\\
D &= \left(1+\frac{6f_1(\mu_1-\mu_{eff})(k_{eff}+2\mu_{eff})}{5\mu_{eff}(3k_{eff}+4\mu_{eff})}d^1+\right.\\
&\left.\frac{6f_2(\mu_2-\mu_{eff})(k_{eff}+2\mu_{eff})}{5\mu_{eff}(3k_{eff}+4\mu_{eff})}d^2+\frac{6f_0(\mu_0-\mu_{eff})(k_{eff}+2\mu_{eff})}{5\mu_{eff}(3k_{eff}+4\mu_{eff})}d^0\right)^{-1}
\end{aligned}
\tag{1}
$$

where m^1, d^1, m^2, d^2, m^0, d^0 are involved in the concentration tensors $\boldsymbol{\alpha}^1$, $\boldsymbol{\alpha}^2$, $\boldsymbol{\alpha}^0$, respectively (see eqs.(69)). These ones cas be deduced from eqs.(39),(40),(41),(42),(43) as:

$$
\begin{aligned}
m^1 &= \left(f_1+f_2m^{21}+\frac{f_0}{f_1+f_2}\left(f_1m^{01}+f_2m^{02}m^{21}\right)\right)^{-1},\\
d^1 &= \left(f_1+f_2d^{21}+\frac{f_0}{f_1+f_2}\left(f_1d^{01}+f_2d^{02}d^{21}\right)\right)^{-1},\\
m^2 &= \left(f_1(m^{21})^{-1}+f_2+\frac{f_0}{f_1+f_2}\left(f_1m^{01}(m^{21})^{-1}+f_2m^{02}\right)\right)^{-1},\\
d^2 &= \left(f_1(d^{21})^{-1}+f_2+\frac{f_0}{f_1+f_2}\left(f_1d^{01}(d^{21})^{-1}+f_2d^{02}\right)\right)^{-1},\\
m^0 &= \left[f_0+f_1\left(\frac{f_1}{f_1+f_2}m^{01}+\frac{f_2}{f_1+f_2}m^{02}m^{21}\right)^{-1}+f_2\left(\frac{f_1}{f_1+f_2}m^{01}(m^{21})^{-1}+\frac{f_2}{f_1+f_2}m^{02}\right)^{-1}\right]^{-1},\\
d^0 &= \left[f_0+f_1\left(\frac{f_1}{f_1+f_2}d^{01}+\frac{f_2}{f_1+f_2}d^{02}d^{21}\right)^{-1}+f_2\left(\frac{f_1}{f_1+f_2}d^{01}(d^{21})^{-1}+\frac{f_2}{f_1+f_2}d^{02}\right)^{-1}\right]^{-1},
\end{aligned}
\tag{2}
$$

with:

$$
\begin{aligned}
m^{21} &= \frac{4\mu_2+3k_1}{4\mu_2+3k_2} \text{ and } d^{21} = \frac{3k_2(2\mu_1+3\mu_2)+4\mu_2(3\mu_1+2\mu_2)}{5\mu_2(3k_2+4\mu_2)},\\
m^{01} &= \frac{4\mu_0+3k_1}{4\mu_0+3k_0} \text{ and } d^{01} = \frac{3k_0(2\mu_1+3\mu_0)+4\mu_0(3\mu_1+2\mu_0)}{5\mu_0(3k_0+4\mu_0)},\\
m^{02} &= \frac{4\mu_0+3k_2}{4\mu_0+3k_0} \text{ and } d^{02} = \frac{3k_0(2\mu_2+3\mu_0)+4\mu_0(3\mu_2+2\mu_0)}{5\mu_0(3k_0+4\mu_0)},
\end{aligned}
\tag{3}
$$

such that according to eq.(40), $\mathbf{w}^{(2/1)} = m^{21}\mathbf{J}+d^{21}\mathbf{K}$, $\mathbf{w}^{(0/1)} = m^{01}\mathbf{J}+d^{01}\mathbf{K}$,

and, $\mathbf{w}^{(0/2)} = m^{02}\mathbf{J} + d^{02}\mathbf{K}$.

Using again eq.(61) applied to eqs.(39),(40),(41),(42),(43) allows us to find the following expressions of n^1, n^2, n^0. These ones represent the thermal contributions present in eqs.(69) to determine the concentration tensors $\boldsymbol{\beta}^1$, $\boldsymbol{\beta}^2$, $\boldsymbol{\beta}^0$. Their expressions are found to be:

$$\begin{aligned} n^1 &= m^1\left(f_2 e^{21} + \frac{f_0}{f_1+f_2}\left(f_1 e^{01} + f_2\left(m^{02}e^{21} + e^{02}\right)\right)\right), \\ n^2 &= m^{21}n^1 - e^{21}, \\ n^0 &= \left(\frac{f_1}{f_1+f_2}\left(m^{01}n^1 - e^{01}\right) + \frac{f_2}{f_1+f_2}\left(m^{02}n^2 - e^{02}\right)\right), \end{aligned} \tag{4}$$

with:

$$\begin{aligned} e^{21} &= \frac{-3\left(k_2\alpha_2 - k_1\alpha_1\right)}{3k_2 + 4\mu_2}, \\ e^{01} &= \frac{-3\left(k_0\alpha_0 - k_1\alpha_1\right)}{3k_0 + 4\mu_0}, \\ e^{02} &= \frac{-3\left(k_0\alpha_0 - k_2\alpha_2\right)}{3k_0 + 4\mu_0}, \end{aligned} \tag{5}$$

where $\mathbf{v}^{(2/1)} = e^{21}\mathbf{I_2}\Theta$, $\mathbf{v}^{(0/1)} = e^{01}\mathbf{I_2}\Theta$, and, $\mathbf{v}^{(0/2)} = e^{02}\mathbf{I_2}\Theta$ following eq.(40).

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